Axiomatic foundations of the universal integral in terms of aggregation functions and preference relations

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Abstract. The concept of universal integral has been recently proposed in order to generalize the Choquet, Shilkret and Sugeno integrals. We present two axiomatic foundations of the universal integral. The first axiomatization is expressed in terms of aggregation functions, while the second is expressed in terms of preference relations.

1 Basic concepts

For the sake of simplicity in this note we present the result in a Multiple Criteria Decision Making (MCDM) setting (for a state of art on MCDM see [1]). Let $N = \{1,\ldots,n\}$ be the set of criteria and let us identify the set of possible alternatives with $[0,1]^n$. For all $\boldsymbol{x} = (x_1 \ldots, x_n) \in [0,1]^n$, the set $\{i \in N \mid x_i \geq t\}, t \in [0,1]$, is briefly indicated with $\{\boldsymbol{x} \geq t\}$. For all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ we say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ and we write $\boldsymbol{x} \succeq \boldsymbol{y}$ if $x_i \geq y_i,\ i = 1,\ldots,n$. An aggregation function $f : [0,1]^n \to \mathbb{R}$ is a function such that, $\inf_{\boldsymbol{x}\in[0,1]^n} f = 0$, $\sup_{\boldsymbol{x}\in[0,1]^n} f = 1$, and $f(\boldsymbol{x}) \geq f(\boldsymbol{y})$ whenever $\boldsymbol{x} \succeq \boldsymbol{y}$ [2]. Let $\mathcal{A}_{[0,1]^n}$ be the set of aggregation functions on $[0,1]^n$.
Let M denotes the set of all capacities m on N, i.e. for all $m \in M$ we have $m : 2^N \to [0,1]$ satisfying the following conditions:

- *boundary conditions*: $m(\emptyset) = 0, m(N) = 1$;
- *monotonicity*: $m(A) \leq m(B)$ for all $\emptyset \subseteq A \subseteq B \subseteq N$.

A *universal integral* [3] is a function $I : M \times [0,1]^n \to [0,1]$ satisfying the following properties:

(UI1) I is non-decreasing in each coordinate,
(UI2) there exists a pseudo-multiplication $\otimes$ (i.e. $\otimes : [0,1]^2 \to [0,1]$ is nondecreasing in its two coordinates and $\otimes(c,1) = \otimes(1,c) = c$) such that for all $m \in M$, $c \in [0,1]$ and $A \subseteq N$,

$$I(m, c\mathbf{1}_A) = \otimes(c, m(A)),$$

(UI3) for all $m_1, m_2 \in M$ and $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, if $m_1(\{\boldsymbol{x} \geq t\}) = m_2(\{\boldsymbol{y} \geq t\})$ for all $t \in]0,1]$, then $I(m_1, \boldsymbol{x}) = I(m_2, \boldsymbol{y})$.

Given a universal integral I with respect to the pseudomultiplication $\otimes$, we shall write

$$I(m, \boldsymbol{x}) = \int_{univ, \otimes} \boldsymbol{x} \, dm$$

for all $m \in M, \boldsymbol{x} \in [0,1]^n$.

2 Axiomatic foundation in terms of aggregation functions

Consider a family $\mathcal{F} \subseteq \mathcal{A}$ with $\mathcal{F} \neq \emptyset$ and consider the following axioms on $\mathcal{F}$:

(A_1) For all $f_1, f_2 \in \mathcal{F}$ and $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ such that for all $t \in [0,1]$

$$f_1\left(\mathbf{1}_{\{\boldsymbol{x} \geq t\}}\right) \geq f_2\left(\mathbf{1}_{\{\boldsymbol{y} \geq t\}}\right),$$

then $f_1(\boldsymbol{x}) \geq f_2(\boldsymbol{y})$;

(A_2) Every $f \in \mathcal{F}$ is idempotent, i.e. for all $c \in [0,1]$ and $f \in \mathcal{F}$,

$$f(c \cdot \mathbf{1}_N) = c;$$

(A_3) For all $m \in M$ there exists $f \in \mathcal{F}$ such that $f(1_A) = m(A)$ for all $A \subseteq N$.

Proposition 1. *Axioms (A_1), (A_2) and (A_3) hold if and only if there exists a universal integral I with a pseudo-multiplication $\otimes_{\mathcal{F}}$ such that, for all $f \in \mathcal{F}$ there exists an $m_f \in M$ for which*

$$f(\boldsymbol{x}) = \int_{univ, \otimes_{\mathcal{F}}} \boldsymbol{x} \, dm_f \qquad \text{for all} \quad \boldsymbol{x} \in [0,1]^n, \ f \in \mathcal{F}.$$

More precisely, for all $f \in \mathcal{F}$ and for all $A \subseteq N$, $m_f(A) = f(\boldsymbol{1}_A)$ and for all $a, b \in [0,1], \otimes_{\mathcal{F}}(a,b) = f(a\boldsymbol{1}_B)$ if $f(\boldsymbol{1}_B) = b$, with $B \subseteq N$.

Remark 1. One can weaken axiom (A_3) as follow.

(A_4) For all $c \in [0,1]$ there exist $A \subseteq N$ and $f \in \mathcal{F}$ such that $f(\mathbf{1}_A) = c$.

In this case above Proposition 1 holds provided that the universal integral is no more defined as a function $I : M \times [0,1]^n \to [0,1]$, but as a function $I : M_{\mathcal{F}} \times [0,1]^n \to [0,1]$ with $M_{\mathcal{F}} \subseteq M$. More precisely, we have $M_{\mathcal{F}} = \{m_f | f \in \mathcal{F}\}$.

3 Axiomatic foundation in terms of preference relations

We consider the following primitives:

- a set of outcomes X,
- a set of binary preference relations $\mathcal{R} = \{\succsim_t, t \in \mathbf{T}\}$ on $X^n, n \in \mathbf{N}$.

In the following

- we shall denote by α the constant vector $[\alpha, \alpha, \ldots, \alpha] \in X^n$, with $\alpha \in X$;
- we shall denote by $\succ_t$ and $\sim_t$ the asymmetric and the symmetric part of $\succsim_t \in \mathcal{R}$, respectively;
- we shall denote by (α_A, β_{N-A}), $\alpha, \beta \in X, A \subset N$, $\boldsymbol{x} \in X^n$ such that $x_i = \alpha$ if $i \in A$ and $x_i = \beta$ if $i \notin A$.

We consider the following axioms:

A1) $\succsim_t$ is a complete preorder on X^n for all $\succsim_t \in \mathcal{R}$.
A2) For all $\alpha, \beta \in X$ and for all $\succsim_t, \succsim_r \in \mathcal{R}$, $\alpha \succsim_t \beta \Rightarrow \alpha \succsim_r \beta$.
A3) X is infinite and there exists a countable subset $A \subseteq X$ such that for all $\succsim_t \in \mathcal{R}$, for all $\alpha, \beta \in X$ for which $\alpha \succ_t \beta$ there is $\gamma \in A$ such that $\alpha \succsim_t \gamma \succsim_t \beta$.
A4) There are $1, 0 \in X$ such that for all $\succsim_t \in \mathcal{R}$ $\mathbf{1} \succ_t \mathbf{0}$ and for all $\boldsymbol{x} \in X^n$,

$$\mathbf{1} \succsim_t \boldsymbol{x} \succsim_t \mathbf{0}.$$

A5) For each $\boldsymbol{x} \in X^n$ and for each $\succsim_t \in \mathcal{R}$, there exists $\alpha \in X$ such that $\boldsymbol{x} \sim_t \alpha$.
A6) For all $\boldsymbol{x}, \boldsymbol{y} \in X^n$, $\succsim_t, \succsim_r, \succsim_s \in \mathcal{R}$,

$$[(1_{\{i \in N: x_i \succsim_t \alpha\}}, 0_{N-\{i \in N: x_i \succsim_t \alpha\}}) \succsim_r \beta \Rightarrow (1_{\{i \in N: y_i \succsim_t \alpha\}}, 0_{N-\{i \in N: y_i \succsim_t \alpha\}}) \succsim_s \beta, \forall \alpha, \beta \in X]$$
$$\Rightarrow$$
$$[\boldsymbol{x} \succsim_r \gamma \Rightarrow \boldsymbol{y} \succsim_s \gamma, \forall \gamma \in X].$$

A7) For all $\mathcal{A} = \{\alpha_1, \ldots, \alpha_{2^n-2}\} \subset X$ there exists $\succsim_t \in \mathcal{R}$ such that for all $\alpha \in \mathcal{A}$ there is A, $\emptyset \subset A \subset N$, for which $\alpha \sim_t 1_A$.

Theorem. Conditions $A1) - A7)$ hold if and only there exist

- a function $u : X \to [0, 1]$,
- a bijection between $\mathcal{R}$ and M for which each $\succsim_t \in \mathcal{R}$ corresponds to one capacity $\mu_t \in M$,
- a pseudo-multiplication $\otimes$,

such that, for all $\boldsymbol{x}, \boldsymbol{y} \in X^n$ and for all $\succsim_t \in \mathcal{R}$

$$\boldsymbol{x} \succsim_t \boldsymbol{y} \Leftrightarrow \int_{univ,\otimes} \mathbf{u(x)} d\mu_t \geq \int_{univ,\otimes} \mathbf{u(y)} d\mu_t,$$

where $\mathbf{u(x)} = [u(x_1), \ldots, u(x_n)]$ and $\mathbf{u(y)} = [u(y_1), \ldots, u(y_n)]$.

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