Can we Improve Understanding of the Financial Market Dependencies in the Crisis by their Decomposition?

Pomůže nám dekompozice závislostí na finančních trzích zlepšit jejich pochopení v krizi?

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Abstract

Study of the financial market dependencies have become one of the most active and successful areas of research in the time series econometrics and economic forecasting during the recent decades. Current financial crisis have shown that understanding of the dependencies in the markets is crucial and it has even boosted the interest of researchers. This work brings new theoretical framework for the realized covariation estimation generalizing the current knowledge and bringing the estimation to the time-frequency domain for the first time. Usage of wavelets allows us to decompose the correlation measures into several investment horizons. Our estimator is moreover able to separate individual jumps, co-jumps and true covariation from the high frequency data, thus brings better understanding of the dependence. The results have crucial impact on the portfolio diversification especially in the crisis years as they point to the strong dynamic relationships at various investment horizons. Results suggest that understanding jumps and co-jumps is important for forecasting the covariance and the correlation as they have large impact on these measures. Our results have significant economic value as wrong assumption about the dependence process will have direct impact on the forecasting and portfolio valuation.

Keywords

correlation, multivariate realized volatility, covariation, jumps, co-jumping, wavelets

JEL Codes

C22, C51, C58, G01, G17

Abstrakt

Studium závislostí na finančních trzích se stalo jednou z nejaktivnějších a nejúspěšnějších oblastí výzkumu v ekonometrii časových řad a v ekonomických prognózách v posledních dekádách. Probíhající finanční krize ukázala, že porozumění závislostem na trzích je klíčové a ještě víc rozproudila zájem výzkumníků o problematiku. Tato práce přináší nový teoretický rámec pro odhad realizované kovariance. Hlavním přínosem této práce je zobecnění stávajících poznatků a možnost studia závislostí v časově-frekvenční doméně. Pomocí našeho odhadu založeného na waveletové analýze můžeme studovat dekomponované korelace dynamicky v čase a na různých časových horizontech zároveň. Náš odhad dále odděluje individuální skoky, společné skoky a skutečnou kovarianci od vysoko frekvenčních dat a tím umožňuje lepší porozumění závislosti. Výsledky mají klíčový dopad na diversifikaci portfolia a to zejména v krizi, jelikož poukazují na silné dynamické vztahy

na různých investičních horizontech. Výsledky naznačují, že pochopení skoků a koordinovaných skoků je důležité pro předpovídání kovariance a korelace, neboť mají velký dopad na tato měření. Naše výsledky jsou ekonomicky důležité, jelikož ukazují, jak velký vliv bude mít nesprávný předpoklad o dynamice závislosti na předpovídání a zhodnocení portfolia.

Klíčová slova
korelace, mnohorozměrná realizovaná volatilita, kovariance, skoky, společné skoky, wavelety

Introduction

One of the most fundamental issues in finance is research of the covariance generating process between asset returns. Demand for accurate covariance estimation is becoming more important for risk measurement and portfolio optimization than ever before. The increasing availability of high-frequency data for a wide range of securities has allowed a shift from parametric conditional covariance estimation based on daily data toward the model-free measurement of so-called "realized quantities" on intraday data. Using a seminal result in semi-martingale process theory, Andersen et al. (2003) show that realized variance becomes a consistent estimator of integrated variance with increasing sampling frequency under the assumption of zero microstructure noise. Barndorff-Nielsen and Shephard (2004) generalize the idea to a multivariate setting of so-called "realized covariation" and provide an asymptotic distribution theory for covariance (and correlation) analysis – again with the assumption of zero microstructure noise.

Although the theory is very appealing and intuitive, it assumes that the observed high-frequency data are the true underlying process. But real-world data are contaminated with microstructure noise and jumps, which makes statistical inference difficult. Realized measures suffer from large bias and inconsistency with the presence of noise and jumps in the observed data. The first approach to dealing with noise actually throws away a large amount of data. While this may not seem to be a logical step, the reason can be found quickly when one looks at the data at various sampling frequencies. The higher the frequency of the data we use (i.e., 1 second, 1 tick), the more microstructure noise they contain and the more biased the estimator is. Thus, a lot of researchers use lower frequencies (i.e., 5 minutes), which results in the throwing away of a very large amount of data directly. This is not an appropriate solution for a statistician to use. In the recent literature, a number of ways have been proposed to restore consistency through subsampling, for example Zhang et al. (2005)'s two-scale realized volatility estimator. Zhang (2011) generalizes these ideas to a multivariate setting and defines a two-scale covariance estimator. Barndorff-Nielsen et al. (2011) achieve positive semi-definiteness of the variance-covariance matrix using multivariate kernel-based estimation.

While inference under noise and jumps in realized variation theory has been widely studied in recent contributions, its generalization to covariation theory is only now emerging in the literature. Together with important contributions by Zhang (2011) and Barndorff-Nielsen et al. (2011), Griffin and Oomen (2011) and Aït-Sahalia et al. (2011) deal with microstructure noise and non-synchronous trading and propose a consistent and efficient estimator of realized covariance. Audrino and Corsi (2010) propose a forecasting model for realized correlations. This research is becoming very active and stands at the frontier of current research in financial econometrics.

In our work, we contribute to the current literature and provide a generalization of multivariate realized covariation theory. The theoretical results for the univariate setting motivate multivariate volatility modeling and forecasting based on realized covariation measures. While most time series models are set in the time domain, we enrich the analysis by the frequency domain. This is enabled by the use of the continuous wavelet transform. It is a logical step to take, as the stock markets are believed to be driven by heterogeneous investment horizons. In our work, we ask if wavelet decomposition can improve our understanding of co-volatility series and hence improve volatility forecasting and risk management.

Another very appealing feature of wavelets is that they can be embedded into stochastic processes, as shown by Antoniou and Gustafson (1999). Thus we can conveniently use them to extend the theory of realized measures. One of the issues with the interpretation of wavelets in economic applications is that they behave like a filter. Thus wavelets can hardly be used for forecasting in econometrics. But in the realized measures, we use wavelets only to decompose the daily variation of the returns using intraday information. Moreover, the approach suggests constructing a model from the wavelet decomposition.

We are not the first to use this idea. Several attempts to use wavelets in the estimation of realized variation have emerged in the past few years. Høg and Lunde (2003) were the first to suggest a wavelet estimator of realized variance. Capobianco (2004), for example, proposes to use a wavelet transform as a comparable estimator of quadratic variation. Subbotin (2008) uses wavelets to decompose volatility into a multi-horizon scale. Next, Nielsen and Frederiksen (2008) compare the finite sample properties of three integrated variance estimators, i.e., realized variance, Fourier and wavelet estimators. They consider several processes generating time series with a long memory, jump processes as well as bid-ask bounce. Gencay et al. (2010) mention the possible use of wavelet multiresolution analysis to decompose realized variance in their paper, while they concentrate on developing much more complicated structures of variance modeling in different regimes through wavelet-domain hidden Markov models.

One remarkable exception which fully completes the current literature on using wavelets in realized variation theory is the work of Fan and Wang (2007), who were the first to use the wavelet-based realized variance estimator and also the methodology for the estimation of jumps from the data. In our work, we generalize the results of Fan and Wang (2007) in several ways. Instead of using the Discrete Wavelet Transform we use the Maximum Overlap Discrete Wavelet Transform (MODWT), which is a more efficient estimator and is not restricted to sample sizes that are powers of two. We also use the Daubechies family of wavelets instead of the Haar type. The most significant contribution is generalization of this approach to covariation and correlation estimation. Moreover, we also present a new theory for estimation of co-jumping in the stock markets.

This paper is organized as follows. Due to the limited space, the first chapter briefly introduces the realized measures of variance-covariance matrix. The second chapter introduces our new wavelet-based covariation theory together with a methodology for detecting multivariate co-jumps using wavelets. Wavelet decomposition is also used to define wavelet-based realized correlation. In the third part, our theory is used to study the dynamics of the dependence in the stock markets while we decompose the dependence into the several investment horizons, individual jumps and co-jumps. Finally, we build a forecasting model based on decomposed measures and study the impact on jumps and co-jumps on the correlation forecasting.

1 Realized Measures

Andersen et al. (2003) suggest estimating the quadratic covariation matrix by taking the outer product of the observed high-frequency return over the period. The realized covariance of the returns process $\mathbf{r}_{t,h}$ over the time interval $[t-h,t]$, for $0 \leq h \leq t \leq T$, is estimated by

$$\widehat{RC}_{t,h} = \sum_{i=1}^{n} \mathbf{r}_{t-h+\left(\frac{i}{n}\right)h} \mathbf{r}'_{t-h+\left(\frac{i}{n}\right)h}, \tag{1}$$

where is the number of observations in $[t-h,t]$. Details of these results can be found in Andersen et al. (2003) and Barndorff-Nielsen and Shephard (2004) who show that the *ex-post* realized covariance $\widehat{RC}_{t,h}$ is an unbiased estimator of the *ex-ante* expected covariation $RC_{t,h}$. With increasing sampling frequency, the realized covariance is, moreover, a consistent estimator of the covariation over any fixed time interval $h > 0$, as $n \to \infty$.

In practice, we observe only discrete prices, thus bias from discretization is unavoidable. Much more damage is caused by market microstructure effects such as price discreteness, bid-ask spread and bid-ask bounce. Thus, when using this estimator in practice, one is left with advice not to sample too often. While the optimal sampling frequency resulting from the vast research on the noise-to-signal ratio, nicely surveyed by Hansen and Lunde (2006), Bandi and Russell (2006), McAleer (2008) and Andersen and Benzoni (2007) can be used, this approach still causes a large amount of available data to be discarded. As in the univariate case of Zhang et al. (2005)'s two-scale realized volatility estimator, multivariate generalization addresses the problem Zhang (2011).

Another significant bias brought into the estimation is caused by jumps. Barndorff-Nielsen and Shephard (2006) introduce a test based on the difference between the bipower variation and the quadratic variation, but the work is currently unfinished. Andersen et al. (2007) and Huang and Tauchen (2005) present a study of multipower variations in order to assess the proportion of the quadratic variation attributable to jumps. Andersen et al. (2007) and Lee and Mykland (2008) introduce two very similar procedures, which compare intraday returns to a local volatility measure. Fan and Wang (2007) develop the wavelet methods for jump estimation. Jiang and Oomen (2008) construct a test based on the hedging error of a variance swap replication strategy. Aït-Sahalia and Jacod (2009) propose an estimator of truncated power variations computed at different sampling frequencies. Finally, Andersen et al. (2009) introduce a test for jumps constructed using the MedRV and MinRV measures. Other tests include Mancini (2009) and Lee and Hannig (2010). The harm imposed by ignoring jumps and co-jumps in assumed price processes can be large, especially with regard to forecasting, option pricing, portfolio risk management and credit risk management. In our work, we will propose a novel method utilizing wavelets to consistently estimate jumps and co-jumps in the data.

One last important assumption about the theory we did not mention is that the data are assumed to be synchronized, meaning that the prices of the assets were collected at the same time stamp. In practice, trading is non-synchronous, delivering fresh prices at irregularly spaced times which differ across stocks. Research of non-synchronous trading has been an active field of financial econometrics in past years - see, for example, Hayashi and Yoshida (2005) and Voev and Lunde (2007). This practical issue induces bias in the estimators and may be partially responsible for the Epps effect a phenomenon of decreasing empirical correlation between the returns of two different stocks with increasing data sampling frequency. In this work, we use refresh time scheme (Barndorff-Nielsen et al., 2011) to synchronize the data.

2 Decomposition of Realized Measures by Wavelets

Let us introduce the decomposition of the realized measure by wavelets. Due to the limited space of this paper, we introduce just the basic idea while we refer reader to the dissertation where the mathematical background is derived Barunik (2011). The realized wavelet covariance (using the MODWT) is a scale by scale decomposition of the realized covariance defined by Definition 1. The realized wavelet covariation of the -th and -th asset return from the m-dimensional vector of returns $\mathbf{r}_{\mathbf{t,h}}$ over $[t-h,t]$, for $0 \leq h \leq t \leq T$, can be defined as

$$\widehat{RC}^{(WRC)}_{(l,q)t,h} = \sum_{j=1}^{J_S+1} \sum_{k=1}^{n} \widetilde{\mathcal{W}}_{(l)j,t-h+\frac{k}{n}h} \widetilde{\mathcal{W}}_{(q)j,t-h+\frac{k}{n}h}, \tag{2}$$

where n is the number of intraday observations over $[t-h,t]$ and J_s is the number of scales considered. $\widetilde{\mathcal{W}}_{(q)j,t-h+\frac{k}{n}h}$ are the MODWT coefficients on scales $j = 1, \dots, Js + 1$, where $Js \leq \log_2 n$. The proof is provided in Appendix A.1.

While this estimator is just decomposition of 1, we need to make adjustments for the noise and jumps. In the Appendix A.1, we describe the methodology of jumps estimation using wavelets. On the jump-adjusted data, our final estimator is defined as follows. Let $\widehat{RC}^{(estimator,J)}_{(l,q)t,h}$ denote an estimator of the realized covariance between the -th and -th asset return on the jump-adjusted observed data, $\mathbf{y_{t,h}}^{(J)} = \mathbf{y}_{t,h} - \widehat{\mathbf{MWJC}}$.. The jump-adjusted wavelet two-scale realized covariance estimator (JWTSCV) is defined as:

$$\widehat{RC}^{(JWTSCV)}_{(l,q)t,h} = c_N \left(\widehat{RC}^{(WRC,J)}_{(l,q)t,h} - \frac{\bar{n}_G}{n_S} \widehat{RC}^{(S,J)}_{(l,q)t,h} \right), \tag{3}$$

where $\widehat{RC}^{(WRC,J)}_{(l,q)t,h} = \frac{1}{G} \sum_{g=1}^{G} \sum_{j=1}^{J_S+1} \sum_{k=1}^{n} \widetilde{\mathcal{W}}_{(l)j,t-h+\frac{k}{n}h} \widetilde{\mathcal{W}}_{(q)j,t-h+\frac{k}{n}h}$ obtained from wavelet coefficient estimates using the MODWT on a grid of size on the jump-adjusted observed data, $\mathbf{y_{t,h}}^{(J)} = \mathbf{y}_{t,h} - \widehat{\mathbf{MWJC}}$., and c_n is a constant that can be tuned for small sample performance. The proof of consistency and unbiasedness of our estimator can be found in Appendix A.3. Estimator 3 converges in probability to the *true* integrated covariance, which is of primary interest in this analysis. Thus we have defined a new wavelet-based Covariation theory which is able to estimate realized covariation consistently in the presence of noise and jumps. In the next section, we use this theory to propose estimators of covariance and realized beta, which are important for financial practitioners.

Once we estimate the variance-covariance matrix, we can easily transform it to the correlation measure which we will use in the empirical part.

3 Decomposition of Stock Market Dynamics

The main motivation of the theory is to bring a new view on the dynamics of the stock markets. The main power of our wavelet-based estimator is that it is able to decompose the realized measures into several investment horizons as well as study the individual jumps and co-jumping. Thus let us have a look at data.

3.1 Data Description

Foreign exchange future contracts are traded on the Chicago Mercantile Exchange (CME) on a 24-hour basis. These markets are among the most liquid, so they are suitable for testing our estimator. We will estimate the realized covariance of British pound (GBP), Swiss franc (CHF) and Euro futures (EUR), while we will focus on the GBP-CHF, GBP-EUR and CHF-EUR futures pairs. After estimating the covariance, we will study the correlations between the currencies. All contracts are quoted in the unit value of the foreign currency in US dollars, which makes them comparable. The cleaned data are available from Tick Data, Inc.[1]

It is important to understand the trading system before we begin the study. In August 2003, CME launched the Globex trading platform, which generated a large increase in the liquidity of currency futures. For the first time ever in a single month, the trading volume on the electronic trading platform exceeded 1 million contracts every day. On Monday, December 18, 2006, the CME Globex(R) electronic platform started offering 23-hours-a-day trading. The weekly trading cycle begins at 5:00 pm on Sunday and ends at 4:00 pm on Friday, while every day the trading is interrupted for one hour from 4:00 pm until 5:00 pm. These changes in the trading system had a dramatic impact on trading activity. For this reason, we restrict ourselves to a sample period extending from January 5, 2007 through November 17, 2010, which contains the most recent financial crisis. The futures contracts we use are automatically rolled over to provide continuous price records, so we do not have to deal with different maturities.

The tick-by-tick transactions are recorded in Chicago Time, referred to as Central Standard Time (CST). Therefore, in a given day, trading activity starts at 5:00 pm CST in Asia, continues in Europe followed by North America, and finally closes at 4:00 pm in Australia. We exclude potential jumps due to the one hour gap in trading from our analysis by redefining the day in accordance with the electronic trading system. Moreover, we eliminate Saturdays and Sundays, US federal holidays, December 24 to December 26, and December 31 to January 2 because of the very low activity on these days, which would bias the estimates. Finally, we are left with 944 days in the sample.

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1 *http://www.tickdata.com/*

For the analysis of relations between the currencies, it is crucial that they are synchronized in time. We use the refresh time scheme to synchronize the data. Looking more closely at the higher frequencies, we find that a large amount of transactions have a common time stamp, so we use the arithmetic average for all observations with the same time stamp. Finally, we redefine the clock according to the refresh time scheme so that we can work with the data that are synchronized. We use the refresh time scheme for each pair separately in order to keep as much data as possible in the analysis.

4 Multivariate Unconditional Volatility Distributions

Having prepared the data, we can proceed to study the dependencies. For each pair, we estimate the covariance and correlation using our jump wavelet two-scale realized covariance estimator (JWTSCV) and for the reference, also the realized covariance (RC) estimator for each futures pair under the study.

Table 1 provides the average estimated covariation and correlation among the three currencies. As the benchmark, we use unconditional open-to-close measures computed as the outer products of the open-to-close returns. Interestingly, the unconditional measures are not far from the realized measures. This seems to be a feature of currency data, as other authors, e.g. Barndorff-Nielsen et al. (2011), have found significant differences on large samples of US stocks.

Table 1: The average covariation ($\times 10^4$) and correlation among the three currencies, GBP-CHF, GBP-EUR and CHF-EUR.

	covariance			correlation		
	GBP-CHF	GBP-EUR	CHF-EUR	GBP-CHF	GBP-EUR	CHF-EUR
RC	0.305	0.384	0.434	0.472	0.605	0.738
JWTSCV	0.249	0.322	0.346	0.506	0.629	0.770
Open-Close	0.245	0.325	0.4217	0.458	0.623	0.787

Source: Author's computations.

All the correlations are positive. The average relationship between the studied currencies is strong, pointing to a strong degree of integration among these European countries. Our findings are consistent with those of Aït-Sahalia et al. (2011), who use the same data set as we do, with the only difference that their data sample ends in June 2009.

The RC estimator shows lower correlation on average. While the correlations are generally strong, it seems that co-jumps do have an impact on the currency data. When compared with the JWTSCV estimating only the dependence of the true, continuous part without jumps, it estimates the correlation to be a little higher. Economically, these differences may lead to improved results in portfolio theory. We will study this impact in more depth in the last part by proposing a forecasting model.

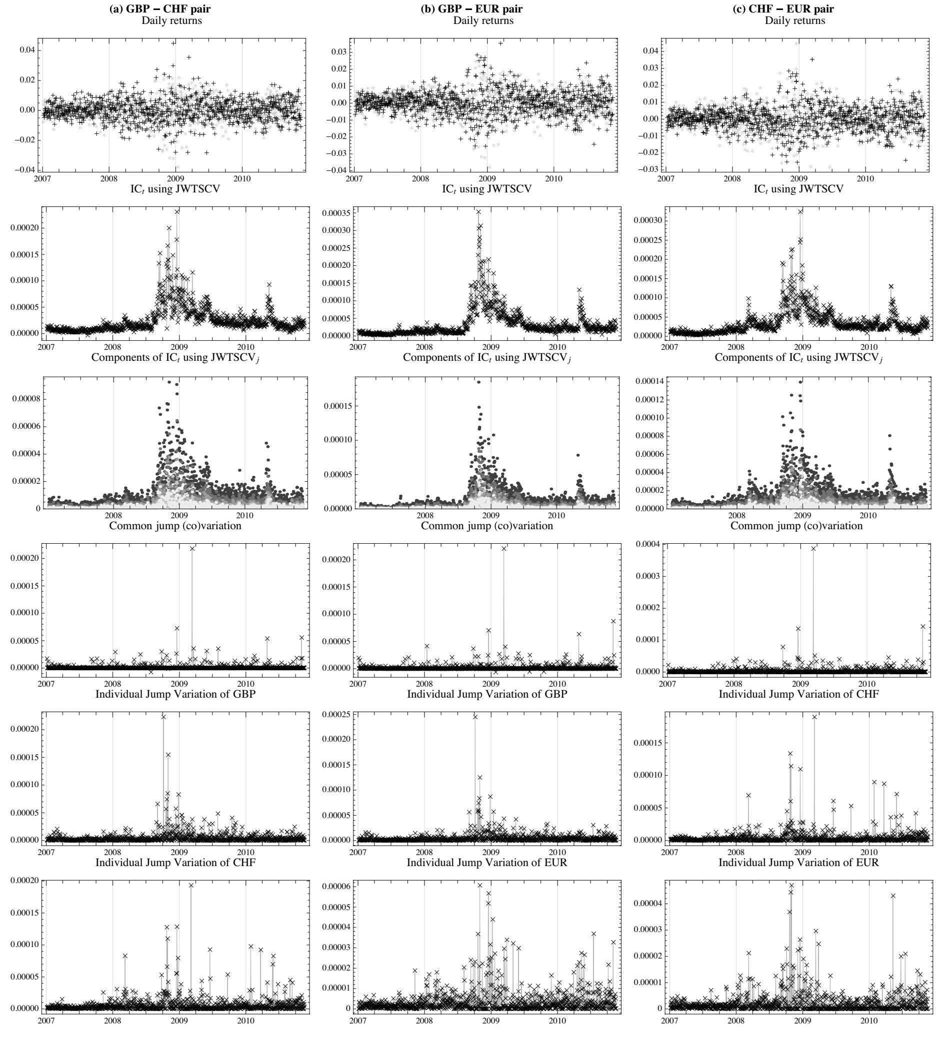
Before we do so, let us look at the decomposed dependencies using wavelets. We decompose the covariance and correlation measures into four scales, corresponding to investment horizons of 5–10 minutes, 10–20 minutes, 20–40 minutes and 40–80 minutes, and the rest (80 minutes up to 1 day). We remind the reader that the sum of these components will always add to the estimator.

5 Dynamics of Decomposed Dependencies

The previous section provided us with a basic statistical overview of the dependence between the currencies. While looking at the averages, we did not show the considerable variation of all the measures. Such variation points to interesting dynamics, which we further uncover. In addition, we take advantage of wavelet theory and study the dynamics of the decomposed measures as well. More specifically, we decompose the covariance and correlation measures into four scales corresponding to investment horizons of 5–10 minutes, 10–20 minutes, 20–40 minutes and 40–80 minutes, and the rest (80 minutes up to 1 day). Finally, we use wavelet theory to disentangle co-jumps and individual jumps from the series.

Figure 1 provides us with the decomposition of the estimated covariance for all the currency pairs. The first row provides the bivariate time series plots and the second row the covariance estimated by our JWTSCV estimator. The third row presents the decomposition of the covariance into the various investment horizons, while the last three rows give estimates of the co-jumps and individual jump variations of both series.

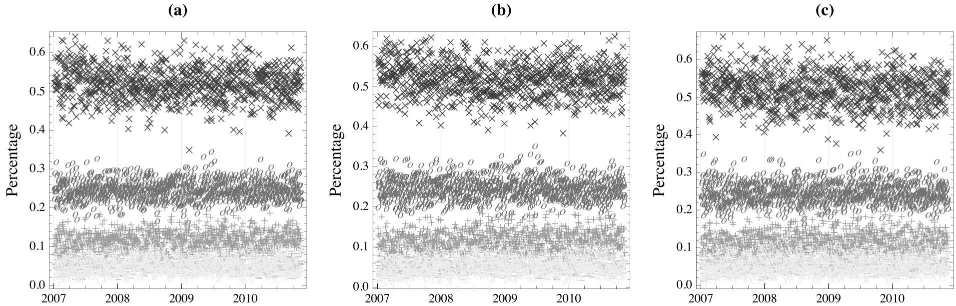
Figure 1: Daily returns, covariation estimated by JWTSCV, decomposition of covariation using $JWTSCV_j$ for $j = 1,\ldots,5$ correspondingcorresponding to investment horizons of 5–10 minutes, 10–20 minutes, 20–40 minutes, 40–80 minutes and 80 minutes up to 1 day, JWTSCV estimated common jump variation, individual jump variations of both time series. (a) GBP-CHF futures pair, (b) GBP-EUR futures pair and (c) CHF-EUR futures pair.



Source: Author's computations.

The most of the covariance comes from the 5–10 minute frequency, which accounts for about 50% of the total covariance, and the 10–20 minute frequency, which accounts for about 25% of the total, which is strikingly similar to the univariate case. The full picture of the contributions for all pairs can be seen in Figure 2.

Figure 2: $JWTSCV_j$, $j = 1,\ldots,5$, contributions of components of integrated covariation CV_t corresponding to investment horizons of 5-10 minutes, 10–20 minutes, 20–40 minutes, 40–80 minutes and 80 minutes up to 1 day. (a) GBP-CHF futures pair, (b) GBP-EUR futures pair and (c) CHF-EUR futures pair.



Source: Author's computations.

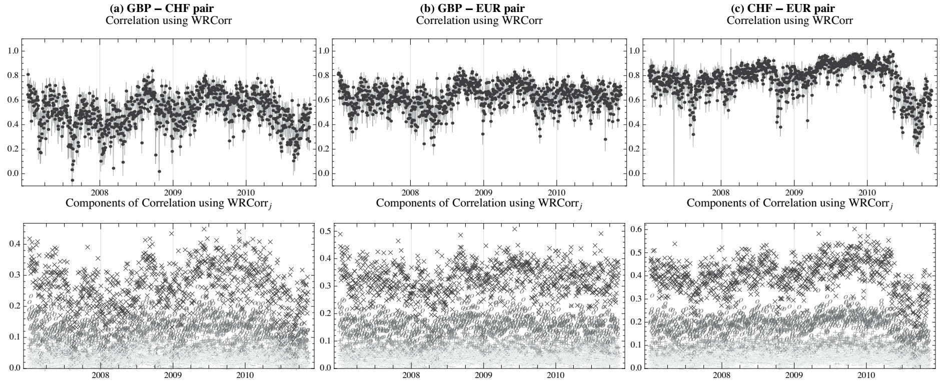
Our method of estimation also allows us to study jumps and co-jumps in the currencies. Interestingly, the jump variation is much stronger than the co-jump variation in the studied currencies. Still, the co-jumps are significant and should not be ignored in any further analysis. These results suggest that if the jumps are ignored, the covariation will be downward biased, as we saw in the previous analysis (Table 1).

Having computed the variances and covariances, we can take a look at the correlation dynamics. Figure 3 presents the estimate of wavelet-based correlation (WRCorr) with 95% confidence intervals, as well as its decomposition. We can see that the correlation of all the pairs vary substantially. While during 2007, the correlation of all three currencies was decreasing, it increased during 2008. At the end of 2008, during the largest stock market falls, which lasted approximately two weeks, the dependence in the currencies weakened. This finding is interesting, as the correlations are expected to grow during large drops. While the currencies show a strong degree of common dependence with the European Union, it seems that the recent financial crisis did not affect the dependence, while it of course substantially increased the variation of all series. Interestingly, the correlation of CHF with both GBP and EUR weakened substantially during 2010.

The decomposition of the correlations again shows an interesting result. Most of the correlation comes from the highest scale of 5–10 minutes. For example, of the total 0.506 average correlation of the GBP-CHF pair, the correlation on the 5–10 minute horizon is 0.26, the correlation on the 10–20 minute horizon is 0.13, and the rest corresponds to 0.06, 0.03 and 0.03 (note that by simply summing these correlations we get the total correlation for the pair).

Figure 4 provides a comparison of the correlation dynamics computed using two estimators: the basic realized correlation and our jump-adjusted wavelet correlation (WRCorr) estimator. It is noticeable that our WRCorr estimator provides an estimate with lower variance (basically due to jumps) and confidence intervals.

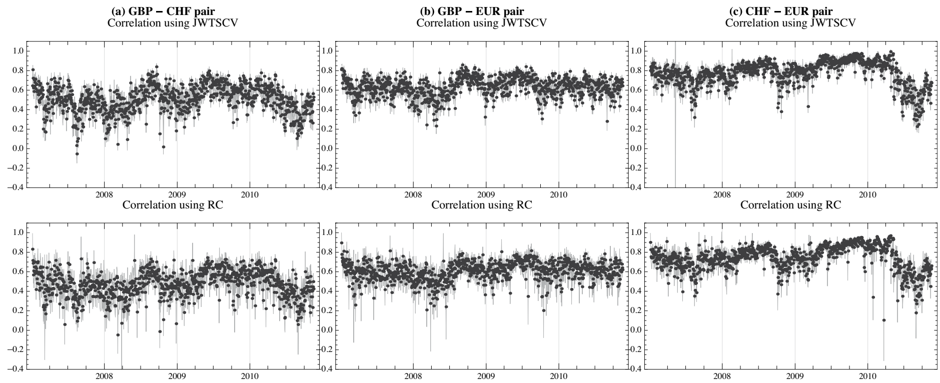
Figure 3: Correlations with 95% confidence interval and decomposition of correlations using *WRCorr*$_j$ for j = 1,…,5 for corresponding to investment horizons of 5–10 minutes, 10–20 minutes, 20–40 minutes, 40–80 minutes and 80 minutes up to 1 day. (a) GBP-CHF futures pair, (b) GBP-EUR futures pair and (c) CHF-EUR futures pair.



Source: Author's computations.

To be precise, the GBP-CHF futures pair, the GBP-EUR futures pair and the CHF-EUR futures pair have average estimated WRCorr correlations of 0.506 (0.069), 0.629 (0.053) and 0.769 (0.051), respectively (95% confidence intervals in parentheses). The average correlations for the same pairs estimated using the standard RC method are 0.47 (0.1), 0.602 (0.086) and 0.738 (0.062), respectively. Even though the correlations change significantly over time, the average correlation estimated using our method is approximately 0.03 larger than that using the simple RC. This result is economically significant and can have direct impact on portfolio diversification. Moreover, our method provides much narrower confidence intervals for the estimates.

Figure 4: Comparison of correlations with 95% confidence interval using in first row and using in second row. (a) GBP-CHF futures pair, (b) GBP-EUR futures pair and (c) CHF-EUR futures pair.



Source: Author's computations.

6 Forecasting Model Based on Decomposed Integrated Covariances

Motivated by the results from the previous analysis, we turn to building a forecasting model for covariances. Since the realized covariances show strong long memory behavior, we make use of this feature to build an ARFIMA-type long memory model. Moreover, we decompose the covariance into several investment horizons and jumps, and forecast the decompositions separately in hope it will bring improvement in forecasting. Forecasting model is described in detail in the 6.4.

6.1 Forecast Evaluation

To analyze the forecast efficiency and information content of the different covariance estimators, we employ the popular approach of Mincer and Zarnowitz (1969) regressions on both the realized covariance and its logarithmic transformation. The regression takes the form:

$$V_{t+1}^{(m)} = \alpha + \beta_1 V_t^{(k)ARFIMA} + \varepsilon_t, \tag{4}$$

with $V_{t+1}^{(m)}$ being the integrated covariance (or its logarithmic transformation) estimated using the th estimator, namely, realized covariance and our jump wavelet two-scale realized covariance estimator. $V_t^{(k)ARFIMA}$ denotes the 1-day ahead forecast $V_{t+1}^{(m)}$ of using the k - th estimator based on ARFIMA (1, d, 1), while we consider the same estimators and independently identically distributed error term. We report in-sample as well as rolling out-of-sample results. If the forecast is unbiased, we expect to α = 0 and β = 1.

After testing the forecasting efficiency of the different covariance estimators, we would also like to test the information content of the wavelet decomposition of the realized covariance. For this purpose, we separately estimate ARFIMA (1, d, 1) for all components $JWTSCV_j$ for j = 1, ...,5 of the realized covariance as well as the estimated jumps. We should note that in the case of logarithmic transformation of the realized covariance, we also take logarithms of the decomposed levels $JWTSCV_j$. After obtaining the forecast for each level, we transform the forecasts back to be able to compare the results.

Finally, we test the information content of the separate decomposed realized covariances by running the following regressions:

$$JWTSCV_{t+1} = \alpha + \beta_1 W_{t,j}^{ARFIMA} + \varepsilon_t, \tag{5}$$

where $W_{t,j}^{ARFIMA}$ denotes the one-day ahead forecasts of the individual components $JWTSCV_j$ for j = 1, ...,5 corresponding to investment horizons of 5–10 minutes, 10–20 minutes, 20–40 minutes, 40–80 minutes and 80 minutes up to 1 day, respectively, and

$$JWTSCV_{t+1} = \alpha + \beta_1 J_t^{ARFIMA} + \varepsilon_t, \tag{6}$$

where J_t^{ARFIMA} denotes the forecasts of the jumps. For now, we consider to include both co-jumps and individual jumps and we will test its separate impact in the following section.

Thus we test the information content of the long memory forecasts of the realized covariance estimators using the coefficient of determination, R^2, of the regression.

6.2 Does Decomposition Bring any Improvement in Covariation Forecasting?

We use the period from January 5, 2007 to December 31, 2009 to perform the estimations of all the models. We refer to this period as the in-sample period and it contains the GBP-CHF, GBP-EUR and CHF-EUR pairs. The year 2010 is saved for comparison of the out-of-sample forecasts, which are done on a rolling basis.

Table 2 presents the results of the realized covariation forecasts. JWTSCV is the easiest to forecast in terms of having the highest for all cases except the GBP-EUR pair, where RC results in a slightly higher . Thus JWTSCV seems to carry the most significant information in comparison with the other estimators. It confirms that the continuous part of the realized covariance has the highest information content.

Table 2: Results for *RCt:* R^2 for the Minzer-Zarnowitz regressions regressing ARFIMA forecasts of RC, JWTSCV and JWTSCV on its estimates.

	in-sample								
	GBP-CHF			GBP-EUR			CHF-EUR		
	RC	JWTSCV	JWTSCV	RC	JWTSCV	JWTSCV	RC	JWTSCV	JWTSCV
RC	0.733	0.738	0.737	0.836	0.842	0.837	0.741	0.747	0.743
JWTSCV	0.773	0.787	0.787	0.862	0.871	0.867	0.796	0.806	0.802
	out-of-sample								
	RC	JWTSCV	JWTSCV	RC	JWTSCV	JWTSCV	RC	JWTSCV	JWTSCV
RC	0.338	0.354	0.322	0.365	0.366	0.365	0.413	0.401	0.383
JWTSCV	0.419	0.402	0.378	0.415	0.402	0.393	0.516	0.468	0.451

Source: Author's computations.

When we decompose the realized covariation, forecast its components individually and then use the sum of the forecasts, it does not seem to bring the improvement in forecasting. Table 3 presents the results of the decomposed models. The separate realized covariances also carry quite large information content, as the first three are able to forecast the realized covariance similarly well. In other words, the 5–10 minute covariation component is able to forecast the total covariation JWTSCV with a similar forecasting power as if the total JWTSCV was used. Thus, even though decomposition does not bring an overall improvement, we can see that the realized covariation at the higher frequency carries the most important information also for forecasting. In other words, the main part of the realized covariation comes from the highest frequency.

Finally, we can see that jumps carry important information which may help to forecast the realized covariation. When we use only jumps to forecast realized covariance, the is relatively high. All the estimated parameters are significantly different from zero and the in-sample fits describe the data well. For reasons of space, we do not provide all the results here and we proceed to test the impact of further decomposition of the jumps into individual jump and co-jump components.

Table 3: Results for *RCt:* R^2 for the Minzer-Zarnowitz regressions regressing ARFIMA forecasts of decomposed covariances.

	in-sample					
	W1	W2	W3	W4	W5	Jumps
GBP-CHF	0.788	0.777	0.749	0.742	0.736	0.593
GBP-EUR	0.864	0.864	0.844	0.829	0.830	0.715
CHF-EUR	0.805	0.796	0.766	0.757	0.755	0.510
	out-of-sample					
	W1	W2	W3	W4	W5	Jumps
GBP-CHF	0.377	0.391	0.384	0.231	0.208	0.134
GBP-EUR	0.394	0.385	0.335	0.276	0.339	0.187
CHF-EUR	0.451	0.449	0.445	0.261	0.322	0.277

Note: W j denotes JW T S CV j , j = 1, . . . , 5 components of realized covariance and Jumps all jumps including co-jumps and individual jumps.
Source: Author's computations.

6.3 Impact of Jumps and Co-jumps on the Covariance Forecasts

We would like to see if further decomposition to co-jumps and individual jumps can help to forecast the realized covariances. For this purpose, we construct an ARFIMA (1, *d*, 1) model for the jump and co-jump components of the realized covariance estimated using our methodology and test for the informational efficiency of each of them to the realized covariance forecast using the encompassing regression:

$$JWTSCV_{t+1} = \alpha + \beta_1 JWTSCV_t^{ARFIMA} + \beta_2 J_{co-jumps,t}^{ARFIMA} + \beta_3 J_{1,t}^{ARFIMA} + \beta_4 J_{2,t}^{ARFIMA} + \varepsilon_t, \quad (7)$$

where $JWTSCV_t^{ARFIMA}$ denotes the one-day ahead forecast of $JWTSCV_{t+1}$ and $J_{co-jumps,t}^{ARFIMA}$ denotes the forecast of co-jumps, while $J_{1,t}^{ARFIMA}$ and $J_{2,t}^{ARFIMA}$ denote the forecasts of individual jumps of both assets in the forecasted pair.

With the help of the encompassing regressions, we can test if jumps contain any information relevant to the covariation forecasts. We will first test the information content of $J_{co-jumps,t}^{ARFIMA}$, $J_{1,t}^{ARFIMA}$ and $J_{2,t}^{ARFIMA}$ separately by setting all other β s to zero. Then, we will add parameters to the regression, starting with α and β_1, and adding β_2, β_3 and β_4 gradually to see if they bring any information, which is not contained in the realized covariation forecast itself. If, for example, common jumps carry information important for the forecast, parameter β_2 will be significantly different from zero, even if parameter β_1 is significantly different from zero.

Table 4 summarizes the results for the realized covariances. Striking evidence of the significance of the co-jumps for the forecasts is found in all cases (in only two cases the parameter has p-values of 0.103 and 0.104, so we can consider it to be marginally significant at the 89% level of significance). The presence of co-jumps in the encompassing regression also significantly improves the R^2 in comparison with the JWTSCV estimate. In the case of CHF-EUR pair, even presence of the EUR individual jumps improves the forecast of realized covariation.

To conclude, we have shown that the decomposition of the realized covariation into a continuous part and co-jumps using our wavelet-based methods can help improve the forecasting significantly. This result has strong economic implications for portfolio valuation as our theory helps to understand the dependencies deeper than standard econometric models.

Table 4: R from encompassing regression of ARFIMA on RC_t estimator JWTSCV, co-jumps (Jcom) and individual jumps (J1 and J2). p-values of estimated parameters in parentheses.

	const.	JWTSCV	Jcom	J1	J2	R2
GBP-CHF	**0.003**(0.000)		**1.342**(0.000)			0.118
	0.003(0.000)			0.677(0.174)		0.015
	0.005(0.000)				-0.286(0.409)	0.006
	0.001(0.002)	**0.722**(0.000)				0.402
	0.001(0.004)	**0.665**(0.000)	**0.608**(0.034)			0.424
	0.001(0.234)	**0.667**(0.000)	0.585(0.103)	0.052(0.913)		0.424
	0.001(0.262)	**0.664**(0.000)	0.586(0.104)	0.065(0.892)	-0.071(0.794)	0.424
	const.	JWTSCV	Jcom	J1	J2	R2
GBP-EUR	**0.004**(0.000)		**1.031**(0.000)			0.104
	0.004(0.000)			0.676(0.138)		0.018
	0.000(0.975)				**2.909**(0.000)	0.160
	0.001(0.000)	**0.694**(0.000)				0.402
	0.001(0.004)	**0.648**(0.000)	**0.627**(0.006)			0.439
	0.002(0.018)	**0.654**(0.000)	**0.697**(0.004)	-0.296(0.434)		0.442
	0.002(0.162)	**0.656**(0.000)	**0.703**(0.012)	-0.300(0.443)	-0.031(0.963)	0.442
	const.	JWTSCV	Jcom	J1	J2	R2
CHF-EUR	**0.003**(0.000)		**1.695**(0.000)			0.267
	0.006(0.000)			**-0.610**(0.054)		0.030
	0.005(0.000)				-0.217(0.696)	0.001
	0.001(0.016)	0.794(0.000)				0.468
	0.001(0.015)	0.670(0.000)	**0.615**(0.019)			0.491
	0.001(0.106)	0.666(0.000)	**0.614**(0.020)	-0.044(0.852)		0.491
	-0.000(0.951)	0.725(0.000)	**0.504**(0.056)	-0.191(0.432)	**0.932**(0.033)	0.511

Source: Author's computations.

6.4 Forecasting of Correlations

While co-jumps cause large bias in the covariance measures, individual jumps may cause bias to the correlation. Thus, we would like to complete our forecasting exercise by creating a forecasting model of the realized correlations, utilizing an ARFIMA (1, *d*, 1) model.

In the previous sections, we have shown that realized correlation estimated using a wavelet-based estimator is much smoother with lower confidence intervals than the correlation estimated using the standard realized variance and covariance measures. Thus we would like to see if our estimate carries better information for forecasting correlations. For this purpose, we again employ encompassing regression. This time, we will test the informational efficiency of each of the two measures. Moreover, we would like to see if decomposition of realized correlation generates any significant improvement. Thus we will forecast the decomposed correlations individually, and then compare the sum of the forecasts with the latter two estimates in the following way:

$$Corr_{t+1} = \alpha + \beta_1 RCorr_t^{ARFIMA} + \beta_2 WRCorr_t^{ARFIMA} + \beta_3 \sum_{i=1}^{5} WRCorr_{t,i}^{ARFIMA} + \varepsilon_t, \qquad (8)$$

where $RCorr_t^{ARFIMA}$ denotes the one-day ahead forecast of correlation using the standard realized correlation, $WRCorr_t^{ARFIMA}$ denotes the forecast using wavelet-based correlation and $\sum_{i=1}^{5} WRCorr_{t,i}^{ARFIMA}$ denotes the sum of the individual forecasts of decomposed correlation using our wavelet estimator.

We also run individual regressions testing the forecasting power of the individual estimators:

$$Corr_{t+1}^{(m)} = \alpha + \beta_1 V_t^{(k)ARFIMA} + \varepsilon_t, \qquad (9)$$

where $Corr_{t+1}^{(m)}$ is the realized correlation estimated using the *m*th estimator, and $V_t^{(k)ARFIMA}$ denotes the one-day ahead forecast of $Corr_{t+1}^{(m)}$ using the th estimator, while we consider the same three estimators, $RCorr_t^{ARFIMA}$, $WRCorr_t^{ARFIMA}$ and $\sum_{i=1}^{5} WRCorr_{t,i}^{ARFIMA} + \varepsilon_t$.

Table 5 in Appendix A.5 summarizes the results of the individual regressions as well as the encompassing regressions for both the in-sample and the out-of-sample periods, which are the same as in the covariance forecasting exercise.

The results from the in-sample fits tell us that our WRCorr is a very efficient estimator for forecasting of realized correlations, as its coefficient is significantly different from zero but is not significantly different from 1, while the forecast is unbiased as the constant coefficient is not significantly different from zero, except in some cases. Moreover, the WRCorr forecasts also carry the highest R^2. The sum of the individual correlation forecasts do not seem to be as efficient as the WRCorr estimator and it also gives slightly biased results. The realized correlation also seems to be quite an efficient and unbiased estimator, even though its coefficient is rather higher than 1 in some cases. It still has the lowest R^2. When looking at the results from the encompassing regressions, we can see that the WRCorr estimator remains the only significant estimator in the regression. Its coefficient is slightly lower than 1, but the coefficients of the other two estimators are not significantly different from zero. This means that these estimators do not generate any other significant information for the correlation forecasts.

When looking at the results for the out-of-sample period, which are much more important as these are the real forecasts, we still have a very similar picture. WRCorr is unaffected in the encompassing regressions, being the only significant estimator. In the individual regressions, the sum of the decomposed forecasts surprisingly seems to be the most efficient estimator, as its coefficient is closest to one, but it has a lower coefficient of determination, R^2, than the WRCorr estimator. To summarize the results from this section, we show that the wavelet-based estimator of the realized correlation is able to bring a significant improvement to the forecasting of correlation.

Conclusions

In this work, we present a new, wavelet-based realized covariation theory. We use wavelets to disentangle jumps from co-jumps, which is crucial in the study of multivariate dependencies. Having defined the estimators of variance and covariance, we also define the transformation of interest for portfolio theory: the wavelet-based realized correlation measure. The main contribution is in providing a new type of multivariate estimators in the time-frequency domain which are able to estimate the dependence of studied assets with highest precision and are unaffected by noise and jumps in the process. Moreover, our theory is able to disentangle jumps and co-jumps from the continuous part of the covariance.

We apply our multivariate theory to study the decomposition of integrated covariation and correlation on the currency markets. Here we note that the theory is able to decompose the realized measures into any arbitrary investment horizon, i.e., from one minute up to one month, when estimating monthly measures. In our analysis performed on forex data, we limit ourselves to illustrating the theory on the decomposition of daily realized measures. Specifically, we decompose the realized covariance and correlation into investment horizons of 5–10 minutes, 10–20 minutes, 20–40 minutes and 40–80 minutes, and the rest (80 minutes up to 1 day). The analysis uncovers interesting dynamics. Most of the action in the stock markets comes from higher frequencies. We find that on average, about 50% of the co-volatility of the forex markets examined is created on the 5–10 minute investment horizon, approximately 25% comes from the 10–20 minutes investment horizon, and only 12%, 7% and 6% correspond to the horizons of 20–40 minutes, 40–80 minutes and the rest (80 minutes up to 1 day), respectively. Note that by adding the contributions of the different investment horizons we always get 100%.

We also bring an important analysis of co-jumping of the currencies. We separate jumps, co-jumps and true covariation between the studied currencies. The results suggest that proper understanding of jumps and co-jumps in a multivariate setting is crucial for studying the dependencies. While individual jumps bring some bias to the covariance, co-jumps introduce large bias into the covariation measure. The impact on correlation is even more crucial. Individual jumps in the processes bring large downward bias to the correlation measure, while co-jumps introduce upward bias with a smaller magnitude.

Finally, we build a forecasting model for covariation and correlation based on wavelet decomposition. Our model outperforms simple realized correlation measure in-sample as well as out-of-sample. As the space of this paper is limited, we do not provide comparison to other methods, but the results can be found in *citation blinded*, where all estimators currently available in the literature are compared and our wavelet-based theory brings the best results. Interesting result is also that we found significant impact on the individual jumps as well as co-jumps on the covariance and correlation forecasts and we find that proper accounting for jumps and co-jumps bring significant improvement in the forecasting of covariance and correlation measures.

In conclusion, this work presents a new theoretical framework generalizing the popular concept of realized covariance. Our results have significant economic value, as a wrong assumption about the dependence process will have a direct impact on the portfolio valuation. The dynamics of the decomposed dependencies reveal interesting results. Our wavelet-based realized theory generates a more precise correlation measure with narrower confidence intervals than the standard realized correlations.

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Appendix A. Technical Part

Appendix A.1 Disentangling jumps from co-jumps

Fan and Wang (2007) first proposed the use of wavelets to estimate jumps in high-frequency data. In this part, we generalize this concept to a multivariate concept. We detect all jumps in the *m* assets separately using wavelet decomposition, and then we estimate the co-jumps. Let us define the procedure.

Definition 1 *Multivariate jump estimation using wavelets*
Let $\widetilde{\mathcal{W}}_{(q)1,k}$ be the 1st level wavelet coefficients of $(y_{(q)t})_{t\in[0,T]}$. If for some $\widetilde{\mathcal{W}}_{(q)1,k}$

$$|\widetilde{\mathcal{W}}_{(q)1,k}| > \frac{median\{|\widetilde{\mathcal{W}}_{(q)1,k}|, k=1,\dots,n\}}{0.6745}\sqrt{2\log n}, \tag{A.1}$$

for $q = 1, \dots, m$ assets, then $\hat{\tau}_{(q)l} = \{k\}$ is the estimated jump location with size $\bar{y}_{(q)\hat{\tau}_{l+}} - \bar{y}_{(q)\hat{\tau}_{l-}}$ (averages over $[\hat{\tau}_{(q)l}, \hat{\tau}_{(q)l} + \delta_n]$ and $[\hat{\tau}_{(q)l}, \hat{\tau}_{(q)l} - \delta_n]$, respectively, with $\delta_n > 0$ being the small neighborhood of the estimated jump location $\hat{\tau}_{(q)l} \pm \delta_n$; 0.6745 is a robust estimate of the standard deviation).
The jump variation of the -th asset is then estimated by the sum of the squares of all its estimated jump sizes:

$$\widehat{MWJC}_{(q)} = \sum_{l=1}^{N_t} (\bar{y}_{(q)\hat{\tau}_{l+}} - \bar{y}_{(q)\hat{\tau}_{l-}})^2. \tag{A.2}$$

Following the theory in Fan and Wang (2007), we can say that $\widehat{MWJC}_{(j)}$ will be a consistent estimator of the jumps for all assets in $\mathbf{p}_t$.

Proposition 1 *Consistency of multivariate wavelet jump estimator*
With

$$plim_{n\to\infty}\widehat{MWJC}_{(q)} = \sum_{l=1}^{N_t} J^2_{(q),l}, \tag{A.3}$$

with the convergence rate $N^{-1/4}$.

Once we have estimated all independent jumps in the studied $\mathbf{p}_t$ vector, we can propose an analysis of co-jumping in the series. The idea is to compare all the jump locations, and those which are the same across all $q = 1, \dots, m$ assets in some small neighborhood will be co-jumps.

Definition 2 *Wavelet co-jump estimation*
Let $\hat{\tau}_{(q)l}$ be the estimated jump locations of $(y_{(q)t})_{t\in[0,T]}$ for all $q = 1, \dots, m$ using Definition 1. Then co-jump location $\hat{\tau}^*_l = \{k\}$ can be estimated as:

$$\hat{\tau}_{(q)l} - \delta_n < \hat{\tau}^*_l < \hat{\tau}_{(q)l} + \delta_n, \quad forall q = 1, \dots, m. \tag{A.4}$$

Co-jumps are particularly important in portfolio theory. For a well diversified large portfolio in the sense of the Arbitrage Pricing Theory, idiosyncratic jumps are diversified away, but common jumps, or co-jumps, remain a problem. Thus in the following subsection, we illustrate our technique on a portfolio multivariate extension.

Appendix A.2 Proof of WRC

The realized covariance for the *l*-th and *q*-th asset return from an *m*-dimensional vector $\mathbf{r}_{t,h}$ over $[t-h,t]$, for $0 \le h \le t \le T$, can be computed using Definition 1

$$\widehat{RC}_{(l,q)t,h} = \sum_{i=1}^{n} r_{(l)t-h+\left(\frac{i}{n}\right)h} r_{(q)t-h+\left(\frac{i}{n}\right)h}, \tag{A.5}$$

while estimator 1 is an unbiased and consistent estimator of realized covariance.
For a particular level *j* we define the realized wavelet covariance over $[t-h,t]$, for $0 \le h \le t \le T$, as the sample covariance between the MODWT wavelet coefficients at level *j*, hence we have:

$$WRCov_{(l,q)t,h,j} = \sum_{t=L_j-1}^{N-1} \widetilde{W}_{(l)j,t-h+\frac{k}{n}h} \widetilde{W}_{(q)j,t-h+\frac{k}{n}h}, \tag{A.6}$$

where we use the $M_j = N - L_j + 1 > 0$ wavelet coefficients at the *j*-th level for both processes which are unaffected by the boundary conditions.
In case $J \to \infty$, the realized covariance $\widehat{RC}_{(l,q)t,h}$ is simply the sum of all wavelet realized covariances:

$$RC_{(l,q)t,h} = \sum_{j=1}^{\infty} WRCov_{(l,q)j,t-h+\frac{k}{n}h}\,. \tag{A.7}$$

Since we have datasets of a finite length, the contribution of the realized covariation of the scaling coefficients is still relatively high (we cannot ignore it), so we use a similar approach as with the wavelet covariance, i.e., the realized wavelet covariance has two parts, the first one being the realized covariance of the MODWT scaling coefficients $\widetilde{V}_{(l)t,h,J}$ and $\widetilde{V}_{(q)t,h,J}$ at the maximum level of decomposition , and the second one being the sum of the realized wavelet covariances up to the maximum level . Thus, for the maximum level of decomposition $J \le log2(N)$ we have:

$$\begin{aligned} RCOV_{(l,q)t,h} &= \sum_{t=L_j-1}^{N-1} \widetilde{V}_{(l)J,t-h+\frac{k}{n}h} \widetilde{V}_{(q)J,t-h+\frac{k}{n}h} + \sum_{j=1}^{J} \sum_{t=L_j-1}^{N-1} \widetilde{W}_{(l)j,t-h+\frac{k}{n}h} \widetilde{W}_{(q)j,t-h+\frac{k}{n}h} \\ &= \sum_{t=L_j-1}^{N-1} \widetilde{V}_{(l)J,t-h+\frac{k}{n}h} \widetilde{V}_{(q)J,t-h+\frac{k}{n}h} + \sum_{j=1}^{J} WRCov_{(lq)j,t-h+\frac{k}{n}h}, \end{aligned} \tag{A.8}$$

where for a specific level we use only the $M_j = N - L_j + 1 > 0$ MODWT wavelet or scaling coefficients unaffected by the boundary conditions.
Denoting by $\widetilde{\mathcal{W}}_{(q)j,k}$ the MODWT coefficients on scales $j = 1,\dots, J_s + 1$, which include both parts of the wavelet covariance, we have

$$\widehat{RC}_{(l,q)t,h}^{(WRC)} = \sum_{j=1}^{J_S+1} \sum_{k=1}^{n} \widetilde{\mathcal{W}}_{(l)j,t-h+\frac{k}{n}h} \widetilde{\mathcal{W}}_{(q)j,t-h+\frac{k}{n}h}, \tag{A.9}$$

where *n* is the number of intraday observations and J_s is the number of scales considered.
Finally, from the presented theory we know that the estimator $\widehat{RC}_{(l,q)t,h}^{(WRC)}$ will converge to the integrated covariation as

$$\widehat{RC}_{(l,q)t,h} = \sum_{i=1}^{n} r_{(l)t-h+\left(\frac{i}{n}\right)h} r_{(q)t-h+\left(\frac{i}{n}\right)h} = \sum_{j=1}^{J_s+1} \sum_{k=1}^{n} \widetilde{\mathcal{W}}_{(l)j,t-h+\frac{k}{n}h} \widetilde{\mathcal{W}}_{(q)j,t-h+\frac{k}{n}h} = \widehat{RC}_{(l,q)t,h}^{(WRC)}. \tag{A.10}$$

Thus, it is an unbiased estimator of integrated covariation:

$$E\left[RC_{(l,q)t,h}|\mathcal{F}_t\right] = E\left[\widehat{RC}_{(l,q)t,h}|\mathcal{F}_t\right] = E\left[\widehat{RC}_{(l,q)t,h}^{(WRC)}|\mathcal{F}_t\right] \tag{A.11}$$

As the wavelet-based covariance estimator is in fact the sample covariance without $1/M$ adjustment, $plim_{n\to\infty}\widehat{RC}_{(l,q)t,h} = plim_{n\to\infty}\widehat{RC}_{(l,q)t,h}^{(WRC)} = \int_{t-h}^{t} \Sigma_{(l,q)s}ds$ and $\widehat{RC}_{(l,q)t,h}^{(WRC)^j}$ provides a consistent estimator with increasing sampling frequency $n \to \infty$.

Appendix A.3 Proof of JWTSCV

The construction of this proof is very similar to the univariate JWTSRV 1.3. As all the theory has been introduced, we just use it to produce a new estimator which combines these ideas. We summarize the logic here.
Generalizing the idea of Fan and Wang (2007), which can consistently estimate the jump variation part, we simply introduce the idea of an $\widehat{MWJC}$ estimator (Definition 1). Jumps estimated using this estimator enable us to work with jump-adjusted data, $\mathbf{y}_{\mathbf{t,h}}^{(J)} = \mathbf{y}t,h$, - $\widehat{\mathbf{MWJC}}$, in a multivariate setting.
We have shown that $\widehat{RC}_{(l,q)t,h}^{(WRC)}$ is able to estimate integrated covariance consistently.
Finally, we plug the wavelet decomposition of the jump-adjusted vector into the TSCV estimator (Zhang, 2011), which is able to estimate realized covariance in the presence of noise.

Appendix A.4 Details of the ARFIMA models for forecasting

If we assume that covariation series belong to the class of ARFIMA processes introduced into econometrics by Granger (1980) then the dth difference of each series is a stationary and invertible ARMA process where parameter d may be any real number such that -1/2 < d < 1/2to ensure stationarity and invertibility. More precisely, $\sigma_{(1)t}\sigma_{(2)t}$ is an ARFIMA (p, d, q)process if it follows:

$$\alpha(L)(1-L)^d(\sigma_{(1)t}\sigma_{(2)t} - \mu) = \beta(L)v_t, \tag{A.12}$$

where $\alpha(z) = 1 - \alpha_1 z - \cdots - \alpha_p z^p$ and $\beta(z) = 1 + \beta_1 z + \cdots + \beta_q z^q$ are polynomials of order p and q, respectively, in the lag operator $L(L\sigma_{(1)t}\sigma_{(2)t} = \sigma_{(1)t-1}\sigma_{(2)t-1})$, which roots strictly outside the unit circle, v_t is *iid* with zero mean and σ_v^2 variance, and $(1 - L)^d$ is defined by its defined by its binomial expansion

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)} L^j, \tag{A.13}$$

using gamma function, Γ(.).

The parameter *d* determines the memory of the process. For $d > 0$, the process is said to have long memory, since its autocorrelations die out at a hyperbolic rate and are no longer absolutely summable, in contrast to the much faster exponential rate in the weak dependence case of $d = 0$, where the process captures the behavior of the short-memory ARMA model.

Once we have estimated the ARFIMA (*p, d, q*) model with the maximum likelihood estimator, forecasting is carried out by extrapolating the estimated model. As in the univariate counterpart, we estimate a simple ARFIMA (*1, d, 1*) model on both the realized covariation and its logarithmic transform.

Appendix A.5

Table 5: R^2 M-Z regression of ARFIMA on Correlations. RCorr denotes realized correlation estimate, WRCorr wavelet-based realized correlation and WRCorr sum of individual forecasts of decomposed correlation. p-values of estimated parameters in parentheses.

GBP-CHF

	in-sample					out-of-sample				
	const.	RCorr	WRCorr	WRCorr	R2	const.	RCorr	WRCorr	WRCorr	R2
RCorr	-0.02(0.47)	**1.04**(0.00)			0.33	0.04(0.46)	**0.83**(0.00)			0.22
WRCorr	-0.00(0.92)		**1.01**(0.00)		0.42	0.04(0.42)		**0.85**(0.00)		0.26
WRCorr	**-0.08**(0.00)			**1.16**(0.00)	0.41	-0.02(0.78)			**0.98**(0.00)	0.24
WRCorr	-0.02(0.55)	0.13(0.29)	**0.91**(0.00)		0.42	0.03(0.63)	0.29(0.40)	**0.61**(0.05)		0.27
WRCorr	0.01(0.76)	0.21(0.14)	**1.17**(0.00)	-0.38(0.21)	0.43	0.09(0.30)	0.44(0.25)	**1.14**(0.07)	-0.79(0.32)	0.27

GBP-EUR

	in-sample					out-of-sample				
	const.	RCorr	WRCorr	WRCorr	R2	const.	RCorr	WRCorr	WRCorr	R2
RCorr	-0.04(0.26)	**1.07**(0.00)			0.28	0.12(0.44)	**0.80**(0.00)			0.08
WRCorr	-0.01(0.75)		**1.02**(0.00)		0.36	**0.23**(0.06)		**0.62**(0.00)		0.07
WRCorr	**-0.20**(0.00)			**1.31**(0.00)	0.33	0.09(0.66)			**0.86**(0.01)	0.06
WRCorr	-0.02(0.64)	0.04(0.72)	**0.99**(0.00)		0.36	-0.05(0.80)	0.13(0.67)	**0.96**(0.03)		0.11
WRCorr	-0.00(0.93)	0.07(0.63)	**1.03**(0.00)	-0.08(0.74)	0.36	0.07(0.79)	0.43(0.48)	**0.97**(0.03)	-0.51(0.57)	0.11

CHF-EUR

	in-sample					out-of-sample				
	const.	RCorr	WRCorr	WRCorr	R2	const.	RCorr	WRCorr	WRCorr	R2
RCorr	-0.00(0.92)	**1.00**(0.00)			0.50	0.06(0.29)	**0.86**(0.00)			0.38
WRCorr	0.00(0.93)		**1.00**(0.00)		0.56	0.08(0.17)		**0.85**(0.00)		0.41
WRCorr	**-0.16**(0.00)			**1.20**(0.00)	0.51	0.02(0.78)			**0.90**(0.00)	0.34
WRCorr	-0.00(0.94)	0.08(0.33)	**0.93**(0.00)		0.57	-0.03(0.64)	0.08(0.77)	**0.96**(0.00)		0.46
WRCorr	0.03(0.36)	0.12(0.18)	**1.04**(0.00)	-0.19(0.19)	0.57	0.00(0.96)	0.28(0.48)	**0.93**(0.00)	-0.22(0.50)	0.46

Source: Author's computations.