

DECOMPOSABLE PSEUDODISTANCES AND APPLICATIONS IN STATISTICAL ESTIMATION

MICHEL BRONIATOWSKI¹ AIDA TOMA² IGOR VAJDA^{†3}

Abstract

The aim of this paper is to introduce new statistical criteria for estimation, suitable for inference in models with common continuous support. This proposal is in the direct line of a renewed interest for divergence based inference tools imbedding the most classical ones, such as maximum likelihood, Chi-square or Kullback Leibler. General pseudodistances with decomposable structure are considered, they allowing to define minimum pseudodistance estimators, without using nonparametric density estimators. A special class of pseudodistances indexed by $\alpha > 0$, leading for $\alpha \downarrow 0$ to the Kulback Leibler divergence, is presented in detail. Corresponding estimation criteria are developed and asymptotic properties are studied. The estimation method is then extended to regression models. Finally, some examples based on Monte Carlo simulations are discussed.

1 Introduction

In parametric estimation, minimum divergence methods, i.e. methods which estimate the parameter by minimizing an estimate of some divergence between the assumed model density and the true density underlying the data, have been extensively studied (see Pardo (2005) and references herein). Generally, in continuous models, the minimum divergence methods have the drawback that it is necessary to use some nonparametric density estimator. In order to remove this drawback, some proposals have been made in literature. Among them, we recall the minimum density power divergence method introduced by Basu et al. (1998), and a minimum divergence method based on duality arguments, independently proposed by Liese and Vajda (2006) and Broniatowski and Keziou (2009). The results obtained in the present paper follow this line of research.

Without referring to all properties of the divergence criteria, we mainly quote their information processing property, i.e. the complete invariance with respect to the

¹Laboratoire de Statistique Théorique et Appliquée, Université Paris 6, Paris, France, e-mail: michel.broniatowski@upmc.fr

²Mathematics Department, Academy of Economic Studies and *Gh. Mihoc - C. Iacob* Institute of Mathematical Statistics and Applied Mathematics, Bucharest, Romania, e-mail: aida_toma@yahoo.com

³Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague, Czech Republic.

statistically sufficient transformations of the observation space. This property is useful but probably not unavoidable in the minimum divergence estimation based on similarity between theoretical and empirical distributions. In this paper we admit general pseudodistances which may not satisfy the information processing property. The definition of the pseudodistance, which is at the start of this work, pertains to the willingness to define a simple frame including all commonly used statistical criteria, from maximum likelihood to the L_2 norm. Such a description is provided in Broniatowski and Vajda (2009). In the present paper we define a class of pseudodistances indexed by $\alpha > 0$, leading for $\alpha \downarrow 0$ to the Kulback Leibler divergence. The peculiar features of these pseudodistances recommend it as an appealing competing choice for defining estimation criteria. We argue that by defining and studying minimum pseudodistances estimators for classical parametric models, respectively for regression models. We present such tools for inference with a special attention to limit properties and robustness, in a similar spirit as in Toma and Broniatowski (2011).

The outline of the paper is as follows. Section 2 introduces decomposable pseudodistances and define minimum pseudodistances estimators. Section 3 presents a special class of minimum pseudodistances estimators. For these estimators we study invariance properties, consistency, asymptotic normality and robustness. The estimation method is applied to linear models for which asymptotic and robustness properties are derived. These results are presented in Section 4. Finally, in order to illustrate the performance of the proposed method in finite samples, we give some examples based on Monte Carlo simulations.

2 Decomposable pseudodistances and estimators

We will consider inference in continuous parametric families, since this is the interesting and complex case with respect to the case of models with finite or countable support. Hence \mathcal{P} is a parametric model with euclidian parameter space Θ and we assume that all the probability measures P_θ in \mathcal{P} share the same support, which is included in \mathbb{R}^d . Every P_θ has a density p_θ with respect to the Lebesgue measure.

We denote by \mathcal{P}_{emp} the class of probability measures induced by samples, namely the class of all probability measures

$$P_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i},$$

where X_1, \dots, X_n is sampled according to a distribution on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, not necessarily in \mathcal{P} . In addition to the previous notation it is useful to introduce a family of measures \mathcal{P}_0 associated to distributions generating the data when studying robustness properties. Often, such a measure is a mixture of some element in \mathcal{P} with a Dirac measure at some point x in \mathbb{R}^d . We also define $\mathcal{P}^+ := \mathcal{P} \cup \mathcal{P}_0$.

Definition 1 We say that $\mathfrak{D} : \mathcal{P} \otimes \mathcal{P}^+ \mapsto \mathbb{R}$ is a pseudodistance between a probability measures $P \in \mathcal{P} = \{P_\theta : \theta \in \Theta\}$ and $Q \in \mathcal{P}^+$ if $\mathfrak{D}(P_\theta, Q) \geq 0$, for all $\theta \in \Theta$ and $Q \in \mathcal{P}^+$ and $\mathfrak{D}(P_\theta, P_{\tilde{\theta}}) = 0$ if and only if $\theta = \tilde{\theta}$.

Definition 2 A pseudodistance \mathfrak{D} on $\mathcal{P} \otimes \mathcal{P}^+$ is called decomposable if there exist functionals $\mathfrak{D}^0 : \mathcal{P} \mapsto \mathbb{R}$, $\mathfrak{D}^1 : \mathcal{P}^+ \mapsto \mathbb{R}$ and measurable mappings

$$\rho_\theta : \mathbb{R}^d \mapsto \mathbb{R}, \quad \theta \in \Theta \quad (2.1)$$

such that for all $\theta \in \Theta$ and $Q \in \mathcal{P}^+$ the expectations $\int \rho_\theta dQ$ exist and

$$\mathfrak{D}(P_\theta, Q) = \mathfrak{D}^0(P_\theta) + \mathfrak{D}^1(Q) + \int \rho_\theta dQ. \quad (2.2)$$

A known class of pseudodistances is that introduced by Basu et al. (1998) and called the class of power divergences. This class corresponds to

$$\mathfrak{D}(P_\theta, Q) = \int \left\{ p_\theta^{\alpha+1} - \left(1 + \frac{1}{\alpha}\right) p_\theta^\alpha q + \frac{1}{\alpha} q^{\alpha+1} \right\} d\lambda \text{ for } \alpha > 0. \quad (2.3)$$

Note that the pseudodistances (2.3) are decomposable in the sense (2.2) with

$$\mathfrak{D}^0(P_\theta) = \int p_\theta^{\alpha+1} d\lambda, \quad \mathfrak{D}^1(Q) = \frac{1}{\alpha} \int q^{\alpha+1} d\lambda \text{ and } \rho_\theta = - \left(1 + \frac{1}{\alpha}\right) p_\theta^\alpha. \quad (2.4)$$

In the next section, we introduce a new class of pseudodistances from which a new statistical criterion for inference is deduced.

Definition 3 We say that a functional $T_{\mathfrak{D}} : \mathcal{Q} \mapsto \Theta$ for $\mathcal{Q} = \mathcal{P}^+ \cup \mathcal{P}_{emp}$ defines a minimum pseudodistance estimator (briefly, $\min \mathfrak{D}$ -estimator) if the pseudodistance $\mathfrak{D}(P_\theta, Q)$ is decomposable on $\mathcal{P} \otimes \mathcal{P}^+$ and the parameters $T_{\mathfrak{D}}(Q) \in \Theta$ minimize $\mathfrak{D}^0(P_\theta) + \int \rho_\theta dQ$ on Θ , in symbols

$$T_{\mathfrak{D}}(Q) = \arg \inf_{\theta} \left[\mathfrak{D}^0(P_\theta) + \int \rho_\theta dQ \right], \quad \text{for all } Q \in \mathcal{Q}. \quad (2.5)$$

In particular, for $Q = P_n \in \mathcal{P}_{emp}$

$$\hat{\theta}_{\mathfrak{D},n} := T_{\mathfrak{D}}(P_n) = \arg \inf_{\theta} \left[\mathfrak{D}^0(P_\theta) + \frac{1}{n} \sum_{i=1}^n \rho_\theta(X_i) \right]. \quad (2.6)$$

Theorem 1 Every $\min \mathfrak{D}$ -estimator

$$\hat{\theta}_{\mathfrak{D},n} = \arg \inf_{\theta} \left[\mathfrak{D}^0(P_\theta) + \frac{1}{n} \sum_{i=1}^n \rho_\theta(X_i) \right] \quad (2.7)$$

is Fisher consistent in the sense that

$$T_{\mathfrak{D}}(P_{\theta_0}) = \theta_0, \quad \text{for all } \theta_0 \in \Theta. \quad (2.8)$$

Proof. Consider arbitrary fixed $\theta_0 \in \Theta$. Then, by assumptions, $\mathfrak{D}^1(P_{\theta_0})$ is a finite constant. Therefore (2.5) together with the definition of pseudodistance implies

$$\begin{aligned} T_{\mathfrak{D}}(P_{\theta_0}) &= \arg \inf_{\theta} \left[\mathfrak{D}^0(P_{\theta}) + \int \rho_{\theta} dP_{\theta_0} \right] \\ &= \arg \inf_{\theta} \left[\mathfrak{D}^0(P_{\theta}) + \mathfrak{D}^1(P_{\theta_0}) + \int \rho_{\theta} dP_{\theta_0} \right] \\ &= \arg \inf_{\theta} \mathfrak{D}(P_{\theta}, P_{\theta_0}) = \theta_0. \end{aligned}$$

The decomposability of a pseudodistance $\mathfrak{D}(P_{\theta}, Q)$ leads to the additive structure of the empirical version

$$\mathfrak{D}(P_{\theta}, P_n) \sim \mathfrak{D}^0(P_{\theta}) + \int \rho_{\theta} dP_n = \mathfrak{D}^0(P_{\theta}) + \frac{1}{n} \sum_{i=1}^n \rho_{\theta}(X_i) \quad (2.9)$$

in the definition (2.7) of the min \mathfrak{D} -estimators, which opens the possibility to apply the methods of the asymptotic theory of M -estimators (cf. Hampel et al. (1986), van der Vaart and Wellner (1996), van der Vaart (1998) or Mieske and Liese (2008)).

3 A special class of minimum pseudodistance estimators

3.1 Definitions and invariance properties

For probability measures $P \in \mathcal{P}$ and $Q \in \mathcal{P}^+$ consider the following family of pseudodistances of orders $\alpha \geq 0$,

$$\mathfrak{R}_{\alpha}(P, Q) = \frac{1}{1+\alpha} \ln \left(\int p^{\alpha} dP \right) + \frac{1}{\alpha(1+\alpha)} \ln \left(\int q^{\alpha} dQ \right) - \frac{1}{\alpha} \ln \left(\int p^{\alpha} dQ \right). \quad (3.1)$$

The following basic condition which guarantees the finiteness of the pseudodistances $\mathfrak{R}_{\alpha}(P, Q)$ is assumed. For some positive β ,

$$p^{\beta}, q^{\beta}, \ln p \in \mathbb{L}_1(Q) \quad \text{for all } P \in \mathcal{P}, Q \in \mathcal{P}^+, \quad (3.2)$$

where $\mathbb{L}_1(Q) := \{f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ such that } \int |f| dQ < \infty\}$.

We then have:

Theorem 2 *Let the condition (3.2) hold for some $\beta > 0$. Then for all $0 < \alpha < \beta$, $\mathfrak{R}_{\alpha}(P, Q)$ defined in (3.1) is a family of pseudodistances decomposable in the sense*

$$\mathfrak{R}_{\alpha}(P, Q) = \mathfrak{R}_{\alpha}^0(P) + \mathfrak{R}_{\alpha}^1(Q) - \frac{1}{\alpha} \ln \left(\int p^{\alpha} dQ \right), \quad (3.3)$$

where

$$\mathfrak{R}_\alpha^0(P) = \frac{1}{1+\alpha} \ln \left(\int p^\alpha dP \right) \quad \text{and} \quad \mathfrak{R}_\alpha^1(Q) = \frac{1}{\alpha(1+\alpha)} \ln \left(\int q^\alpha dQ \right) \quad (3.4)$$

and the limit relation holds

$$\mathfrak{R}_\alpha(P, Q) \rightarrow \mathfrak{R}_0(P, Q) := \int \ln q dQ - \int \ln p dQ \quad \text{for } \alpha \downarrow 0. \quad (3.5)$$

Proof. Under (3.2), the expressions $\ln(\int q^\alpha dQ)$, $\ln(\int p^\alpha dQ)$ and $\int \ln p dQ$ appearing in (3.1) and (3.5) are finite so that the expressions $\mathfrak{R}_\alpha(P, Q)$ and $\mathfrak{R}_0(P, Q)$ are well defined. Recall that, for arbitrary arguments $s, t > 0$ and fixed parameters $a, b > 0$ with the property $1/a + 1/b = 1$ it holds

$$st \leq \frac{s^a}{a} + \frac{t^b}{b} \quad (3.6)$$

with equality if and only if $s^a = t^b$. Indeed, from the strict concavity of the logarithmic function we deduce the inequality

$$\ln(st) = \frac{1}{a} \ln s^a + \frac{1}{b} \ln t^b \leq \ln \left(\frac{s^a}{a} + \frac{t^b}{b} \right)$$

and the stated condition for equality.

Taking $\alpha > 0$ and substituting

$$s = \frac{p^\alpha}{\left(\int p^{\alpha a} d\lambda \right)^{1/a}}, \quad t = \frac{q}{\left(\int q^b d\lambda \right)^{1/b}} \quad \text{with} \quad a = \frac{1+\alpha}{\alpha}, \quad b = 1+\alpha$$

in the inequality (3.6), and integrating both sides by λ , we obtain the inequality

$$\int p^\alpha q d\lambda \leq \left(\int p^{1+\alpha} d\lambda \right)^{\alpha/(1+\alpha)} \left(\int q^{1+\alpha} d\lambda \right)^{1/(1+\alpha)}$$

with equality if and only if $p^{\alpha a} = q^b$ λ -a.s., i.e. if and only if $p = q$ λ -a.s. Since the expression (3.1) satisfies for $\alpha > 0$ the relation

$$\mathfrak{R}_\alpha(P, Q) = \frac{1}{\alpha} \left\{ \ln \left[\left(\int p^{1+\alpha} d\lambda \right)^{\alpha/(1+\alpha)} \left(\int q^{1+\alpha} d\lambda \right)^{1/(1+\alpha)} \right] - \ln \int p^\alpha q d\lambda \right\}, \quad (3.7)$$

we see that $\mathfrak{R}_\alpha(P, Q)$ is a pseudodistance on the space $\mathcal{P} \otimes \mathcal{P}^+$. The decomposability in the sense of (3.3) on this space is obvious and the limit relation

$$\mathfrak{R}_0(P, Q) = \lim_{\alpha \downarrow 0} \mathfrak{R}_\alpha(P, Q)$$

results as follows:

$$\begin{aligned}
\lim_{\alpha \downarrow 0} \mathfrak{R}_\alpha(P, Q) &= \\
&= \lim_{\alpha \downarrow 0} \frac{1}{\alpha + 1} \ln \left(\int p^\alpha dP \right) + \frac{1}{\alpha(\alpha + 1)} \ln \left(\int q^\alpha dQ \right) - \frac{1}{\alpha} \ln \left(\int p^\alpha dQ \right) \\
&= \lim_{\alpha \downarrow 0} \frac{1}{\alpha + 1} \left[\ln \left(\int p^\alpha dP \right) - \ln \left(\int q^\alpha dP \right) \right] + \frac{1}{\alpha} \left[\ln \left(\int q^\alpha dQ \right) - \ln \left(\int p^\alpha dQ \right) \right] \\
&= \lim_{\alpha \downarrow 0} \frac{1}{\alpha} \ln \frac{\int q^\alpha dQ}{\int p^\alpha dQ} = \int \ln \frac{q}{p} dQ = \mathfrak{R}_0(P, Q).
\end{aligned}$$

Similarly as earlier in this paper, we are interested in the estimators obtained by replacing the hypothetical probability measure P_{θ_0} in the \mathfrak{R}_α -pseudodistances $\mathfrak{R}_\alpha(P_\theta, P_{\theta_0})$ by the empirical measure P_n . Consider therefore the family of minimum pseudodistance estimators of orders $0 \leq \alpha \leq \beta$ (in symbols, $\min \mathfrak{R}_\alpha$ -estimators) defined as $\hat{\theta}_n = T_\alpha(P_n)$ for $T_\alpha(Q) \in \Theta$ with $Q \in \mathcal{Q} = \mathcal{P}^+ \cup \mathcal{P}_{\text{emp}}$ defined by

$$T_\alpha(Q) = \begin{cases} \arg \inf_\theta \left[\frac{1}{1+\alpha} \ln \left(\int p_\theta^\alpha dP_\theta \right) - \frac{1}{\alpha} \ln \left(\int p_\theta^\alpha dQ \right) \right] & \text{if } 0 < \alpha \leq \beta \\ \arg \inf_\theta - \int \ln p_\theta dQ & \text{if } \alpha = 0. \end{cases} \quad (3.8)$$

The upper formula is equivalent to

$$T_\alpha(Q) = \arg \sup_\theta \frac{\int p_\theta^\alpha dQ}{C_\alpha(\theta)} \quad (3.9)$$

where

$$C_\alpha(\theta) = \left(\int p_\theta^{1+\alpha} d\lambda \right)^{\alpha/(1+\alpha)}. \quad (3.10)$$

Hence, alternatively, we can write

$$\hat{\theta}_n = \begin{cases} \arg \sup_\theta \left[-\frac{1}{\alpha+1} \ln \left(\int p_\theta^\alpha dP_\theta \right) + \frac{1}{\alpha} \ln \left(\frac{1}{n} \sum_{i=1}^n p_\theta^\alpha(X_i) \right) \right] & \text{if } 0 < \alpha \leq \beta \\ \arg \sup_\theta \frac{1}{n} \sum_{i=1}^n \ln p_\theta(X_i) & \text{if } \alpha = 0 \end{cases} \quad (3.11)$$

or

$$\hat{\theta}_n = \begin{cases} \arg \sup_\theta C_\alpha(\theta)^{-1} \frac{1}{n} \sum_{i=1}^n p_\theta^\alpha(X_i) & \text{if } 0 < \alpha \leq \beta \\ \arg \sup_\theta \frac{1}{n} \sum_{i=1}^n \ln p_\theta(X_i) & \text{if } \alpha = 0. \end{cases} \quad (3.12)$$

Note that, for $\alpha \downarrow 0$, the upper criterion function in (3.11) tends to the lower maximum likelihood criterion. Indeed,

$$\lim_{\alpha \rightarrow 0} \left[-\frac{1}{\alpha + 1} \ln \left(\int p_\theta^\alpha dP_\theta \right) + \frac{1}{\alpha} \ln \left(\frac{1}{n} \sum_{i=1}^n p_\theta^\alpha(X_i) \right) \right] = \frac{1}{n} \sum_{i=1}^n \ln p_\theta(X_i)$$

by the l'Hospital rule.

In the following, we give some invariance properties of $\min \mathfrak{R}_\alpha$ -estimators.

If the statistical model $\langle (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)); \mathcal{P} = (P_\theta : \theta \in \Theta) \rangle$ is reparametrized by $\vartheta = \vartheta(\theta)$, then the new $\min \mathfrak{R}_\alpha$ -estimators $\tilde{\vartheta}_n$ are related to the original $\hat{\theta}_n$ by $\tilde{\vartheta}_n = \vartheta(\hat{\theta}_n)$. If the observations $x \in \mathcal{X}$ are replaced by $y = T(x)$, where $T : (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \mapsto (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a measurable statistic with the inverse T^{-1} , then the densities

$$\tilde{p}_\theta = \frac{d\tilde{P}_\theta}{d\tilde{\lambda}}$$

in the transformed model $\tilde{\mathcal{P}} = (\tilde{P}_\theta = P_\theta T^{-1} : \theta \in \Theta)$ with respect to σ -finite dominating measure $\tilde{\lambda} = \lambda T^{-1}$ are related to the original densities p_θ by

$$\tilde{p}_\theta(y) = p_\theta(T^{-1}y) \mathcal{J}_T(y), \quad (3.13)$$

where $\mathcal{J}_T(y) = d\lambda T^{-1}/d\tilde{\lambda}$ is a generalized Jacobian of the statistic T . If λ is the Lebesgue measure and the inverse mapping $H = T^{-1}$ is differentiable, then $\mathcal{J}_T(y)$ is the determinant

$$\mathcal{J}_T(y) = \left| \frac{d}{dy} H(y) \right|.$$

The $\min \mathfrak{R}_\alpha$ -estimators are in general not equivariant with respect to invertible transformations T of observations, unless $\alpha = 0$.

Theorem 3 *The $\min \mathfrak{R}_\alpha$ -estimators $\tilde{\theta}_n$ in the above considered transformed model coincide with the original $\min \mathfrak{R}_\alpha$ -estimators $\hat{\theta}_n$, if the Jacobian \mathcal{J}_T of transformation is a nonzero constant on the transformed observation space. Thus, the $\min \mathfrak{R}_\alpha$ -estimators are equivariant under linear statistics $Tx = ax + b$.*

Proof. For $\alpha = 0$ the $\min \mathfrak{R}_\alpha$ -estimator coincides with the maximum likelihood estimator, whose equivariance is well known. For $\alpha > 0$, by (3.13) and (3.12),

$$\begin{aligned} \tilde{\theta}_n &= \arg \sup_{\theta} C_\alpha(\theta)^{-1} \frac{1}{n} \sum_{i=1}^n \tilde{p}_\theta^\alpha(TX_i) \\ &= \arg \sup_{\theta} C_\alpha(\theta)^{-1} \frac{1}{n} \sum_{i=1}^n p_\theta^\alpha(X_i) \mathcal{J}_T^\alpha(TX_i). \end{aligned}$$

Comparing with (3.12) it follows that $\tilde{\theta}_n = \hat{\theta}_n$ if $y \mapsto \mathcal{J}_T(y)$ is a nonzero constant. If $\alpha = 0$, then the estimator coincides with the maximum likelihood estimator and its equivariance is well known.

3.2 Limit properties of $\min \mathfrak{R}_\alpha$ -estimators

Define

$$R_\alpha(\theta_0) := \sup_{\theta} \int h(x, \theta) dP_{\theta_0}(x)$$

where

$$h(x, \theta) := \frac{p_\theta^\alpha(x)}{C_\alpha(\theta)}.$$

By the Fisher consistency of the functional T_α defined in (3.9), it holds

$$\arg \sup_{\theta} \int h(x, \theta) dP_{\theta_0}(x) = \theta_0$$

and θ_0 is the only optimizer in the above expression, since $\mathfrak{R}_\alpha(P_\theta, P_{\theta_0}) = 0$ implies $\theta = \theta_0$.

Define the estimate of $R_\alpha(\theta_0)$ through

$$\widehat{R}_\alpha(\theta_0) := \sup_{\theta} \int h(x, \theta) dP_n = \sup_{\theta} \frac{1}{n} \sum_{i=1}^n h(X_i, \theta),$$

where the θ_0 indicates that the sampling is i.i.d. under P_{θ_0} . The $\min \mathfrak{R}_\alpha$ -estimator $\widehat{\theta}_n$ is then defined through

$$\widehat{\theta}_n = \arg \sup_{\theta} \frac{1}{n} \sum_{i=1}^n h(X_i, \theta).$$

This optimum need not be uniquely defined.

The usual regularity properties of the model will be assumed throughout the rest of the paper, namely: (i) The density $p_\theta(x)$ has continuous partial derivatives with respect to θ up to 3th order (for all x λ -a.e.). (ii) There exists a neighborhood $N(\theta_0)$ of θ_0 such that the first, the second and the third order partial derivatives (w.r.t. θ) of $h(x, \theta)$ are dominated on $N(\theta_0)$ by some P_{θ_0} -integrable functions. (iii) The integrals $\int (\partial^2 / \partial \theta^2) h(x, \theta_0) dP_{\theta_0}(x)$ and $\int (\partial / \partial \theta) h(x, \theta_0) (\partial / \partial \theta)^t h(x, \theta_0) dP_{\theta_0}(x)$ exist.

Theorem 4 *Assume that the above conditions hold.*

- (a) *Let $B := \{\theta \in \Theta; \|\theta - \theta_0\| \leq n^{-1/3}\}$. Then, as $n \rightarrow \infty$, with probability one, the function $\theta \mapsto \frac{1}{n} \sum_{i=1}^n h(X_i, \theta)$ attains a local maximal value at some point $\widehat{\theta}_n$ in the interior of B , which implies that the estimate $\widehat{\theta}_n$ is $n^{1/3}$ -consistent.*
- (b) *$\sqrt{n}(\widehat{\theta}_n - \theta_0)$ converges in distribution to a centered multivariate normal random variable with covariance matrix*

$$V = S^{-1}MS^{-1} \tag{3.14}$$

with $S := -\int (\partial^2 / \partial \theta^2) h(x, \theta_0) dP_{\theta_0}(x)$ and $M := \int (\partial / \partial \theta) h(x, \theta_0) (\partial / \partial \theta)^t h(x, \theta_0) dP_{\theta_0}(x)$.

(c) $\sqrt{n} \left(\widehat{R}_\alpha(\theta_0) - R_\alpha(\theta_0) \right)$ converges in distribution to a centered normal variable with variance $\sigma^2(\theta_0) = \int h(x, \theta_0)^2 dP_{\theta_0}(x) - \left(\int h(x, \theta_0) dP_{\theta_0}(x) \right)^2$.

Proof. (a) A simple calculus give

$$\int (\partial/\partial\theta)h(x, \theta_0)dP_{\theta_0}(x) = 0 \quad (3.15)$$

and

$$\int (\partial^2/\partial\theta^2)h(x, \theta_0)dP_{\theta_0}(x) = -S. \quad (3.16)$$

Observe that the matrix S is symmetric and positive definite.

Let $U_n := \frac{1}{n} \sum_{i=1}^n (\partial/\partial\theta)h(X_i, \theta_0)$, and use (3.15) in connection with the central limit theorem to see that

$$\sqrt{n}U_n \rightarrow \mathcal{N}(0, M). \quad (3.17)$$

Also, let $V_n := \frac{1}{n} \sum_{i=1}^n (\partial^2/\partial\theta^2)h(X_i, \theta_0)$, and use (3.16) in connection with the law of large numbers to conclude that

$$V_n \rightarrow -S \quad (a.s.). \quad (3.18)$$

Now, for any $\theta = \theta_0 + un^{-1/3}$ with $|u| \leq 1$, consider a Taylor expansion of $\frac{1}{n} \sum_{i=1}^n h(X_i, \theta)$ in θ around θ_0 , and use the hypothesis to see that

$$\sum_{i=1}^n h(X_i, \theta) - \sum_{i=1}^n h(X_i, \theta_0) = n^{2/3}u^t U_n + 2^{-1}n^{1/3}u^t V_n u + O(1) \quad (a.s.)$$

uniformly on u with $|u| \leq 1$. Now, use (3.18) and the fact that $U_n = O(n^{-1/2}(\log \log n)^{1/2})$ (a.s.) to conclude that

$$\sum_{i=1}^n h(X_i, \theta) - \sum_{i=1}^n h(X_i, \theta_0) = O(n^{1/6}(\log \log n)^{1/2}) - 2^{-1}u^t S u n^{1/3} + O(1) \quad (a.s.)$$

uniformly on u with $|u| \leq 1$. Hence, uniformly on the surface of the ball B (i.e., uniformly on u with $|u| = 1$), we have

$$nP_n h(\theta) - nP_n h(\theta_0) \leq O(n^{1/6}(\log \log n)^{1/2}) - 2^{-1}cn^{1/3} + O(1) \quad (a.s.) \quad (3.19)$$

where c is the smallest eigenvalue of the matrix S . Note that c is positive since S is positive definite (it is symmetric, positive and non singular). In view of (3.19), by the continuity of $\theta \mapsto \sum_{i=1}^n h(X_i, \theta) - \sum_{i=1}^n h(X_i, \theta_0)$ and since it takes value zero on $\theta = \theta_0$ and is asymptotically nonpositive, it holds that as $n \rightarrow \infty$, with probability one, $\theta \mapsto \frac{1}{n} \sum_{i=1}^n h(X_i, \theta)$ attains its maximum value at some point $\widehat{\theta}_n$ in the interior of the ball B , and therefore the estimate $\widehat{\theta}_n$ satisfies $\|\widehat{\theta}_n - \theta_0\| = O(n^{-1/3})$.

(b) Using the fact that $\frac{1}{n} \sum_{i=1}^n (\partial/\partial\theta)h(X_i, \hat{\theta}_n) = 0$ and a Taylor expansion of $\frac{1}{n} \sum_{i=1}^n (\partial/\partial\theta)h(X_i, \theta)$ in $\hat{\theta}_n$ around θ_0 , we obtain

$$0 = \frac{1}{n} \sum_{i=1}^n (\partial/\partial\theta)h(X_i, \hat{\theta}_n) = \frac{1}{n} \sum_{i=1}^n (\partial/\partial\theta)h(X_i, \theta_0) + (\hat{\theta}_n - \theta_0)^t \sum_{i=1}^n (\partial^2/\partial\theta^2)h(X_i, \theta_0) + o_p(n^{-1/2}).$$

Hence,

$$\sqrt{n} (\hat{\theta}_n - \theta_0) = -V_n^{-1} \sqrt{n} U_n + o_p(1). \quad (3.20)$$

Using (3.17) and (3.18) and Slutsky theorem, we conclude then

$$\sqrt{n} (\hat{\theta} - \theta_0) \rightarrow \mathcal{N}(0, V)$$

where $V = S^{-1}MS^{-1}$.

(c) A Taylor expansion of $\hat{R}_\alpha(\theta_0) = \frac{1}{n} \sum_{i=1}^n h(X_i, \theta)$ in $\hat{\theta}_n$ around θ_0 , using the fact that $\int (\partial/\partial\theta)h(x, \theta_0)dP_{\theta_0}(x) = 0$, gives

$$\hat{R}_\alpha(\theta_0) = \frac{1}{n} \sum_{i=1}^n h(X_i, \theta_0) + o_p(n^{-1/2}). \quad (3.21)$$

Hence,

$$\sqrt{n} (\hat{R}_\alpha(\theta_0) - R_\alpha(\theta_0)) = \sqrt{n} \left[\frac{1}{n} \sum_{i=1}^n h(X_i, \theta_0) - \int h(x, \theta_0)dP_{\theta_0}(x) \right] + o_p(1),$$

which by the central limit theorem, converges in distribution to a centered normal variable with variance $\sigma^2(\theta_0) = \int h(x, \theta_0)^2 dP_{\theta_0}(x) - (\int h(x, \theta_0)dP_{\theta_0}(x))^2$.

3.3 Robustness results

We recall that a map T defined on a set of probability measures and the parameter space valued is a statistical functional corresponding to an estimator $\hat{\theta}_n$ of the parameter θ_0 , whenever $T(P_n) = \hat{\theta}_n$. The influence function of the functional T in P measures the effect on T of adding a small mass at x and is defined as

$$\text{IF}(x; T, P) = \lim_{\varepsilon \rightarrow 0} \frac{T(\tilde{P}_{\varepsilon x}) - T(P)}{\varepsilon},$$

where $\tilde{P}_{\varepsilon x} = (1 - \varepsilon)P + \varepsilon\delta_x$. When the influence function is bounded, the corresponding estimator is called B-robust. The gross error sensitivity of T at P is defined by

$$\gamma^*(T, P) := \sup_x \|\text{IF}(x; T, P)\|, \quad (3.22)$$

the supremum being taken over all x where $\text{IF}(x; T, P)$ exists.

The statistical functional corresponding to the min \mathfrak{R}_α -estimator is

$$T_\alpha(Q) := \arg \sup_{\theta} \frac{\int p_\theta^\alpha dQ}{C_\alpha(\theta)}, \quad (3.23)$$

where $C_\alpha(\theta) := (\int p_\theta^{\alpha+1} d\lambda)^{\frac{\alpha}{\alpha+1}}$. Derivation w.r.t. θ shows that $T_\alpha(Q)$ is a solution of the equation

$$\int [p_\theta^{\alpha-1} \dot{p}_\theta - c_\alpha(\theta) p_\theta^\alpha] dQ = 0, \quad (3.24)$$

where $c_\alpha(\theta) := \frac{\int p_\theta^\alpha \dot{p}_\theta d\lambda}{\int p_\theta^{\alpha+1} d\lambda}$ and \dot{p}_θ is the derivative with respect to θ of p_θ .

Theorem 5 *The influence function of T_α in P_θ is given by*

$$\text{IF}(x; T_\alpha, P_\theta) = M_\alpha(\theta)^{-1} [p_\theta^{\alpha-1}(x) \dot{p}_\theta(x) - c_\alpha(\theta) p_\theta^\alpha(x)], \quad (3.25)$$

where

$$M_\alpha(\theta) = \int p_\theta^{\alpha-1} \dot{p}_\theta \dot{p}_\theta^t d\lambda - \frac{\int p_\theta^\alpha \dot{p}_\theta d\lambda (\int p_\theta^\alpha \dot{p}_\theta d\lambda)^t}{\int p_\theta^{\alpha+1} d\lambda}. \quad (3.26)$$

Proof. Consider the contaminated model $\tilde{P}_{\varepsilon x} = (1 - \varepsilon)P_\theta - \varepsilon\delta_x$. Then

$$\begin{aligned} & (1 - \varepsilon) \int \{p_{T_\alpha(\tilde{P}_{\varepsilon x})}^{\alpha-1} \dot{p}_{T_\alpha(\tilde{P}_{\varepsilon x})} - c_\alpha(T_\alpha(\tilde{P}_{\varepsilon x})) p_{T_\alpha(\tilde{P}_{\varepsilon x})}^\alpha\} dP_\theta + \\ & + \varepsilon \{p_{T_\alpha(\tilde{P}_{\varepsilon x})}^{\alpha-1}(x) \dot{p}_{T_\alpha(\tilde{P}_{\varepsilon x})}(x) - c_\alpha(T_\alpha(\tilde{P}_{\varepsilon x})) p_{T_\alpha(\tilde{P}_{\varepsilon x})}^\alpha(x)\} = 0. \end{aligned}$$

Derivating with respect to ε in the above display and taking the derivative in $\varepsilon = 0$, after some calculations, we obtain

$$\text{IF}(x; T_\alpha, P_\theta) = M_\alpha(\theta)^{-1} [p_\theta^{\alpha-1}(x) \dot{p}_\theta(x) - c_\alpha(\theta) p_\theta^\alpha(x)], \quad (3.27)$$

where $M_\alpha(\theta)$ is given by the formula (3.26).

Beside the influence function, the breakdown point provides information about the robustness of an estimator. The breakdown point of an estimator $\hat{\theta}_n$ of a parameter θ_0 is the largest amount of contamination that the data may contain, such that $\hat{\theta}_n$ still gives some information about θ_0 . Following Maronna et al. (2006) (p.58), the asymptotic contamination breakdown point of an estimator $\hat{\theta}_n$ at P_{θ_0} , denoted by $\varepsilon^*(\hat{\theta}_n, \theta_0)$, is the largest $\varepsilon^* \in (0, 1)$ such that for $\varepsilon < \varepsilon^*$, $T((1 - \varepsilon)P_{\theta_0} + \varepsilon P)$ as function of P remains bounded and also bounded away from the boundary $\partial\Theta$ of Θ . Here, $T((1 - \varepsilon)P_{\theta_0} + \varepsilon P)$ is the asymptotic value of the estimator at $(1 - \varepsilon)P_{\theta_0} + \varepsilon P$ by means of the convergence in probability. The definition means that there exists a bounded and closed set $K \subset \Theta$ such that $K \cap \partial\Theta = \emptyset$ and

$$T((1 - \varepsilon)P_{\theta_0} + \varepsilon P) \in K, \text{ for all } \varepsilon < \varepsilon^* \text{ and all } P.$$

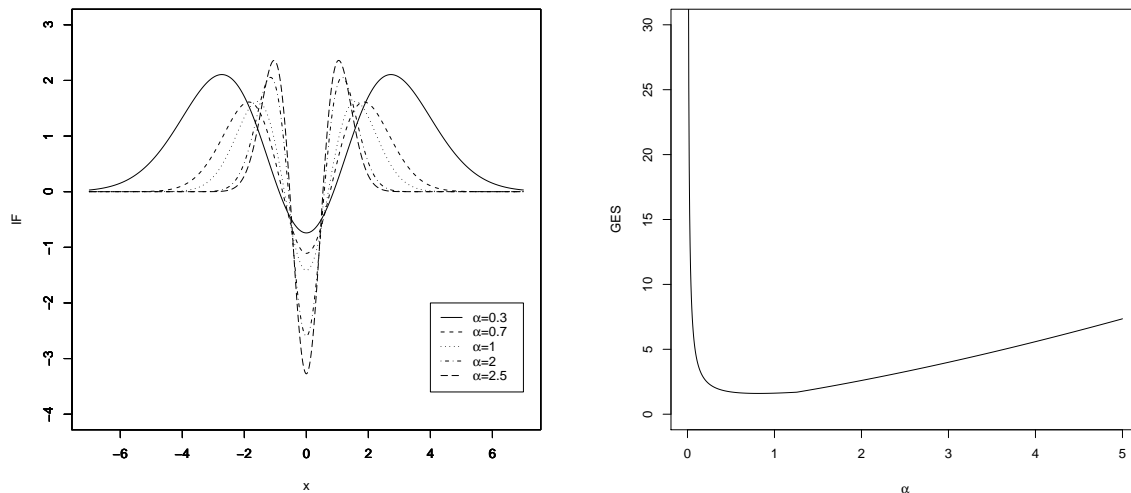


Figure 1: Influence functions and gross error sensitivity of $\min \mathfrak{R}_\alpha$ -estimators of the scale parameter $\sigma = 1$ from the normal model, when $m = 0$ is known

3.3.1 Scale models

(a) *Standard deviation of univariate normal.* Consider the scale normal model with known mean m and take $\theta = \sigma$. The influence function (3.25) takes on the form

$$\text{IF}(x; T_\alpha, P_\sigma) = \frac{\sigma(\alpha + 1)^{5/2}}{2} \left[\left(\frac{x - m}{\sigma} \right)^2 - \frac{1}{\alpha + 1} \right] \exp \left(-\frac{\alpha}{2} \left(\frac{x - m}{\sigma} \right)^2 \right). \quad (3.28)$$

The gross error sensitivity of T_α in P_σ is given by

$$\gamma^*(T_\alpha, P_\sigma) = \max \left\{ \frac{\sigma(\alpha + 1)^{3/2}}{2}, \frac{\sigma(\alpha + 1)^{5/2}}{\alpha} \exp \left(-\frac{3\alpha + 2}{2(\alpha + 1)} \right) \right\}, \quad (3.29)$$

independently upon the value of m .

Figure 1 presents influence functions $\text{IF}(x; T_\alpha, P_\sigma)$, in the case of the scale normal model with the known mean $m = 0$, when $\sigma = 1$. For the same model, the gross error sensitivity of the $\min \mathfrak{R}_\alpha$ -estimator, as function of α , is represented. The gross error sensitivity attains its minimum value $\gamma^*(T_\alpha, P_\sigma) = 1.600413$, for $\alpha = 0.81648$. This means that the $\min \mathfrak{R}_\alpha$ -estimator corresponding to $\alpha = 0.81648$ is the most B-robust estimator within the class of the $\min \mathfrak{R}_\alpha$ -estimators of $\sigma = 1$.

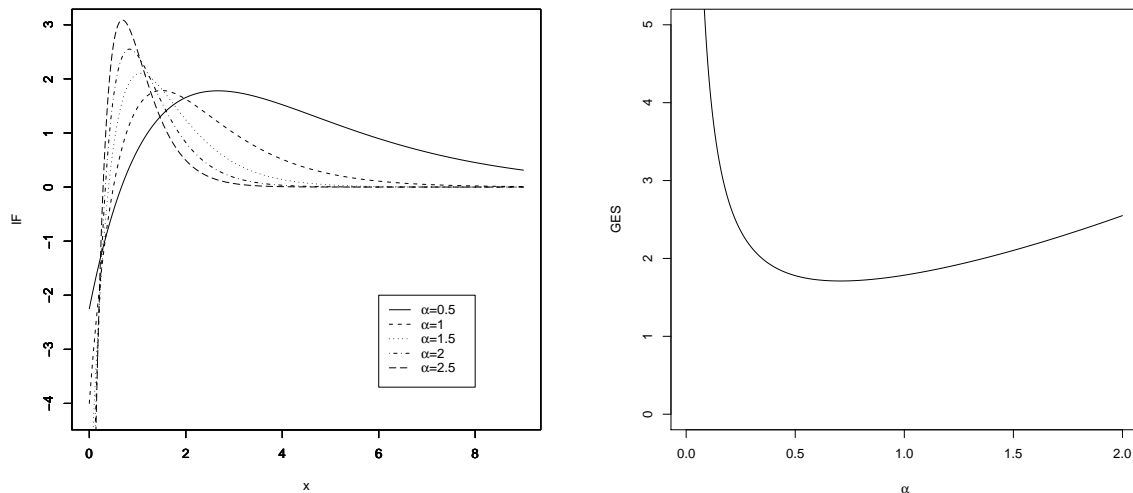


Figure 2: Influence functions and gross error sensitivity of $\min \mathfrak{R}_\alpha$ -estimators of the scale parameter $\theta = 1$ from the exponential model

On the other hand, the asymptotic relative efficiency of T_α is

$$\text{ARE}(T_\alpha, P_\sigma) = \frac{2(2\alpha + 1)^{5/2}}{(\alpha + 1)^3(3\alpha^2 + 4\alpha + 2)}. \quad (3.30)$$

Results for different values of α are given in the first row of Table 1. Note that, when α increases, the asymptotic relative efficiency of the estimator decreases. Therefore, positive values of α close to zero will ensure high efficiency and in the meantime the B-robustness of the estimator. For example, the asymptotic relative efficiency of the estimator is 0.975 for $\alpha = 0.1$, respectively 0.919 for $\alpha = 0.2$. For both cases, the $\min \mathfrak{R}_\alpha$ -estimator is B-robust.

(b) *Exponential model.* Consider the exponential model with density $p_\theta(x) = \frac{1}{\theta} \exp(-\frac{x}{\theta})$, $x \geq 0$. The influence function of a $\min \mathfrak{R}_\alpha$ -estimator of the parameter θ is

$$\text{IF}(x; T_\alpha, P_\theta) = \theta(\alpha + 1)^3 \left(\frac{x}{\theta} - \frac{1}{\alpha + 1} \right) \exp\left(-\frac{\alpha x}{\theta}\right) \quad (3.31)$$

and the corresponding gross error sensitivity is

$$\gamma^*(T_\alpha, P_\theta) = \theta \frac{(\alpha + 1)^3}{\alpha} \exp\left(-\frac{2\alpha + 1}{\alpha + 1}\right). \quad (3.32)$$

In Figure 2, for different values of α , influence functions of $\min \mathfrak{R}_\alpha$ -estimators of the parameter $\theta = 1$ from the exponential model are represented. The gross error sensitivity (3.32) as function of α is also represented. The most B-robust estimator over the class

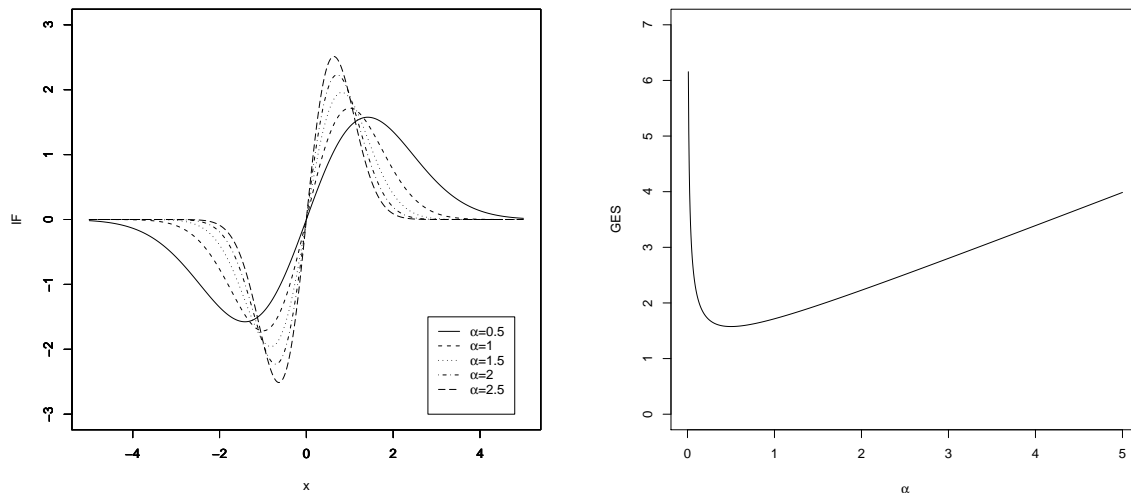


Figure 3: Influence functions and gross error sensitivity of $\min \mathfrak{R}_\alpha$ -estimators of the location parameter $m = 0$ from the normal model, when $\sigma = 1$ is known

of $\min \mathfrak{R}_\alpha$ -estimators of $\theta = 1$ is associated to $\alpha = 0.707$, case in which the gross error sensitivity takes on the value 1.710.

The asymptotic relative efficiency of T_α in P_θ is given by

$$\text{ARE}(T_\alpha, P_\theta) = \frac{(2\alpha + 1)^3}{(\alpha + 1)^4(2\alpha^2 + 2\alpha + 1)}. \quad (3.33)$$

As in the case of the scale normal model, the efficiency remains high for small α . Thus, positive values of α close to zero will ensure high efficiency and the B-robustness of the estimation procedure.

3.3.2 Location models

(c) *Mean of univariate normal.* Letting p_θ be the $\mathcal{N}(m, \sigma)$ density with known σ , the influence function of a $\min \mathfrak{R}_\alpha$ -estimator of the location parameter $\theta = m$ is

$$\text{IF}(x; T_\alpha, P_m) = (\alpha + 1)^{3/2}(x - m) \exp\left(-\frac{\alpha}{2} \left(\frac{x - m}{\sigma}\right)^2\right) \quad (3.34)$$

and the gross error sensitivity is

$$\gamma^*(T_\alpha, P_m) = (\alpha + 1)^{3/2} \frac{\sigma}{\sqrt{\alpha}} \exp(-1/2), \quad (3.35)$$

Table 1.

Asymptotic relative efficiencies of the min \mathfrak{R}_α -estimators

	α :	0.00	0.02	0.05	0.10	0.20	0.25	0.5	1.00
Model									
Normal σ		1.00000	0.99884	0.99321	0.97543	0.91922	0.88527	0.70572	0.43301
Exponential(θ)		1.00000	0.99846	0.99096	0.96741	0.89412	0.85070	0.63209	0.33750
Normal m		1.00000	0.99942	0.99660	0.98762	0.95862	0.94060	0.83805	0.64951
Mean of $\mathcal{N}_2(m, V)$		1.00000	0.99923	0.99547	0.98353	0.94521	0.92160	0.79012	0.56250
Mean of $\mathcal{N}_3(m, V)$		1.00000	0.99903	0.99434	0.97946	0.93199	0.90297	0.74493	0.48713
Mean of $\mathcal{N}_4(m, V)$		1.00000	0.99884	0.99321	0.97541	0.91896	0.88473	0.70233	0.42187

independently from the value of m .

In Figure 3, for $\sigma = 1$ and different values of α , influence functions of min \mathfrak{R}_α -estimators of the location parameter $m = 0$ are represented. The gross error sensitivity (3.35) as function of α is also represented. Note that, when $\sigma = 1$, the most B-robust estimator over the class of min \mathfrak{R}_α -estimators of m is associated to $\alpha = 0.4999836$, regardless the value of m .

The ψ -function of a min \mathfrak{R}_α -estimator of the parameter m , given by the formula,

$$\psi_\alpha(x, m) = \alpha(\alpha + 1)^{\frac{\alpha}{2(\alpha+1)}} \sigma^{-\frac{3\alpha+2}{\alpha+1}} (\sqrt{2\pi})^{-\frac{\alpha}{\alpha+1}} (x - m) \exp\left(-\frac{\alpha}{2} \left(\frac{x - m}{\sigma}\right)^2\right) \quad (3.36)$$

is redescending w.r.t. x . Then, the asymptotic breakdown point of the corresponding estimator is 0.5 according to the results regarding redescending M-estimators of location parameters presented in Marona et al. (2006), p.59.

The asymptotic relative efficiency is

$$\text{ARE}(T_\alpha, P_m) = \frac{(2\alpha + 1)^{3/2}}{(\alpha + 1)^3}. \quad (3.37)$$

Efficiency calculations are presented in the third row of Table 1. Small α min \mathfrak{R}_α estimation continues to retain high efficiency.

(d) *Mean of multivariate normal.* The family is $\mathcal{N}_p(m, V)$. The influence function of a min \mathfrak{R}_α -estimator of the mean vector m , when V is known, is

$$\text{IF}(x; T_\alpha, P_m) = (\sqrt{\alpha + 1})^{p+2} (x - m) \exp\left(-\frac{\alpha}{2} (x - m)^t V^{-1} (x - m)\right). \quad (3.38)$$

This is a bounded function w.r.t. x , meaning that all min \mathfrak{R}_α -estimators of the mean vector m are B-robust. In Figure 4 the norm of the influence function (3.38), when $m = (0, 0)^t$, $V = \text{diag}(2, 1)$ and $\alpha = 0.2$, is represented.

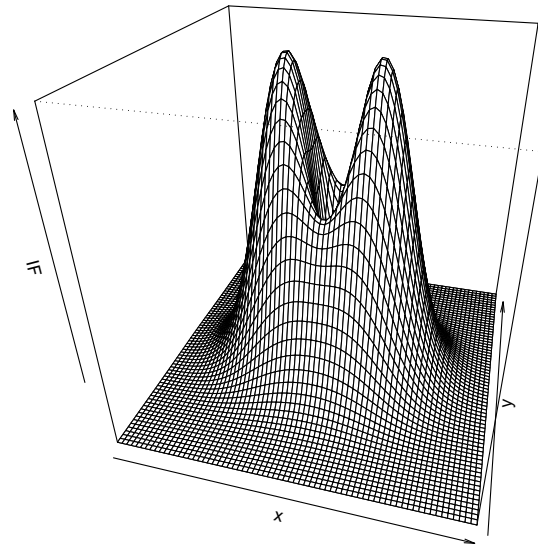


Figure 4: The norm of the influence function of a $\min \mathfrak{R}_\alpha$ -estimator of $m = (0, 0)^t$, when $V = \text{diag}(2, 1)$

The limiting covariance matrix of $n^{1/2}$ times the min \mathfrak{R}_α -estimator of m , when V is known, can be shown to be

$$\left(\frac{\alpha + 1}{\sqrt{2\alpha + 1}}\right)^{p+2} V. \quad (3.39)$$

Then, the asymptotic relative efficiency of a min \mathfrak{R}_α -estimator of m is

$$\left(\frac{\sqrt{2\alpha + 1}}{\alpha + 1}\right)^{p+2}. \quad (3.40)$$

Thus, one loses efficiency for increasing p if α is kept fixed. In Table 1 efficiencies for some values of α and p are presented. Again, small values of α ensure high efficiency and B-robustness of the estimations.

4 min \mathfrak{R}_α -estimators in regression models

Suppose we have i.i.d. $(p + 1)$ -dimensional random vectors (X_i, Y_i) , $i = 1, \dots, n$, satisfying the linear relation

$$Y_i = \beta^t X_i + U_i, \quad (4.1)$$

where the U_i 's are i.i.d. with $\mathcal{N}(0, \sigma)$ and independent of the X_i 's. X_i and β are p -dimensional column vectors with coordinates (X_{i1}, \dots, X_{ip}) and $(\beta_1, \dots, \beta_p)$, respectively. Call \mathbf{X} the $n \times p$ matrix with elements X_{ij} and assume that the distribution of X is not concentrated on any subspace, i.e. $P(a^t X = 0) < 1$, for all $a \neq 0$. This condition implies that the probability that \mathbf{X} has full rank tends to one when $n \rightarrow \infty$ and holds for example if the distribution of X has density.

Let P_σ be the probability measure associated to a random variable $\mathcal{N}(0, \sigma)$ and $P_n(\beta)$ be the empirical measure based on the sample U_1, \dots, U_n , where $U_i = Y_i - \beta^t X_i$, $i = 1, \dots, n$.

The \mathfrak{R}_α pseudodistance between P_σ and the probability measure Q is

$$\begin{aligned} \mathfrak{R}_\alpha(P_\sigma, Q) &= \frac{1}{\alpha + 1} \ln \left(\int p_\sigma^\alpha(x) dP_\sigma(x) \right) + \frac{1}{\alpha(\alpha + 1)} \ln \left(\int q^\alpha(x) dQ(x) \right) - \\ &\quad - \frac{1}{\alpha} \ln \left(\int p_\sigma^\alpha(x) dQ(x) \right). \end{aligned} \quad (4.2)$$

The estimators of the parameters β and σ are defined by minimizing the following

empirical version of the pseudodistance (4.2),

$$\begin{aligned}
& \mathcal{R}_n(P_\sigma, P_n(\beta)) = \\
&= \frac{1}{\alpha+1} \ln \int p_\sigma^{\alpha+1} d\lambda + \frac{1}{\alpha(\alpha+1)} \ln \int \left(\frac{1}{n} \sum_{j=1}^n \delta_{x-U_j} \right)^\alpha dP_n(\beta) - \frac{1}{\alpha} \ln \int p_\sigma^\alpha dP_n(\beta) \\
&= \frac{1}{\alpha+1} \ln \int p_\sigma^{\alpha+1} d\lambda + \frac{1}{\alpha(\alpha+1)} \ln \left[\frac{1}{n} \sum_{i=1}^n \left(\frac{1}{n} \sum_{j=1}^n \delta_{U_i-U_j} \right)^\alpha \right] - \frac{1}{\alpha} \ln \left[\frac{1}{n} \sum_{i=1}^n p_\sigma^\alpha(U_i) \right] \\
&= \frac{1}{\alpha+1} \ln \int p_\sigma^{\alpha+1} d\lambda + \frac{1}{\alpha+1} \ln \left(\frac{1}{n} \right) - \frac{1}{\alpha} \ln \left[\frac{1}{n} \sum_{i=1}^n p_\sigma^\alpha(U_i) \right], \tag{4.3}
\end{aligned}$$

obtained by replacing in (4.2) Q with $P_n(\beta)$ and $q(x)$ with $\hat{q}(x) = \frac{1}{n} \sum_{j=1}^n \delta_{x-U_j}$.

Since the middle term in the above display does not depend on β or σ , the estimators $\hat{\beta}$ and $\hat{\sigma}$ are defined by

$$\arg \inf_{\beta, \sigma} \left[\frac{1}{\alpha+1} \ln \int p_\sigma^{\alpha+1} d\lambda - \frac{1}{\alpha} \ln \left(\frac{1}{n} \sum_{i=1}^n p_\sigma^\alpha(Y_i - \beta^t X_i) \right) \right] \tag{4.4}$$

or equivalently as

$$\arg \sup_{\beta, \sigma} \sum_{i=1}^n \frac{p_\sigma^\alpha(Y_i - \beta^t X_i)}{\left[\int p_\sigma^{\alpha+1} d\lambda \right]^{\frac{\alpha}{\alpha+1}}}. \tag{4.5}$$

A simple calculation shows that

$$\left[\int p_\sigma^{\alpha+1} d\lambda \right]^{\frac{\alpha}{\alpha+1}} = (\sigma\sqrt{2\pi})^{-\frac{\alpha^2}{\alpha+1}} (\sqrt{\alpha+1})^{-\frac{\alpha}{\alpha+1}} \tag{4.6}$$

and (4.5) writes as

$$\arg \sup_{\beta, \sigma} \sum_{i=1}^n \sigma^{-\frac{\alpha}{\alpha+1}} \exp \left(-\frac{\alpha}{2} \left(\frac{Y_i - \beta^t X_i}{\sigma} \right)^2 \right). \tag{4.7}$$

Derivating with respect to β and σ , we see that the estimators $\hat{\beta}$ and $\hat{\sigma}$ are solutions of the system:

$$\sum_{i=1}^n \exp \left(-\frac{\alpha}{2} \left(\frac{Y_i - \beta^t X_i}{\sigma} \right)^2 \right) \left(\frac{Y_i - \beta^t X_i}{\sigma} \right) X_i = 0 \tag{4.8}$$

$$\sum_{i=1}^n \exp \left(-\frac{\alpha}{2} \left(\frac{Y_i - \beta^t X_i}{\sigma} \right)^2 \right) \left[\left(\frac{Y_i - \beta^t X_i}{\sigma} \right)^2 - \frac{1}{\alpha+1} \right] = 0. \tag{4.9}$$

Note that, for $\alpha = 0$, the above system corresponds to the system that define the least square estimators of β and σ .

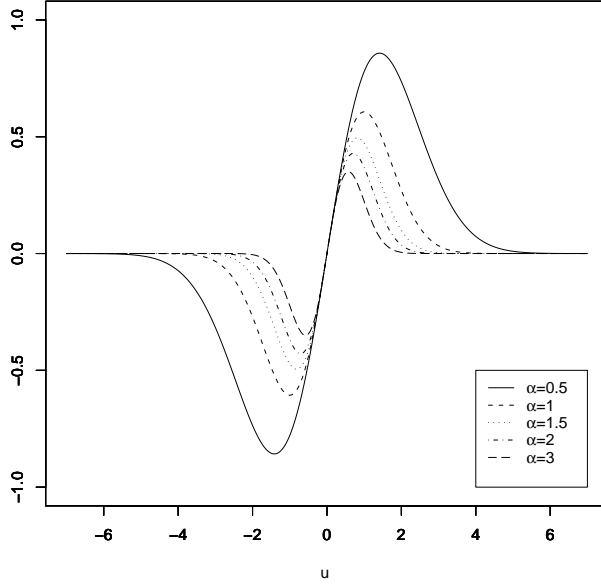


Figure 5: The function $\phi(u)$ corresponding to the regression M-estimators, for different values of α .

The system formed by (4.8) and (4.9) can be written as

$$\sum_{i=1}^n \Psi(Z_i, \xi) = 0, \quad (4.10)$$

where $Z_i = (X_i, Y_i)$, ξ is the $(p+1)$ -dimensional vector with coordinates (β, σ) and

$$\Psi(Z_i, \xi) = \left(\phi \left(\frac{Y_i - \beta^t X_i}{\sigma} \right) X_i, \chi \left(\frac{Y_i - \beta^t X_i}{\sigma} \right) \right)^t \quad (4.11)$$

with

$$\phi(u) = \exp \left(-\frac{\alpha}{2} u^2 \right) u \quad \text{and} \quad \chi(u) = \left[u^2 - \frac{1}{\alpha + 1} \right] \exp \left(-\frac{\alpha}{2} u^2 \right). \quad (4.12)$$

The redescending nature of the functions ϕ and χ can be seen in Figure 5 and in Figure 6.

Let $\hat{\xi} = (\hat{\beta}, \hat{\sigma})$. The asymptotic normality of the M-estimator $\hat{\xi}$ can be established by using similar conditions with those from Theorem 4. Such conditions are satisfied by the function

$$h(z, \xi) := \sigma^{-\frac{\alpha}{\alpha+1}} \exp \left(-\frac{\alpha}{2} \left(\frac{y - \beta^t x}{\sigma} \right)^2 \right) \quad (4.13)$$

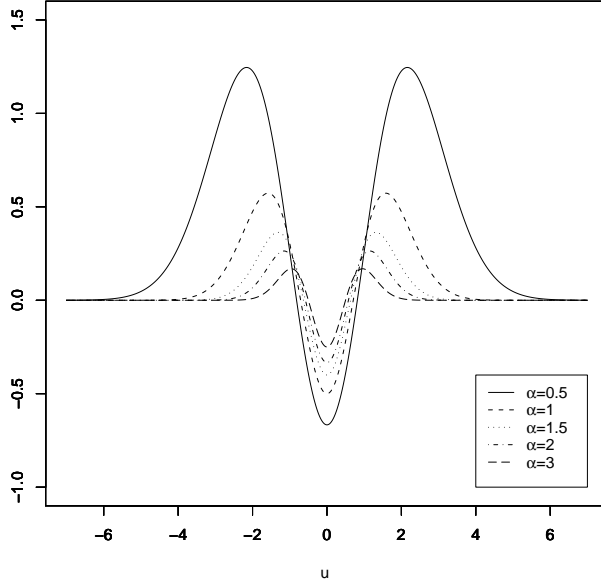


Figure 6: The function $\chi(u)$ corresponding to the regression M-estimators, for different values of α .

associated to the M-estimator $\widehat{\xi}$, reason for which we obtain

$$\sqrt{n}(\widehat{\xi} - \xi) \rightarrow \mathcal{N}_{p+1}(0, S^{-1}M(S^{-1})^t) \quad (4.14)$$

where

$$M = E\Psi(Z, \xi)\Psi(Z, \xi)^t \text{ and } S = E\dot{\Psi}(Z, \xi), \quad (4.15)$$

$\dot{\Psi}$ being the matrix with entries $\dot{\Psi}_{jk} := \frac{\partial \Psi_j}{\partial \xi_k}$.

After some calculations we find that the matrices M and S are

$$M = \begin{pmatrix} \frac{\sigma^2}{(2\alpha+1)^{3/2}}V_X & 0 \\ 0 & \frac{\sigma^2(3\alpha^2+4\alpha+2)}{(2\alpha+1)^{5/2}(\alpha+1)^2} \end{pmatrix} \quad (4.16)$$

and respectively

$$S = - \begin{pmatrix} \frac{1}{(\alpha+1)^3}V_X & 0 \\ 0 & \frac{2}{(\alpha+1)^{5/2}} \end{pmatrix} \quad (4.17)$$

where $V_X = EXX^T$.

Thus $\widehat{\xi}$ is asymptotically normal distributed with the asymptotic covariance matrix

$$\sigma^2 \begin{pmatrix} \frac{(\alpha+1)^3}{(2\alpha+1)^{3/2}}V_X^{-1} & 0 \\ 0 & \frac{(\alpha+1)^3(3\alpha^2+4\alpha+2)}{4(2\alpha+1)^{5/2}} \end{pmatrix}. \quad (4.18)$$

It follows that $\widehat{\beta}$ and $\widehat{\sigma}$ are asymptotically independent and the asymptotic covariance matrix of $\widehat{\beta}$ is $\sigma^2 \frac{(\alpha+1)^3}{(2\alpha+1)^{3/2}} V_X^{-1}$.

Denote by T and S the statistical functionals corresponding to the estimators $\widehat{\beta}$ and $\widehat{\sigma}$, respectively. For a given probability measure P , these functionals are defined through the solutions of the system

$$\int \Psi(z, T(P), S(P)) dP = 0, \quad (4.19)$$

where

$$\Psi(z, \xi) = \left(\phi \left(\frac{y - \beta^t x}{\sigma} \right) x, \chi \left(\frac{y - \beta^t x}{\sigma} \right) \right). \quad (4.20)$$

The influence functions of the functionals T and S are given in the following theorem:

Theorem 6 *The influence functions of the functionals associated to the min \mathfrak{R}_α -estimators of β and σ are*

$$\begin{aligned} \text{IF}(x_0, y_0; T, P_\xi) &= \sigma(\alpha + 1)^{3/2} \exp \left(-\frac{\alpha}{2} \left(\frac{y_0 - \beta^t x_0}{\sigma} \right)^2 \right) \left(\frac{y_0 - \beta^t x_0}{\sigma} \right) V_X^{-1} x_0 \\ \text{IF}(x_0, y_0; S, P_\xi) &= \frac{(\alpha + 1)^{5/2}}{2} \exp \left(-\frac{\alpha}{2} \left(\frac{y_0 - \beta^t x_0}{\sigma} \right)^2 \right) \left[\left(\frac{y_0 - \beta^t x_0}{\sigma} \right)^2 - \frac{1}{\alpha + 1} \right], \end{aligned}$$

P_ξ being the probability measure associated to Z .

Proof. The system (4.19) can be written as

$$\begin{aligned} \int \phi \left(\frac{y - T(P)^t x}{S(P)} \right) x dP(x, y) &= 0 \\ \int \chi \left(\frac{y - T(P)^t x}{S(P)} \right) dP(x, y) &= 0. \end{aligned} \quad (4.21)$$

We consider the contaminated model $\widetilde{P}_{\varepsilon, x_0, y_0} = (1 - \varepsilon)P_\xi + \varepsilon\delta_{(x_0, y_0)}$, where (x_0, y_0) is an arbitrary point from $\mathbb{R}^p \times \mathbb{R}$. For this model, the system (4.21) writes as

$$\begin{aligned} (1 - \varepsilon) \int \phi \left(\frac{y - T(\widetilde{P}_{\varepsilon, x_0, y_0})^t x}{S(\widetilde{P}_{\varepsilon, x_0, y_0})} \right) x dP_\xi(x, y) + \varepsilon \phi \left(\frac{y_0 - T(\widetilde{P}_{\varepsilon, x_0, y_0})^t x_0}{S(\widetilde{P}_{\varepsilon, x_0, y_0})} \right) x_0 &= 0 \\ (1 - \varepsilon) \int \chi \left(\frac{y - T(\widetilde{P}_{\varepsilon, x_0, y_0})^t x}{S(\widetilde{P}_{\varepsilon, x_0, y_0})} \right) dP_\xi(x, y) + \varepsilon \chi \left(\frac{y_0 - T(\widetilde{P}_{\varepsilon, x_0, y_0})^t x_0}{S(\widetilde{P}_{\varepsilon, x_0, y_0})} \right) &= 0. \end{aligned}$$

Derivating with respect to ε and taking the derivative in $\varepsilon = 0$, after some calculations, we find

$$\begin{aligned} \text{IF}(x_0, y_0; T, P_\xi) &= \sigma(\alpha + 1)^{3/2} \phi\left(\frac{y_0 - \beta^t x_0}{\sigma}\right) V_X^{-1} x_0 \\ &= \sigma(\alpha + 1)^{3/2} \exp\left(-\frac{\alpha}{2} \left(\frac{y_0 - \beta^t x_0}{\sigma}\right)^2\right) \left(\frac{y_0 - \beta^t x_0}{\sigma}\right) V_X^{-1} x_0 \end{aligned}$$

and

$$\begin{aligned} \text{IF}(x_0, y_0; S, P_\xi) &= \frac{(\alpha + 1)^{5/2}}{2} \chi\left(\frac{y_0 - \beta^t x_0}{\sigma}\right) \\ &= \frac{(\alpha + 1)^{5/2}}{2} \exp\left(-\frac{\alpha}{2} \left(\frac{y_0 - \beta^t x_0}{\sigma}\right)^2\right) \left[\left(\frac{y_0 - \beta^t x_0}{\sigma}\right)^2 - \frac{1}{\alpha + 1}\right]. \end{aligned}$$

Since χ is redescending, the estimator $\hat{\sigma}$ has the influence function bounded and hence is B-robust. On the other hand, $\text{IF}(x_0, y_0; T, P_\xi)$ will tend to infinity only when x_0 tends to infinity and $\left|\frac{y_0 - \beta^t x_0}{\sigma}\right| \leq k$, for some k . This means that large outliers have no influence on the estimates.

5 Simulation results

In this section we present some simulation studies in order to illustrate the performance of $\min \mathfrak{R}_\alpha$ -estimators in finite samples.

First, we considered the scale normal model with known mean. We estimated the scale parameter σ by using the $\min \mathfrak{R}_\alpha$ -estimator which is obtained as solution of the equation

$$\sum_{i=1}^n \left[\left(\frac{X_i - m}{\sigma} \right)^2 - \frac{1}{\alpha + 1} \right] \exp\left(-\frac{\alpha}{2} \left(\frac{X_i - m}{\sigma} \right)^2\right) = 0, \quad (5.1)$$

m being the known mean.

To make some comparisons, we also considered the minimum density power divergence estimator of Basu et al. (1998) (in the present paper we will denote it by $\min \mathfrak{D}_\alpha$ -estimator). For the scale normal model, this estimator is solution of the equation

$$\frac{\alpha}{(\alpha + 1)^{3/2}} + \frac{1}{n} \sum_{i=1}^n \left[\left(\frac{X_i - m}{\sigma} \right)^2 - 1 \right] \exp\left(-\frac{\alpha}{2} \left(\frac{X_i - m}{\sigma} \right)^2\right) = 0, \quad (5.2)$$

In a first Monte Carlo experiment, 5000 samples of size $n = 100$ were generated from the scale normal model $\mathcal{N}(0, 1)$ with mean $m = 0$ known, $\sigma = 1$ being the parameter to be estimated. In a second Monte Carlo experiment we generated 5000 samples with

100 observations, for each sample 95 observations being generated from $\mathcal{N}(0, 1)$ and 5 from $\mathcal{N}(2, 1)$, and then we generated 5000 samples with 100 observations, for each sample 90 observations being generated from $\mathcal{N}(0, 1)$ and 10 from $\mathcal{N}(2, 1)$. For each sample we computed $\min \mathfrak{R}_\alpha$ -estimators and $\min \mathfrak{D}_\alpha$ -estimators corresponding to $\alpha \in \{0.02, 0.05, 0.1, 0.2, 0.25, 0.5, 1\}$ and the MLE for $\alpha = 0$.

In Table 2 we present the mean estimated scale $\widehat{\sigma}$ and simulation based estimates of the MSE defined by

$$\widehat{\text{MSE}} := \frac{1}{n_s} \sum_{i=1}^{n_s} (\widehat{\sigma}_i - \sigma)^2 \quad (5.3)$$

where n_s denotes the number of samples (5000 in our study) and $\widehat{\sigma}_i$ represents an estimate of $\sigma = 1$ obtained on the basis of the i th sample.

As it can be seen, both the $\min \mathfrak{R}_\alpha$ -estimators and $\min \mathfrak{D}_\alpha$ -estimators perform well under the model. Under contamination, the $\min \mathfrak{R}_\alpha$ -estimator with $\alpha = 1$ gives the best results in terms of robustness, while keeping small empirical MSE. However, the $\min \mathfrak{R}_\alpha$ -estimators, as well as the $\min \mathfrak{D}_\alpha$ -estimators, exhibit outlier resistance properties even for small values of α . For example, in the case of 5% contamination, the estimates of $\sigma = 1$ obtained for $\alpha = 0.2$ are 1.07494, respectively 1.07562, fairly close to the estimates obtained for $\alpha = 1$. In this case, the $\min \mathfrak{R}_\alpha$ -estimator combines robustness with the asymptotic relative efficiency 0.91922.

Similar results are presented in Table 3 and Table 4, where 5% or 10% from data come from the contaminating distribution $\mathcal{N}(0, 3)$ or from δ_{10} . When the contaminating distribution is δ_{10} , the $\min \mathfrak{R}_\alpha$ -estimators have strong robustness properties, and this can also be explained by the influence function which is redescending, as it can be seen in Figure 1.

In the second example, our estimation method is applied to the location normal model $\mathcal{N}(0, 1)$, $\sigma = 1$ being known. Here we compute the $\min \mathfrak{R}_\alpha$ -estimates as solutions of equation

$$\sum_{i=1}^n (X_i - m) \exp \left(-\frac{\alpha}{2} \left(\frac{X_i - m}{\sigma} \right)^2 \right) = 0. \quad (5.4)$$

We consider the case of no outliers and the cases of 5% or 10% outliers coming from the model $\mathcal{N}(2, 1)$. The results are given in Table 5. Again, the choice $\alpha = 0.2$ provides robustness and high efficiency of the estimation procedure. When the outliers come from δ_{10} , we obtain very good results in terms of robustness, even for very small values of α , as it can be seen in Table 6. These results are in accordance with the redescending nature of the influence functions represented in Figure 3.

Our examples show that increasing α leads to estimators which are far more robust than the maximum likelihood estimator. The simulation results suggest that α between 0.1 and 0.25 provides competitive estimators in terms of robustness and efficiency.

Table 2.

Simulation results for $\min \mathfrak{R}_\alpha$ -estimators, $\min \mathfrak{D}_\alpha$ -estimators and MLE of the parameter $\sigma = 1$ when data are generated from the model $\mathcal{N}(0, 1)$, when 95 data are generated from the model $\mathcal{N}(0, 1)$ and 5 data from $\mathcal{N}(2, 1)$, respectively when 90 data are generated from the model $\mathcal{N}(0, 1)$ and 10 data from $\mathcal{N}(2, 1)$.

	no outliers		5% outliers		10% outliers	
	$\hat{\sigma}$	$\widehat{\text{MSE}}$	$\hat{\sigma}$	$\widehat{\text{MSE}}$	$\hat{\sigma}$	$\widehat{\text{MSE}}$
MLE						
$\alpha = 0$	0.99763	0.00503	1.09289	0.01446	1.17999	0.03888
$\min \mathfrak{R}_\alpha$						
$\alpha = 0.02$	0.99987	0.00501	1.09216	0.01420	1.17886	0.03827
$\alpha = 0.05$	1.00022	0.00504	1.08902	0.01357	1.17445	0.03663
$\alpha = 0.1$	1.00069	0.00514	1.08398	0.01272	1.16712	0.03412
$\alpha = 0.2$	1.00122	0.00545	1.07494	0.01162	1.15310	0.02999
$\alpha = 0.25$	1.00137	0.00566	1.07100	0.01131	1.14659	0.02835
$\alpha = 0.5$	1.00142	0.00710	1.05610	0.01122	1.11981	0.02348
$\alpha = 1$	0.99956	0.01173	1.03931	0.01494	1.08746	0.02315
$\min \mathfrak{D}_\alpha$						
$\alpha = 0.02$	0.99977	0.00463	1.09233	0.01398	1.17940	0.03828
$\alpha = 0.05$	1.00012	0.00467	1.08926	0.01336	1.17505	0.03665
$\alpha = 0.1$	1.00060	0.00477	1.08434	0.01252	1.16784	0.03415
$\alpha = 0.2$	1.00123	0.00508	1.07562	0.01144	1.15420	0.03007
$\alpha = 0.25$	1.00146	0.00527	1.07194	0.01113	1.14802	0.02848
$\alpha = 0.5$	1.00217	0.00648	1.05945	0.01093	1.12494	0.02390
$\alpha = 1$	1.00326	0.00882	1.05181	0.01251	1.10781	0.02259

Table 3.

Simulation results for $\min \mathfrak{R}_\alpha$ -estimators, $\min \mathfrak{D}_\alpha$ -estimators and MLE of the parameter $\sigma = 1$ when data are generated from the model $\mathcal{N}(0, 1)$, when 95 data are generated from the model $\mathcal{N}(0, 1)$ and 5 data from $\mathcal{N}(0, 3)$, respectively when 90 data are generated from the model $\mathcal{N}(0, 1)$ and 10 data from $\mathcal{N}(0, 3)$.

	no outliers		5% outliers		10% outliers	
	$\hat{\sigma}$	$\widehat{\text{MSE}}$	$\hat{\sigma}$	$\widehat{\text{MSE}}$	$\hat{\sigma}$	$\widehat{\text{MSE}}$
MLE						
$\alpha = 0$	0.99794	0.00498	1.17726	0.04887	1.33251	0.13507
$\min \mathfrak{R}_\alpha$						
$\alpha = 0.02$	0.99749	0.00493	1.15713	0.03788	1.30542	0.11284
$\alpha = 0.05$	0.99783	0.00496	1.13024	0.02669	1.26450	0.08485
$\alpha = 0.1$	0.99827	0.00505	1.09683	0.01692	1.20663	0.05362
$\alpha = 0.2$	0.99874	0.00536	1.06176	0.01059	1.13522	0.02695
$\alpha = 0.25$	0.99884	0.00557	1.05226	0.00955	1.11441	0.02152
$\alpha = 0.5$	0.99869	0.00709	1.03035	0.00905	1.06610	0.01367
$\alpha = 1$	0.99670	0.01198	1.01659	0.01356	1.03857	0.01598
$\min \mathfrak{D}_\alpha$						
$\alpha = 0.02$	0.99870	0.00487	1.15614	0.03690	1.30385	0.11189
$\alpha = 0.05$	0.99914	0.00489	1.12922	0.02596	1.26309	0.08411
$\alpha = 0.1$	0.99973	0.00497	1.09665	0.01677	1.20657	0.05379
$\alpha = 0.2$	1.00051	0.00525	1.06368	0.01083	1.13856	0.02810
$\alpha = 0.25$	1.00078	0.00543	1.05509	0.00982	1.11932	0.02285
$\alpha = 0.5$	1.00143	0.00659	1.03786	0.00906	1.07956	0.01526
$\alpha = 1$	1.00203	0.00891	1.03484	0.01101	1.07089	0.01588

Table 4.

Simulation results for $\min \mathfrak{R}_\alpha$ -estimators, $\min \mathfrak{D}_\alpha$ -estimators and MLE of the parameter $\sigma = 1$ when the data are generated from the model $\mathcal{N}(0, 1)$, when 95 data are generated from the model $\mathcal{N}(0, 1)$ and 5 data from δ_{10} .

	no outliers		5% outliers	
	$\hat{\sigma}$	$\widehat{\text{MSE}}$	$\hat{\sigma}$	$\widehat{\text{MSE}}$
MLE				
$\alpha = 0$	0.99859	0.00497	2.43937	2.07260
$\min \mathfrak{R}_\alpha$				
$\alpha = 0.02$	0.99731	0.00494	2.28640	1.65613
$\alpha = 0.05$	0.99773	0.00497	1.95068	0.90877
$\alpha = 0.1$	0.99828	0.00506	1.03369	0.01115
$\alpha = 0.2$	0.99900	0.00538	0.99922	0.00575
$\alpha = 0.25$	0.99923	0.00559	0.99913	0.00592
$\alpha = 0.5$	0.99968	0.00708	0.99947	0.00747
$\alpha = 1$	0.99875	0.01176	0.99832	0.01238
$\min \mathfrak{D}_\alpha$				
$\alpha = 0.02$	0.99567	0.00506	2.28566	1.65427
$\alpha = 0.05$	0.99602	0.00509	1.94971	0.90702
$\alpha = 0.1$	0.99646	0.00517	1.03533	0.01155
$\alpha = 0.2$	0.99700	0.00544	1.00358	0.00585
$\alpha = 0.25$	0.99716	0.00563	1.00522	0.00602
$\alpha = 0.5$	0.99755	0.00678	1.01521	0.00748
$\alpha = 1$	0.99818	0.00905	1.03344	0.01082

Table 5.

Simulation results for $\min \mathfrak{R}_\alpha$ -estimators and MLE of the parameter $m = 0$ when data are generated from the model $\mathcal{N}(0, 1)$, when 95 data are generated from the model $\mathcal{N}(0, 1)$ and 5 data from $\mathcal{N}(2, 1)$, respectively when 90 data are generated from the model $\mathcal{N}(0, 1)$ and 10 data from $\mathcal{N}(2, 1)$.

	no outliers		5% outliers		10% outliers	
	\widehat{m}	$\widehat{\text{MSE}}$	\widehat{m}	$\widehat{\text{MSE}}$	\widehat{m}	$\widehat{\text{MSE}}$
MLE						
$\alpha = 0$	0.00158	0.01004	0.10161	0.02054	0.20116	0.05063
$\min \mathfrak{R}_\alpha$						
$\alpha = 0.02$	0.00075	0.01003	0.09723	0.01956	0.19573	0.04853
$\alpha = 0.05$	0.00072	0.01006	0.09257	0.01873	0.18769	0.04554
$\alpha = 0.1$	0.00067	0.01015	0.08568	0.01768	0.17557	0.04141
$\alpha = 0.2$	0.00059	0.01045	0.07457	0.01640	0.15539	0.03549
$\alpha = 0.25$	0.00055	0.01065	0.07007	0.01605	0.14698	0.03339
$\alpha = 0.5$	0.00036	0.01194	0.05427	0.01583	0.11651	0.02774
$\alpha = 1$	0.00004	0.01550	0.03900	0.01864	0.08580	0.02675

Table 6.

Simulation results for $\min \mathfrak{R}_\alpha$ -estimators and MLE of the parameter $m = 0$ when data are generated from the model $\mathcal{N}(0, 1)$, when 95 data are generated from the model $\mathcal{N}(0, 1)$ and 5 data from δ_{10} , respectively when 90 data are generated from the model $\mathcal{N}(0, 1)$ and 10 data from δ_{10} .

	no outliers		5% outliers		10% outliers	
	\widehat{m}	$\widehat{\text{MSE}}$	\widehat{m}	$\widehat{\text{MSE}}$	\widehat{m}	$\widehat{\text{MSE}}$
MLE						
$\alpha = 0$	0.00227	0.00999	0.50186	0.26125	1.00183	1.01247
$\min \mathfrak{R}_\alpha$						
$\alpha = 0.02$	0.00018	0.01021	0.20421	0.05278	0.44055	0.20618
$\alpha = 0.05$	0.00023	0.01023	0.04819	0.01342	0.10345	0.02288
$\alpha = 0.1$	0.00033	0.01033	0.00499	0.01088	0.00999	0.01159
$\alpha = 0.2$	0.00054	0.01064	0.00106	0.01111	0.00147	0.01166
$\alpha = 0.25$	0.00065	0.01085	0.00112	0.01132	0.00150	0.01188
$\alpha = 0.5$	0.00118	0.01219	0.00158	0.01275	0.00194	0.01336
$\alpha = 1$	0.00198	0.01582	0.00230	0.01658	0.00271	0.01740

References

- [1] A. Basu, I. R. Harris, N.L. Hjort and M. C. Jones (1998). “Robust and efficient estimation by minimizing a density power divergence,” *Biometrika*, vol. 85, No. 3, pp. 549–559.
- [2] M. Broniatowski and A. Keziou (2006). “Minimization of ϕ -divergences on sets of signed measures,” *Studia Scientiarum Mathematica Hungarica*, vol. 43, pp. 403–442.
- [3] M. Broniatowski and A. Keziou (2009). “Parametric estimation and tests through divergences and the duality technique,” *Journal of Multivariate Analysis*, vol. 100, No. 1, pp. 16–31.
- [4] M. Broniatowski and I. Vajda (2009). “Several applications of divergence criteria in continuous families”, arXiv:0911.0937v1.
- [5] F. R. Hampel, E. M. Ronchetti, P. J. Rousseeuw and W. A. Stahel (1986). *Robust Statistics: The Approach Based on Influence Functions*, New York: Wiley.
- [6] F. Liese and I. Vajda, (1987). *Convex Statistical Distances*, Leipzig: Teubner.
- [7] F. Liese and I. Vajda, (2006). “On divergences and informations in statistics and information theory, *IEEE Transactions on Information Theory*, vol. 52, No. 10, pp. 4394–4412.
- [8] R.A. Maronna, R.D. Martin and V.J. Yohai, (2006). *Robust Statistics - Theory and Methods*, Wiley, New York.
- [9] C. Miescke and F. Liese, (2008). *Statistical Decision Theory*, Berlin: Springer.
- [10] L. Pardo, (2006). *Statistical Inference Based on Divergence Measures*, Chapman & Hall.
- [11] A. Toma, M. Broniatowski (2011). “Dual divergence estimators and tests: Robustness results”, *Journal of Multivariate Analysis*, vol. 102, No. 1, pp. 20–36
- [12] I. Vajda, (2008). *Modifications of Divergence Criteria for Applications in Continuous Families*. Research Report No. 2230, Institute of Information Theory and Automation, Prague, November 2008.
- [13] A. W. van der Vaart and J. A. Wellner (1996). *Weak Convergence and Empirical Processes*, Berlin: Springer.
- [14] A. W. van der Vaart (1998). *Asymptotic Statistics*, Cambridge University Press, Cambridge.