Combination and Soft-Normalization of Belief Functions on MV-Algebras

Tommaso Flaminio[1], Lluís Godo[1], and Tomáš Kroupa[2]

[1] IIIA Artificial Intelligence Research Institute (CSIC)
Campus UAB s/n, Bellaterra 08193, Spain
{tommaso,godo}@iiia.csic.es
[2] Institute of Information Theory and Automation of the ASCR
Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
kroupa@utia.cas.cz

Abstract. Extending the notion of belief functions to fuzzy sets leads to the generalization of several key concepts of the classical Dempster-Shafer theory. In this paper we concentrate on characterizing normalized belief functions and their fusion by means of a generalized Dempster rule of combination. Further, we introduce soft-normalization that arises by either rising up the usual level of contradiction above 0, or by decreasing the classical level of normalization below 1.

1 Introduction

The *Dempster-Shafer theory of evidence* [4,13] is a generalization of Bayesian probability theory that allows to combine all the available informations about a given event E into a unique one. The theory shows how all the available evidences can be used to evaluate the degree of belief of E via a *belief function bel*. In fact, in the classical setting, pieces of evidence are encoded by means of subsets of a fixed domain X called the *frame of discernment.* To each piece of evidence (i.e. to each subset of X) is attached a weight (called *mass* in Dempster-Shafer theory) that is given by a probability distribution m defined over the powerset 2^X. If a subset is assigned a strictly positive mass, it is called a *focal element.*

Specifically, our belief is encoded by a mass assignment $m : 2^X \to [0,1]$, that is, $\sum_{B\in 2^X} m(B) = 1$ and $m(\emptyset) = 0$. Its associated *belief function* $bel : 2^X \to [0,1]$ attaches to each $A \in 2^X$ the sum of the masses of those pieces of evidence supporting A, i.e.

$$bel(A) = \sum_{B\subseteq A} m(B). \tag{1}$$

It is worth noticing that, since every mass m is a probability distribution over 2^X, the belief of A can be equivalently defined as

$$bel(A) = P_m(\beta_A) \tag{2}$$

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where P_m is the probability measure defined over $2^{(2^X)}$ and β_A is the characteristic function[1] of the inclusion set $\{B \in 2^X : B \subseteq A\}$.

Recently, several generalizations of belief function theory to the algebraic setting of MV-algebras of continuous fuzzy sets have been proposed [9,5]. The soft-computational setting of fuzzy sets and the related algebraic framework open the door to the generalization of the key concepts that form the basis of classical Dempster-Shafer theory. In this paper, after some needed preliminaries on MV-algebras of fuzzy sets and finitely additive measures on them, called *states*, we first recall those generalized notions of belief functions. For the particular class of belief functions whose focal elements are crisp, we also study their Möbious transform. Then, always in the generalized setting of MV-algebras of continuous fuzzy sets, we discuss the notion of normalized belief function and characterize it in terms of the support of the state underlying it. Finally, after recaling some generalized forms of the Dempster rule of combination (not only conjunctive), we consider a notion of *soft-normalization* that arises by either rising up above 0 the usual levels of contradiction, or by decreasing the classical level of normalization below 1.

2 MV-Algebras of Fuzzy Sets and States

MV-algebras were introduced by Chang [1] as the equivalent algebraic semantics for the infinite-valued Łukasiewicz calculus. They are algebraic structures $M = (M, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following requirements: the reduct $(M, \oplus, 0)$ is a commutative monoid, and for every $a, b \in M$, the following equations hold: $\neg\neg a = a$, $a \oplus \neg 0 = \neg 0$ and $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

It is well known [2] that the class of MV-algebras forms an algebraic variety. Moreover, in every MV-algebra the following operations are definable: $a \odot b$ is $\neg(\neg a \oplus \neg b)$; $a \Rightarrow b$ is $\neg a \oplus b$; $a \vee b$ is $(a \Rightarrow b) \Rightarrow b$, $a \wedge b$ is $\neg(\neg a \wedge \neg b)$, and the constant 1 stands for $\neg 0$. In every MV-algebra M, a partial order relation is defined as follows: for every $a, b \in M$, $a \leq b$ iff $a \Rightarrow b = 1$. An MV-algebra is said to be linearly ordered (or an MV-chain), if the order $\leq$ is linear.

Example 1. (1) Every Boolean algebra is an MV-algebra. Moreover, for every MV-algebra M, the set of its idempotent elements $B(M) = \{a \in M : a \oplus a = a\}$ is the domain of the largest Boolean subalgebra of M, the so called *Boolean skeleton* of M.

(2) Consider the real unit interval $[0, 1]$ equipped with Łukasiewicz operations: for every $a, b \in [0, 1]$,

$$a \oplus b = \min\{1, a + b\}, \qquad \neg a = 1 - a.$$

Then the structure $[0, 1]_{MV} = ([0, 1], \oplus, \neg, 0)$ is an MV-chain. Chang theorem [1,2] says that an equation holds in $[0, 1]_{MV}$ iff it holds in every MV-algebra.

[1] Throughout the paper, we make no formal distinction between a set and its characteristic function.

It is worth noticing that in $[0,1]_{MV}$ the above introduced operations have the following form:

$$a \odot b = \max\{0, a+b-1\},\; a \Rightarrow b = \min\{1, 1-a+b\},$$
$$a \vee b = \max\{a,b\}, \qquad a \wedge b = \min\{a,b\}.$$

(3) For every $n \in \mathbb{N}$, consider the class F_n of n-place McNaughton functions, i.e. functions from $[0,1]^n$ into $[0,1]$ which are continuous, piecewise linear, each piece having integer coefficient. The algebra $(F_n, \oplus, \neg, \overline{0})$ with operations $\oplus$ and $\neg$ defined pointwise, and where $\overline{0}$ here denotes the zero-constant function, is an MV-algebra that coincides with the free MV-algebra over n generators. We will henceforth denote this algebra by Free(n).

An *MV-clan over a set* X is a collection of functions from X into $[0,1]$ (i.e. a set of fuzzy subsets of X) that contains the zero-constant function and that is closed under the finitary pointwise application of $\oplus$ and $\neg$ as defined in $[0,1]_{MV}$. We will denote by $[0,1]^X$ the clan of all functions from X into $[0,1]$. A clan $M \subseteq [0,1]^X$ is said to be *separating* if for every $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exists a function $f \in M$ such that $f(x_1) \neq f(x_2)$. Clearly, $[0,1]^X$ is separating, and it is well known that for every $n \in \mathbb{N}$, Free(n) is a separating MV-clan as well (cf. [2, §3.6]).

Whenever X is finite, we will call $[0,1]^X$ a *finite domain* MV-clan. Finite domain MV-clans will play a central role in this paper. The following notion of *state* is the MV-counterpart of the notion of a finitely-additive probability measure on a Boolean algebra.

Definition 1 ([11]). *Let M be an MV-algebra. A* state *on M is a map* $\mathbf{s}: M \to [0,1]$ *satisfying* $\mathbf{s}(1) = 1$*, and* $\mathbf{s}(a \oplus b) = \mathbf{s}(a) + \mathbf{s}(b)$ *whenever* $a \odot b = 0$*. A state* $\mathbf{s}$ *is said to be* faithful *if* $\mathbf{s}(x) = 0$ *implies* $x = 0$.

The following theorem, independently proved in [8] and [12], shows an integral representation of states by Borel probability measures defined on the σ-algebra $\mathfrak{B}(X)$ of Borel subsets of X, where X is any compact Hausdorff topological space.

Theorem 1. *Let $M \subseteq [0,1]^X$ be a separating clan of continuous functions over a compact Hausdorff space X. Then there is a one-to-one correspondence between the states on M and the regular Borel probability measures on $\mathfrak{B}(X)$. In particular, for every state* $\mathbf{s}$ *on M, there exists a unique regular Borel probability measure μ on $\mathfrak{B}(X)$ such that for every $f \in M$,*

$$\mathbf{s}(f) = \int_X f \, \mathrm{d}\mu. \tag{3}$$

3 Belief Functions on MV-Algebras of Fuzzy Sets

In what follows we will assume X to be a finite set.

3.1 Crisp-Focal Belief Functions

In [9], the author proposes the following generalization of belief functions. Let $M = [0,1]^X$ be a finite domain MV-clan and consider, for every $f : X \to [0,1]$, the map $\hat{\rho}_f : 2^X \to [0,1]$ defined as follows: for every $B \subseteq X$,

$$\hat{\rho}_f(B) = \min\{f(x) : x \in B\}. \tag{4}$$

Remark 1. Notice that $\hat{\rho}_f$ generalizes β_A in the following sense: whenever $A \in B(M) = 2^X$, then $\hat{\rho}_A = \beta_A$. Namely, for every $A \in B(M)$, $\hat{\rho}_A(B) = 1$ if $B \subseteq A$, and $\hat{\rho}_A(B) = 0$, otherwise.

Definition 2. *A map* $\hat{\mathbf{b}} : M \to [0,1]$ *is called a* crisp-focal belief function *whenever there is a state* $\hat{\mathbf{s}} : [0,1]^{(2^X)} \to [0,1]$ *such that* $\hat{\mathbf{s}}(\{\emptyset\}) = 0$ *and, for every* $f \in M$,

$$\hat{\mathbf{b}}(f) = \hat{\mathbf{s}}(\hat{\rho}_f). \tag{5}$$

With X being finite, Theorem 1 yields a unique probability measure $\mu : 2^{(2^X)} \to [0,1]$ such that $\hat{\mathbf{s}}(\hat{\rho}_f) = \sum_{C \in 2^X} \hat{\rho}_f(C) \cdot \mu(\{C\})$. Moreover, it is easy to see that, for every $C \subseteq 2^X$, $\mu(\{C\}) = \hat{\mathbf{s}}(\{C\})$. Since $\mu(\{\emptyset\}) = 0$, probability measure μ induces a mass assignment m such that $m(C) = \mu(\{C\})$.

In Dempster-Shafer theory, given a belief function $bel : 2^X \to [0,1]$, the mass m that defines bel can be recovered from bel by Möbius transform:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} bel(B).$$

In case of crisp-focal belief functions, the situation is analogous.

Proposition 1. *Let* $\hat{\mathbf{b}} : [0,1]^X \to [0,1]$ *be a crisp-focal belief function, defined as* $\hat{\mathbf{b}}(f) = \hat{\mathbf{s}}(\rho_f)$ *for some state* $\hat{\mathbf{s}}$ *on* $[0,1]^{2^X}$ *such that* $\hat{\mathbf{s}}(\{\emptyset\}) = 0$ *and* $\hat{\mathbf{s}}(\{C\}) > 0$ *iff* $C(x) \in \{0,1\}$, *where* $C \neq \emptyset$. *Then*

$$\hat{\mathbf{s}}(\{A\}) = m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \hat{\mathbf{b}}(B)$$

for each $A \subseteq X$.

Proof. Definition (5) directly gives that $\hat{\rho}_A(C), \in \{0,1\}$ for each pair of crisp sets $A, C \subseteq X$ and thus

$$\hat{\mathbf{b}}(A) = \sum_{C \in 2^X} \hat{\rho}_A(C) \cdot \hat{\mathbf{s}}(\{C\}) = \sum_{B \subseteq A} \hat{\mathbf{s}}(\{B\}) = \sum_{B \subseteq A} m(B).$$

This implies that the restriction of $\hat{\mathbf{b}}$ to 2^X is a classical belief function. See [10] for further details. □

As a corollary, observe that, in the hypothesis of the above proposition, the values $\hat{\mathbf{b}}(f)$ for non-crisp $f \in [0,1]^X$ are necessarily determined by the values of $\hat{\mathbf{b}}$ over the crisp sets of 2^X. Indeed, in [9] it shown that, for any $f \in [0,1]^X$, $\hat{\mathbf{b}}(f)$ is in fact the Choquet integral of f with respect to the restriction of $\hat{\mathbf{b}}$ over 2^X.

Moreover, this shows another characterization of crisp-focal belief functions. Indeed, a function $bel : [0,1]^X \to [0,1]$ is a crisp-focal belief function iff its restriction on crisp sets 2^X is a total monotone function, i.e., for every natural n and every $A_1, \ldots, A_n \in 2^X$, the following inequality holds:

$$bel\left(\bigvee_{i=1}^{n} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot bel\left(\bigwedge_{k \in I} A_k\right).$$

3.2 General Belief Functions

The definition introduced in [5] generalizes crisp-focal belief function by introducing, for every $f \in M$, a map ρ_f associating with each *fuzzy set* $g \in M$ the degree of inclusion of g into f. Specifically, let $M = [0,1]^X$ be a finite domain MV-clan, and consider, for every $f \in M$, the map $\rho_f : M \to [0,1]$ defined as follows: for every $g \in M$,

$$\rho_f(g) = \min\{g(x) \Rightarrow f(x) : x \in X\}. \tag{6}$$

The choice of $\Rightarrow$ in the above definition is due to the MV-algebraic setting, but other choices could be made in other fuzzy logics.

Those mappings ρ_f can be regarded as *generalized inclusion operators* between fuzzy sets (cf. [5] for further details). For every $f \in \{0,1\}^X$ (i.e. whenever f is identified with a vector in $[0,1]^X$ with *integer* components), the map $\rho_f : [0,1]^X \to [0,1]$ is a pointwise minimum of finitely many linear functions with integer coefficients, and hence ρ_f is a non-increasing McNaughton function [2].

Lemma 1. *The MV-algebra $\mathcal{R}_{\mathbf{2}}$ generated by the set $\varrho_{\mathbf{2}} = \{\rho_a : a \in \{0,1\}^X\}$ coincides with* Free(n)*, where n is the cardinality of X.*

Proof. By [3, Theorem 3.13], if a variety $\mathbb{V}$ of algebras is generated by an algebra A, then the free algebra over a cardinal $n > 0$ is, up to isomorphisms, the subalgebra of A^{A^X} generated by the projection functions $\theta_i : A^X \to A$. Therefore, in order to prove our claim it suffices to show that the projection functions $\theta_1, \ldots, \theta_n$ belong to $\varrho_{\mathbf{2}}$.

Consider, for every $i = 1, \ldots, n$ the point $\bar{i} \in \{0,1\}^X$ such that

$$\bar{i}(j) = \begin{cases} 0, & \text{if } j = i \\ 1, & \text{otherwise.} \end{cases}$$

Then $\rho_{\bar{i}} = 1 - \theta_i$. In fact, for every $b \in [0,1]^X$, and for every $i, j \in X$ such that $j \neq i$, we have $b(j) \to \bar{i}(j) = 1$, and $b(i) \to \bar{i}(i) = 1 - b(i)$, so that $1 - \rho_{\bar{i}(b)} = \theta_i(b) = b(i)$. This actually shows that the MV-algebra $\mathcal{R}_{\mathbf{2}}^{\neg}$ generated by the set $\neg\varrho_{\mathbf{2}} = \{1 - \rho_a : a \in \{0,1\}^X\}$ is isomorphic to Free(n). Clearly $\mathcal{R}_{\mathbf{2}}$ and $\mathcal{R}_{\mathbf{2}}^{\neg}$ are isomorphic through the map $g : a \in \mathcal{R}_{\mathbf{2}} \mapsto 1 - a \in \mathcal{R}_{\mathbf{2}}^{\neg}$. □

Therefore, if we consider the MV-algebra $\mathcal{R}$ generated by $\varrho = \{\rho_f : f \in [0,1]^X\}$ we obtain a semisimple MV-algebra that properly extends Free(n), and whose elements are continuous functions from $[0,1]^X$ into $[0,1]$. This implies, in particular, that $\mathcal{R}$ is separating.

Definition 3. *Let X be a finite set and let $M = [0,1]^X$. A map $\mathbf{b} : M \to [0,1]$ will be called a* belief function *on the finite domain MV-clan M provided there exists a state $\mathbf{s} : \mathcal{R} \to [0,1]$ such that for every $a \in M$,*

$$\mathbf{b}(a) = \mathbf{s}(\rho_a). \tag{7}$$

We will denote by $Bel(M)$ the class of all the belief functions over a finite domain MV-clan M.

Note that if $\mathbf{s}$ is such that the set $\{f \in M \mid \mathbf{s}(\{f\}) > 0\}$ is countable, then the above expression yields

$$\mathbf{b}(a) = \sum_{f \in M} \rho_a(f) \cdot \mathbf{s}(\{f\}).$$

As in the previous section, we will identify the mass of a belief function $\mathbf{b}$ with the unique Borel regular probability measure μ over $\mathfrak{B}([0,1]^X)$ that represents the state $\mathbf{s}$ via Theorem 1.

Since belief functions on $[0,1]^X$ are defined as states on $\mathcal{R}$ and different states $\mathbf{s}_1$ and $\mathbf{s}_2$ determine different belief functions $\mathbf{b}_1$ and $\mathbf{b}_2$, the set $Bel([0,1]^X)$ of belief functions on $[0,1]^X$ is in 1-1 correspondence with the set $\mathcal{S}(\mathcal{R})$ of all states on $\mathcal{R}$. Hence $Bel([0,1]^X)$ is a compact convex subset of $[0,1]^{[0,1]^X}$. Therefore Krein-Mil'man theorem shows that $Bel([0,1]^X)$ is in the closed convex hull of its extremal points. The following result characterizes the extremal points of $Bel([0,1]^X)$.

Proposition 2. *For every $x \in [0,1]^X$, the belief function $\mathbf{b}_x$ defined by*

$$\mathbf{b}_x(f) = \mathbf{s}_x(\rho_f) = \rho_f(\{x\}), \quad f \in [0,1]^X, \tag{8}$$

is an extremal point of $Bel([0,1]^X)$.

Proof. A belief function $\mathbf{b} \in Bel([0,1]^X)$ is extremal iff its state assignment is extremal in $\mathcal{S}(\mathcal{R})$. In fact $\mathbf{s}$ is not extremal iff there exist $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S}(\mathcal{R})$ and a real number $\lambda \in (0,1)$ such that $\mathbf{s} = \lambda \mathbf{s}_1 + (1-\lambda)\mathbf{s}_2$. In particular, for every $a \in [0,1]^X$,

$$\mathbf{b}(a) = \mathbf{s}(\rho_a) = \lambda \mathbf{s}_1(\rho_a) + (1-\lambda)\mathbf{s}_2(\rho_a) = \lambda \mathbf{b}_1(a) + (1-\lambda)\mathbf{b}_2(a),$$

whence $\mathbf{b}$ would not be extremal as well.

□

As we recalled above, $\mathcal{R}$ is separating. Therefore from Proposition 2 the extreme points of its state space are MV-homomorphisms $\mathbf{s}_x$, for each $x \in [0,1]^X$. Hence the following holds due to (8).

Theorem 2. *Every belief function is a pointwise limit of a convex combination of some elements $\rho_{.}(x^1),\ldots, \rho_{.}(x^k)$, where $x^1,\ldots,x^k \in [0,1]^X$.*

3.3 On Normalized Belief Functions

In classical Dempster-Shafer theory, the notion of focal element is crucial for classifying belief functions. Whenever $X = \{1, \ldots, n\}$ is a finite set, the Boolean algebra 2^X is finite, and hence the mass assignment $m : 2^X \to [0,1]$ defines obviously only finitely many focal elements. On the other hand, the MV-algebra $[0,1]^X$ has uncountably many elements, and hence we cannot find in general a mass assignment μ defined over $\mathfrak{B}([0,1]^X)$ that defines a belief function $\mathbf{b}$ through (10) which is supported by a finite set. This observation leads to the following definition.

Definition 4. *Let $\mathcal{K}$ be the set of all compact subsets of a finite domain MV-clan $[0,1]^X$. For every regular Borel probability measure μ defined on $\mathfrak{B}([0,1]^X)$, we call the set*

$$\mathrm{spt}\mu = \bigcap \{K | K \in \mathcal{K}, \mu(K) = 1\}$$

the support of μ.

By Theorem 1 we can regard $\mathrm{spt}\mu$ as the support of the state $\mathbf{s}$ defined from μ via (3). In particular, the following holds:

$$\mathbf{b}(a) = \int_{[0,1]^X} \rho_a \, \mathrm{d}\mu = \int_{\mathrm{spt}\mu} \rho_a \, \mathrm{d}\mu. \tag{9}$$

Therefore, for a belief function $\mathbf{b}$ on $[0,1]^X$ whose state assignment $\mathbf{s}$ is characterized through (3) by a regular Borel probability measure μ, we will henceforth refer to $\mathrm{spt}\mu$ as the set of focal elements of $\mathbf{b}$. We restrict our attention to those belief functions on $[0,1]^X$ such that their state assignment $\mathbf{s}$ on Free(n) satisfies the condition

$$\mathbf{b}(\overline{0}) = \mathbf{s}(\rho_{\overline{0}}) = 0. \tag{10}$$

Proposition 3. *The set $\mathcal{S}_0$ of all states on $\mathcal{R}$ satisfying (10) is a nonempty compact convex subset of $[0,1]^{\mathcal{R}}$ considered with its product topology.*

Proof. $\mathcal{S}_0$ is nonempty: let $\mathbf{s}_1$ be defined by

$$\mathbf{s}_1(f) = f(1, \ldots, 1),$$

for every $f \in \mathcal{R}$. This gives $\mathbf{s}_1(\rho_{\overline{0}}) = \rho_{\overline{0}}(1, \ldots, 1) = 0$ and thus $\mathbf{s}_1 \in \mathcal{S}_0$. Let $\mathbf{s}, \mathbf{s}' \in \mathcal{S}_0$ and $\alpha \in (0,1)$. Then the function $\mathcal{R} \to [0,1]$ given by

$$\alpha\mathbf{s} + (1-\alpha)\mathbf{s}'$$

is a state on $\mathcal{R}$ which clearly satisfies (10). Hence $\mathcal{S}_0$ is a convex subset of the product space $[0,1]^{\mathcal{R}}$. Since the space $[0,1]^{\mathcal{R}}$ is compact, we only need to show that $\mathcal{S}_0$ is closed (in its subspace product topology). To this end, consider a convergent sequence $(\mathbf{s}_m)_{m\in\mathbb{N}}$ in $\mathcal{S}_0$ whose limit is $\mathbf{s}$. As the set of all states on $\mathcal{R}$ is closed, $\mathbf{s}$ is a state. That $\mathbf{s}$ satisfies (10) follows from $\mathbf{s}(\rho_{\overline{0}}) = \lim_{m\to\infty} \mathbf{s}_m(\rho_{\overline{0}}) = 0$. □

The family of states $\mathcal{S}_0$ can be characterized by employing integral representation of states. Namely, we will show that a state assignment $\mathbf{s} \in \mathcal{S}_0$ iff $\mathbf{s}$ is "supported" by normal fuzzy sets in $[0,1]^X$, i.e. fuzzy sets $f \in [0,1]^X$ such that $f(x) = 1$ for some $x \in X$. We will denote by $\mathcal{NF}(X)$ the set of normalized fuzzy sets from $[0,1]^X$, i.e.

$$\mathcal{NF}(X) = \{f \in [0,1]^X \mid f(x) = 1 \text{ for some } x \in X\}.$$

Proposition 4. *Let* $\mathbf{s}$ *be a state assignment on* $\mathcal{R}$ *and* μ *be the regular Borel probability measure associated with* $\mathbf{s}$*. Then* $\mathrm{spt}\mu \subseteq \mathcal{NF}(X)$ *if and only if* $\mathbf{s} \in \mathcal{S}_0$*.*

Proof. Let μ be a probability measure on Borel subsets of $[0,1]^X$ such that $\mathrm{spt}\mu \subseteq \mathcal{NF}(X)$. Put

$$\mathbf{s}(f) = \int_{[0,1]^X} f \, \mathrm{d}\mu, \quad f \in \mathcal{R}. \tag{11}$$

Since $\rho_{\overline{0}}(x) = 0$ for each $x \in \mathrm{spt}\mu$, it follows that

$$\mathbf{s}(\rho_{\overline{0}}) = \int_{\mathrm{spt}\mu} \rho_{\overline{0}} \, \mathrm{d}\mu = 0,$$

hence $\mathbf{s} \in \mathcal{S}_0$. Conversely, assume that

$$\mathbf{s}(\rho_{\overline{0}}) = \int_{[0,1]^X} \rho_{\overline{0}} \, \mathrm{d}\mu = 0,$$

which implies $\rho_{\overline{0}} = 0$ μ-almost everywhere over $[0,1]^X$. Since $\rho_{\overline{0}}(x) = 0$ iff $x \in [0,1]^X$ is such that $x_i = 1$, for some $i \in X$, we obtain $\mu(\mathcal{NF}(X)) = 1$. □

In particular, every state assignment of a generalized belief function in the sense of [9] belongs to $\mathcal{S}_0$.

4 Generalized Dempster Rule of Combination

In [5] the authors present a way to generalize the well-known Dempster rule to combine the information carried by two belief functions $\mathbf{b}_1, \mathbf{b}_2 \in Bel(M)$, into a belief function $\mathbf{b}_{1,2} \in Bel(M)$. In this section we will recall the basic steps of that construction, and we also add some remarks. We start with an easy result about the definition of states in a product space.

Proposition 5. *For every MV-algebra* N*, and for every pair of states* $\mathbf{s}_1, \mathbf{s}_2 : N \to [0,1]$*, there exists a state* $\mathbf{s}_{1,2}$ *defined on the direct product* $N \times N$ *such that for every* $(b,c) \in N \times N$*,* $\mathbf{s}_{1,2}(b,c) = \mathbf{s}_1(b) \cdot \mathbf{s}_2(c)$*.*

Let now $M = [0,1]^X$, and let $\mathcal{R}$ be as defined in Section 3. Also let $\mathbf{s}_1, \mathbf{s}_2$ be two states on $\mathcal{R}$ such that $\mathbf{b}_1(f) = \mathbf{s}_1(\rho_f)$ and $\mathbf{b}_2(f) = \mathbf{s}_2(\rho_f)$ for all $f \in M$. Furthermore, let $\mu_1, \mu_2 : \mathfrak{B}(M) \to [0,1]$ be the two regular probability measures of support $\mathrm{spt}\mu_i$ (for $i = 1,2$), such that for $i = 1,2$,

$$\mathbf{s}_i(f) = \int_{\mathrm{spt}\mu_i} f \, \mathrm{d}\mu_i.$$

Take the mapping $\mu_{1,2} : \mathfrak{B}(M \times M) \to [0,1]$ to be the product measure on Borel subsets generated by $M \times M$. Let $\mathbf{s}_{1,2}$ be a state on $[0,1]^{M\times M}$ defined by integrating measurable functions $M \times M \to [0,1]$ with respect to μ. If there exist $g, h : M \to [0,1]$ and f such that $f(x,y) = g(x) \cdot h(y)$, then Proposition 5 yields $\mathbf{s}_{1,2}(f) = \mathbf{s}_1(g) \cdot \mathbf{s}_2(h)$.

Finally, for every $f \in M$, consider the map $\rho_f^{\wedge} : M \times M \to [0,1]$ defined by $\rho_a^{\wedge}(b,c) = \rho_a(b \wedge c)$. Then we are ready to define the following combination of belief functions.

Definition 5 (Generalized Dempster rule). *Given* $\mathbf{b}_1, \mathbf{b}_2 \in Bel(M)$ *as above, define its* min*-conjunctive combination* $\mathbf{b}_{1,2} : M \to [0,1]$ *as follows: for all* $a \in M$*,*

$$\mathbf{b}_{1,2}(a) = \mathbf{s}_{1,2}(\rho_a^{\wedge}). \tag{12}$$

Regarding the support of the combined measure, it is worth noticing that by [6, Theorem 417C (v)], $\text{spt}\mu_{1,2} = \text{spt}\mu_1 \times \text{spt}\mu_2$, and hence, whenever μ_1 and μ_2 are normalized in the sense that their support is included into $\mathcal{NF}(X)$, $\text{spt}\mu_{1,2} \subseteq \mathcal{NF}(X)$ as well. Therefore, by Proposition 4 one might deduce that, if $\mathbf{b}_1$ and $\mathbf{b}_2$ are normalized belief functions, then $\mathbf{b}_{1,2}$ is normalized as well. The following example shows that it is not the case, since in the definition of $\mathbf{b}_{1,2}$, together with the product measure $\mu_{1,2}$ we also use the map $\rho^{\wedge}$ which, in fact, is not a genuine fuzzy-inclusion operator.

Example 2. Consider two belief functions $\mathbf{b}_1$ and $\mathbf{b}_2$ on $[0,1]^2$ with masses concentrated as follows:

$$\mu_1(1,0) = 1/4;\ \mu_1(1,1) = 3/4;\ \mu_2(0,1) = 1/2;\ \mu_2(1,1) = 1/2.$$

Then, the product measure $\mu_{1,2}$ has support in the cartesian product of the supports of the two masses, $\{((1,0),(0,1)), ((1,0),(1,1)), ((1,1),(0,1)), ((1,1),(1,1))\}$, and it takes values

$$\mu_{1,2}((1,0),(0,1)) = 1/8,\ \mu_{1,2}((1,0),(1,1)) = 1/8,\ \mu_{1,2}((1,1),(0,1)) = 3/8,$$
$$\mu_{1,2}((1,1),(1,1)) = 3/8.$$

So, $\mu_{1,2}$ is normalized in the sense that each of its focal elements can be regarded as a normalized vector of $[0,1]^4$. On the other hand, $\mathbf{b}_{1,2}$ is not normalized because $(0,0) = (1,0) \wedge (0,1)$, $\rho_{(0,0)}(0,0) = 1$, and hence

$$\mathbf{b}(0,0) = \sum_{b \wedge c = (0,0)} \rho_{(0,0)}((0,0)) \cdot \mu_1(b) \cdot \mu_2(c) = \rho_{(0,0)}(0,0) \cdot \mu_1(1,0) \cdot \mu_2(0,1) = 1/8 > 0.$$

The above min-conjunctive combination can easily be extended to well-known MV-operations on fuzzy sets, such as max-disjunction $\vee$, strong conjunction $\odot$ and strong disjunction $\oplus$, by defining $(b_1 \circledast b_2)(a) = s_{1,2}(\rho_a^{\circledast})$, for $\circledast$ being one of these operations, and defining $\rho_a^{\circledast}(b,c) = \rho_a(b \circledast c)$. In this generalized case, the map $\mathbf{b}_{1,2}^{\circledast}$ resulting from the respective combination rule will be called the $\circledast$*-combination of* $\mathbf{b}_1$ *and* $\mathbf{b}_2$.

Whenever the supports of μ_1 and μ_2 are countable, it is easy to prove that $\mathbf{b}^{\circledast}_{1,2}$ is a belief function in the sense of Definition 3. In fact, in this case Definition 5 yields $\mathbf{b}^{\circledast}_{1,2}(a) = \sum_{b,c\in M} \rho_a(b \circledast c) \cdot \mu_1(\{b\}) \cdot \mu_2(\{c\})$. Notice that the above expression reduces to

$$\mathbf{b}_{1,2}(a) = \sum_{d\in M} \sum_{b,c\in M, b\circledast c=d} \rho_a(d) \cdot (\mu_1(\{b\}) \cdot \mu_2(\{c\})) = \sum_{d\in M} \rho_a(d) \cdot \mu^*(\{d\}),$$

where

$$\mu^*(\{d\}) = \sum_{b,c\in M, b\circledast c=d} \mu_1(\{b\}) \cdot \mu_2(\{c\})$$

is indeed a mass assignment and hence $\mathbf{b}^{\circledast}_{1,2} \in Bel(M)$.

Therefore, turning back to the above Example 2 and Proposition 4, there exists a mass $\mu \neq \mu_{1,2}$ for $\mathbf{b}^{\circledast}_{1,2}$ such that $\mathrm{spt}\mu \not\subseteq \mathcal{NF}(X)$.

5 Soft Normalization for Mass Assignments

The *height* of a fuzzy set $f \in [0,1]^X$ is defined in the literature as

$$h(f) = \max\{f(x) : x \in X\}. \tag{13}$$

The value $h(f)$ can be interpreted as the degree of normalization of f. As a matter of fact, a fuzzy set f is called normalized whenever $h(f) = 1$, otherwise it is called non-normalized. A non-normalized fuzzy set represents a partially inconsistent information.

Consider now a belief function $\mathbf{b}$ defined by a state with support $\mathrm{spt}\mu$. Assume there exists a focal element $f \in \mathrm{spt}\mu$ with $\mu(\{f\}) > 0$ that is a non-normalized fuzzy set.[2] This means that $\mathbf{b}$ is associating a positive degree of evidence to a (partially) inconsistent information, which is reflected on the value that $\mathbf{b}$ assigns to the $\overline{0}$. Indeed, in such a case we have $\rho_{\overline{0}}(f) > 0$, and hence $\mathbf{b}(\overline{0}) \geq \rho_{\overline{0}}(f)\cdot\mu(\{f\}) > 0$. And in fact it is easy to see that the more inconsistent are the focal elements of $\mathbf{b}$, the greater is the value $\mathbf{b}(\overline{0})$. When events and focal elements are crisp sets (and hence the unique possible not-normalized focal element is $\overline{0}$), normalization consists in redistributing the mass that μ assigns to $\overline{0}$ to the other focal elements of μ (if any).

Dealing with fuzzy focal elements, allows us to introduce a notion of *soft normalization* for belief functions. In particular, it allows a softer redistribution of the masses, depending on two thresholds.

Definition 6. *A mass assignment* $\mu : [0,1]^X \to [0,1]$ *is said to be* α*-normalized provided that* $\inf\{h(f) : f \in \mathrm{spt}\mu\} = \alpha$.

In other words, a mass is α-normalized provided that each focal element of μ has at least height α. In particular, for a belief function $\mathbf{b}$ we define the *degree of normalization* of $\mathbf{b}$ as the value

[2] Notice that if $\mathrm{spt}\mu$ is not countable, the condition $f \in \mathrm{spt}\mu$ does not guarantee $\mu(\{f\}) > 0$.

$$\inf\{h(f) : f \in \mathrm{spt}\mu\},$$

where μ is the mass associated to $\mathbf{b}$.

In what follows we assume masses μ such that their supports sptμ are countable. Let now $\mu : [0,1]^X \to [0,1]$ be an α-normalized mass assignment, and assume that there exists a focal element g for m such that $h(g) = \beta > \alpha$.

The mass μ can be renormalized to the higher degree β by defining a new mass μ^β as follows: for every $f \in \mathcal{F}(X)$,

$$\mu^\beta(\{f\}) = \begin{cases} 0, & \text{if } h(f) < \beta \\ \frac{\mu(\{g\})}{1-K}, & \text{otherwise.} \end{cases} \tag{14}$$

where $K = \sum_{h(l)<\beta} \mu(\{l\})$.

The idea of this *β-normalization*, similarly to the classical normalization, consists in fixing the value β as a new level of consistency for the mass we are considering. Since $\alpha < \beta \leq 1$, the class of focal elements of height lower then β is not empty. Then the process of β-normalization consists in redistributing all the mass $K = \sum_{h(l)<\beta} \mu(\{l\})$, which μ assigns to the fuzzy sets of height lower than β, to those focal elements of height greater of (or equal to) β.

Clearly a mass μ can be renormalized, up to the maximum value

$$\beta_{max} = \sup\{h(f) : f \in \mathrm{spt}(\mu)\}.$$

Consider two belief functions $\mathbf{b}_1$ and $\mathbf{b}_2$ with associated masses μ_1 and μ_2 respectively, also we assume for simplicity sptμ_1 and sptμ_2 to be countable. Let $\mathbf{b}_{1,2}^{\circledast}$ the belief function defined by the $\circledast$-combination of $\mathbf{b}_1$ and $\mathbf{b}_2$ as we introduced in Section 4. Then the focal elements of $\mathbf{b}_{1,2}^{\circledast}$ forms the following set:

$$\{f \circledast g : f \in \mathrm{spt}\mu_1 \text{ and } g \in \mathrm{spt}\mu_2\}.$$

Therefore for each focal element $f \circledast g$ of $\mathbf{b}_{1,2}^{\circledast}$, its height is easily calculated as $h(f \circledast g) = \max\{f(x) \circledast g(x) : x \in X\}$. Therefore the level to which a $\circledast$ combined belief function $\mathbf{b}_{1,2}^{\circledast}$ allows to be normalized can be similarly calculated by the height of the focal elements of the combining functions $\mathbf{b}_1$ and $\mathbf{b}_2$. It is worth to point out that, whenever $\circledast$ is a conjunctive operation (like a t-norm for instance), $h(f \circledast g) \leq \min\{h(f), h(g)\}$.

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