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–Brief Paper–

ALTERNATIVE METHOD OF SOLUTION OF THE REGULATOR EQUATION: L^2-SPACE APPROACH

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ABSTRACT

An alternative method for the proof of solvability of the differential equation that is a part of the regulator equation which arises from the solution of the output regulation problem. The proof uses the L^2-space based theory of solutions of partial differential equations for the case of the linear output regulation problem. In the nonlinear case, a sequence of linear equations is defined so that their solutions converge to the solution of the nonlinear problem. This is proved using the Banach Contraction Theorem.

Key Words: Output regulation problem, partial differential equations.

I. INTRODUCTION

First, the Output regulation problem is briefly presented. Let the following systems be defined as

$$\dot{x} = F(x) + G(x)u,\ x(0) = x(0),\ y = h(x) \tag{1}$$

$$\dot{v} = Sv,\ v(0) = v_0,\ reference = Qv. \tag{2}$$

where $x(t) \in R^n; u(t), y(t) \in R,\ S \in R^{\mu\times\mu},\ Q \in R^{1\times\mu}$. Functions $F, G: R^n \to R^n,\ F(0) = 0,\ h: R^n \to R$, $h(0) = 0$ are supposed to be sufficiently smooth. The system (1) is the controlled system (the plant) while (2) represents the exosystem.

The plant can be defined in a more general way, [1]. However, the restriction made above allows to significantly simplify the notation, especially those following the equation (10). As the case considered here covers a fairly large number of practical applications this assumption might be seen as not overly restrictive.

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A crucial role in the theory is played by the zero-error manifold. It is a μ-dimensional manifold in the $n+1$-dimensional Euclidean space. It can be described as the graph of functions $\mathbf{x}: R^\mu \to R^n,\ c: R^\mu \to R$ solving the *regulator equation*:

$$\frac{\partial \mathbf{x}(v)}{\partial v} Sv = F(\mathbf{x}(v)) + G(\mathbf{x}(v))c(v) \tag{3}$$

$$0 = h(\mathbf{x}(v)) - Qv. \tag{4}$$

If the state of the exosystem at the time t equals $v(t)$ then the tracking error is zero provided the equalities $x(t) = \mathbf{x}(v(t)), u(t) = c(v(t))$ hold.

The regulator equation consists of two parts: a system of partial differential equations (PDE) (3) and the algebraic equation (4). The equation (3) exhibits some features that can be considered 'nonstandard' in the common theory of PDE: It is a first-order PDE. Its solution is sought on the whole domain R^μ. This implies no boundary condition is given, it is replaced by the condition $\mathbf{x}(0) = 0$. There are several methods to solve the regulator equation (3), (4). One is based on undetermined Taylor series, see [1] and references therein. For the MIMO case results, see also [2], robust case is treated in the recent paper [3], neural networks solution of the regulator equation is described in [4]. A finite elements-based method appeared [5–7]. An algorithm is described in [5, 6]: the plant (1) is stabilized using a state feedback, then feedforward c is fixed, the differential equation (3) is solved, which in general does not guarantee validity of the algebraic equation (4).

Therefore, the error made in (4) is measured using a cost functional. This value is used to find another feedforward to decrease this error. In the sequel, it is assumed (1) has already been stabilized.

The papers [5, 6] present a condition of convergence of the above iterative algorithm, however, they do not deal with the question of solvability of (3) in the setting suitable for the iterative algorithm as the asymptotic expansion-based local estimates are not sufficient for this goal. The purpose of this paper (which is based on the conference paper [8]) is to fill this gap - to prove existence of its solution. In other words, the result of [6] was 'provided the equation (3) has a solution for each c on a domain Ω, the output regulation problem is solvable'. This paper adds 'there exists a domain Ω such that (3) has a solution for each c on Ω'. This together constitutes an alternative proof of existence of a solution of the output regulation problem to the one presented in [1]. Moreover, it has advantages as [6] provides useful global error estimates and serves as a base for modern numerical methods like the finite-element method. This paper also uncovers a strong link between the modern (L^p-based) PDE theory and output regulation theory as conditions for existence of a solution of a first-order PDE ([9]) are naturally satisfied by stability of the plant and neutral stability of the exosystem. For the sake of clarity, the linear case is treated first in the next section, these results are used while studying the nonlinear case later on.

Another approach to solve the regulator equation is presented in [4]. It is based on a design of neural networks that approximate the solution.

Notation used: $L^2(\Omega)$ is the space of scalar square-integrable functions defined on the domain Ω, It is equipped with the usual norm (denoted by $\|.\|$). The space $(L^2(\Omega))^n = \{(\varphi_1,\ldots,\varphi_n)|\varphi_i \in L^2(\Omega)\forall i = 1,\ldots,n\}$. For a more detailed description, see [10] and references therein. All equalities between functions are considered in the 'almost everywhere' sense, all functions are supposed to be measurable. The gradient of the function $f(v)$ is denoted by $\nabla f(v)$ or $\frac{\partial f}{\partial v}$.

II. LINEAR CASE

In this case, the plant is described by the equation $\dot{x} = Ax + Bu,\ y = Hx$ with matrices A, B, H having suitable dimensions. (As noted in the Introduction, the plant is stabilized by a state feedback, thus A is supposed to be Hurwitz with real eigenvalues. Under usual controllability assumptions, this can be achieved. The eigenvalues of the matrix A are also considered to be design parameters.) In this case, the equations (3), (4) attain the form

$$\frac{\partial \mathbf{x}(v)}{\partial v} Sv = A\mathbf{x}(v) + Bc(v), \tag{5}$$

$$0 = H\mathbf{x}(v) - Qv. \tag{6}$$

To solve the above equation numerically, one has to restrict oneself to the solution on a bounded domain which will be denoted by Ω. The question that arises now is the following: *Exists there a bounded domain* $0 \in \Omega \subset R^\mu$ *and functions* $\mathbf{x} \in (L^2(\Omega))^n$, $c \in L^2(\Omega)$ *such that the equation* (5) *is satisfied*? *Is the condition* $\mathbf{x}(0) = 0$ *satisfied*; *if so*, *in what sense*? To give the answer the following assumptions are essential and are thus supposed throughout the following text:

- **L1**: the matrix A has real negative eigenvalues.
- **L2**: there exist a diagonal matrix D and a regular matrix T such that $A = T^{-1}DT$.
- **L3**: all eigenvalues of S are simple, have zero real part and there exists a smooth Lyapunov function $V: R^\mu \to [0, +\infty)$ such that $V(0) = 0$, $V(v) > 0$ for $v \neq 0$, $\nabla V(v).Sv = 0$.

The equation (5) can be rewritten as $T\frac{\partial \mathbf{x}}{\partial v}Sv = DT\mathbf{x}(v) + TBc(v)$. Having changed the variables $\xi(v) = T\mathbf{x}(v)$ one observes that $T\frac{\partial \mathbf{x}}{\partial v} = \frac{\partial \xi(v)}{\partial v}$. To sum up, (5) is transformed into $\frac{\partial \xi}{\partial v}Sv = D\xi(v) + TBc(v)$. Considering the function c to be fixed, the above system is composed of n equations such that the unknown function ξ_i appears only in the i-th equation which then reads (denote the i-th diagonal element of D by d_{ii}):

$$\frac{\partial \xi_i}{\partial v} Sv = d_{ii}\xi_i(v) + (TBc(v))_i \tag{7}$$

A theorem guaranteeing existence of a solution of a PDE of this type can be found in [9], Lemma 1.6. Before citing it one has to deal with boundary conditions.

Assume for now a domain $\Omega \subset R^\mu$ as described above is given. Denote the element of the matrix S on the i,j position by S_{ij}. Moreover, let $\theta: R^\mu \to R^\mu$ be defined as

$$\theta(v) = \left(\sum_{k=1}^{\mu} S_{1k}v_k, \ldots, \sum_{k=1}^{\mu} S_{\mu k}v_k\right) = Sv.$$

Note that for every $i \in \{1,\ldots,n\}$: $\frac{\partial \xi_i}{\partial v}Sv = \theta(v)\nabla\xi_i$. For each $v \in \partial\Omega$ let $n(v)$ denote the outward normal to the domain Ω at the point v. As in [9], denote also $\Gamma_- = \{v \in \partial\Omega | \theta(v).n(v) < 0\}$.

Remark II.1. With the assumptions, $\operatorname{div}\theta(v) = 0$.

Proof. From the definition of θ one has $\operatorname{div}\theta(v) = \sum_{j=1}^{\mu} \frac{\partial}{\partial v_j}(\sum_{k=1}^{\mu} S_{jk} v_k) = \sum_{j=1}^{\mu} S_{jj} = \text{Trace}\, S$. Thanks to L3 one has Trace $S = 0$. □

The lemma guaranteeing existence of a solution of (7) is cited here without proof (see [9], Lemma 1.6.):

Lemma II.1. Let $\Omega \subset R^{\mu}$ be a domain with smooth boundary and let the functions $b : \bar{\Omega} \to R^{\mu}$, $\beta : \bar{\Omega} \to R$ be from $C^1(\bar{\Omega})$ and let $\varphi \in L^2(\Omega)$. Assume there exists a constant $\omega > 0$ such that

$$\beta(v) - \tfrac{1}{2}\operatorname{div}\, b(v) \geq \omega \quad \forall v \in \Omega. \tag{8}$$

Let $\xi_0 \in L^2(\Gamma_-)$. Then there is a function $\xi \in L^2(\Omega)$ such that $\nabla\xi(v)$ exists that solves the equation

$$b(v)\nabla\xi(v) + \beta(v)\xi(v) = \varphi(v) \quad \text{in } \Omega,$$

$$\xi(v) = 0 \quad \text{on } \bar{\Gamma}_-.$$

Remark II.2. Lemma II.1 is applied to the equation (7) as follows: the function b, resp. β in Lemma II.1 correspond to the function θ, resp. d_{ii} as defined above. Hence, (8) now reads $-d_{ii} - \frac{1}{2}\operatorname{div}\theta(v) \geq \omega\ \forall v \in \Omega$. This holds thanks to Remark II.1 and due to $d_{ii} < 0$.

Lemma II.2. Under L3, there exists a set of domains with smooth boundary Ω_α, $\alpha > 0$ such that

$$0 \in \Omega_\alpha \quad \forall \alpha > 0$$

$$\Omega_\alpha \subset \Omega_\beta \quad \text{if } \alpha < \beta$$

$$n(v).\theta(v) = 0 \quad \forall v \in \partial\Omega_\alpha.$$

Hence, $\Gamma_- = \emptyset$ for every Ω_α. Consequently, the boundary condition is not needed for the solution of (7).

Proof. Let $\alpha > 0$, $\Gamma_\alpha = \{v \in R^{\mu} | V(v) = \alpha\}$, $\Omega_\alpha = \{v \in R^{\mu} : V(v) < \alpha\}$. Then the normal vector $n(v)$ is parallel to the vector $\nabla V(v)$ for every $v \in \Gamma_\alpha$. On the other hand, $\nabla V(v).\theta(v) = 0$ thus also $\Gamma_- = \emptyset$. Γ_α is smooth due to the implicit function theorem. □

Let us multiply the equation (7) by ξ_i, integrate it over the domain Ω_α and use the Stokes theorem on the term containing the derivatives of ξ_i. Taking into account that $n(v).\theta(v) = 0$ on $\partial\Omega_\alpha$ and $\operatorname{div}\theta = 0$ in Ω_α as shown above, one arrives at

$$\int_{\Omega_\alpha} -d_{ii}\xi^2 dv = \int_{\Omega_\alpha} \xi_i (TB)_i c\, dv. \tag{9}$$

Using the Schwarz inequality $\|\xi_i\| \leq \frac{\|(TB)_i\|}{-d_{ii}}\|c\|$ on Ω_α. As the function $\mathbf{x}$ is a linear combination of the functions ξ_i, one has $\|\mathbf{x}\| \leq C\|c\|$ for some $C > 0$. Note that $C \to 0$ if $|d_{ii}| \to +\infty$ for all $i = 1, \ldots, n$ and also, the smaller measure of Ω_α, the smaller value of C.

Theorem II.1. There exists an open bounded domain $\Omega \subset R^{\mu}$ with smooth boundary such that $0 \in \Omega$ such that for every function $c \in L^2(\Omega)$ there exists a uniquely determined function $\mathbf{x} \in L^2(\Omega)$ satisfying (3) in Ω while no boundary condition is to be defined on $\partial\Omega$. Moreover, there exists a constant $C > 0$ (independent of c) such that $\|\mathbf{x}\| \leq C\|c\|$.

Uniqueness follows from the above estimate combined with linearity, the remaining part was proved above.

The next task is to verify the condition $\mathbf{x}(0) = 0$. As all the involved functions are elements of the space $L^2(\Omega)$ one cannot speak about their function values. Instead of it one can use the expression

$$L(\mathbf{x}, 0) = \lim_{t \to 0+} L(\mathbf{x}, t),$$

$$L(\mathbf{x}, t) = \frac{1}{\text{meas} B_t} \int_{B_t} \mathbf{x}(v)\, dv$$

where the symbol B_t denotes the μ-dimensional open ball with radius t and the center at the origin. Hence:

Lemma II.3. Assume there exists $M > 0$ such that for every $t >$ there exists $\alpha_t > 0$ such that $B_t \subset \Omega_{\alpha_t} \subset B_{Mt}$. Let $L(c^2, 0) = 0$. Then $L(\mathbf{x}, 0) = 0$.

Denote $\tilde{L}(\mathbf{x}, \alpha) = \frac{1}{\text{meas}\,\Omega_\alpha}\int_{\Omega_\alpha} |\mathbf{x}| dv$. Then $L(\mathbf{x}, t) \leq (L(|\mathbf{x}|^2, t)^{\frac{1}{2}} \leq (\frac{\text{meas}\,\Omega_{\alpha_t}}{\text{meas}\,B_t})^{\frac{1}{2}} (\tilde{L}(|\mathbf{x}|^2, \alpha_t)^{\frac{1}{2}} \leq C(\frac{\text{meas}\,\Omega_{\alpha_t}}{\text{meas}\,B_t})^{\frac{1}{2}} (\tilde{L}(c^2, \alpha_t)^{\frac{1}{2}} \leq C(\frac{\text{meas}\,B_{Mt}}{\text{meas}\,B_t})^{\frac{1}{2}} (L(c^2, Mt))^{\frac{1}{2}}$. The latter converges to 0 with $t \to 0$. (The first inequality is the Jensen's inequality.)

Remark II.3. Let the functions c, $\mathbf{x}$ be continuous with $c(0) = 0$. Then $\xi(0) = 0$. Consequently $\mathbf{x}(0) = 0$.

If a function φ is continuous at 0 $L(\varphi, 0) = \varphi(0)$ holds.

III. NONLINEAR CASE

It is assumed the equation (3) is solved on the domain Ω satisfying properties in Lemma II.2.

Originating in (1), denote by A the Jacobi matrix of the function F (evaluated at the origin) and let $B = G(0)$. Define also the functions $f : R^n \to R^n$ and $g : R^n \to R^n$ by $f(x) = F(x) - Ax$, $g(x) = G(x) - B$.

The equation (3) can be written in form

$$\frac{\partial \mathbf{x}}{\partial v} Sv = A\mathbf{x} + Bc + f(\mathbf{x}) + g(\mathbf{x})c. \tag{10}$$

The following assumption will be assumed in the following text.

Assumption N1. There exist positive constants $K_f, K_g, \tilde{K}_g$ such that for every $x_1, x_2 \in R^n$ the conditions (a) and either (b_1) or (b_2) hold.

(a) $\|f(x_1) - f(x_2)\| \leq K_f \|x_1 - x_2\|$,

(b_1) $\|g(x_1) - g(x_2)\| \leq K_g \|x_1 - x_2\|$, $\|g(x)\| \leq \tilde{K}_g$,

(b_2) $\|g(x_1) - g(x_2)\| \leq K_g \|x_1 - x_2\|$, $\|g(x)\| \leq \tilde{K}_g \|x\|$.

Remark III.1. Let $x \in L^2(\Omega)$. The condition (a) implies $f(x) \in L^2(\Omega)$. Similarly, (b_1) and $c \in L^2(\Omega)$ imply $g(x)c(x) \in L^2(\Omega)$. Finally, the same holds under (b_2) and $c \in L^\infty(\Omega)$. See [10] for details.

Proposition III.1. There exists a domain Ω such as in Theorem II.1 such that for every $\tilde{\mathbf{x}} \in (L^2(\Omega))^n$, $c \in L^2(\Omega)$ there exists a function $\mathbf{x} \in (L^2(\Omega))^n$ solving

$$\frac{\partial \mathbf{x}}{\partial v} S v = A\mathbf{x} + Bc + f(\tilde{\mathbf{x}}) + g(\tilde{\mathbf{x}})c. \tag{11}$$

Lemma III.1. Let $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2 \in (L^2(\Omega))^n, c \in L^2(\Omega)$ be given. Denote by $\mathbf{x}_i$, $i \in \{1, 2\}$ the solution of the equation (11) with $\tilde{\mathbf{x}}$ replaced by $\tilde{\mathbf{x}}_i$. Then, there exist positive constants $K_1 = CK_f, K_2 = CK_g$ such that $\|\mathbf{x}_1 - \mathbf{x}_2\| \leq (K_1 + K_2\|c\|)\|\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\|$.

Proposition III.1 with Theorem III.1 yield the proof.

Theorem III.1. If $K_1 + K_2\|c\| < 1$ holds then a unique solution of (3) exists.

Proof. Let $c \in L^2(\Omega)$. Lemma III.1 and (N) imply the mapping $\Pi : L^2(\Omega) \to L^2(\Omega)$ defined by $\Pi(\tilde{\mathbf{x}}) = \mathbf{x}$ where $\mathbf{x}$ solves (11) is a contraction. Let $\mathbf{x}_0$ solve (5). Moreover, let $\mathbf{x}_i$ solve (11) with $\tilde{\mathbf{x}}$ replaced by $\mathbf{x}_{i-1}$, $i = 1, 2, \ldots$. Then, the sequence $\mathbf{x}_i$ converges to a function $\mathbf{x}$ strongly in $L^2(\Omega)$. It remains to prove that the function $\mathbf{x}$ solves (3). Note that there exists the solution $\mathbf{z}$ of the equation $\frac{\partial \mathbf{z}}{\partial v} S v = A\mathbf{z} + Bc + f(\mathbf{x}) + g(\mathbf{x})c$. To prove that $\mathbf{z} = \mathbf{x}$, subtract this from (11) with $\tilde{\mathbf{x}}$ replaced by $\mathbf{x}_{i-1}$:

$$\frac{\partial(\mathbf{z} - \mathbf{x}_i)}{\partial v} S v = A(\mathbf{z} - \mathbf{x}_i) + (f(\mathbf{x}) - f(\mathbf{x}_{i-1})) + (g(\mathbf{x}) - g(\mathbf{x}_{i-1}))c.$$

This implies $\|\mathbf{z} - \mathbf{x}_i\| \leq (K_1 + K_2\|c\|)\|\mathbf{x} - \mathbf{x}_{i-1}\|$, the limit for $i \to +\infty$ here gives $\mathbf{z} = \mathbf{x}$. □

Example. Let $\varphi(t) = \exp(-1/t^2)$ if $t \neq 0$, $\varphi(0) = 0$. Consider the system $\dot{x}_1 = -x_1 + (1 - \varphi(x_2))c$, $\dot{x}_2 = \varphi(x_1) - x_2$ with the exosystem defined as $\dot{v}_1 = v_2$, $\dot{v}_2 = -v_1$, we aim $v_1 = x_1$. Then, the Taylor series-based method yields $c = v_1 + v_2$ as if no nonlinearities are present which leads to a significant error in the solution of the regulator equation. The figure below shows the residua (differences in the left- and right hand sides) in the second equation of (3) if first v_1 and, second, the finite-element solution of (3) is substituted for $\mathbf{x}$. In the Taylor expansion-based method, the residua are much higher than those of caused by the finite elements, moreover, they exhibit a clear pattern showing how precision decreases with increasing distance from the origin. Note that to apply the finite elements, estimates of the solution derived here are necessary. More can be found in [6].

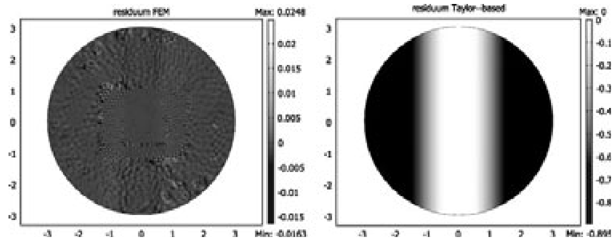

Fig. 1. Accuracy of approximations of (3).

IV. CONCLUSIONS

A proof of solvability of the differential part of the regulator equation suitable for the optimization-based iterative method [6, 5] was presented (Fig. 1).

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