Symmetries of Quasi-Values

Ales A. Kubena[1] and Peter Franek[2]

[1] Institute of Information Theory and Automation of the ASCR,
Pod Vodarenskou vezi 4, 182 08, Prague, Czech Republic
kubena@utia.cas.cz,
[2] Institute of Information Technologies, Czech Technical University, Thakurova 9,
Prague 160 00, Czech Republic
peter.franek@fit.cvut.cz

Abstract. According to Shapley's game-theoretical result, there exists a unique game-value of finite cooperative games that satisfies axioms on additivity, efficiency, null-player property and symmetry. The original setting requires symmetry with respect to arbitrary permutations of players. We analyze the consequences of weakening the symmetry axioms and study quasi-values that are symmetric with respect to permutations from a group $G \leq S_n$. We classify all the permutation groups G that are large enough to assure a unique G-symmetric quasi-value, as well as the structure and dimension of the space of all such quasi-values for a general permutation group G.
We show how to construct G-symmetric quasi-values algorithmically by averaging certain basic quasi-values (marginal operators).

1 Introduction

A cooperative game is an assignment of a real number to each subset of a given set of players Ω. This illustrates an economic situation where a coalition profit depends on the involved players in a generally non-aditive way. Several approaches deal with the question of redistributing the generated profit to the individual players in a stable or in a "fair" way. The mathematical theory of cooperative games was developed in forties by Neumann and Morgenstern [17]. Values of games provide a tool for evaluating the contibutions of the individual players such that certain natural axiomas are satisfied. A value is a function from cooperative games on a fixed played set Ω to $\mathbb{R}^{\Omega}$ satisfying certain natural properties. The most famous value is the Shapley value introduced in 1953 [22] that exists and is unique for all finite sets Ω.

There exist many axiomatic systems on game-values such that the Shapley value is their only solution: the original Shapley's axiomatics [22], Neyman's [18], Young's [24], van den Brink's [3] and Kar's axiomatics [15]. One of its important characteristics is the symmetry with respect to any permutation of players. This means, roughly speaking, that the value of a player is calculated only from his contributions to various coalitions and not from his identity. One may consider this to represent the *equality* of all payers. However, this is probably not a realistic assumption in many real-world situations where personal friendships and linkage play a major role. Some examples of values with restricted symmetry were studied, such as the *Owen value* [20] or the *weighted Shapley value* in [14], and the formal concept of *quasi-value*, where one completely relaxes any symmetry requirement, was introduced by Gilboa and Monderer in 1991 [10]. It is known that for a particular player set, there exists an infinite number of quasi-values.

In this work, we analyze one particular way of weakening the symmetry axiom. We suppose that a group G of permutations of Ω is given and define a *G-symmetric quasi-value* to be any quasi-value symmetric to all permutations in G. Informally, the equality of players is restricted to a group of permutations of players, not necessarily to all permutations. The group G expresses the measure of symmetry. If G is the full symmetry group, then the only G-symmetric quasi-value is the Shapley value; if G is the trivial group, then it carries no symmetry requirement and each quasi-value is G-symmetric. Our contribution is the classification of all permutation groups G of finite sets of players Ω for which there exists a unique G-symmetric quasi-value. It turns out that while in the infinite setting for non-atomic games, one may reduce the group of symmetries in a number of ways [16, 19], in the finite setting, only few subgroups of the full permutation group assure uniqueness. Even if the group G acts transitively on Ω (i.e. for any two players a, b, there exists a permutation $\pi \in G$ such that $\pi(a) = b$), there may still exist many G-symmetric quasi-values different from the Shapley value. We also calculate the dimension of the space of all G-symmetric quasi-values for a general permutation group G.

In the second section, we give the formal definitions of Shapley value, G-symmetric quasi-value and some necessary definitions from group theory including our definition of a super-transitive group action. In the third section, we show that the space of all G-symmetric quasi-values is an affine subspace of the vector space of all values and derive a formula for its dimension. We further classify all permutation groups G such that there exists a unique G-symmetric quasivalue. In the fourth section, we give some examples of G-symmetric quasi-values and show how more examples can be constructed by averaging the marginal operators. The last section (Appendix) contains the proof of an auxiliary statement from group theory that we use in the proof of Theorem 3. We postpone this technical issue to the end in order to keep the rest of the text fluent.

2 Definitions and notation

2.1 Cooperative games

Let Ω be a set of players. In this paper, we always suppose that Ω is finite.

Definition 1. *A* cooperative game *is a function* $v : 2^\Omega \to \mathbb{R}$ *such that* $v(\emptyset) = 0$. *A cooperative game is* additive*, if for all* $T, R \in 2^\Omega$, $R \cap T = \emptyset$ *implies* $v(R \cup T) = v(R) + v(T)$. *We denote by* Γ *the set of all cooperative games and* Γ_1 *the set of all additive cooperative games. A* game value *is an operator* $\psi : \Gamma \to \Gamma_1$. *For a game value* ψ *and* $i \in \Omega$, *we define* $\psi_i(v) := \psi(v)(\{i\})$.

For each game v, $\psi(v)$ is uniquelly determined by the numbers $\psi_i(v)$.

Shapley theorem [22] proves the existence and uniqueness of a game-value operator φ assuming it satisfies the following four axioms:

1. *Linearity*: $\varphi(\alpha v + \beta w) = \alpha\varphi(v) + \beta\varphi(w)$ for all $v, w \in \Gamma$ and $\alpha, \beta \in \mathbb{R}$.
2. *Null-player property*: if $i \in \Omega$ is a "null-player", i.e. $\forall R \subseteq \Omega\ v(R \cup \{i\}) = v(R)$, then $\varphi_i(v) = 0$.
3. *Efficiency*: $\sum_i \varphi_i(v) = v(\Omega)$ for all games v.
4. *Symmetry* (sometimes called *anonymity*): $\varphi(\pi \cdot v) = \pi \cdot \varphi(v)$ for every permutation π of Ω, where the game $\pi \cdot v$ is defined by $(\pi \cdot v)(R) := v(\pi^{-1}(R))$ for any $R \subseteq \Omega$.

The value defined by these axioms is called *Shapley value*. Axioms 1-4 are independent. Gilles [11] and Schmeidler [5] give examples of values satisfying any 3 of them and not the 4th.

Any game value satisfying axioms 1, 2 and 3 is called a *quasi-value*. In the original economic interpretation, the fourth axiom (Symmetry) is an expression of equality of all the participating players. It can be formulated in a more elegant way by the commutativity of the following diagram.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Gamma_1 \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi} \\ \Gamma & \xrightarrow{\varphi} & \Gamma_1 \end{array} \tag{1}$$

Axiom 4 requires that it commutes for each permutation of players π.

The following definition introduces the main object of our study.

Definition 2. *Let G by a group of permutations of Ω. A* G-symmetric quasi-value *is a game value that satisfies axioms 1, 2, 3 and such that $\varphi(\pi \cdot v) = \pi \cdot \varphi(v)$ for every permutation $\pi \in G$. In other words, diagram* (1) *commutes for all $\pi \in G$.*

Throughout this work, we will need the following standard basis of the space of cooperative games, introduced in Shapley's original paper [22].

Definition 3. *The* unanimity basis *is the basis $\{u_R\}_{\emptyset \neq R \subseteq \Omega}$ of the vector space of all cooperative games over the set Ω defined by $u_R(S) = 1$ if $R \subseteq S$ and* 0 *otherwise.*

2.2 Group theory

We say that a group G *acts* on the set X, if G is a subgroup of the group S_X of permutations of X. Any set $G \cdot x$ is called an *orbit*, or a G-orbit of x. The set of all G-orbits is denoted by X/G. The action of G on X is *transitive*, if for each $x, y \in X$, there exists a $g \in G$ such that $g \cdot x = y$. The *stabilizer* of a subset $A \subseteq X$ is the subgroup G_A of all elements $g \in G$ such that $g \cdot A \subseteq A$. For a subgroup H of G, $g \cdot H$ denotes a *left* and $H \cdot g$ a *right coset* of H and any group $H' = g^{-1}Hg$ is *conjugate* to H.

We introduce here a definition that will help us to describe a property of permutation groups we will need later.

Definition 4. *Let G be a group acting on a set X. We say that the action is a* supertransitive action*, if the stabilizer G_A of any subset $A \subseteq X$ acts transitively on A. A permutation group $G \subseteq S_n$ is* supertransitive*, if the stabilizer G_A acts transitively on each $A \subseteq \{1, \ldots, n\}$.*

For any n, S_{n-1} may be embedded into S_n as a set of permutations preserving one element. However, for $n = 6$, there exists an embedding of S_5 into S_6 different from the standard one. This embedding $S_5 \hookrightarrow S_6$ may be realized as the action of the projective linear group $PGL(2,5)$ on the projective line over $\mathbb{Z}_5$. The reader may find the details in the literature [7, p. 60-61], [4]. We will call this embedding an *exotic embedding*. It is well known that such a nonstandard embedding is only one up to conjugation by an element of S_6. In this paper, we only need the property that the image of the exotic embedding is a super-transitive subgroup of S_6. This is proved in the appendix.

3 Dimension of *G*-symmetric quasi-values

If a quasi-value is symmetric with respect to a set of permutation, it is also symmetric with respect to any permutation they generate in S_Ω, hence the set of all symmetries of a quasi-value is always a group. For a finite set Ω and a group $G \subseteq S_\Omega$ of permutations we denote by $\mathcal{A}_G$ the set of all G-symmetric quasi-values.

First we show how to represent $\mathcal{A}_G$ as a space of matrices. Each game value φ can be represented as a map from Γ to $\mathbb{R}^\Omega$ by the natural identification $\Gamma_1 \simeq \mathbb{R}^\Omega$. Choosing the unanimity basis on Γ (Def. 3) and the canonical basis $(e_i)_{i\in\Omega}$ on $\mathbb{R}^\Omega$, we may represent linear game values as matrices of the size $|\Omega| \times (2^{|\Omega|} - 1)$. The null player property applied to the unanimity basis implies $\varphi(u_R)(\{i\}) = 0$ for each $i \notin R$, because such player i doesn't contribute to any coalition in the game u_R. As a consequence, a matrix A with elements $(a_{iR})_{i\in\Omega,\, R\subseteq\Omega}$ corresponds to a linear game value satisfying the null-player-property iff $a_{iR} = 0$ for all pairs (i, R) such that $i \notin R$.

Further, the game value satisfies the efficiency axiom iff for any nonempty $R \subseteq \Omega$, $\varphi(u_R)(\Omega) = 1$, which translates to a constraint on matrix coefficients $\sum_{i\in R} a_{iR} = 1$ for each $\emptyset \neq R \subseteq \Omega$.

The G-symmetry of a game value requires $\varphi(g \cdot v) = g \cdot (\varphi(v))$ for any game v and permutation $g \in G$, the action of G on Γ defined by (**??**). An element u_R from the unanimity basis satisfies $(g \cdot u_R)(S) = u_R(g^{-1}(S)) = u_{gR}(S)$, so the unanimity basis is invariant with respect to the group action and $g \cdot u_R = u_{gR}$. The symmetry axiom is equivalent to

$$((g \cdot \varphi)(u_R))(\{i\}) = (\varphi(u_{gR}))(\{i\}),$$

for all $i \in \Omega$ and $R \subseteq \Omega$. The left-hand side is equal to $\varphi(u_R)(\{g^{-1}i\})$. In the matrix representation of φ, the symmetry axiom translates to the condition $a_{(g^{-1}i)\,R} = a_{i\,(gR)}$, or simply $a_{iR} = a_{(gi)\,(gR)}$ for all $i \in \Omega$ and $R \subseteq \Omega$.

Summarising this, we have the following.

Observation 1 *Choosing the unanimity basis of Γ and the canonical basis of $\mathbb{R}^{\Omega} \simeq \Gamma_1$, $\mathcal{A}_G$ may be identified with a set of matrices $A = (a_{iR})$ with elements satisfying the following equations:*

- $a_{iR} = 0$ *if* $i \notin R$,
- *The sum of elements in each column is* 1,
- *Matrix elements a_{iR} are constant on the orbits of the G-action $g \cdot (i, R) = (gi, gR)$.*

All these conditions are linear equations for matrix elements a_{iR} and they are all satisfied for the Shapley value. So, $\mathcal{A}_G$ is nonempty affine space.

Theorem 2. *Let $X = \{(i, R);\, i \in R \subseteq \Omega\}$, $\chi = \{R;\, \emptyset \neq R \subseteq \Omega\}$ and let $G \subseteq S_G$ be a group of permutations acting on sets X and χ, extending naturally its action on Ω. Then the dimension of $\mathcal{A}_G$ is $|X/G| - |\chi/G|$. Explicitly it can also be expressed as*

$$\dim \mathcal{A}_G = (\frac{dZ_G}{dx_1} - Z_G)|_{(2,2\ldots 2)} + 1 \tag{2}$$

where Z_G is the cycle index of the group G [8, p. 85]

$$Z_G(x_1 \ldots x_n) = \frac{1}{|G|} \sum_{\pi \in G} x_1^{j_1(\pi)} \cdots x_n^{j_n(\pi)}, \tag{3}$$

$j_k(\pi)$ denotes the number of cycles of length k in the permutation π.

Proof. We will call the G-orbits of X “orbits” and the G-orbits of χ “metaorbits” and identify elements of $\mathcal{A}_G$ with matrices as described in 1.

Let $p : X \to \chi$ be the map $(i, R) \to R$. For any $x := (i, R) \in X$ and $g \in G$, $p(gx) \in g(p(x))$. For any $R \subseteq \Omega$, the stabilizer G_R acts on R and R splits into k_R orbits $\{R_1, \ldots, R_{k_R}\}$ with respect to this action. If $R' = gR$ is on the same metaorbit, then the stabilizer of R' is gG_Rg^{-1} and g maps any G_R-orbit $R_i \subseteq R$ bijectively onto a $G_{R'}$-orbit $R'_i \subseteq R'$. So, $k_R = k_{R'}$ and $|R_i| = |R'_i|$ for $i = 1, \ldots, k_R$. For a meta-orbit m, we define $k_m := k_R$ for any $R \in m$ and $l_{mi} = |R_i|$ for $i = 1, \ldots, k_m$. These numbers are independent on the choice of R.

We will say that a metaorbit m *contains* an orbit Gx, if $p(x) \in m$. A metaorbit m contains k_m orbits. For any metaorbit $m \in \chi/G$ containing the orbits $\{o_1, \ldots, o_{k_m}\}$, we may choose real numbers c_{mi} such that $\sum_{i=1}^{k_m} c_{mi} l_{mi} = 1$ with $k_m - 1$ degrees of freedom. Choosing such numbers c_{mi} for all metaorbits m gives

$$\sum_{m \in \chi/G} (k_m - 1) = \sum_{m \in M} k_m - |\chi/G| = |X/G| - |\chi/G|$$

degrees of freedom. Any such choice of c_{mi} defines a matrix of game-value operator given by

$$a_{iR} = \begin{cases} c_{mi} \text{ if } i \in R_i \subseteq R \in m \\ 0 \text{ if } i \notin R \end{cases}$$

These are exactly matrices A constant on the orbits of X satisfying $\sum_i a_{iR} = 1$ for all R and $a_{iR} = 0$ for all $i \notin R$. The number of degrees of freedom for the choice of c_{mi} is equal to the dimension of $\mathcal{A}_G$. This proves the first part.

Burnside lemma [21, p. 58] enables to express the number of orbits of a group action in an explicit way. If a finite group H acts on a finite set Y, then

$$|Y/H| = \frac{1}{|H|} \sum_{h \in H} |\{y \in Y;\, h(y) = y\}|. \tag{4}$$

A permutation $\pi \in G$ fixes those sets $R \subseteq \Omega$ that don't split any cycle of π. There exists $2^{\#\, cycles(\pi)}$ such sets, $2^{\#\, cycles(\pi)} - 1$ of them nonempty. So,

$$|\chi/G| = \big(\frac{1}{|G|} \sum_{\pi} 2^{\#\, cycles(\pi)}\big) - 1.$$

Elements of X fixed by π are pairs (i, R) such that $i \in R$, $\pi(i) = i$ and $\pi(R) = R$. There exists $\#\, fixedpoints(\pi) {*} 2^{\#\, cycles(\pi)-1}$ such pairs. We derived the following equation:

$$\dim \mathcal{A}_G = \frac{1}{|G|}\big(\sum_{\pi \in G} (\#\text{fixedpoints}(\pi) * 2^{\# cycles(\pi)-1}) - \sum_{\pi \in G} 2^{\# cycles(\pi)}\big) + 1.$$

The statement of the theorem follows from this by a direct calculation. □

The cycle index Z_G is known in a more explicit form than (3) for many subgroups of S_n and it has also been generalized and computed for finite classical groups [9].

Further, we will show for which groups G the dimension of $\mathcal{A}_G$ is zero, i.e. for which G the only quasi-value they contain is the Shapley value. In Section 2.2, we defined a group $G \subseteq S_\Omega$ to be supertransitive, if the stabilizer G_R acts transitively on R for each subset $R \subseteq \Omega$. In other words, if for each R and each $i, j \in R$, there exists a $g \in G$ such that $g(R) = R$ and $g \cdot i = j$. We will show that this condition is equivalent to the existence of a unique G-symmetric quasi-value.

Theorem 3. *Let Ω be finite and $G \leq S_\Omega$. There exists a unique G-symmetric quasi-value if and only if G acts supertransitively on Ω. Equivalently, this is if and only one of the following conditions is satisfied:*

- *$G = S_\Omega$, the full symmetric group*
- *$|\Omega| > 3$ and $G = A_\Omega$, the alternating group*
- *$|\Omega| = 6$ and G is the image of an exotic embedding $S_5 \hookrightarrow S_6$ (see Section 2.2).*

Proof. We will work with the matrix representation of $\mathcal{A}_G$, described in Observation 1. Let (a_{iR}) be a matrix representing a value in $\mathcal{A}_G$.

If the action of G on Ω is supertransitive, then for each $\emptyset \neq R \subseteq \Omega$ and each $i, j \in R$, there exists an element $g \in G$ such that $g(i) = j$ and $g(R) = R$. For a particular $\emptyset \neq R \subseteq \Omega$, all elements $\{(i, R);\ i \in R\}$ lie on the same G-orbit of G, so all these elements are equal. The null-player property implies that $a_{iR} = 0$ for $i \notin R$ and together with the efficiency condition we obtain that for each $i \in R$, $a_{iR} = 1/|R|$. This implies uniqueness.

If the action of G on Ω is not supertransitive, then there exists a nonempty subset $\tilde{R} \subseteq \Omega$ such that the stabilizer $G_{\tilde{R}}$ has not a transitive action on $\tilde{R}$. So, $\tilde{R}$ contains at least two $G_{\tilde{R}}$-orbits. We may define the matrix a_{iR} as follows. In the matrix column corresponding to $\tilde{R}$ we choose $a_{i\tilde{R}} = 0$ if $i \notin \tilde{R}$ and the other elements $a_{j\tilde{R}}$ arbitrary, constant on $G_{\tilde{R}}$-orbits and such that $\sum_j a_{j\tilde{R}} = 1$. For all R' on the G-orbit of R, we define the coefficients $a_{iR'}$ in a unique way so that they are constant on the G-orbits and the remaining matrix elements may be equal to elements of the original Shapley matrix. In this way, we may construct an infinite number of different G-symmetric quasi-values which proves that $\dim \mathcal{A}_G \geq 1$.

For the classification part, it remains to prove that the groups listed in the theorem are exactly the groups acting supertransitively on $\{1, \ldots, n\}$. The proof of this is technical and we postpone it to the Appendix (Chapter 5). □

4 Consequences

4.1 Examples

First we will give some examples of groups and G-symmetric quasi-values. In all these examples, we assume that the player set Ω consists of n players.

Example 1. Let $G = \{\mathrm{id}\}$ be the trivial group. In this case, any quasi-value is G-symmetric. Consider a selector $\gamma : 2^\Omega \to \Omega$ with $\gamma(R) \in R$ for all $\emptyset \neq R \subseteq \Omega$. Now we define the value ψ as

$$\psi_i(v) = \sum_{i=\gamma(R)} \Delta_v(R) \tag{5}$$

where $\Delta_v(R) \in \mathbb{R}$ is a *Harsanyi dividend* of the coalition $R \subseteq \Omega$ defined by $\Delta_v(R) = \sum_{T \subseteq R} (-1)^{|R|-|T|} v(T)$. This value satisfies efficiency, null player property and linearity. However, the number of maps $\gamma : 2^\Omega \to \Omega$ satisfying $\gamma(R) \in R$ is much larger, so many of the quasi-values defined by (5) are affine dependent. The cycle index of the trivial group is $Z(x_1) = x_1^n$ and substituting into (2) yields $\dim \mathcal{A}_G = n2^{n-1} - 2^n + 1$. For $n \geq 4$, this is strictly smaller than $n! - 1$ which implies that marginal vectors are affine dependent. It was shown in [6] that such values satisfy the axioms for quasi-values. [3]

[3] In the matrix representation, such values correspond to matrices $a_{i\,R} = \delta_{i\gamma(R)}$.

Example 2. ("Caste system") The set Ω is split into k nonempty disjoint subsets ("castes") $\Omega_1, \ldots, \Omega_k$ and G is chosen so that it guarantees equality within each Ω_i. Formally, $G = \{\pi \in S_\Omega;\ \forall i\ \pi(\Omega_i) = \Omega_i\}$.

Some examples of G-symmetric quasivalues have been described in the literature. The *Owen value*, first described in [20], can be obtained as the expected value of *marginal vectors* (see Section 4.2), if we first randomely choose an order of the castes and then the order of the players within each caste. Another related concept is the *weighted Shapley value*, studied by Kalai and Samet in [14]. Here an order of the castes is given and within each caste, the profit is diveded among players proportional to their *weights*. In the case of equal weights of all players, the weighted Shapley value is symmetric with respect to all G-permutations.

The cycle index is $Z_G = \prod_{r=1}^k Z_{S_{\Omega_r}}$. We know from the proof of Theorem 2 that the number of metaorbits is $|\chi/G| = \frac{1}{|G|}\sum_g 2^{\#cycles(g)}$. In particular, for $G = S_n$, $|\chi/G| = n+1$, because metaorbits of S_n are $O_s = \{R \subseteq \Omega;\ |R| = s\}$ for $s = 0, 1, \ldots, n$. This enables as to calculate

$$Z_{S_n}|_{(2,\ldots,2)} = \frac{1}{n!}\sum_\pi 2^{j_1+\ldots+j_n} = \frac{1}{n!}\sum_\pi 2^{\#cycles(\pi)} = |\chi/S_n| = n+1.$$

If $G = S_n$, then the Shapley value is the only game value, so it follows from Theorem 2 that $(\frac{dZ_{S_n}}{dx_1} - Z_{S_n})|_{(2,\ldots,2)} + 1 = 0$ and $\frac{dZ_{S_n}}{dx_1}|_{(2,\ldots,2)} = n$. So, for $G = \prod_{r=1}^k S_{\Omega_r}$

$$\frac{dZ_G}{dx_1}|_{(2,2\ldots2)} = (\sum_{r=1}^k \frac{dZ_{S_{\Omega_r}}}{dx_1}\prod_{s\neq r} Z_{S_{\Omega_s}})|_{(2,2\ldots2)} = \sum_{r=1}^k |\Omega_r| \prod_{s\neq r}(1+|\Omega_s|)$$

and

$$\dim \mathcal{A}_G = (\sum_{r=1}^k \frac{|\Omega_r|}{1+|\Omega_r|} - 1)\prod_{r=1}^k (1+|\Omega_r|) + 1.$$

For the case of two castes $k = 2$ this simplifies to $|\Omega_1| \times |\Omega_2|$.

Example 3. (Cyclic group) This example illustrates that transitive group action does not imply a unique G-symmetric quasi-value. If G is the cyclic group $C_n \subseteq S_n$, the cycle index is $Z_{C_n} = \frac{1}{n}\sum_{f|n}\phi(f)x_f^{n/f}$, where $\phi(f)$ is the Euler totient function $\phi(f) = p_1^{k_1-1}(p_1-1)\ldots p_r^{k_r-1}(p_r-1)$, where $f = p_1^{k_1}\ldots p_r^{k_r}$ is the prime number decomposition.[8, p. 86]. Substituting into the formula in Theorem 2 gives

$$\dim \mathcal{A}_G = 2^{n-1} - \frac{1}{n}\sum_{f|n}\phi(f)2^{n/f} + 1.$$

In the case of $n = 3$, the dimension turns out to be $2^2 - \frac{1}{3}(2^3 + 2\times 2) + 1 = 1$, so there exists a one-dimensional space of quasi-values symmetric with respect to cyclic permutations of players.

4.2 Shapley-value as an expected value of non-uniformly distributed marginal vectors

Suppose that $\Omega = \{1, 2, \dots, n\}$, i.e. an order is given on the set of player. For a game $v \in \Gamma$ and a permutation $\pi \in S_n$, we may define a quasi-value m_π by $(m_\pi)(v)_{\pi(1)} = v(\pi(1))$ and

$$(m_\pi(v))_{\pi(i)} = v(\{\pi(1), \pi(2), \dots, \pi(i)\}) - v(\{\pi(1), \pi(2), \dots, \pi(i-1)\})$$

for $i = 2, \dots, n$. We call m_π the *marginal operator* and $m_\pi(v)$ the *marginal vector* [p. 19][2]. It corresponds to a situation where the players arrive in the order $\pi(1), \pi(2), \dots, \pi(n)$ and each player is assigned the value of his or her contribution to the coalition of all players that have arrived before. The evaluation of m_π on a game u_R from the unanimity basis is $m_\pi(u_R)(\{\pi(i)\}) = u_R(\pi(1), \dots, \pi(i)) - u_R(\pi(1), \dots, \pi(i-1))$ which is equal to 1 if and only if $\pi(i) \in R$ and $\pi(j) \notin R$ for $j > i$ and 0 otherwise. After the identification 1, we can represent m_π is as a matrix

$$(m_\pi)_{iR} = \begin{cases} 1 \text{ iff } i \in R \text{ and } \pi^{-1}(i) = \max \pi^{-1}(R) \\ 0 \ \text{ otherwise.} \end{cases}$$

A theorem of Weber [23] shows that if π is a random permutation taken from a uniform distribution on S_n then for any game v, the expected value of a marginal operator m_π is exactly the Shapley value. This can be generalized to the following statement.

Proposition 1. *Let G be a subgroup of S_n and A^π be a probability distributioin on S_n constant on the right cosets $\{G \cdot \pi\}_\pi$, i.e. $A^\pi = A^{g\pi}$ for all $g \in G$ and $\pi \in S_n$. Then $\sum A^\pi\, m_\pi \in \mathcal{A}_G$ is a G-symmetric quasi-value.*

Proof. We will show that the identity holds if evaluated on games from the unanimity basis of Γ. For the game u_R (Definition 3), we start with the following equation:

$$(g \cdot m_\pi)(u_R) = m_{g\,\pi}(u_{gR}). \tag{6}$$

To prove this, we evaluate both sides on $\{i\}$ and rewrite the left-hand side to the equivalent equation

$$(m_\pi(u_R))(\{g^{-1}(i)\}) = (m_{g\,\pi}(u_{gR}))(\{i\}).$$

Both sides are equal to 1 if and only if $\pi^{-1}(g^{-1}(i)) = \max \pi^{-1}(R)$ and 0 otherwise, which proves (6) for all $R \subseteq \Omega$, $i \in \Omega$ and $g \in G$. The G-symmetry of $\sum_{\pi \in S_n} A^\pi m_\pi$ follows from

$$\begin{aligned}\Big(g \cdot \sum_{\pi \in S_n} A^\pi m_\pi\Big)(u_R) &= \sum_{\pi \in S_n} A^\pi (g \cdot m_\pi)(u_R) = \sum_{\pi \in S_n} A^\pi m_{g\pi}(u_{gR}) = \\ &= \sum_{\pi \in S_n} A^{g\pi} m_{g\,\pi}(u_{gR}) = \sum_{g\,\pi = \pi' \in S_n} A^{\pi'} m_{\pi'}(g \cdot u_R) = \Big(\big(\sum_{\pi' \in S_n} A^{\pi'} m_{\pi'}\big) \cdot g\Big)(u_R)\end{aligned}$$

where we used (6) in the second and $A^\pi = A^{g\pi}$ in the third equality. □

An immediate consequence of the classification Theorem 3 is that for $|\Omega| > 3$ any quasi-value symmetric with respect to the alternating group A_n is already the Shapley value. It follows from the last proposition that $\sum_\pi A^\pi m_\pi$ is the Shapley value not only for $A^\pi = \frac{1}{n!}$ but also for $A^\pi = \frac{s}{n!}$ for π even and $A^\pi = \frac{2-s}{n!}$ for π odd, $s \in [0, 2]$. In fact, there are many other possibilities how to express the Shapley value as a convex combination of marginal operators. The space of all quasi-values on Ω is $(n2^{n-1} - 2^n + 1)$-dimensional and the set of all probability distributions on S_n is a $(n! - 1)$-dimensional convex region in $\mathbb{R}^{n!}$, so there are at least $n! - n2^n + 2^{n-1} - 2$ degrees of freedom for the choice of a distribution A^π such that $\sum_\pi A^\pi m_\pi =$ Shapley.

Exponentially many (with respect to n) of these probability distributions A^π can be constructed as follows. Choose $\Omega_0 \subseteq \Omega$, $|\Omega_0| > 3$ and define S_0 to be a group of all permutations π acting identically on $\Omega \setminus \Omega_0$. Choose $\alpha \in (0, 2)$ and define a probability distribution on S_n by

$$A^\pi(\Omega_0) = \begin{cases} \frac{1}{n!} \text{ if } \pi \notin S_0 \\ \frac{\alpha}{n!} \text{ if } \pi \in S_0 \text{ and } \pi \text{ is even} \\ \frac{2-\alpha}{n!} \text{ if } \pi \in S_0 \text{ and } \pi \text{ is odd} \end{cases}$$

One can verify that the corresponding expected value of marginal operators m_π is the Shapley value. For a set $\{\Omega_1, \Omega_2, \ldots, \Omega_k\}$ s.t. $\Omega_i \nsubseteq \Omega_j$ for all i and j, the vectors $(A^\pi(\Omega_i) - \frac{1}{n!})_i \in \mathbb{R}^{n!}$ are linearly independent and the distributions $(A^\pi(\Omega_i))_i$ are affine independent.

5 Appendix

Here we finish the proof of Theorem 3 by the classification of supertransitive groups. Our proof is based on a classification of set-transitive permutation groups given by Beamont and Petrson in 1955 [1]. Another proof of the supertransitive groups classification was given by Michal Jordan on mathoverflow [13].

Theorem 4. *G is a supertransitive subgroup of S_n if and only if one of the following conditions holds:*

- *G is the full symmetric group S_n for some n,*
- *G is the alternating group A_n for $n > 3$,*
- *G is conjugate to the image of an exotic embedding of S_5 to S_6.*

Proof. Let $G \subseteq S_n$ be a group of permutations acting supertransitively on $\{1, \ldots, n\}$. This means that the stabilizer of each $A \subseteq \{1, \ldots, n\}$ acts transitively on A. Let $B \subseteq \{1, \ldots, n\}$ and $i, j \notin B$. Then G acts transitively on $B \cup \{i, j\}$ and there exists a permutation $\pi \in G$ taking $B \cup \{i\}$ to $B \cup \{j\}$ such that $\pi(j) = i$. This implies that for each A and B s.t. $|A| = |B| > 1$, there exists a permutation $\pi \in G$ s.t. $\pi(A) = B$. If $|A| = |B| = 1$, the same is true because super-transitivity implies transitivity. We have shown that if the action of G is super-transitive, it is also set-transitive.

If G has a supertransitive action on $\{1, \ldots, n\}$, then its order has to be divisible by each $k \leq n$, because each k-element set A is isomorphic to G/G_A, hence $|G| = |A| \times |G_A|$. So, G has to be divisible by the lowest common multiple of $\{1, \ldots, n\}$.

Beamont and Petrson classified all set-transitive permutation groups in [1]. It follows that such subgroups of S_n are exactly the full symmetric group S_n for any n, the alternating group A_n for $n > 2$ and 5 exceptions. The first and second exceptions are subgroups of S_5 of order 10, resp. 20. These groups cannot have a supertransitive action on $\{1, \ldots, 5\}$, because the lowest common multiple of $\{1, \ldots, 5\}$ is 60. Two other exceptions in Beamont's classification are subgroups of S_9 of orders 504 and 1512. These numbers are not divisible by the lowest common multiple of $\{1, \ldots, 9\}$ so we can exclude them as well. The last exception is a subgroup of S_6 of order 120. This groups is equivalent to the exotic embedding of S_5 to S_6 and we will show that it acts supertransitively on S_6.

In [12], the authors realize this group action on $\{1, \ldots, 6\}$ as the conjugate action of S_5 on its six Sylow 5-subgroups. Using this realisation, we may show that this action is super-transitive by direct calculation. Let as denote the Sylow 5-subgroups by $I = \langle(12345)\rangle$, $II = \langle(12354)\rangle$, $III = \langle(12435)\rangle$, $IV = \langle(12453)\rangle$, $V = \langle(12534))\rangle$ and $\langle(12543)\rangle$. The conjugate action of S_5 acts transitively on $\{I, \ldots, VI\}$. An elementary calculation shows that the image of a product of two disjoint transpositions in S_5 is the product of two disjoint transpositions in S_6, e.g. $(1,2)(3,4) \in S_5 \mapsto (I, V)\,(III, VI)$ in the above realisation. So, for any permutation $(a_I, \ldots, a_{VI})$ of $(I, \ldots, VI)$, there exists an element in S_5 interchanging A_I and A_{II}, A_{III} and A_{IV} and fixing A_V and A_{VI}. This immediately implies both 2-, 3-, 4- and 5-supertransitivity, so this subgroup of S_6 is super-transitive.

It remains to prove that A_n is supertransitive if and only if $n > 3$. First note that $A_2 = \{id\}$, reps. $A_3 = \langle(123)\rangle$ are not super-transitive, because no element of these groups takes 1 to 2 and preserves $\{1, 2\}$. Let $n > 3$ and $A \subseteq \{1, \ldots, n\}$ be a k-set. If $k < n - 1$, then any permutation of A can be extended to an even permutation of $\{1, \ldots, n\}$. If $k = n - 1 > 2$, then for each $i, j \in A$, there exists an even permutation of A taking i to j and hence the complement of i to the complement of j. This completes the proof. $\square$

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