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Compositional models and conditional independence in evidence theory

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ABSTRACT

The goal of the paper is twofold. The first is to show that some of the ideas for representation of multidimensional distributions in probability and possibility theories can be transferred into evidence theory. Namely, we show that multidimensional basic assignments can be rather efficiently represented in a form of so-called compositional models. These models are based on the iterative application of the operator of composition, whose definition for basic assignments as well as its properties are presented. We also prove that the operator of composition in evidence theory is in a sense generalization of its probabilistic counterpart.

The second goal of the paper is to introduce a new definition of conditional independence in evidence theory and to show in what sense it is superior to that formerly introduced by other authors.

1. Introduction—motivation

Any application of AI models to problems of practice must cope with two basic issues: uncertainty and multidimensionality. In present time we can say that a "classical" solution to these problems is offered by *probabilistic graphical Markov models*. In these models, the problem of multidimensionality is solved with the help of the notion of conditional independence, which enables factorization of a multidimensional probability distribution into small parts, usually marginal or conditional low-dimensional distributions (or generally into low-dimensional factors). Such a factorization not only decreases the storage requirements for representation of a multidimensional distribution but it usually also induces possibility to employ efficient computational procedures.

About 10 years ago, as an alternative to graphical Markov models we introduced *compositional models* [12], in which multidimensional probability distributions were assembled from a system of low-dimensional ones by application of a special operator of composition. Later we introduced compositional models also within the framework of possibility theory [22], which meant that we had to define an operator of composition for possibilistic distributions as well. Naturally, (computational) efficiency of all these models also takes advantage of properties of conditional independence.

The research results presented in the current paper were motivated by Didier Dubois, who asked us once whether it was possible to define an operator of composition for belief functions as well. The importance of such a question is apparent. It is enough to realize the fact that we need efficient methods for representation of probabilistic and possibilistic distributions, which require an exponential number of parameters. Thus, we have an even greater need of efficient methods for representation of a belief function, which cannot be represented by a point function (distribution). For such a representation we need a set function, and thus its space requirements are superexponential. To avoid these problems, several techniques have been developed in the past [3,20]. In this context we have to keep in mind that while multidimensionality in probability and

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possibility theories usually involves hundreds of variables, for belief functions several tens of variables bring enormous computational problems.

In the paper we define the operator of composition for basic assignments (Section 2), study its basic properties (Section 3), and describe how it can be used to represent multidimensional basic assignments (Section 5). In agreement with the fact that in probability and possibility theories the operator of composition is closely connected with the notion of conditional independence, its definition for basic assignments inspired us to revise the notion of conditional independence in evidence theory (Section 4). The paper is concluded with results showing the relationship between operators of composition in probability and evidence theories (Section 6).

2. Basic notions

The aim of this section is to introduce a notation and briefly overview basic notions from evidence theory. Its last part is devoted to the definition of the operator of composition.

2.1. Set notation

For an index set $N = \{1, 2, \ldots, n\}$ let $\{X_i\}_{i \in N}$ be a system of variables, each X_i having its values in a finite set $\mathbf{X}_i$. In this paper we will deal with *multidimensional frame of discernment*

$$\mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_n,$$

and its *subframes* (for $K \subseteq N$)

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i.$$

When dealing with groups of variables on these subframes, X_K will denote a group of variables $\{X_i\}_{i \in K}$ throughout the paper.

A *projection* of $x = (x_1, x_2, \ldots, x_n) \in \mathbf{X}_N$ into $\mathbf{X}_K$ will be denoted $x^{\downarrow K}$, i.e. for $K = \{i_1, i_2, \ldots, i_k\}$

$$x^{\downarrow K} = (x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \in \mathbf{X}_K.$$

Analogously, for $M \subset K \subseteq N$ and $A \subset \mathbf{X}_K$, $A^{\downarrow M}$ will denote a *projection* of A into $\mathbf{X}_M$[1]:

$$A^{\downarrow M} = \{y \in \mathbf{X}_M | \exists x \in A : y = x^{\downarrow M}\}.$$

In addition to the projection, in this text we will need also an opposite operation, which will be called a join. By a *join*[2] of two sets $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L (K, L \subseteq N)$ we will understand a set

$$A \bowtie B = \{x \in \mathbf{X}_{K \cup L} : x^{\downarrow K} \in A \;\&\; x^{\downarrow L} \in B\}.$$

Let us note that if K and L are disjoint, then

$$A \bowtie B = A \times B,$$

and if $K = L$

$$A \bowtie B = A \cap B.$$

In view of this paper it is important to realize that if $x \in A \bowtie B$ (assuming still that $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ it means that $x \in \mathbf{X}_{K \cup L}$) then $x^{\downarrow K} \in A$ and $x^{\downarrow L} \in B$ (and, naturally, $x^{\downarrow K \cap L} \in A^{\downarrow K \cap L} \cap B^{\downarrow K \cap L}$). However, and it is important to keep this in mind, it does not mean, analogous to Cartesian product, that for $C \subseteq \mathbf{X}_{K \cup L}$ it holds that $C = C^{\downarrow K} \bowtie C^{\downarrow L}$. In this case, from the mentioned properties one can immediately see that generally $C \subseteq C^{\downarrow K} \bowtie C^{\downarrow L}$. For example, considering for $i = 1, 2, 3$, $\mathbf{X}_i = \{a_i, \bar{a}_i\}$ and $C = \{a_1 a_2 a_3, \bar{a}_1 a_2 a_3, a_1 a_2 \bar{a}_3\}$ one gets

$$C^{\downarrow \{1,2\}} \bowtie C^{\downarrow \{2,3\}} = \{a_1 a_2, \bar{a}_1 a_2\} \bowtie \{a_2 a_3, a_2 \bar{a}_3\} = \{a_1 a_2 a_3, \bar{a}_1 a_2 a_3, a_1 a_2 \bar{a}_3, \bar{a}_1 a_2 \bar{a}_3\} \supsetneq C.$$

Let us mention that the sets $C \subseteq \mathbf{X}_{K \cup L}$ for which $C = C^{\downarrow K} \bowtie C^{\downarrow L}$ are called *Z-layered rectangles* (for $Z = \mathbf{X}_{K \cap L}$) in [4].

2.2. Independence in evidence theory

In evidence theory (or Dempster–Shafer theory) two measures are used to model the uncertainty: belief and plausibility measures. Both of them can be defined with the help of another set function called a *basic* (*probability* or *belief*) *assignment* m on $\mathbf{X}_N$, i.e.

$$m : \mathscr{P}(\mathbf{X}_N) \to [0, 1],$$

---

[1] Let us remark that we do not exclude situations when $M = \emptyset$. In this case $A^{\downarrow \emptyset} = \emptyset$.

[2] This term and notation are taken from the theory of relational databases [2].

where $\mathscr{P}(\mathbf{X}_N)$ is power set of $\mathbf{X}_N$ and $\sum_{A\subseteq \mathbf{X}_N} m(A)=1$. Furthermore, we assume that $m(\emptyset)=0$. A set $A\in\mathscr{P}(X_N)$ is a *focal element* if $m(A)>0$.

In addition to *belief* and *plausibility* measures, which will not be discussed in this paper, also *commonality function* can be obtained from basic assignment m:

$$Q(A)=\sum_{B\subseteq \mathbf{X}_N: A\subseteq B} m(B).$$

This notion plays an important role in the definition of so-called (conditional) non-interactivity of variables.

For a basic assignment m on $\mathbf{X}_K$ and $M\subset K$ a *marginal basic assignment* of m is defined (for each $B\subseteq \mathbf{X}_M$):

$$m^{\downarrow M}(A)=\sum_{B\subseteq \mathbf{X}_K: B^{\downarrow M}=A} m(B).$$

Analogously, $Q^{\downarrow M}$ will denote the respective marginal commonality function.

Having two basic assignments m_1 and m_2 on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively, $(K,L\subseteq N)$, we say that these assignments are *projective* if

$$m_1^{\downarrow K\cap L}=m_2^{\downarrow K\cap L},$$

which occurs if and only if there exists a basic assignment m on $\mathbf{X}_{K\cup L}$ such that both m_1 and m_2 are marginal assignments of m.

Let us close this section by recalling the notion of independence.[3]

Definition 1. Let m be a basic assignment on $\mathbf{X}_M$ and $K,L\subset M$ be disjoint. We say that groups of variables X_K and X_L are *independent with respect to basic assignment m* (in notation $K\perp\!\!\!\perp L\ [m]$) if

$$m^{\downarrow K\cup L}(A)=m^{\downarrow K}(A^{\downarrow K})\cdot m^{\downarrow L}(A^{\downarrow L}) \tag{1}$$

for all $A\subseteq \mathbf{X}_{K\cup L}$ for which $A=A^{\downarrow K}\times A^{\downarrow L}$, and $m(A)=0$ otherwise.

Lemma 1. *Let $K,\ L$ be disjoint, then $K\perp\!\!\!\perp L\ [m]$ iff $Q^{\downarrow K\cup L}(A)=Q^{\downarrow K}(A^{\downarrow K})\cdot Q^{\downarrow L}(A^{\downarrow L})$ for all $A\subseteq \mathbf{X}_{K\cup L}$.*

Proof. First assume $K\perp\!\!\!\perp L\ [m]$ and compute for any $A\subseteq \mathbf{X}_{K\cup L}$

$$\begin{aligned}Q^{\downarrow K\cup L}(A)&=\sum_{B\subseteq \mathbf{X}_{K\cup L}:A\subseteq B} m^{\downarrow K\cup L}(B)=\sum_{\substack{B\subseteq \mathbf{X}_{K\cup L}:A\subseteq B\\ B=B^{\downarrow K}\times B^{\downarrow L}}} m^{\downarrow K\cup L}(B)=\sum_{C\subseteq \mathbf{X}_K:A^{\downarrow K}\subseteq C}\ \sum_{D\subseteq \mathbf{X}_L:A^{\downarrow L}\subseteq D} m^{\downarrow K}(C)\cdot m^{\downarrow L}(D)\\ &=\left(\sum_{C\subseteq \mathbf{X}_K:A^{\downarrow K}\subseteq C} m^{\downarrow K}(C)\right)\left(\sum_{D\subseteq \mathbf{X}_L:A^{\downarrow L}\subseteq D} m^{\downarrow L}(D)\right)=Q^{\downarrow K}(A^{\downarrow K})\cdot Q^{\downarrow L}(A^{\downarrow L}),\end{aligned}$$

which finishes the first part of the proof.

Now assume that $Q^{\downarrow K\cup L}(A)=Q(A^{\downarrow K})\cdot Q(A^{\downarrow L})$ for all $A\subseteq \mathbf{X}_{K\cup L}$. To show that $K\perp\!\!\!\perp L\ [m]$ we have to show that $m(A)=0$ for all $A\neq A^{\downarrow K}\times A^{\downarrow L}$ and that for $A=A^{\downarrow K}\times A^{\downarrow L}$ equality (1) holds. So, first, assume that $A\neq A^{\downarrow K}\times A^{\downarrow L}$, then (in the following computations we use only definition of the commonality function and the trivial fact that $A\subseteq A^{\downarrow K}\times A^{\downarrow L}$)

$$\begin{aligned}m^{\downarrow K\cup L}(A)&=\sum_{B\subseteq \mathbf{X}_{K\cup L}:A\subseteq B} m^{\downarrow K\cup L}(B)-\sum_{B\subseteq \mathbf{X}_{K\cup L}:A\subsetneq B} m^{\downarrow K\cup L}(B)=Q^{\downarrow K\cup L}(A)-\sum_{B\subseteq \mathbf{X}_{K\cup L}:A\subsetneq B} m^{\downarrow K\cup L}(B)\\ &\leqslant Q^{\downarrow K\cup L}(A)-\sum_{B\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}\times A^{\downarrow L}\subseteq B} m^{\downarrow K\cup L}(B)=Q^{\downarrow K\cup L}(A)-Q^{\downarrow K\cup L}(A^{\downarrow K}\times A^{\downarrow L})=Q^{\downarrow K\cup L}(A)-Q^{\downarrow K\cup L}(A^{\downarrow K})\cdot Q(A^{\downarrow L})=0.\end{aligned}$$

Since it is clear that

$$m^{\downarrow K\cup L}(\mathbf{X}_{K\cup L})=m^{\downarrow K\cup L}(\mathbf{X}_K\times\mathbf{X}_L)=Q^{\downarrow K\cup L}(\mathbf{X}_K\times\mathbf{X}_L)=Q^{\downarrow K}(\mathbf{X}_K)\cdot Q^{\downarrow L}(\mathbf{X}_L)=m^{\downarrow K}(\mathbf{X}_K)\cdot m^{\downarrow L}(\mathbf{X}_L),$$

we can prove validity of equality (1) for all A for which $A=A^{\downarrow K}\times A^{\downarrow L}$ by mathematical induction. Consider such a set A and assume that validity of equality (1) has already been proven for all $B\subseteq \mathbf{X}_{K\cup L}$, for which $B=B^{\downarrow K}\times B^{\downarrow L}$ and $|B|>|A|$. Now we can compute

[3] Couso et al. [5] call this independence *independence in random sets*, Klir [15] *non-interactivity*.

$$
\begin{aligned}
m^{\downarrow K\cup L}(A) &= \sum_{B\subseteq \mathbf{X}_{K\cup L}: A\subseteq B} m^{\downarrow K\cup L}(B) - \sum_{B\subseteq \mathbf{X}_{K\cup L}: A\subsetneqq B} m^{\downarrow K\cup L}(B) = Q^{\downarrow K\cup L}(A) - \sum_{\substack{B\subseteq \mathbf{X}_{K\cup L}: A\subsetneqq B\\ B=B^{\downarrow K}\times B^{\downarrow L}}} m^{\downarrow K\cup L}(B)\\
&= Q^{\downarrow K}(A^{\downarrow K})\cdot Q^{\downarrow L}(A^{\downarrow L}) - \sum_{C\subseteq \mathbf{X}_K: A^{\downarrow K}\subsetneqq C}\ \sum_{D\subset \mathbf{X}_L: A^{\downarrow L}\subsetneqq D} m^{\downarrow K\cup L}(C\times D) - \sum_{C\subseteq \mathbf{X}_K: A^{\downarrow K}\subsetneqq C} m^{\downarrow K\cup L}(C\times A^{\downarrow L}) - \sum_{D\subset \mathbf{X}_L: A^{\downarrow L}\subsetneqq D} m^{\downarrow K\cup L}(A^{\downarrow K}\times D)\\
&= \left(\sum_{C\subseteq \mathbf{X}_K: A^{\downarrow K}\subseteq C} m^{\downarrow K}(C)\right)\left(\sum_{D\subset \mathbf{X}_L: A^{\downarrow L}\subseteq D} m^{\downarrow L}(D)\right) - \sum_{C\subseteq \mathbf{X}_K: A^{\downarrow K}\subsetneqq C}\ \sum_{D\subset \mathbf{X}_L: A^{\downarrow L}\subsetneqq D} m^{\downarrow K}(C)\cdot m^{\downarrow L}(D) - \sum_{C\subseteq \mathbf{X}_K: A^{\downarrow K}\subsetneqq C} m^{\downarrow K}(C)\cdot m^{\downarrow L}(A^{\downarrow L})\\
&\quad - \sum_{D\subset \mathbf{X}_L: A^{\downarrow L}\subsetneqq D} m^{\downarrow K}(A^{\downarrow K})\cdot m^{\downarrow L}(D)\\
&= m^{\downarrow K}(A^{\downarrow K})\cdot m^{\downarrow L}(A^{\downarrow L}). \qquad \square
\end{aligned}
$$

Let us note that there exist numerous independence concepts within the broader framework of imprecise probabilities, e.g. epistemic irrelevance, epistemic independence or strong independence [5,7,17], but their application usually leads to models beyond the framework of evidence theory (cf., e.g. [24]).

2.3. Operator of composition

Let K and L be two subsets of N. At this moment we do not pose any restrictions on K and L; they may be but need not be disjoint, one may be a subset of the other. We even admit that one or both of them are empty.[4] Let m_1 and m_2 be basic assignments on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively.

Our goal is to define a new basic assignment on $\mathbf{X}_{K\cup L}$, denoted $m_1 \triangleright m_2$, which will contain all of the information contained in m_1 and as much as possible of information of m_2 (for the exact meaning see properties (ii) and (iii) of Lemma 2). The required property is met by the following definition.

Definition 2. For two arbitrary basic assignments m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ a *composition* $m_1 \triangleright m_2$ is defined for all $C \subseteq \mathbf{X}_{K\cup L}$ by one of the following expressions:

[a] if $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) > 0$ and $C = C^{\downarrow K} \bowtie C^{\downarrow L}$ then

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K})\cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})};$$

[b] if $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) = 0$ and $C = C^{\downarrow K} \times \mathbf{X}_{L\setminus K}$ then

$$(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow K});$$

[c] in all other cases

$$(m_1 \triangleright m_2)(C) = 0.$$

Remark. Notice what this definition yields in the following simple special situations:

- if $K \cap L = \emptyset$ then

$$m_1 \triangleright m_2(C) = m_1(C^{\downarrow K})\cdot m_2(C^{\downarrow L})$$

  for $C = C^{\downarrow K} \times C^{\downarrow L}$, and $m_1 \triangleright m_2(C) = 0$ otherwise; i.e. $m_1 \triangleright m_2(C)$ is basic assignment of independent groups of variables X_K and X_L (see Definition 1);
- if $K \supseteq L$ then $m_1 \triangleright m_2 = m_1$.

Let us finish this section with a simple example illustrating the application of the particular cases of Definition 2.

Example 1. Let for $i = 1,2,3,\ \mathbf{X}_i = \{a_i, \bar{a}_i\}$ and let us consider the following basic assignments m_1 and m_2 on $\mathbf{X}_1 \times \mathbf{X}_2$ and $\mathbf{X}_2 \times \mathbf{X}_3$, respectively:

[4] Notice that basic assignment m on $\mathbf{X}_\emptyset$ is defined $m(\emptyset) = 1$. Let us note that this is the only case when we accept $m(\emptyset) > 0$, otherwise $m(\emptyset) = 0$ according to the classical definitions of basic assignment, see [19].

Table 1
Composed basic assignments.

A	$(m_1 \triangleright m_2)(A)$	$(m_2 \triangleright m_1)(A)$
$\mathbf{X}_1 \times \mathbf{X}_2 \times \{a_3\}$	0.3	0.5
$\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$	0.3	0.5
$\mathbf{X}_1 \times \{a_2\} \times \mathbf{X}_3$	0.4	0

$$m_1(\mathbf{X}_1 \times \{a_2\}) = 0.4,$$
$$m_1(\mathbf{X}_1 \times \mathbf{X}_2) = 0.6,$$
$$m_2(\mathbf{X}_2 \times \{a_3\}) = 0.5,$$
$$m_2(\mathbf{X}_2 \times \mathbf{X}_3) = 0.5$$

(the values of both basic assignments m_1 and m_2 on the remaining subsets being zero). From Definition 2 one can immediately see that formula in case [a] can assign a positive value to $(m_1 \triangleright m_2)(A)$ and/or $(m_2 \triangleright m_1)(A)$ only for those $A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ for which

$$A^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \{a_2\} \quad \text{or } A^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \mathbf{X}_2,$$

and

$$A^{\downarrow\{2,3\}} = \mathbf{X}_2 \times \{a_3\} \quad \text{or } A^{\downarrow\{2,3\}} = \mathbf{X}_2 \times \mathbf{X}_3.$$

There are only two such sets

$$\mathbf{X}_1 \times \mathbf{X}_2 \times \{a_3\} \quad \text{and} \quad \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3.$$

For these sets we get

$$(m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{a_3\}) = \frac{m_1(\mathbf{X}_1 \times \mathbf{X}_2) \cdot m_2(\mathbf{X}_2 \times \{a_3\})}{m_2^{\downarrow\{2\}}(\mathbf{X}_2)} = \frac{0.6 \cdot 0.5}{1} = 0.3,$$
$$(m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) = \frac{m_1(\mathbf{X}_1 \times \mathbf{X}_2) \cdot m_2(\mathbf{X}_2 \times \mathbf{X}_3)}{m_2^{\downarrow\{2\}}(\mathbf{X}_2)} = \frac{0.6 \cdot 0.5}{1} = 0.3$$

and similarly

$$(m_2 \triangleright m_1)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{a_3\}) = \frac{m_2(\mathbf{X}_2 \times \{a_3\}) \cdot m_1(\mathbf{X}_1 \times \mathbf{X}_2)}{m_1^{\downarrow\{2\}}(\mathbf{X}_2)} = \frac{0.5 \cdot 0.6}{0.6} = 0.5,$$
$$(m_2 \triangleright m_1)(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) = \frac{m_2(\mathbf{X}_2 \times \mathbf{X}_3) \cdot m_1(\mathbf{X}_1 \times \mathbf{X}_2)}{m_1^{\downarrow\{2\}}(\mathbf{X}_2)} = \frac{0.5 \cdot 0.6}{0.6} = 0.5.$$

From case [b] of Definition 2 we will get yet another focal element for $m_1 \triangleright m_2$, namely

$$A = \mathbf{X}_1 \times \{a_2\} \times \mathbf{X}_3,$$

for which

$$A^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \{a_2\} \quad \text{and} \quad A^{\downarrow\{3\}} = \mathbf{X}_3.$$

For this set, since $m_2^{\downarrow\{2\}}\ (A^{\downarrow\{2\}}) = 0$ and $A^{\downarrow\{3\}} = \mathbf{X}_3$, we get

$$(m_1 \triangleright m_2)(\mathbf{X}_1 \times \{a_2\} \times \mathbf{X}_3) = m_1(\mathbf{X}_1 \times \{a_2\}) = 0.4.$$

Notice that when computing a composition $m_2 \triangleright m_1$, case [b] of Definition 2 does not assign a positive value to any subset A of $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ since if $m_2^{\downarrow\{2\}}(A^{\downarrow\{2\}}) > 0$ then also $m_1^{\downarrow\{2\}}(A^{\downarrow\{2\}}) > 0$.

Both the composed basic assignments $m_1 \triangleright m_2$ and $m_2 \triangleright m_1$ are outlined in Table 1 (recall once more that for all other $A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ different from those included in Table 1 both assignments equal 0). It is also evident from the table, that the operator $\triangleright$ is not commutative.

3. Basic properties of the operator

3.1. Background

First of all it is necessary to realize that the operator of composition substantially differs from any other rule of combination published before. Whether considering classical Dempster's rule of combination [9,14] or its cautious versions introduced by Dubois et al. [11] or Denœux [10], they, in a way, describe the process of combination of information sources, for which the term *information fusion* is usually used. In contrast to this, the composition introduced in the previous section describes the process, when global model is assembled from a number of local models. So it rather corresponds to what is often called *knowledge integration*.

Before presenting formal properties of the operator of composition let us present (based on a legitime request of the anonymous referees) some ideas in the background. Definition 2 was inspired by the simple formula defining the operator of composition in probability theory [12]

$$p(x,y) \triangleright q(y,z) = p(x,y) \cdot q(z|y) = \frac{p(x,y) \cdot q(y,z)}{q(y)}.$$

It should be stressed, however, that this ratio had been used in many formulae of probability theory before. Let us mention here at least two of its applications clarifying the properties of the operator.

In most of basic textbooks on probability theory there appears a simple formula (by some of authors called a *chain rule*)

$$p(x_1,x_2,x_3,\ldots,x_s) = p(x_1) \cdot p(x_2|x_1) \cdot p(x_3|x_1,x_2) \cdots p(x_s|x_1,\ldots,x_{s-1}),$$

which can be generalized for an arbitrary partition $I_1,I_2,\ldots,I_r$ of the index set $\{1,2,\ldots,s\}$ as

$$p(x_1,x_2,x_3,\ldots,x_s) = p(x_i)_{i\in I_1} \cdot p((x_i)_{i\in I_2}|(x_i)_{i\in I_1}) \cdot p((x_i)_{i\in I_3}|(x_i)_{i\in I_1\cup I_2}) \cdots p((x_i)_{i\in I_r}|(x_i)_{i\in I_1\cup\cdots\cup I_{r-1}}),$$

or, using the operator of composition

$$p(x_1,x_2,x_3,\ldots,x_s) = p(x_i)_{i\in I_1} \triangleright p((x_i)_{i\in I_1\cup I_2}) \triangleright p((x_i)_{i\in I_1\cup I_2\cup I_3}) \triangleright \cdots \triangleright p((x_i)_{i\in I_1\cup\cdots\cup I_r}).$$

Considering a partition $I_1,I_2,\ldots,I_r$ of the index set $\{1,2,\ldots,s\}$ and a system of index sets $J_1,J_2,\ldots,J_r$ such that for all $j=1,\ldots,r$

$$I_j \subseteq J_j \subseteq I_1 \cup \cdots \cup I_j,$$

Perez introduced in [18] an approximation (so-called *dependence structure simplification* approximation) by the formula that can be written using the operator $\triangleright$ in the form

$$p(x_1,x_2,x_3,\ldots,x_s) = p((x_i)_{i\in J_1}) \triangleright p((x_i)_{i\in J_2}) \triangleright p((x_i)_{i\in J_3}) \triangleright \cdots \triangleright p((x_i)_{i\in J_r}).$$

In the cited paper, when studying properties of these approximations, he took advantage of the fact that they have a specific dependence structure following from one of the basic properties of the operator of composition (expressed here in its simplest form):

for probability distribution $p(x,y) \triangleright q(y,z)$, variables X and Z are conditionally independent given Y.

Another field of application of the studied operator appears when one needs to get a projection of a distribution $p(x,y)$ into a set of distributions with a given marginal $q(x)$, i.e. when one needs to find a distribution from the set

$$\{\hat{p}(x,y) : \hat{p}(x) = q(x)\}$$

as close as possible to $p(x,y)$. It was shown by Csiszár [6] that when measuring the distance of distributions with the help of Kullback–Leibler divergence then the required projection is exactly $q(x) \triangleright p(x,y)$. This fact was intuitively exploited as early as in 1940 in Iterative Proportional Fitting Procedure by Deming and Stephan [8]. All this led us to another requirement concerning the composition: it should preserve the first operand.

3.2. Formal properties

Lemma 2. *For arbitrary two basic assignments m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ the following properties hold true:*

(i) *$m_1 \triangleright m_2$ is a basic assignment on $\mathbf{X}_{K\cup L}$;*
(ii) *$(m_1 \triangleright m_2)^{\downarrow K} = m_1$;*
(iii) *$m_1 \triangleright m_2 = m_2 \triangleright m_1 \iff m_1^{\downarrow K\cap L} = m_2^{\downarrow K\cap L}$;*
(iv) *if $K \subseteq L$ then $m_2^{\downarrow K} \triangleright m_2 = m_2$.*

Proof. ad (ii). To prove equality (ii) we have to prove that for any $B \subseteq \mathbf{X}_K$

$$\sum_{A\subseteq \mathbf{X}_{K\cup L}: A^{\downarrow K}=B} (m_1 \triangleright m_2)(A) = m_1(B). \tag{2}$$

Since, due to Definition 2, $(m_1 \triangleright m_2)(C) = 0$ for any $C \subseteq \mathbf{X}_{K\cup L}$, for which $C \neq C^{\downarrow K} \bowtie C^{\downarrow L}$, we see that

$$\sum_{A\subseteq \mathbf{X}_{K\cup L}: A^{\downarrow K}=B} (m_1 \triangleright m_2)(A) = \sum_{\substack{A\subseteq \mathbf{X}_{K\cup L}: A^{\downarrow K}=B \\ A=A^{\downarrow K}\bowtie A^{\downarrow L}}} (m_1 \triangleright m_2)(A) = \sum_{\substack{C\subseteq \mathbf{X}_L \\ C^{\downarrow K\cap L}=B^{\downarrow K\cap L}}} (m_1 \triangleright m_2)(B \bowtie C).$$

To prove formula (2), we have to distinguish two situations depending on the value of $m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})$. If this value is positive then

$$\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B}(m_1 \triangleright m_2)(A) = \sum_{\substack{C\subseteq \mathbf{X}_L\\ C^{\downarrow K\cap L}=B^{\downarrow K\cap L}}} \frac{m_1(B)\cdot m_2(C)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})} = \frac{m_1(B)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})}\sum_{\substack{C\subseteq \mathbf{X}_L\\ C^{\downarrow K\cap L}=B^{\downarrow K\cap L}}} m_2(C) = \frac{m_1(B)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})} m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L}) = m_1(B).$$

If $m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L}) = 0$ then, according to Definition 2 (case [b]), there exists only one $A\subseteq \mathbf{X}_{K\cup L}$ for which $A^{\downarrow K}=B$ and $(m_1 \triangleright m_2)(A)$ may be positive; namely $A = B\times \mathbf{X}_{L\setminus K}$. Therefore

$$\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B}(m_1 \triangleright m_2)(A) = (m_1 \triangleright m_2)(B\times \mathbf{X}_{L\setminus K}) = m_1(B).$$

ad (i). To prove that $m_1 \triangleright m_2$ is a basic assignment on $\mathbf{X}_{K\cup L}$ we have to show that for each $A\subseteq \mathbf{X}_{K\cup L}$ value $(m_1 \triangleright m_2)(A)$ is non-negative (which is evident) and that the sum of all these values equals 1. The latter holds true, too, because (using equality (2))

$$\sum_{A\subseteq \mathbf{X}_{K\cup L}}(m_1 \triangleright m_2)(A) = \sum_{B\subseteq \mathbf{X}_K}\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B}(m_1 \triangleright m_2)(A) = \sum_{B\subseteq \mathbf{X}_K} m_1(B) = 1.$$

ad (iii). Let us first assume that $m_1^{\downarrow K\cap L} = m_2^{\downarrow K\cap L}$. To prove that $m_1 \triangleright m_2 = m_2 \triangleright m_1$ consider an arbitrary $A\subseteq \mathbf{X}_{K\cup L}$. If $A \neq A^{\downarrow K}\bowtie A^{\downarrow L}$ then both $(m_1 \triangleright m_2)(A)$ and $(m_2 \triangleright m_1)(A)$ equal 0 (due to Definition 2). Therefore we have to prove the equality only for $A = A^{\downarrow K}\bowtie A^{\downarrow L}$.

If $m_1^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L}) > 0$ then

$$(m_1 \triangleright m_2)(A) = \frac{m_1(A^{\downarrow K})\cdot m_2(A^{\downarrow L})}{m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L})} = \frac{m_1(A^{\downarrow K})\cdot m_2(A^{\downarrow L})}{m_1^{\downarrow K\cap L}(A^{\downarrow K\cap L})} = (m_2 \triangleright m_1)(A).$$

If $m_1^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = 0$, then both $m_1(A^{\downarrow K})$ and $m_2(A^{\downarrow L})$ must equal 0 and therefore (according to [b] of Definition 2) $(m_1 \triangleright m_2)(A) = (m_2 \triangleright m_1)(A) = 0$.

To prove the other side of the equivalence (i.e. $m_1 \triangleright m_2 = m_2 \triangleright m_1$ implies $m_1^{\downarrow K\cap L} = m_2^{\downarrow K\cap L}$) it is enough to realize that if $m_1^{\downarrow K\cap L} \neq m_2^{\downarrow K\cap L}$ then also $m_1 \triangleright m_2 \neq m_2 \triangleright m_1$ because, due to already proven item (ii) of this assertion, $m_1^{\downarrow K\cap L} = (m_1 \triangleright m_2)^{\downarrow K\cap L}$ and $m_2^{\downarrow K\cap L} = (m_2 \triangleright m_1)^{\downarrow K\cap L}$.

ad (iv). This property follows directly from previously proven items (iii) and (ii). □

For a binary operator, natural questions arises: is this operator commutative, associative and idempotent? The answers to these questions for operator of composition are simple, based on the properties proven in Lemma 2. From properties (i) and (ii) it follows that the operator is idempotent. On the other hand, from Example 1 one can immediately see that this operator is not commutative. However, and it should be stressed, property (iii) says that the operator is commutative for projective basic assignments.

How is it with the associativity of the operator of composition? As it is shown in the following simple example, generally the operator is not associative. However, similarly to commutativity, there are special situations under which the operator becomes associative.

Example 2. Let X_1 and X_2 be two variables with values in $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, $\mathbf{X}_i = \{a_i, \bar{a}_i\},\ i = 1, 2$, and let $m_1,\ m_2$ and m_3 be three basic assignments on $\mathbf{X}_1,\ \mathbf{X}_2$ and $\mathbf{X}_1\times\mathbf{X}_2$, respectively, defined as follows:

$$\begin{aligned}
&m_1(\{a_1\}) = 0.5,\\
&m_1(\mathbf{X}_1) = 0.5,\\
&m_2(\{a_2\}) = 0.5,\\
&m_2(\mathbf{X}_2) = 0.5,
\end{aligned}$$

and

$$\begin{aligned}
&m_3(\{a_1, a_2\}) = 0.5,\\
&m_3(\mathbf{X}_1\times\mathbf{X}_2) = 0.5.
\end{aligned}$$

Due to (i) and (ii) of Lemma 2 one has

$$(m_1 \triangleright m_2) \triangleright m_3 = m_1 \triangleright m_2,$$

and therefore

$$\begin{aligned}
&(m_1 \triangleright m_2) \triangleright m_3(\{a_1, a_2\}) = 0.25,\\
&(m_1 \triangleright m_2) \triangleright m_3(\{a_1\}\times\mathbf{X}_2) = 0.25,\\
&(m_1 \triangleright m_2) \triangleright m_3(\mathbf{X}_1\times\{a_2\}) = 0.25,\\
&(m_1 \triangleright m_2) \triangleright m_3(\mathbf{X}_1\times\mathbf{X}_2) = 0.25.
\end{aligned}$$

On the other hand (see that all the focal elements of $m_1 \triangleright m_2$ are computed with the help of case [a] of Definition 2)

$$m_2 \triangleright m_3(\{a_1, a_2\}) = 0.5,$$
$$m_2 \triangleright m_3(\mathbf{X}_1 \times \mathbf{X}_2) = 0.5,$$

and therefore

$$m_1 \triangleright (m_2 \triangleright m_3)(\{a_1, a_2\}) = 0.5,$$
$$m_1 \triangleright (m_2 \triangleright m_3)(\mathbf{X}_1 \times \mathbf{X}_2) = 0.5,$$

i.e., operator $\triangleright$ is not associative.

To illustrate special cases under which the associativity holds for the operator of composition let us present the following assertion (since we do not need it in this text we do not prove it here).

Lemma 3. *Let $m_1,\ m_2,\ m_3$ be basic assignments on $\mathbf{X}_K,\ \mathbf{X}_L$ and $\mathbf{X}_M$, respectively, such that for all $C \subseteq \mathbf{X}_{K\cup L\cup M}$*

$$m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) \iff m_3^{\downarrow K\cap M}(C^{\downarrow K\cap M}).$$

If $K \supseteq L \cap M$ then

$$(m_1 \triangleright m_2) \triangleright m_3 = m_1 \triangleright (m_2 \triangleright m_3).$$

4. Conditional independence

Before starting a deeper study of the concept of conditional independence in this section, let us stress that it is a crucial notion in most approaches to multidimensional modeling. As it was suggested in the remark in Section 2.3, in the case when basic assignments are defined on non-overlapping subframes of discernment their composition is a basic assignment of independent (and also non-interactive [15]) groups of variables. More precisely, for m_1 and m_2 defined on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively, if $K \cap L = \emptyset$ then $K \perp\!\!\!\perp L\ [m_1 \triangleright m_2]$. In this section we will deal with two generalizations of this concept.

Ben Yaghlane et al. [4] generalized the notion of non-interactivity in the following way: Let m be a basic assignment on $\mathbf{X}_N$ and $K, L, M \subset N$ be disjoint, $K \neq \emptyset \neq L$. Groups of variables X_K and X_L are *conditionally non-interactive*[5] *given X_M with respect to m* if and only if the equality

$$Q^{\downarrow K\cup L\cup M}(A) \cdot Q^{\downarrow M}(A^{\downarrow M}) = Q^{\downarrow K\cup M}(A^{\downarrow K\cup M}) \cdot Q^{\downarrow L\cup M}(A^{\downarrow L\cup M}) \tag{3}$$

holds for any $A \subseteq \mathbf{X}_{K\cup L\cup M}$.

The cited authors proved [4] that conditional non-interactivity satisfies the so-called semigraphoid properties,[6] usually taken as sound properties of a conditional independence relation.

Nevertheless, this notion of independence does not seem to be appropriate for construction of multidimensional models. As it was shown by Studeny [21], it is *not consistent with marginalization*. The exact meaning of this statement can be seen from the following simple example (suggested by Studeny, as cited by Ben Yaghlane et al. in [4]).

Example 3. Let $X_1,\ X_2$ and X_3 be three binary variables with values in $\mathbf{X}_1 = \{a_1, \bar{a}_1\},\ \mathbf{X}_2 = \{a_2, \bar{a}_2\},\ \mathbf{X}_3 = \{a_3, \bar{a}_3\}$ and m_1 and m_2 be two basic assignments on $\mathbf{X}_1 \times \mathbf{X}_3$ and $\mathbf{X}_2 \times \mathbf{X}_3$, respectively, both of them having only two focal elements:

$$\begin{aligned} &m_1(\{(a_1, \bar{a}_3), (\bar{a}_1, \bar{a}_3)\}) = m_1(\{(a_1, \bar{a}_3), (\bar{a}_1, a_3)\}) = 0.5,\\ &m_2(\{(a_2, \bar{a}_3), (\bar{a}_2, \bar{a}_3)\}) = m_2(\{(a_2, \bar{a}_3), (\bar{a}_2, a_3)\}) = 0.5. \end{aligned} \tag{4}$$

Since their marginals are projective

$$m_1^{\downarrow 3}(\{\bar{a}_3\}) = m_2^{\downarrow 3}(\{\bar{a}_3\}) = 0.5,$$
$$m_1^{\downarrow 3}(\{a_3, \bar{a}_3\}) = m_2^{\downarrow 3}(\{a_3, \bar{a}_3\}) = 0.5,$$

there exists (at least one) common extension of both of them, but none of them is such that it would imply conditional non-interactivity of X_1 and X_2 given X_3. Namely, the application of the equality (3) to basic assignments m_1 and m_2 leads to the following values of the joint "basic assignment":

$$\bar{m}(\mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{a}_3\}) = 0.25,$$
$$\bar{m}(\mathbf{X}_1 \times \{a_2\} \times \{\bar{a}_3\}) = 0.25,$$
$$\bar{m}(\{a_1\} \times \mathbf{X}_2 \times \{\bar{a}_3\}) = 0.25,$$
$$\bar{m}(\{(a_1, a_2, \bar{a}_3), (\bar{a}_1, \bar{a}_2, a_3)\}) = 0.5,$$
$$\bar{m}(\{(a_1, a_2, \bar{a}_3)\}) = -0.25,$$

which is outside of evidence theory.

[5] Let us note that the definition presented in [4] is based on conjunctive Dempster's rule, but the authors proved its equivalence with (3).

[6] The reader not familiar with semigraphoid axioms is referred to Theorem 1, where they are formulated for the notion of conditional independence.

Therefore, instead of the already mentioned conditional non-interactivity, we propose to use the following notion of conditional independence.

Definition 3. Let m be a basic assignment on $\mathbf{X}_N$ and $K, L, M \subset N$ be disjoint, $K \neq \emptyset \neq L$. We say that groups of variables X_K and X_L are conditionally independent given X_M with respect to m (and denote it by $K \perp\!\!\!\perp L|M\ [m]$), if the equality

$$m^{\downarrow K\cup L\cup M}(A) \cdot m^{\downarrow M}(A^{\downarrow M}) = m^{\downarrow K\cup M}(A^{\downarrow K\cup M}) \cdot m^{\downarrow L\cup M}(A^{\downarrow L\cup M}) \tag{5}$$

holds for any $A \subseteq \mathbf{X}_{K\cup L\cup M}$ such that $A = A^{\downarrow K\cup M} \bowtie A^{\downarrow L\cup M}$, and $m(A) = 0$ otherwise.

Notice that for $M = \emptyset$ the concept coincides with Definition 1, which enables us to use the term conditional independence. Let us also note that (5) resembles, from the formal point of view, the definition of stochastic conditional independence [16].

Before formulating an important theorem justifying the above introduced definition, let us formulate and prove an assertion concerning set joins.

Lemma 4. *Let $K \cap L \subseteq M \subseteq L \subseteq N$. Then for any $C \subseteq \mathbf{X}_{K\cup L}$ the following condition* (a) *holds if and only if both conditions* (b) *and* (c) *hold true.*

(a) $C = C^{\downarrow K} \bowtie C^{\downarrow L}$;
(b) $C^{\downarrow K\cup M} = C^{\downarrow K} \bowtie C^{\downarrow M}$;
(c) $C = C^{\downarrow K\cup M} \bowtie C^{\downarrow L}$.

Proof. Before proving the required implications let us realize that evidently

$$x \in C \Rightarrow (x^{\downarrow K} \in C^{\downarrow K}\ \&\ x^{\downarrow L} \in C^{\downarrow L}),$$

which means that $C \subseteq C^{\downarrow K} \bowtie C^{\downarrow L}$. Therefore $C = C^{\downarrow K} \bowtie C^{\downarrow L}$ is equivalent to

$$\forall x \in \mathbf{X}_{K\cup L} \quad (x^{\downarrow K} \in C^{\downarrow K}\ \&\ x^{\downarrow L} \in C^{\downarrow L} \Rightarrow x \in C).$$

(a) $\Rightarrow$ (b). Consider $x \in \mathbf{X}_{K\cup M}$, such that $x^{\downarrow K} \in C^{\downarrow K}$ and $x^{\downarrow M} \in C^{\downarrow M}$. Since $x^{\downarrow M} \in C^{\downarrow M}$ there must exist (at least one) $y \in C^{\downarrow L}$, for which $y^{\downarrow M} = x^{\downarrow M}$. Now construct $z \in \mathbf{X}_{K\cup L}$ for which $z^{\downarrow K} = x^{\downarrow K}$ and $z^{\downarrow L} = y$ (it is possible because $y^{\downarrow M} = x^{\downarrow M}$). From this construction we see that $z^{\downarrow K\cup M} = x$. Therefore $z^{\downarrow K} = x^{\downarrow K} \in C^{\downarrow K}$ and $z^{\downarrow L} = y \in C^{\downarrow L}$ from which, because we assume that (a) holds, we get that $z \in C$, and therefore also $x = z^{\downarrow K\cup M} \in C^{\downarrow K\cup M}$.
(a) $\Rightarrow$ (c). Consider now $x \in \mathbf{X}_{K\cup L}$, for which its projections $x^{\downarrow K\cup M} \in C^{\downarrow K\cup M}$ and $x^{\downarrow L} \in C^{\downarrow L}$. From $x^{\downarrow K\cup M} \in C^{\downarrow K\cup M}$ we immediately get that $x^{\downarrow K} \in C^{\downarrow K}$, which in combination with $x^{\downarrow L} \in C^{\downarrow L}$ (due to the assumption (a)) yields that $x \in C$.
(b) & (c) $\Rightarrow$ (a). Consider $x \in \mathbf{X}_{K\cup L}$ such that $x^{\downarrow K} \in C^{\downarrow K}$ and $x^{\downarrow L} \in C^{\downarrow L}$. From the last property one gets also $x^{\downarrow M} \in C^{\downarrow M}$, which, in combination with $x^{\downarrow K} \in C^{\downarrow K}$ gives, because (b) holds true, that $x^{\downarrow K\cup M} \in C^{\downarrow K\cup M}$. And the last property in combination with $x^{\downarrow L} \in C^{\downarrow L}$ yields the required $x \in C$. $\square$

Theorem 1. *Conditional independence $K \perp\!\!\!\perp L|M\ [m]$ satisfies semigraphoid properties (for I, K, L, M disjoint):*

(A1) $K \perp\!\!\!\perp L|M\ [m] \Rightarrow L \perp\!\!\!\perp K|M\ [m]$,
(A2) $K \perp\!\!\!\perp L \cup M|I\ [m] \Rightarrow K \perp\!\!\!\perp M|I\ [m]$,
(A3) $K \perp\!\!\!\perp L \cup M|I\ [m] \Rightarrow K \perp\!\!\!\perp L|M \cup I\ [m]$,
(A4) $K \perp\!\!\!\perp L|M \cup I\ [m] \wedge K \perp\!\!\!\perp M|I\ [m] \Rightarrow K \perp\!\!\!\perp L \cup M|I\ [m]$.

Proof. To simplify the formulae we will omit in the proof symbol $\cup$ and use, for example, KLM instead of $K \cup L \cup M$.
ad (A1). The validity of the implication immediately follows from the commutativity of multiplication.
ad (A2). The assumption $K \perp\!\!\!\perp LM|I\ [m]$ means that for any $A \subseteq \mathbf{X}_{KLMI}$ such that $A = A^{\downarrow KI} \bowtie A^{\downarrow LMI}$ the equality

$$m^{\downarrow KLMI}(A) \cdot m^{\downarrow I}(A^{\downarrow I}) = m^{\downarrow KI}(A^{\downarrow KI}) \cdot m^{\downarrow LMI}(A^{\downarrow LMI}) \tag{6}$$

holds, and if $A \neq A^{\downarrow KI} \bowtie A^{\downarrow LMI}$, then $m(A) = 0$. Let us prove first that also for any $B \subseteq \mathbf{X}_{KMI}$ such that $B = B^{\downarrow KI} \bowtie B^{\downarrow MI}$, the equality

$$m^{\downarrow KMI}(B) \cdot m^{\downarrow I}(B^{\downarrow I}) = m^{\downarrow KI}(B^{\downarrow KI}) \cdot m^{\downarrow MI}(B^{\downarrow MI}) \tag{7}$$

is valid. To do so, let us compute

$$\begin{aligned} m^{\downarrow KMI}(B) \cdot m^{\downarrow I}(B^{\downarrow I}) &= \sum_{\substack{A \subseteq \mathbf{X}_{KLMI} \\ A^{\downarrow KMI} = B^{\downarrow KI} \bowtie B^{\downarrow MI}}} m^{\downarrow KLMI}(A) \cdot m^{\downarrow I}(A^{\downarrow I}) = \sum_{\substack{A \subseteq \mathbf{X}_{KLMI}, A = A^{\downarrow KI} \bowtie A^{\downarrow LMI} \\ A^{\downarrow KMI} = B^{\downarrow KI} \bowtie B^{\downarrow MI}}} m^{\downarrow KLMI}(A) \cdot m^{\downarrow I}(A^{\downarrow I}) \\ &= \sum_{\substack{A \subseteq \mathbf{X}_{KLMI}, A = A^{\downarrow KI} \bowtie A^{\downarrow LMI} \\ A^{\downarrow KMI} = B^{\downarrow KI} \bowtie B^{\downarrow MI}}} m^{\downarrow KI}(A^{\downarrow KI}) \cdot m^{\downarrow LMI}(A^{\downarrow LMI}) = m^{\downarrow KI}(A^{\downarrow KI}) \cdot \sum_{\substack{C \subseteq \mathbf{X}_{LMI} \\ C^{\downarrow MI} = B^{\downarrow MI}}} m^{\downarrow LMI}(C) = m^{\downarrow KI}(B^{\downarrow KI}) \cdot m^{\downarrow MI}(B^{\downarrow MI}), \end{aligned}$$

as

$$m^{\downarrow I}(B^{\downarrow I}) = m^{\downarrow I}(A^{\downarrow I}),$$
$$m^{\downarrow KI}(B^{\downarrow KI}) = m^{\downarrow KI}(A^{\downarrow KI}).$$

So, to finish this step it remains to prove that if $B \neq B^{\downarrow KI} \bowtie B^{\downarrow MI}$ then $m^{\downarrow KMI}(B) = 0$. Also, in this case

$$m^{\downarrow KMI}(B) = \sum_{\substack{A \subseteq \mathbf{X}_{KLMI} \\ A^{\downarrow KMI} = B}} m^{\downarrow KLMI}(A),$$

but since $B = A^{\downarrow KMI} \neq A^{\downarrow KI} \bowtie A^{\downarrow MI}$ then, because of Lemma 4, also $A \neq A^{\downarrow KI} \bowtie A^{\downarrow LMI}$ for any A such that $A^{\downarrow KMI} = B$. But for these A, $m^{\downarrow KLMI}(A) = 0$ and therefore also $m^{\downarrow KMI}(B) = 0$.

ad (A3). Again, let us suppose validity of $K \perp\!\!\!\perp LM|I\ [m]$, i.e., for any $A \subseteq \mathbf{X}_{KLMI}$ such that $A = A^{\downarrow KI} \bowtie A^{\downarrow LMI}$ equality (6) holds, and $m^{\downarrow KLMI}(A) = 0$ otherwise. Our aim is to prove that for any $C \subseteq \mathbf{X}_{KLMI}$ such that $C = C^{\downarrow KMI} \bowtie C^{\downarrow LMI}$, the equality

$$m^{\downarrow KLMI}(C) \cdot m^{\downarrow MI}(C^{\downarrow MI}) = m^{\downarrow KMI}(C^{\downarrow KMI}) \cdot m^{\downarrow LMI}(C^{\downarrow LMI}) \tag{8}$$

is satisfied as well, and $m^{\downarrow KI}(C) = 0$ otherwise. Let C be such that $m^{\downarrow I}(C^{\downarrow I}) > 0$. Since we assume that $K \perp\!\!\!\perp LM|I\ [m]$ holds, we have for such a C

$$m^{\downarrow KLMI}(C) \cdot m^{\downarrow I}(C^{\downarrow I}) = m^{\downarrow KI}(C^{\downarrow KI}) \cdot m^{\downarrow LMI}(C^{\downarrow LMI}),$$

and therefore we can compute

$$\begin{aligned} m^{\downarrow KLMI}(C) \cdot m^{\downarrow MI}(C^{\downarrow MI}) &= m^{\downarrow KLMI}(C) \cdot m^{\downarrow I}(C^{\downarrow I}) \cdot \frac{m^{\downarrow MI}(C^{\downarrow MI})}{m^{\downarrow I}(C^{\downarrow I})} = m^{\downarrow KI}(C^{\downarrow KI}) \cdot m^{\downarrow LMI}(C^{\downarrow LMI}) \cdot \frac{m^{\downarrow MI}(C^{\downarrow MI})}{m^{\downarrow I}(C^{\downarrow I})} \\ &= \frac{m^{\downarrow KI}(C^{\downarrow KI}) \cdot m^{\downarrow MI}(C^{\downarrow MI})}{m^{\downarrow I}(C^{\downarrow I})} \cdot m^{\downarrow LMI}(C^{\downarrow LMI}) = m^{\downarrow KMI}(C^{\downarrow KMI}) \cdot m^{\downarrow LMI}(C^{\downarrow LMI}), \end{aligned}$$

where the last equality is satisfied due to (A2) and the fact that $m^{\downarrow I}(C^{\downarrow I}) > 0$. If $m^{\downarrow I}(C^{\downarrow I}) = 0$ then also $m^{\downarrow KMI}(C^{\downarrow KMI}) = 0$, $m^{\downarrow LMI}(C^{\downarrow LMI}) = 0$ and $m^{\downarrow KLMI}(C) = 0$ and therefore (8) also holds true.

It remains to be proven that $m(C) = 0$ for all $C \neq C^{\downarrow KMI} \bowtie C^{\downarrow LMI}$. But in this case, as a consequence of Lemma 4, also $C \neq C^{\downarrow KI} \bowtie C^{\downarrow LMI}$ and therefore $m(C) = 0$ due to the assumption.

ad (A4). First, supposing $K \perp\!\!\!\perp L|MI\ [m]$ and $K \perp\!\!\!\perp M|I\ [m]$ let us prove that for any $A \subseteq \mathbf{X}_{KLMI}$ such that $A = A^{\downarrow KI} \bowtie A^{\downarrow LMI}$ the equality (6) holds. Since from $A = A^{\downarrow KI} \bowtie A^{\downarrow LMI}$ it also follows due to Lemma 4 that $A = A^{\downarrow KMI} \bowtie A^{\downarrow LMI}$, and therefore (since we assume $K \perp\!\!\!\perp L|MI\ [m]$)

$$m^{\downarrow KLMI}(A) \cdot m^{\downarrow MI}(A^{\downarrow MI}) = m^{\downarrow KMI}(A^{\downarrow KMI}) \cdot m^{\downarrow LMI}(A^{\downarrow LMI}). \tag{9}$$

Now, let us further assume that $m^{\downarrow MI}(A^{\downarrow MI}) > 0$ (and thus also $m^{\downarrow I}(A^{\downarrow I}) > 0$). Since from $A = A^{\downarrow KI} \bowtie A^{\downarrow LMI}$ Lemma 4 implies $A^{\downarrow KMI} = A^{\downarrow KI} \bowtie A^{\downarrow MI}$, one gets from $K \perp\!\!\!\perp M|I\ [m]$ that

$$m^{\downarrow KMI}(A^{\downarrow KMI}) \cdot m^{\downarrow I}(A^{\downarrow I}) = m^{\downarrow KI}(A^{\downarrow KI}) \cdot m^{\downarrow MI}(A^{\downarrow MI}),$$

which, in combination with equality (9), yields

$$m^{\downarrow KLMI}(A) \cdot m^{\downarrow MI}(A^{\downarrow MI}) = \frac{m^{\downarrow KI}(A^{\downarrow KI}) \cdot m^{\downarrow MI}(A^{\downarrow MI})}{m^{\downarrow I}(A^{\downarrow I})} \cdot m^{\downarrow LMI}(A^{\downarrow LMI}),$$

which is (for positive $m^{\downarrow MI}(A^{\downarrow MI})$) evidently equivalent to (6). If, on the other hand, $m^{\downarrow MI}(A^{\downarrow MI}) = 0$, then also $m^{\downarrow LMI}(A^{\downarrow LMI}) = 0$ and $m^{\downarrow KLMI}(A) = 0$ and both sides of (6) equal 0.

It remains to prove that $m^{\downarrow KLMI}(A) = 0$ for all $A \neq A^{\downarrow KI} \bowtie A^{\downarrow LMI}$. But $m^{\downarrow KLMI}(A) = 0$ because Lemma 4 says that either $A \neq A^{\downarrow KMI} \bowtie A^{\downarrow LMI}$ (and therefore $m^{\downarrow KLMI}(A) = 0$ from the assumption that $K \perp\!\!\!\perp L|MI\ [m]$) or $A^{\downarrow KMI} \neq A^{\downarrow KI} \bowtie A^{\downarrow MI}$ (and then $m^{\downarrow KMI}(A^{\downarrow KMI}) = 0$ due to the assumption $K \perp\!\!\!\perp M|I\ [m]$, and therefore also $m^{\downarrow KLMI}(A) = 0$). □

Remark. As the introduced notion generalizes the probabilistic notion of conditional independence, we do not expect that it satisfies—for general basic assignments—the following property

(A5) $K \perp\!\!\!\perp L|M \cup I\ [m] \wedge K \perp\!\!\!\perp M|L \cup I\ [m] \Rightarrow K \perp\!\!\!\perp L \cup M|I\ [m]$.

This is why we do not consider in this paper so-called *graphoid axioms* (i.e. (A1)–(A5)) studied by other authors, as, e.g. in [1,4].

Theorem 2. *Let m be a joint basic assignment on $\mathbf{X}_M$, $K, L \subseteq M$. Then $(K \setminus L) \perp\!\!\!\perp (L \setminus K) | (K \cap L)$ $[m]$ if and only if*

$$m^{\downarrow K\cup L}(A) = (m^{\downarrow K} \triangleright m^{\downarrow L})(A)$$

for any $A \subseteq \mathbf{X}_{K\cup L}$.

Proof. Let $X_{K\setminus L}$ and $X_{L\setminus K}$ be conditionally independent given $X_{K\cap L}$ with respect to a basic assignment m, and $A \subseteq \mathbf{X}_{K\cup L}$ be such that $m(A^{\downarrow K\cap L}) > 0$. Then Definition 3 guarantees that $A = A^{\downarrow K} \bowtie A^{\downarrow L}$ and

$$m^{\downarrow K\cup L}(A) \cdot m^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L}), \tag{10}$$

and therefore

$$m^{\downarrow K\cup L}(A) = \frac{m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L})}{\cdot m^{\downarrow K\cap L}(A^{\downarrow K\cap L})} = (m^{\downarrow K} \triangleright m^{\downarrow L})(A).$$

If $m(A^{\downarrow K\cap L}) = 0$ then also $m(A^{\downarrow K}) = 0,\ m(A^{\downarrow L}) = 0$ and $m(A) = 0$ and therefore $(m^{\downarrow K} \triangleright m^{\downarrow L})(A)$ can be obtained via [b] or [c] depending of whether $A = A^{\downarrow K} \times \mathbf{X}_{L\setminus K}$ or not.

Let, on the other hand, $m(A) = (m^{\downarrow K} \triangleright m^{\downarrow L})(A)$ for any $A \subseteq \mathbf{X}_{K\cup L}$. First let us show that (10) holds for all $A = A^{\downarrow K} \bowtie A^{\downarrow L}$. If $m(A^{\downarrow K\cap L}) > 0$ then multiplying both sides of the formula (case [a]) by $m(A^{\downarrow K\cap L})$ we obtain the equality (10). If $m(A^{\downarrow K\cap L}) = 0$ then also $m(A^{\downarrow K}) = 0,\ m(A^{\downarrow L}) = 0$ and $m(A) = 0$ and therefore both sides of (10) equal 0. If $A \neq A^{\downarrow K} \bowtie A^{\downarrow L}$, case [c] of Definition 2 is applied, and therefore $m(A) = (m^{\downarrow K} \triangleright m^{\downarrow L})(A) = 0$. □

Example 2. (Continued) Let us go back to the problem of finding a common extension of basic assignments m_1 and m_2 defined by (4). Theorem 2 says that for basic assignment $\hat{m} = m_1 \triangleright m_2$ with the following two focal elements

$$\hat{m}(\mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{a}_3\}) = 0.5,$$
$$\hat{m}(\{(a_1, a_2, \bar{a}_3), (\bar{a}_1, \bar{a}_2, a_3)\}) = 0.5,$$

variables X_1 and X_3 are conditionally independent given X_2.

5. Compositional models

5.1. Iterative application of the operator $\triangleright$

In this section we want to recall the fact that the operator of composition was originally designed to create multidimensional models from a system of oligodimensional (low-dimensional) ones. From this point of view those properties are of importance which enable us to construct multidimensional basic assignments, to recognize when two different expressions define the same basic assignments, and which enable us to use the multidimensional models for inference. The situation is strongly influenced by the fact that the introduced operator of composition $\triangleright$ is neither commutative, nor associative. Therefore we will concentrate our attention on those properties which make it possible to exchange the order of the arguments without changing the resulting model. In this paper we are presenting only one—the most important assertion of this type—which will be necessary in the proof of Theorem 3.

Lemma 5. *Let $m_1,\ m_2$ and m_3 be basic assignments on $\mathbf{X}_{K_1},\ \mathbf{X}_{K_2}$ and $\mathbf{X}_{K_3}$, respectively. Then*

$$K_1 \supseteq K_2 \cap K_3 \Rightarrow (m_1 \triangleright m_2) \triangleright m_3 = (m_1 \triangleright m_3) \triangleright m_2.$$

Proof. The goal is to prove that for any $C \subseteq \mathbf{X}_{K_1\cup K_2\cup K_3}$

$$((m_1 \triangleright m_2) \triangleright m_3)(C) = ((m_1 \triangleright m_3) \triangleright m_2)(C). \tag{11}$$

We have to distinguish five special cases.

A. $C \neq C^{\downarrow K_1} \bowtie C^{\downarrow K_2} \bowtie C^{\downarrow K_3}$.
This is the simplest situation because in this case both sides of formula (11) equal 0 due to Definition 2 (case [c]).

B. $C = C^{\downarrow K_1} \bowtie C^{\downarrow K_2} \bowtie C^{\downarrow K_3}$ & $m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2}) > 0,\ m_3^{\downarrow K_1\cap K_3}(C^{\downarrow K_1\cap K_3}) > 0$.
In this case it is enough to realize that (under the given assumptions) $K_3 \cap (K_1 \cup K_2) = K_3 \cap K_1$ and, analogously, $K_2 \cap (K_1 \cup K_3) = K_2 \cap K_1$. Then we see that both sides of formula (11) again coincide:

$$((m_1 \triangleright m_2) \triangleright m_3)(C) = \frac{m_1(C^{\downarrow K_1}) \cdot m_2(C^{\downarrow K_2})}{m_2^{\downarrow K_2\cap K_1}(C^{\downarrow K_2\cap K_1})} \cdot \frac{m_3(C^{\downarrow K_3})}{m_3^{\downarrow K_3\cap(K_1\cup K_2)}(C^{\downarrow K_3\cap(K_1\cup K_2)})},$$

$$((m_1 \triangleright m_3) \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K_1}) \cdot m_3(C^{\downarrow K_3})}{m_3^{\downarrow K_3\cap K_1}(C^{\downarrow K_3\cap K_1})} \cdot \frac{m_2(C^{\downarrow K_2})}{m_2^{\downarrow K_2\cap(K_1\cup K_3)}(C^{\downarrow K_2\cap(K_1\cup K_3)})}.$$

C. $C = C^{\downarrow K_1} \bowtie C^{\downarrow K_2} \bowtie C^{\downarrow K_3}$ & $m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2}) > 0,\ m_3^{\downarrow K_1\cap K_3}(C^{\downarrow K_1\cap K_3}) = 0.$
In this case, if $C^{\downarrow K_3 \setminus K_1} \neq \mathbf{X}_{K_3\setminus K_1}$ then both sides of formula (11) equal 0, because, due to Definition 2, both assignments $m_1 \triangleright m_2$ and $(m_1 \triangleright m_3) \triangleright m_2$ equal 0. Therefore, consider $C = C^{\downarrow K_1} \bowtie C^{\downarrow K_2} \bowtie \mathbf{X}_{K_3\setminus K_1}$. For this we get from Definition 2

$$((m_1 \triangleright m_2) \triangleright m_3)(C) = (m_1 \triangleright m_2)(C^{\downarrow K_1\cup K_2}).$$

For the right-hand side of formula (11) we get

$$(m_1 \triangleright m_3)(C^{\downarrow K_1\cup K_3}) = m_1(C^{\downarrow K_1})$$

and therefore

$$((m_1 \triangleright m_3) \triangleright m_2)(C) = (m_1 \triangleright m_2)(C^{\downarrow K_1\cup K_2}).$$

D. $C = C^{\downarrow K_1} \bowtie C^{\downarrow K_2} \bowtie C^{\downarrow K_3}$ & $m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2}) = 0,\ m_3^{\downarrow K_1\cap K_3}(C^{\downarrow K_1\cap K_3}) > 0.$
The proof is analogous to that under item C.

E. $C = C^{\downarrow K_1} \bowtie C^{\downarrow K_2} \bowtie C^{\downarrow K_3}$ & $m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2}) = 0,\ m_3^{\downarrow K_1\cap K_3}(C^{\downarrow K_1\cap K_3}) = 0.$
It is obvious from Definition 2 that both sides of formula (11) equal 0 for all C but for $C = C^{\downarrow K_1} \bowtie \mathbf{X}_{K_2\setminus K_1} \bowtie \mathbf{X}_{K_3\setminus K_1}$. For this special case, however,

$$((m_1 \triangleright m_2) \triangleright m_3)(C) = m_1(C^{\downarrow K_1}),$$
$$((m_1 \triangleright m_3) \triangleright m_2)(C) = m_1(C^{\downarrow K_1}). \qquad \square$$

Let us formulate an important property generalizing (iv) of Lemma 2.

Lemma 6. *Let m_1 and m_2 be basic assignments on $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively, and $K_2 \supseteq L \supseteq K_1 \cap K_2$. Then*

$$m_1 \triangleright m_2 = (m_1 \triangleright m_2^{\downarrow L}) \triangleright m_2.$$

Proof. The goal of this proof is to show that

$$(m_1 \triangleright m_2)(C) = ((m_1 \triangleright m_2^{\downarrow L}) \triangleright m_2)(C)$$

holds true for any $C \subseteq \mathbf{X}_{K_1\cup K_2}$. The proof will be performed in three steps corresponding to cases [a], [b], and [c] of Definition 2.

ad [a]. Assume that $C = C^{\downarrow K_1} \bowtie C^{\downarrow K_2}$ and $m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2}) > 0$. From this and Lemma 4 we get that also $C^{\downarrow K_1\cup L} = C^{\downarrow K_1} \bowtie C^{\downarrow L}$, and therefore (since $K_1 \cap K_2 = K_1 \cap L$)

$$(m_1 \triangleright m_2^{\downarrow L})(C^{\downarrow K_1\cup L}) = \frac{m_1(C^{\downarrow K_1}) \cdot m_2^{\downarrow L}(C^{\downarrow L})}{m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2})}.$$

In the rest of this step we have to distinguish between two situations depending whether $m_2^{\downarrow L}(C^{\downarrow L})$ equals 0 or not.

If $m_2^{\downarrow L}(C^{\downarrow L}) > 0$ (realize that in this case also $m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2}) > 0$) then

$$((m_1 \triangleright m_2^{\downarrow L}) \triangleright m_2)(C) = \frac{(m_1 \triangleright m_2^{\downarrow L})(C^{\downarrow K_1\cup L}) \cdot m_2(C^{\downarrow K_2})}{m_2^{\downarrow L}(C^{\downarrow L})} = \frac{\frac{m_1(C^{\downarrow K_1}) \cdot m_2^{\downarrow L}(C^{\downarrow L})}{m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2})} \cdot m_2(C^{\downarrow K_2})}{m_2^{\downarrow L}(C^{\downarrow L})} = \frac{m_1(C^{\downarrow K_1}) \cdot m_2(C^{\downarrow K_2})}{m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2})} = (m_1 \triangleright m_2)(C).$$

If $m_2^{\downarrow L}(C^{\downarrow L}) = 0$ then, according to Definition 2, either

$$((m_1 \triangleright m_2^{\downarrow L}) \triangleright m_2)(C) = (m_1 \triangleright m_2^{\downarrow L})(C^{\downarrow K_1\cup L}),$$

in case that $C = C^{\downarrow K_1\cup L} \bowtie \mathbf{X}_{K_2\setminus L}$, or

$$((m_1 \triangleright m_2^{\downarrow L}) \triangleright m_2)(C) = 0,$$

in the opposite case. However, in this case also

$$(m_1 \triangleright m_2^{\downarrow L})(C^{\downarrow K_1\cup L}) = \frac{m_1(C^{\downarrow K_1}) \cdot m_2^{\downarrow L}(C^{\downarrow L})}{m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2})} = 0,$$

and therefore $((m_1 \triangleright m_2^{\downarrow L}) \triangleright m_2)(C) = 0$ independently of whether $C^{\downarrow K_2\setminus L} = \mathbf{X}_{K_2\setminus L}$ or not. Regarding the fact that in the considered situation (i.e., $m_2^{\downarrow L}(C^{\downarrow L}) = 0$) also $m_2(C^{\downarrow K_2}) = 0$, and therefore also

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K_1}) \cdot m_2(C^{\downarrow K_2})}{m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2})} = 0,$$

we have finished the first step of the proof.

ad [b]. Now we assume that $C = C^{\downarrow K_1} \bowtie \mathbf{X}_{K_2\setminus K_1}$, and that $m_2^{\downarrow K_1\cap K_2}(C^{\downarrow K_1\cap K_2}) = 0$. In this case, naturally, also $m_2^{\downarrow L}(C^{\downarrow L}) = 0$ and $C = C^{\downarrow K_1} \bowtie \mathbf{X}_{L\setminus K_1} \bowtie \mathbf{X}_{K_2\setminus L}$. Therefore, according to case [b] of Definition 2,

$$(m_1 \triangleright m_2^{\downarrow L})(C^{\downarrow K_1 \cup L}) = m_1(C^{\downarrow K_1}),$$

and for the same reasons also

$$((m_1 \triangleright m_2^{\downarrow L}) \triangleright m_2)(C) = (m_1 \triangleright m_2^{\downarrow L})(C^{\downarrow K_1 \cup L}) = m_1(C^{\downarrow K_1}).$$

In this case $(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow K_1})$ as well, and we have finished the second step of the proof.

ad [c]. The last step is trivial. In this case, as the reader can immediately see, both $((m_1 \triangleright m_2^{\downarrow L}) \triangleright m_2)(C)$ and $(m_1 \triangleright m_2)(C)$ equal 0; therefore, they are equal to each other. □

Lemma 7. *Let m_1 and m_2 be basic assignments on $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively. Then*

$$K_1 \cup K_2 \supseteq L \supseteq K_1 \Rightarrow (m_1 \triangleright m_2)^{\downarrow L} = m_1 \triangleright m_2^{\downarrow K_2 \cap L}.$$

Proof. First consider $B \subseteq \mathbf{X}_L$ such that $m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2}) > 0$. For this B we get

$$\begin{aligned}(m_1 \triangleright m_2)^{\downarrow L}(B) &= \sum_{A \subseteq \mathbf{X}_{K_1 \cup K_2}: A^{\downarrow L} = B} (m_1 \triangleright m_2)(A) = \sum_{A \subseteq \mathbf{X}_{K_1 \cup K_2}: A^{\downarrow L} = B} \frac{m_1(A^{\downarrow K_1}) \cdot m_2(A^{\downarrow K_2})}{m_2^{\downarrow K_1 \cap K_2}(A^{\downarrow K_1 \cap K_2})} = \sum_{C \subseteq \mathbf{X}_{K_2}: C^{\downarrow L \cap K_2} = B^{\downarrow L \cap K_2}} \frac{m_1(B^{\downarrow K_1}) \cdot m_2(C)}{m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2})} \\ &= \frac{m_1(B^{\downarrow K_1})}{m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2})} \sum_{C \subseteq \mathbf{X}_{K_2}: C^{\downarrow L \cap K_2} = B^{\downarrow L \cap K_2}} m_2(C) = \frac{m_1(B^{\downarrow K_1}) m_2^{\downarrow L \cap K_2}(B^{\downarrow L \cap K_2})}{m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2})} = (m_1 \triangleright m_2^{\downarrow L \cap K_2})(B).\end{aligned}$$

If $m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2}) = 0$ for some $B \subseteq \mathbf{X}_L$, then there is only one $A \subseteq \mathbf{X}_{K_1 \cup K_2}$ such that $A^{\downarrow K_1} = B^{\downarrow K_1}$ for which $(m_1 \triangleright m_2)(A)$ may be positive, namely $A^* = B^{\downarrow K_1} \bowtie \mathbf{X}_{K_2 \setminus K_1}$ with $(m_1 \triangleright m_2)(A^*) = m_1(B^{\downarrow K_1})$. Thus if $B = B^{\downarrow K_1} \bowtie \mathbf{X}_{L \setminus K_1}$,

$$(m_1 \triangleright m_2)^{\downarrow L}(B) = \sum_{A \subseteq \mathbf{X}_{K_1 \cup K_2}: A^{\downarrow L} = B} (m_1 \triangleright m_2)(A) = (m_1 \triangleright m_2)(A^*) = m_1(B^{\downarrow K_1}) = (m_1 \triangleright m_2^{\downarrow K_2 \cap L})(A^{*\downarrow L}) = (m_1 \triangleright m_2^{\downarrow K_2 \cap L})(B).$$

If $B \neq B^{\downarrow K_1} \bowtie \mathbf{X}_{L \setminus K_1}$ and $m_2^{\downarrow K_1 \cap K_2}(B^{\downarrow K_1 \cap K_2}) = 0$ then

$$(m_1 \triangleright m_2)^{\downarrow L}(B) = 0 = (m_1 \triangleright m_2^{\downarrow K_2 \cap L})(B). \qquad \square$$

The following theorem shows that, in certain circumstances, computation of a marginal from a composed basic assignment may be very simple.

Theorem 3. *Let m_1 and m_2 be basic assignments on $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively. Then*

$$K_1 \cup K_2 \supseteq L \supseteq K_1 \cap K_2 \Rightarrow (m_1 \triangleright m_2)^{\downarrow L} = m_1^{\downarrow K_1 \cap L} \triangleright m_2^{\downarrow K_2 \cap L}.$$

Proof. In addition to the properties presented in preceding lemmata we will also use an obvious fact which directly follows from Definition 2:

$$(m_1 \triangleright m_2)^{\downarrow L} = ((m_1 \triangleright m_2)^{\downarrow K_1 \cup L})^{\downarrow L},$$

$$\begin{aligned}(m_1 \triangleright m_2)^{\downarrow L} &= ((m_1 \triangleright m_2)^{\downarrow K_1 \cup L})^{\downarrow L} \\ &= (m_1 \triangleright m_2^{\downarrow K_2 \cap L})^{\downarrow L} \quad \text{application of Lemma 7} \\ &= ((m_1^{\downarrow K_1 \cap K_2} \triangleright m_1) \triangleright m_2^{\downarrow K_2 \cap L})^{\downarrow L} \quad \text{application of property (iv) of Lemma 2} \\ &= ((m_1^{\downarrow K_1 \cap K_2} \triangleright m_2^{\downarrow K_2 \cap L}) \triangleright m_1)^{\downarrow L} \quad \text{application of Lemma 5} \\ &= (m_1^{\downarrow K_1 \cap K_2} \triangleright m_2^{\downarrow K_2 \cap L}) \triangleright m_1^{\downarrow K_1 \cap L} \quad \text{application of Lemma 7} \\ &= (m_1^{\downarrow K_1 \cap K_2} \triangleright m_1^{\downarrow K_1 \cap L}) \triangleright m_2^{\downarrow K_2 \cap L} \quad \text{application of Lemma 5} \\ &= m_1^{\downarrow K_1 \cap L} \triangleright m_2^{\downarrow K_2 \cap L} \quad \text{application of property (iv) of Lemma 2.} \qquad \square\end{aligned}$$

The following simple example demonstrates, that the condition on set inclusion is substantial.

Example 3. Let $X_1,\ X_2$ and X_3 be three variables with values in $\mathbf{X}_1,\ \mathbf{X}_2$ and $\mathbf{X}_3$, respectively, $\mathbf{X}_i = \{a_i, \bar{a}_i\},\ i = 1, 2, 3$, and let m_1 and m_2 be two basic assignments on $\mathbf{X}_1 \times \mathbf{X}_3$ and $\mathbf{X}_2 \times \mathbf{X}_3$, respectively, defined as follows:

$$m_1(\{a_1, a_3\}) = 0.5,$$
$$m_1(\mathbf{X}_1 \times \mathbf{X}_3) = 0.5.$$

and

$$m_2(\{a_2, a_3\}) = 0.5,$$
$$m_2(\mathbf{X}_2 \times \mathbf{X}_3) = 0.5.$$

Applying Definition 2 one gets

$$m_1 \triangleright m_2(\{a_1, a_2, a_3\}) = 0.5,$$
$$m_1 \triangleright m_2(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) = 0.5.$$

from which

$$(m_1 \triangleright m_2)^{\downarrow\{1,2\}}(\{a_1, a_2\}) = 0.5,$$
$$(m_1 \triangleright m_2)^{\downarrow\{1,2\}}(\mathbf{X}_1 \times \mathbf{X}_2) = 0.5$$

immediately follows. On the other hand

$$m_i^{\downarrow\{i\}}(\{a_i\}) = 0.5,$$
$$m_i^{\downarrow\{i\}}(\mathbf{X}_i) = 0.5,$$

for $i = 1, 2$, and therefore

$$m_1^{\downarrow\{1\}} \triangleright m_2^{\downarrow\{2\}}(\{a_1, a_2\}) = 0.25,$$
$$m_1^{\downarrow\{1\}} \triangleright m_2^{\downarrow\{2\}}(\{a_1\} \times \mathbf{X}_2) = 0.25,$$
$$m_1^{\downarrow\{1\}} \triangleright m_2^{\downarrow\{2\}}(\mathbf{X}_1 \times \{a_2\}) = 0.25,$$
$$m_1^{\downarrow\{1\}} \triangleright m_2^{\downarrow\{2\}}(\mathbf{X}_1 \times \mathbf{X}_2) = 0.25,$$

i.e., the equality in Theorem 3 is not generally valid.

5.2. Generating sequences

In this part of the text we will consider a system of low-dimensional basic assignments $m_1, m_2, \ldots, m_n$ defined on $\mathbf{X}_{K_1}, \mathbf{X}_{K_2}, \ldots, \mathbf{X}_{K_n}$, respectively. Composing them together by multiple application of the operator of composition, one gets multidimensional basic assignments on $\mathbf{X}_{K_1 \cup K_2 \cup \cdots \cup K_n}$. However, since we know (from what has been shown in the preceding sections) that the operator of composition is neither commutative nor associative, we have to properly specify what we understand by saying "composing them together".

To avoid using too many brackets let us make the following convention. Whenever we put down the expression $m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n$ we will understand that the operator of composition is performed successively from left to right[7]:

$$m_1 \triangleright m_2 \triangleright \ldots \triangleright m_n = (\cdots((m_1 \triangleright m_2) \triangleright m_3) \triangleright \cdots) \triangleright m_n.$$

Therefore, when we want to describe a multidimensional model that is a composition of many low-dimensional basic assignments, it is enough to specify an ordered sequence of these assignments; we will say that a *generating sequence* $m_1, m_2, \ldots, m_n$ represents multidimensional basic assignments $m_1 \triangleright m_2 \triangleright \cdots \triangleright m_n$.

Example 4. In this simple example we will show that the ordering in which the basic assignments are considered is substantial. Consider three variables $X_1,\ X_2$ and X_3 with values in $\mathbf{X}_1,\ \mathbf{X}_2$ and $\mathbf{X}_3$, respectively, $\mathbf{X}_i = \{a_i, \bar{a}_i\},\ i = 1, 2, 3$. Let $m_1,\ m_2$ and m_3 be three basic assignments on $\mathbf{X}_1 \times \mathbf{X}_2,\ \mathbf{X}_2 \times \mathbf{X}_3$ and $\mathbf{X}_1 \times \mathbf{X}_3$, respectively, each m_i having only one focal element A_i

$$A_1 = \{(a_1, a_2), (\bar{a}_1, \bar{a}_2)\},$$
$$A_2 = \{(a_2, a_3), (\bar{a}_2, \bar{a}_3)\},$$
$$A_3 = \mathbf{X}_1 \times \mathbf{X}_3,$$

i.e. $m_i(A_i) = 1$.

These basic assignments are pairwise projective (any one-dimensional marginal has only one focal element, namely $\mathbf{X}_i$), but the sequence is not perfect (cf. definition following this example). Therefore, application of the operator of composition to different orderings of these three basic assignments leads to different joint basic assignments on $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$. Each of these composed basic assignments has again only one focal element, namely

$$A_1 \bowtie A_2 = \{(a_1, a_2, a_3), (\bar{a}_1, \bar{a}_2, \bar{a}_3)\}$$

[7] Naturally, if we want to change the ordering in which the operators are to be performed we will do so with the help of brackets.

for $m_1,\ m_2,\ m_3$ and $m_2,\ m_1,\ m_3$,

$$\{(a_1,a_2),(\bar{a}_1,\bar{a}_2)\}\times\mathbf{X}_3$$

for $m_1,\ m_3,\ m_2$,

$$\mathbf{X}_1\times\{(a_2,a_3),(\bar{a}_2,\bar{a}_3)\}$$

for $m_2,\ m_3,\ m_1$ and, finally,

$$\mathbf{X}_1\times\mathbf{X}_2\times\mathbf{X}_3$$

for $m_3,\ m_1,\ m_2$ and $m_3,\ m_2,\ m_1$.

When representing knowledge in a specific area of interest, a special role is played by the so-called *perfect sequences*, i.e., generating sequences $m_1,m_2,\ldots,m_n$, for which

$$\begin{aligned}
&m_1\triangleright m_2 = m_2\triangleright m_1,\\
&m_1\triangleright m_2\triangleright m_3 = m_3\triangleright(m_1\triangleright m_2),\\
&\vdots\\
&m_1\triangleright m_2\triangleright\cdots\triangleright m_n = m_n\triangleright(m_1\triangleright\cdots\triangleright m_{n-1}).
\end{aligned}$$

The property explaining why we call these sequences perfect is expressed in the following assertion.

Theorem 4. *A generating sequence $m_1,m_2,\ldots,m_n$ is perfect if and only if all $m_1,m_2,\ldots,m_n$ are marginal assignments of the multidimensional assignment $m_1\triangleright m_2\triangleright\cdots\triangleright m_n$:*

$$(m_1\triangleright m_2\triangleright\cdots\triangleright m_n)^{\downarrow K_j} = m_j,$$

for all $j=1,\ldots,n$.

Proof. The fact that all assignments m_j from a perfect sequence are marginals of $(m_1\triangleright m_2\triangleright\cdots\triangleright m_n)$ follows from the fact that $(m_1\triangleright\cdots\triangleright m_j)$ is marginal to $(m_1\triangleright\cdots\triangleright m_n)$ (due to (ii) of Lemma 2), and m_j is marginal to $m_j\triangleright(m_1\triangleright\cdots\triangleright m_{j-1}) = m_1\triangleright\cdots\triangleright m_j$.

Suppose now that for all $j=1,\ldots,n$, m_j are marginal assignments to $m_1\triangleright\cdots\triangleright m_n$. This means that all of the assignments from the sequence are pairwise projective, and that each m_j is projective with any marginal assignment of $m_1\triangleright\cdots\triangleright m_n$, and consequently also with $m_1\triangleright\cdots\triangleright m_{j-1}$. Hence we get that

$$m_j^{\downarrow K_j\cap(K_1\cup\cdots\cup K_{j-1})} = (m_1\triangleright\cdots\triangleright m_{j-1})^{\downarrow K_j\cap(K_1\cup\cdots\cup K_{j-1})}$$

for all $j=2,\ldots,n$, which is equivalent, due to property (iii) of Lemma 2, to the fact that

$$m_1\triangleright m_2\triangleright\cdots\triangleright m_j = m_j\triangleright(m_1\triangleright\cdots\triangleright m_{j-1}),$$

which corresponds to the definition of perfect sequence. $\square$

Let us interpret this assertion in the language of artificial intelligence. If low-dimensional assignments $m_1,m_2,\ldots,m_n$ correspond to pieces of local knowledge, then the global knowledge represented by multidimensional assignment $m_1\triangleright m_2\triangleright\cdots\triangleright m_n$ contains all these pieces of local knowledge. The next theorem shows that each generating sequence defining a compositional model $m_1\triangleright\cdots\triangleright m_n$ can be transformed into a perfect sequence without changing the represented multidimensional assignment. In other words, any basic assignment representable by a generating sequence $m_1,m_2,\ldots,m_n$ can also be represented by a perfect sequence $\hat{m}_1,\hat{m}_2,\ldots,\hat{m}_n$. First, we shall formulate this property just for two basic assignments, then it will be generalized to an arbitrary generating sequence.

Lemma 8. *Let $m_1,\ m_2$ be basic assignments on $\mathbf{X}_{K_1},\ \mathbf{X}_{K_2}$, respectively. Then*

$$m_1\triangleright m_2 = m_1\triangleright(m_1^{\downarrow K_1\cap K_2}\triangleright m_2).$$

Proof. Due to (ii) of Lemma 2, assignments m_1 and $(m_1^{\downarrow K_1\cap K_2}\triangleright m_2)$ are projective and therefore (due to property (iii) of the same lemma), these arguments may be commuted

$$m_1\triangleright(m_1^{\downarrow K_1\cap K_2}\triangleright m_2) = (m_1^{\downarrow K_1\cap K_2}\triangleright m_2)\triangleright m_1.$$

The last expression meets the assumptions of Lemma 5 and therefore we can exchange the second and third arguments, from which the required expression is obtained by application of (iv) of Lemma 2:

$$(m_1^{\downarrow K_1\cap K_2}\triangleright m_2)\triangleright m_1 = (m_1^{\downarrow K_1\cap K_2}\triangleright m_1)\triangleright m_2 = m_1\triangleright m_2. \qquad\square$$

Theorem 5. *For any generating sequence $m_1, m_2, \ldots, m_n$ the sequence $\hat{m}_1, \hat{m}_2, \ldots, \hat{m}_n$ computed by the following process*

$$\begin{aligned}
\hat{m}_1 &= m_1,\\
\hat{m}_2 &= \hat{m}_1^{\downarrow K_2\cap K_1} \triangleright m_2,\\
\hat{m}_3 &= (\hat{m}_1 \triangleright \hat{m}_2)^{\downarrow K_3\cap(K_1\cup K_2)} \triangleright m_3,\\
&\vdots\\
\hat{m}_n &= (\hat{m}_1 \triangleright \cdots \triangleright \hat{m}_{n-1})^{\downarrow K_n\cap(K_1\cup\cdots\cup K_{n-1})} \triangleright m_n
\end{aligned}$$

is perfect and

$$m_1 \triangleright \cdots \triangleright m_n = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_n.$$

Proof. The perfectness of the sequence $\hat{m}_1, \ldots, \hat{m}_n$ follows immediately from property (ii) of Lemma 2 and from the definition of this sequence as

$$\hat{m}_i^{\downarrow K_i\cap(K_1\cup\cdots\cup K_{i-1})} = (\hat{m}_1 \triangleright \cdots \triangleright \hat{m}_{i-1})^{\downarrow K_i\cap(K_1\cup\cdots\cup K_{i-1})}$$

yields equivalence

$$\hat{m}_1 \triangleright \hat{m}_2 \triangleright \cdots \triangleright \hat{m}_i = \hat{m}_i \triangleright (\hat{m}_1 \triangleright \cdots \triangleright \hat{m}_{i-1}).$$

So, we have only to prove that

$$m_1 \triangleright \cdots \triangleright m_n = \hat{m}_1 \triangleright \cdots \triangleright \hat{m}_n.$$

We will do so by mathematical induction. Since $m_1 = \hat{m}_1$ by definition, it is enough to show that

$$m_1 \triangleright \cdots \triangleright m_i = \hat{m}_1 \triangleright \cdots \triangleright \hat{m}_i$$

also implies

$$m_1 \triangleright \cdots \triangleright m_{i+1} = \hat{m}_1 \triangleright \cdots \triangleright \hat{m}_{i+1}.$$

In the following computations we will use the fact that due to Theorem 3

$$(\hat{m}_1 \triangleright \cdots \triangleright \hat{m}_i)^{\downarrow K_{i+1}\cap(K_1\cup\cdots\cup K_i)} \triangleright m_{i+1} = ((\hat{m}_1 \triangleright \cdots \triangleright \hat{m}_i) \triangleright m_{i+1})^{\downarrow K_{i+1}},$$

and afterwards we will employ Lemma 8

$$\begin{aligned}
\hat{m}_1 \triangleright \cdots \triangleright \hat{m}_{i+1} &= \hat{m}_1 \triangleright \cdots \triangleright \hat{m}_i \triangleright ((\hat{m}_1 \triangleright \cdots \triangleright \hat{m}_i)^{\downarrow K_{i+1}\cap(K_1\cup\cdots\cup K_i)} \triangleright m_{i+1}) = \hat{m}_1 \triangleright \ldots \triangleright \hat{m}_i \triangleright ((\hat{m}_1 \triangleright \cdots \triangleright \hat{m}_i) \triangleright m_{i+1})^{\downarrow K_{i+1}}\\
&= \hat{m}_1 \triangleright \cdots \triangleright \hat{m}_i \triangleright m_{i+1} = m_1 \triangleright \cdots \triangleright m_i \triangleright m_{i+1},
\end{aligned}$$

where the last modification is an application of the inductive assumption. □

6. Bayesian basic assignments

As already mentioned in Section 1, the operator of composition was originally designed for probability theory. Let us recall this definition.

Definition 4. Consider two arbitrary probability distributions p_1 and p_2 defined on $\mathbf{X}_{K_1}$, $\mathbf{X}_{K_2}$, respectively, ($K_1 \neq \emptyset \neq K_2$). If $p_1^{\downarrow K_1\cap K_2}$ is dominated by $p_2^{\downarrow K_1\cap K_2}$, i.e.,

$$\forall z \in \mathbf{X}_{K_1\cap K_2} \quad p_2^{\downarrow K_1\cap K_2}(z) = 0 \Rightarrow p_1^{\downarrow K_1\cap K_2}(z) = 0,$$

then $p_1 \triangleright p_2$ is for all $x \in \mathbf{X}_{K\cup L}$ defined by the expression[8]

$$(p_1 \triangleright p_2)(x) = \frac{p_1(x^{\downarrow K_1}) \cdot p_2(x^{\downarrow K_2})}{p_2^{\downarrow K_1\cap K_2}(x^{\downarrow K_1\cap K_2})}.$$

Otherwise the composition $p_1 \triangleright p_2$ remains undefined.

[8] In case of necessity we define $\frac{0 \cdot 0}{0} = 0$.

A basic assignment m degenerates into a probability distribution if all of its focal elements are singletons (in other words: $m(A) > 0 \Rightarrow |A| = 1$). In agreement with [19] we will call such assignments *Bayesian basic assignments*. It would be inconsistent if the operator of composition we have introduced in this paper did not coincide with the probabilistic one when applied to Bayesian basic assignments. Fortunately, that is not the case.

Lemma 9. *Let m_1 and m_2 be Bayesian basic assignments on $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively, for which*

$$m_2^{\downarrow K_1 \cap K_2}(\{z\}) = 0 \Rightarrow m_1^{\downarrow K_1 \cap K_2}(\{z\}) = 0 \tag{12}$$

for any $z \in \mathbf{X}_{K_1 \cap K_2}$. Let p_1 and p_2 be probabilistic distributions on $\mathbf{X}_{K_1}$ and $\mathbf{X}_{K_2}$, respectively, such that

$$m_1(\{x^{\downarrow K_1}\}) = p_1(x^{\downarrow K_1}),$$
$$m_2(\{x^{\downarrow K_2}\}) = p_2(x^{\downarrow K_2}),$$

for any $x \in \mathbf{X}_{K_1 \cup K_2}$. Then $m_1 \triangleright m_2$ is a Bayesian basic assignment and

$$(m_1 \triangleright m_2)(\{x\}) = (p_1 \triangleright p_2)(x)$$

for all $x \in \mathbf{X}_{K_1 \cup K_2}$.

Proof. To prove that basic assignment $m_1 \triangleright m_2$ is Bayesian, it is enough to show that if $A \subseteq \mathbf{X}_{K_1 \cup K_2}$ is not a singleton then $(m_1 \triangleright m_2)(A) = 0$.

Consider any $A \subseteq \mathbf{X}_{K_1 \cup K_2}$ that is not a singleton. Therefore there must exist two different elements $x, y \in A$. Since $x \neq y$ then either $x^{\downarrow K_1} \neq y^{\downarrow K_1}$ or $x^{\downarrow K_2} \neq y^{\downarrow K_2}$ (or both). Therefore, either $A^{\downarrow K_1}$ or $A^{\downarrow K_2}$ is not a singleton and therefore $m_1(A^{\downarrow K_1}) \cdot m_2(A^{\downarrow K_2}) = 0$. This means that if $m_2^{\downarrow K_1 \cap K_2}(A^{\downarrow K_1 \cap K_2}) > 0$ (notice that $A^{\downarrow K_1 \cap K_2}$ can be a singleton) then, due to Definition 2, $(m_1 \triangleright m_2)(A) = 0$.

If $m_2^{\downarrow K_1 \cap K_2}(A^{\downarrow K_1 \cap K_2}) = 0$ then, because we assume the validity of implication (12),$m_1^{\downarrow K_1 \cap K_2}(A^{\downarrow K_1 \cap K_2}) = 0$ and therefore also $m_1(A^{\downarrow K_1}) = 0$. Therefore, according to Definition 2, $(m_1 \triangleright m_2)(A) = 0$ as well. So, we have proven that $m_1 \triangleright m_2$ is Bayesian.

Now, consider a singleton $\{x\}$ for some $x \in \mathbf{X}_{K_1 \cup K_2}$. If $m_2^{\downarrow K_1 \cap K_2}(\{x\}^{\downarrow K_1 \cap K_2}) = p_2(x^{\downarrow K_1 \cap K_2}) > 0$, point [a] of Definition 2 yields

$$(m_1 \triangleright m_2)(\{x\}) = \frac{m_1(\{x\}^{\downarrow K_1}) \cdot m_2(\{x\}^{\downarrow K_2})}{m_2^{\downarrow K_1 \cap K_2}(\{x\}^{\downarrow K_1 \cap K_2})} = \frac{p_1(x^{\downarrow K_1}) \cdot p_2(x^{\downarrow K_2})}{p_2^{\downarrow K_1 \cap K_2}(x^{\downarrow K_1 \cap K_2})} = (p_1 \triangleright p_2)(x)$$

by Definition 4. Similarly, if $m_2^{\downarrow K_1 \cap K_2}(\{x\}^{\downarrow K_1 \cap K_2}) = p_2(x^{\downarrow K_1 \cap K_2}) = 0$, we get according to point[9] [c] of Definition 2

$$(m_1 \triangleright m_2)(\{x\}) = 0,$$

and according to Definition 4

$$(p_1 \triangleright p_2)(x) = \frac{p_1(x^{\downarrow K_1}) \cdot p_2(x^{\downarrow K_2})}{p_2^{\downarrow K_1 \cap K_2}(x^{\downarrow K_1 \cap K_2})} = \frac{0 \cdot 0}{0} = 0,$$

which finishes the proof. □

The reader should, however, notice that the definition of the operator of composition for Bayesian basic assignments is not fully equivalent to the definition of composition for probabilistic distributions. They equal each other only if the probabilistic version is defined. This is anchored in Lemma 9 by assuming the implication (12). In case it does not hold, the probabilistic operator is not defined, even though its belief version introduced in this paper is always defined. Nevertheless, in this case, the result is not a Bayesian assignment. We will illustrate this fact with the aid of a simple example.

Example 5. Let $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ be as in the previous example and consider the following Bayesian basic assignments m_1 and m_2 on $\mathbf{X}_1 \times \mathbf{X}_2$ and $\mathbf{X}_2 \times \mathbf{X}_3$, respectively:

$$m_1(\{(a_1, a_2)\}) = m_1(\{(a_1, \bar{a}_2)\}) = m_1(\{(\bar{a}_1, a_2)\}) = m_1(\{(\bar{a}_1, \bar{a}_2)\}) = 0.25,$$
$$m_2(\{(a_2, a_3)\}) = m_2(\{(a_2, \bar{a}_3)\}) = 0.5,$$
$$m_2(\{(\bar{a}_2, a_3)\}) = m_2(\{(\bar{a}_2, \bar{a}_3)\}) = 0.$$

(Naturally, since m_1 and m_2 are Bayesian, $m_1(A) = m_2(A) = 0$ for any $A \subseteq \mathbf{X}_{\{1,2,3\}}$ for which $|A| > 1$.)

Let us compute $m_1 \triangleright m_2$ for all singletons $\{x_1 x_2 x_3\} \in \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$. If $x_2 = a_2$, then

$$(m_1 \triangleright m_2)(\{(x_1, a_2, x_3)\}) = \frac{m_1(\{(x_1, a_2)\}) \cdot m_2(\{(a_2, x_3)\})}{m_2^{\downarrow 2}(\{a_2\})} = \frac{0.25 \cdot 0.5}{1} = 0.125.$$

[9] Notice that for singleton $\{x\} \subseteq \mathbf{X}_{K_1 \cup K_2}$, $\{x\} = \{x\}^{\downarrow K_1} \bowtie \{x\}^{\downarrow K_2}$ but $\{x\} \neq \{z\}^{\downarrow K_1} \times \mathbf{X}_{K_2 \setminus K_1}$.

For a singleton $\{(x_1, \bar{a}_2, x_3)\}$ we get

$$(m_1 \triangleright m_2)(\{(x_1, \bar{a}_2, x_3)\}) = 0,$$

because $m_2^{\downarrow 2}(\{\bar{a}_2\}) = 0$. In this case, however, we get

$$(m_1 \triangleright m_2)(\{(x_1, \bar{a}_2)\} \times \mathbf{X}_3) = m_1(\{(x_1, \bar{a}_2)\}) = 0.25.$$

In other words, there are six focal elements of $m_1 \triangleright m_2$, namely, four singletons:

$$\{(x_1, a_2, x_3)\} \quad \text{for } x_1 \in \mathbf{X}_1,\ x_3 \in \mathbf{X}_3,$$

and two two-element sets

$$\{(x_1, \bar{a}_2)\} \times \mathbf{X}_3 \quad \text{for } x_1 \in \mathbf{X}_1.$$

Let us remark that in contrast to $m_1 \triangleright m_2,\ m_2 \triangleright m_1$ is a Bayesian basic assignment, because whenever $m_1^{\downarrow\{2\}}(x_2) = 0$ then also $m_2^{\downarrow\{2\}}(x_2) = 0$. Basic assignment $m_1 \triangleright m_2$ has four focal elements:

$$(m_2 \triangleright m_1)(\{(a_1, a_2, a_3)\}) = (m_2 \triangleright m_1)(\{(a_1, a_2, \bar{a}_3)\}) = (m_2 \triangleright m_1)(\{(\bar{a}_1, a_2, a_3)\}) = (m_2 \triangleright m_1)(\{(\bar{a}_1, a_2, \bar{a}_3)\}) = 0.25.$$

7. Conclusions

In this paper we have introduced a new definition of conditional independence for basic assignments (and thus also for belief functions, though we even have not recalled the notion of a belief function in the paper). The new concept is closely related to previously defined notions of conditional non-interactivity of Ben Yahlane et al. [4] but these two notions are not equivalent to each other. They coincide only for unconditional independence and for conditional independence of Bayesain basic assignments. In general, although each of these concepts meets semigraphoid axioms, they differ from each other. Since the newly introduced notion does not suffer from the drawback explained in Example 3, we believe it better corresponds to the requirements laid on the notion of conditional independence.

The newly introduced definition was motivated by the operator of composition, which was for belief functions (or, more precisely, for basic assignments) originally introduced in [13]. This operator, which was designed for construction of multidimensional models, was formerly introduced in probability theory [12] and later also in possibility theory [22]. Since it is well known that probability and possibility theories are (in a way) special cases of evidence theory, a legitimate question arises whether the operator of composition introduced for basic assignments corresponds in these special cases to the operators introduced in the respective theories. While for probability theory the positive answer was presented in Section 6, for possibility theory, i.e., for consonant bodies of evidence, the situation is quite different. The situation is much more complicated, since the operator of composition in possibility theory is parameterized by a continuous t-norm. One could hardly expect that the possibilistic operator of composition would be a special case of the one for basic assignments for any continuous t-norm. Nevertheless, it should hold for one of them and if it were so, this relationship would help us to distinguish among the t-norms (and consequently also among resulting models). Unfortunately, the situation is substantially different. If we compose (following Definition 2) two basic assignments corresponding to consonant non-vacuous bodies of evidence on different frames of discernment, the resulting basic assignment *never* corresponds to a consonant body of evidence. In other words: application of Definition 2 to possibility distributions leads to results beyond the possibilistic framework. This is so because the independence concept in evidence theory does not preserve consonancy. For more details see [23].

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