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RESEARCH REPORT

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A causal model of price and volume on market with a market maker

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1 Introduction

One of the key roles in price formation at today's financial markets is played by market makers (MMs) - agents who are obliged to set buying and selling qoutes (bid and ask) and trade for the prices they set. Clearly, as other economic agents, MMs are profit maximizers. The economic analysis of their behariour is, however, quite complicated since the decision problems they face are dynamic by nature hence intractable (see[2] and the citation therein).

In the present paper, we suggest a rather simple version of such a decision problem. In particular, we assume the MM to maximizehis discounted consumption while keeping the probability of the bancrupcy (i.e. running out of the money or the traded asset) at a prescribed, perhaps very small level. We do not give analytic solution of the problom but we prove that the prices set by the MM depend - out of all the past information - only on the amount of the asset held by MM and on his uncertainty concerning the fair price.

After a nomral approximation of (Compound Poisson) bought and sold anounts of an asset we are able to determine a distribution of the process of midpoint prices and the inventory of the MM (i.e. total number of assets held) which we, after a local linearization of optimal strategies, validate by means of ten seconds high frequency data and estimate its parameters. As we used OLS, our estimates are both consistent and asyptotically normal. We also show a benchmark models assuming irrationality of the MM's and/or the liquidity takers, may be rejected in favour of our model.

2 The setting

Let there be two types of agents: the market makers posting quotes (the best bid and ask), and the informed traders.

In our model, there is a single (representative) market maker, i.e., agent who, at each $t \in \mathbb{N}$,[1] sets the log-quotes a_t and b_t (the actual best ask and best bid are then $A_t = e^{a_t}$, $B_t = e^{b_t}$,respectively) in order to maximize their discounted overall consumption.

In reaction to the quotes, the traders post market orders, i.e. requests to buy or to sell a certain amount of the asset for the ask price, bid price, respectively. The numbers of buy and sell market orders arrived from time $t-1$ to t, are Poisson with intensities depending solely on a distance of the corresponding log-quote to a log-fair price $\pi_t \in \mathbb{R}$, in particular, the intensities are

$$\lambda(a_{t-1} - \pi_{t-1}), \qquad \lambda(\pi_{t-1} - b_{t-1}),$$

respectively where

$$\lambda(z) = \begin{cases} r(1 - z/D) & z \le D \\ 0 & z > D \end{cases}, \qquad r > 0, \quad D > 0$$

The sizes of the orders are random, with common distribution $\mathcal{P}$, independent each on other and on the numbers of the order arrived. We denote μ the mean and s the raw second moment of P (i.e. the variance of $\mathcal{P}$ is $s - \mu^2$).

The (log)fair price π follows a possibly non-normal non-homogenous random walk with

$$\mathbb{E}\Delta\pi_t = 0,$$

We distinguish three possible degrees of information, available to the MM:

(I) The MM is fully informed, i.e. the values of π are observable to him

[1] Other trading frequencies may be modelled by scaling of the time.

(P) The MM is partially informed, i.e. he observes a proxy

$$e_t = \pi_t + \zeta_t$$

for some ζ_t, $\mathbb{E}(\zeta_t) = 0$

(U) The MM is uninformed.

Denoting X_{t+1} and Y_{t+1} the total volume of the buy market orders, sell market orders, respectively, which have arrived since t to $t+1$, and denoting

$$\Xi_t = (\pi_0, X_1, Y_1, e_1, \pi_1, \ldots, X_t, Y_t, e_t, \pi_t) \tag{1}$$

all the (historical) information relevent for the market (given (I), we can put $e_t = \pi_t$, given (U), we can assume an infinite variance of e_t by definition), our setting may be formally described as follows:

(D1) $X_t|\Xi_t \sim \mathrm{CP}\left(\lambda(a_{t-1} - \pi_{t-1}), \mathcal{D}\right)$,

(D2) $Y_t|\Xi_{t-1}, \sim \mathrm{CP}\left(\lambda(\pi_{t-1} - b_{t-1}), \mathcal{D}\right)$,

(I1) $\Delta\pi_t$, ζ_t and (Ξ_{t-1}, X_t, Y_t) are mutually independent.

(I2) X_t is conditionally independent of Y_t given $a_{t-1} - \pi_{t-1}, \pi_{t-1} - b_{t-1}$

Here, $\mathrm{CP}(\kappa, \mathcal{Q})$ denotes the compound Poisson distribution with intensity κ and summands' distribution $\mathcal{Q}$.

Note that the functions $\bar{d}(p) = \mu\kappa(p - \pi_t)$ and $\bar{s}(p) = \mu\lambda(\pi_t - p)$, are equal to the expected volumes of buy market orders, sell market orders, respectively, arriving between t and $t{+}1$; therefore, $\bar{d}$ and $\bar{s}$ may be interpreted as (expected) demand curve, supply curve, respectively. Moreover, since

$$\pi_i = \arg\max_p [\bar{d}(p) \wedge \bar{s}(p)]$$

we may regard π_t as the equilibrium price. The expected overall traded volume between t and $t+1$ is then given by

$$R = \bar{d}(\pi_t) = \bar{s}(\pi_t) = \mu r;$$

thus, constant R may be called *preliquidity.* Finally, as

$$\bar{d}(p) \wedge \bar{s}(p) \geq 0 \Leftrightarrow p \in (\pi_t - D, \pi_t - D)$$

the parameter D may be named *prespread.*

3 The MM's decision problem

Let us turn out to the study of the MM holding M_0 units of cash and N_0 units of the traded asset at the time 0. Denote C_t the MM's consumption at t. Naturally, the increments of the cash holding, asset holding, respectively, are

$$\Delta M_t = e^{a_{t-1}} X_t - e^{b_{t-1}} Y_t - C_{t-1}, \qquad \Delta N_t = Y_t - X_t, \qquad t > 0. \tag{2}$$

We assume that the consuption C_t may be also negative, i.e. it is allowed to MM to "put its own money into the bussiness" if needed. Moreover, we allow the MM to borrow stocks for a single period.

As we have already premised, we assume the MM to maximize his discounted consumption at a time t so that both the probability of running out of the money (i.e. a negative M_{t+1}) and the probability of depleeting the asset (i.e. a negative N_{t+1}) is less than a prescribed level.

As the MM possibly does not know the values of the fair price, his information set at the time t consists of

$$\xi_t = (e_1, X_1, Y_1, e_2, X_2, Y_2, \dots e_t, X_t, Y_t)$$

where $e_t = \pi_t$ in case of (I) and e_t has an infinite variance by definition in case of (U).

Definition 1. The decision problem, solved by the MM at each $t \in \mathbb{N} \cup \{0\}$, is given by

$$V_t(\xi_t) = \sup_{a_\tau, b_\tau, C_\tau, t \le \tau \le T} \mathbb{E}\left[\sum_{\tau=t}^{T-1} e^{-\rho(\tau-t)} C_\tau + e^{-\rho(T-t)}(M_T + e^{\pi_T} N_T)|\xi_t\right] \tag{3}$$

such that, for all $t \le \tau < T$,

$$(a_\tau, b_\tau, C_\tau)\text{is } \sigma(\xi_\tau) \text{ measurable,}$$

$$\begin{aligned} a_\tau &\ge b_\tau & \mathcal{A}(\tau) \\ \mathbb{P}\left[M_{\tau+1} < 0|\xi_\tau\right] &\le \gamma, & \mathcal{M}(\tau) \\ \mathbb{P}\left[N_{\tau+1} < 0|\xi_\tau\right] &\le \gamma, & \mathcal{N}(\tau) \end{aligned}$$

$$\tau \ge t.$$

Here, T is a time horizon fulfilling $t \le T \le \infty$, ρ is a discount factor and γ is a prechosen probability level.

4 Approximation

Denote

$$h_\tau = \mathbb{E}(\pi_\tau|\xi_\tau), \qquad P_\tau = \frac{a_\tau + b_\tau}{2}$$

the expected (log)fair price, (log)midpoint price, respectievely, and define

$$\delta_\tau = P_\tau - h_\tau, \qquad \sigma_\tau = \frac{a_\tau - b_\tau}{2}.$$

ther relative price, half-spread respectively.

Our next aim is to describe the dyna_mics of h_τ and the distribution of $h'_\tau s$ observation error

$$\eta_\tau = h_\tau - \pi_\tau$$

Because, in cases (U) and (P), the dynamics of h comes out as non-linear and the distribution of η_τ as non-normal, an approximation is needed. To this end, denote

$$v_{\eta,\tau} = \operatorname{var}(\eta_\tau|\xi_\tau),$$

and observe that, once

$$r\text{is high enough}$$

and

$$a_t - h_t + \sqrt{v_{\eta,t}} = \sigma_t + \delta_t + \sqrt{v_{\eta,t}} \ll D, \qquad h_t - b_t + \sqrt{v_\eta} = \sigma_t - \delta_t + \sqrt{v_{\eta,t}} \ll D, \tag{4}$$

we may approximate the Compound Poisson conditional distributions of X_t and Y_t by normal one with first and second moments matching. From the Chebyshev inequality and from (4), we further get that the probability of $a_t - h_t + \eta_t > D$, $h_t - b_t - \eta_t > D$ is very small (similarly b_t) so we may take λ as linear. Therefore, we keep assuming **(I1)** and **(I2)** but we approximate

(A1) $X_t|\Xi_{t-1}\dot\sim\mathcal{N}\left(R\left(1-\frac{a_{t-1}-\pi_{t-1}}{D}\right),\frac{s}{\mu}R\left(1-\frac{a_{t-1}-\pi_{t-1}}{D}\right)\right)$

(A2) $Y_t|\Xi_{t-1},\dot\sim\mathcal{N}\left(R\left(1-\frac{\pi_{t-1}-b_{t-1}}{D}\right),\frac{s}{\mu}R\left(1-\frac{\pi_{t-1}-b_{t-1}}{D}\right)\right)$

Further, for each t, we assume

$$\Delta\pi_t\sim\mathcal{N}(0,v_\pi)$$
$$\zeta_t\sim\mathcal{N}(0,v_\zeta)$$

Unfortunatelly, even given this approximation, we would not get analytical formulas for the conditional distribution of η_t given ξ_t which we will need to describe the dynamics of the price-volume process (the reason being the conditional variance of both X_t and Y_t dependent on η_t). One way to overcome this is to approximate its conditional density; however, since the formulas, resulting from that approach, would still be quite complex hence difficult to work with further, we rather assume that, instead of both the values X_t and Y_t, the MM takes into account only the value the increase of the inventory

$$\Delta N_t=Y_t-X_t,$$

whose conditional variance does not depend on η_t. This approximation could be justified by the fact that the loss of information given such a simplification is not large.[2]Hence, we assume

(A3) $\xi_t=(e_1,\Delta N_1,\dots,e_t,\Delta N_t)$

until the end of the paper.

Proposition 2. *Given (I1),(I2),(A1)-(A3) and if*

$$\eta_{t-1}|\xi_{t-1}\sim\mathcal{N}(0,v_{\eta,t-1})$$

then

$$\Delta h_t=-c_N(k^{-1}\Delta N_t-\delta_{t-1})+c_e(e_t-h_{t-1})$$

where

$$k=2\frac{R}{D}$$
$$c_N=c_N(v_{\eta,t-1},\sigma_{t-1})=\frac{v_{\eta,t-1}v_\zeta}{u_t},$$
$$c_e=c_e(v_{\eta,t-1},\sigma_{t-1})=\frac{v_{\eta,t-1}v_{\theta,t}+v_\pi v_{\theta,t}+v_{\eta,t-1}v_\pi}{u_t}$$
$$u_t=u(v_{\eta,t-1},\sigma_{t-1})=v_{\eta,t-1}v_\pi+v_{\eta,t-1}v_{\theta,t}+v_\pi v_{\theta,t}+v_{\eta,t-1}v_\zeta+v_{\theta,t}v_\zeta$$
$$v_{\theta,t}=v_\theta(\sigma_{t-1})=\frac{s}{k\mu}\left(D-2\sigma_{t-1}\right)$$

and

$$\eta_t|\xi_t\sim\mathcal{N}(0,v_{\eta,t}),$$

where

$$\begin{aligned}v_{\eta,t}&=v_\eta(v_{\eta,t-1},\sigma_{t-1})\\&=v_{\eta,t-1}+v_\pi-\frac{1}{u_t}(v_{\eta,t-1}^2v_\zeta+v_{\eta,t-1}^2v_{\theta,t}+v_{\eta,t-1}v_\pi v_{\theta,t}\\&\qquad+v_{\eta,t-1}^2v_\pi+v_{\eta,t-1}v_{\theta,t}v_\pi+v_\pi^2v_{\theta,t}+v_{\eta,t-1}v_\pi^2).\end{aligned}$$

[2]As, due to (4), the variances of Y_t and X_t are similar and their relation to e_t is analogous, their contribution to the information about π_t is similar too so the same absolute value of coeffitients in an eventual linear estimate of π_t are defensitble; moreover, as they depend on π_t reverse way, the signs of the coeffitient should clearly be opposite.

Proof. Denote

$$\vartheta_t = \frac{\Delta N_t}{k} - \delta_{t-1} - \eta_{t-1}.$$

Given (P), we have

$$\begin{aligned}
&\vartheta_t|\Xi_{t-1},\zeta_t,\Delta\pi_t = \vartheta_t|\Xi_{t-1}\\
&\dot{\sim}\frac{\mathcal{N}\left(R\left[\left(1-\frac{\pi_{t-1}-b_{t-1}}{D}\right)-\left(1-\frac{a_{t-1}-\pi_{t-1}}{D}\right)\right],\frac{s}{\mu}R\left[\left(1-\frac{a_{t-1}-\pi_{t-1}}{D}\right)+\left(1-\frac{\pi_{t-1}-b_{t-1}}{D}\right)\right]\right)}{k}\\
&\qquad - \delta_{t-1} - \eta_{t-1}\\
&= \frac{\mathcal{N}\left(R\left(-\frac{2\pi_{t-1}-(b_{t-1}+a_{t-1})}{D}\right),\frac{s}{\mu}R\left(2-\frac{a_{t-1}-b_{t-1}}{D}\right)\right)}{k} - \delta_{t-1} - \eta_{t-1}\\
&= \frac{\mathcal{N}\left(k\left(\eta_{t-1}+\delta_{t-1}\right),k\frac{s}{\mu}\left(D-2\sigma_{t-1}\right)\right)}{k} - \delta_{t-1} - \eta_{t-1} = \mathcal{N}\left(0,v_{\theta,t}\right) \sim \vartheta_t|\sigma_{t-1},\\
&\qquad v_{\theta,t} = \frac{s}{k\mu}\left(D-2\sigma_{t-1}\right)
\end{aligned}$$

(the last "$\sim$" follows from the fact that the the parameters of the conditional distribution of $\vartheta_t|\Xi_{t-1}$ are measurable with respect to $\sigma(\sigma_{t-1})$) from which and the assumptions it is clear that

$$\vartheta_t,\zeta_t,\Delta\pi_t,\eta_{t-1}|\Xi_{t-1} \sim \mathbb{N}(0,\mathrm{diag}(v_{\vartheta,t},v_{\zeta,t},v_{\pi,t},v_{\eta,t-1}))$$

Now, denote $z_t = k^{-1}\Delta N_t$. Since

$$\pi_t = h_{t-1} - \eta_{t-1} + \Delta\pi_t$$
$$z_t = \delta_{t-1} + \eta_{t-1} + \vartheta_t,$$

$$e_t = \pi_t + \zeta_t = h_{t-1} - \eta_{t-1} + \Delta\pi_t + \zeta_t$$

we are geting that (π_t, z_t, e_t) given $\xi_{t-1}$is normal with mean $(h_{t-1},\delta_{t-1},h_{t-1})$ and variance

$$\begin{pmatrix} v_{\eta,t-1}+v_\pi & -v_{\eta,t-1} & v_{\eta,t-1}+v_\pi\\ -v_{\eta,t-1} & v_{\theta,t}+v_{\eta,t-1} & -v_{\eta,t-1}\\ v_{\eta,t-1}+v_\pi & -v_{\eta,t-1} & v_\pi+v_\zeta+v_{\eta,t-1}\end{pmatrix}$$

so, by textbook formula,

$$\pi_t|z_t,e_t,\xi_{t-1}$$

is normal with mean

$$\begin{aligned}
&h_{t-1} + (-v_{\eta,t-1},v_{\eta,t-1}+v_\pi)\begin{pmatrix} v_{\theta,t}+v_{\eta,t-1} & -v_{\eta,t-1}\\ -v_{\eta,t-1} & v_\pi+v_\zeta+v_{\eta,t-1}\end{pmatrix}^{-1}\begin{bmatrix} z_t-\delta_{t-1}\\ \hat{e}_t\end{bmatrix}\\
&= h_{t-1} + \frac{1}{u_t}(-v_{\eta,t-1},v_{\eta,t-1}+v_\pi)\begin{pmatrix} v_\pi+v_\zeta+v_{\eta,t-1} & v_{\eta,t-1}\\ v_{\eta,t-1} & v_{\theta,t}+v_{\eta,t-1}\end{pmatrix}\begin{bmatrix} z_t-\delta_{t-1}\\ e_t-h_{t-1}\end{bmatrix}\\
&= h_{t-1} + \frac{1}{u_t}[-v_{\eta,t-1}(v_\pi+v_\zeta+v_{\eta,t-1})\\
&+ v_{\eta,t-1}(v_{\eta,t-1}+v_\pi), -v_{\eta,t-1}^2 + (v_{\eta,t-1}+v_\pi)(v_{\theta,t}+v_{\eta,t-1})]\begin{bmatrix} z_t-\delta_{t-1}\\ e_t-h_{t-1}\end{bmatrix}\\
&= h_{t-1} + \frac{1}{u_t}(-v_{\eta,t-1}v_\zeta, v_{\eta,t-1}v_{\theta,t}+v_\pi v_{\theta,t}+v_{\eta,t-1}v_\pi)\begin{bmatrix} z_t-\delta_{t-1}\\ e_t-h_{t-1}\end{bmatrix}
\end{aligned}$$

and variance

$$v_{\eta,t} = v_{\eta,t-1} + v_\pi - \frac{1}{u_t}(v_{\eta,t-1}^2 v_\zeta + v_{\eta,t-1}^2 v_{\theta,t} + v_{\eta,t-1} v_\pi v_{\theta,t} + v_{\eta,t-1}^2 v_\pi + v_{\eta,t-1} v_{\theta,t} v_\pi + v_\pi^2 v_{\theta,t} + v_{\eta,t-1} v_\pi^2)$$

$\square$

Untill the end of the paper assume the case (P), i.e., that the mm is partially informed.

5 The optimal decision

Before proceeding, note that, as h_τ is (by definition) ξ_τ-measurable, any strategy of (3) may be alternatively expressed by $(\delta_\tau, \sigma_\tau, C_\tau)$.

The following Proposition shows how the optimal dicesion depends on the past of the MM's information.

Proposition 3. *Denote $(\delta_\tau, \sigma_\tau, C_\tau)_{\tau \geq t}$, be optimal solution of (3). Then,*

(i) if

- *T is finite*

or

- *$T = \infty$ and additional constraints*

$$\delta_\tau \in [-D_0, D_0], \qquad \sigma_\tau \leq S_0, \qquad\qquad C(\tau)$$

are added to the problem,

then

- *$V_t(\xi_t) = V(M_t, N_t, v_{\eta,t}, h_t, T-t)$ for some function V*
- *$\delta_\tau = \delta(N_\tau, v_{\eta,\tau}, T-\tau)$,$\sigma_t = \sigma(N_\tau, v_{\eta,\tau}, T-\tau)$ for some functions δ, σ*
- *$C_\tau = C(M_t, N_\tau, v_{\eta,\tau}, h_\tau)$ for some function C.*

(ii) For $T = \infty$ and given constraint $\mathcal{C}$,

$$V(M, N, v, h) = \frac{1}{1 - e^{-\rho}} M_\tau + e^{h_\tau} W(N, v)$$

where W is given by a "Bellman-like" equation

$$\begin{aligned} W(n, v) \quad &= \quad \sup_{\delta,\sigma} \{\phi(\delta, \sigma) + \frac{1}{e^\rho - 1} R[e^{\delta+\sigma}(1 - \frac{\delta + \sigma}{D}) - e^{\delta-\sigma}(\frac{\delta - \sigma}{D} - 1)] \\ &\qquad + \mathbb{E}[e^{w(\delta,\sigma,\Delta N)} W(n + \Delta N, v_\eta(v, \sigma))]\} \\ &\quad s.t. \\ &\qquad \delta \in [-D_0, D_0], \\ &\qquad \sigma \in [0, S_0] \\ &\qquad \mathbb{P}(\Delta N + N < 0) \leq \gamma \end{aligned}$$

Here,

- $\Delta N \sim \mathcal{N}\left(k\delta, k^2(v + v_\theta(v,\sigma))\right)$
- $k, R, \rho, D, \gamma, D_0, S_0$ *are previously defined constants*
- $c_N, c_e, v_\eta, v_\theta$ *are previously defined functions*
- $\phi(\sigma,\delta) = \sup\{x; \Psi(x;\delta,\sigma) \leq \gamma\}$,
  $\Psi(z;\delta,\sigma) = \mathbb{E}_{\eta\sim\mathcal{N}(0,v)}\varphi\left(\frac{e^{-\delta}z - e^{\sigma}R(D-(\sigma+\delta+\eta)) - e^{-\sigma}(D-(\sigma-\delta-\eta))}{\sqrt{\frac{Rs}{\mu}[e^{2\sigma}(D-(\sigma+\delta+\eta)) + e^{-2\sigma}(D-(\sigma-\delta-\eta))]}}\right)$
- $w(\delta,\sigma,\Delta N) = \frac{c_e(v,\sigma)}{v+v_\theta(v,\sigma)}(k^{-1}\Delta N - \delta) + \frac{1}{2}c_e(v,\sigma)^2\left(v + v_\pi + v_\zeta - \frac{k^2v^2}{v+v_\theta(v,\sigma)}\right) - c_N(v,\sigma)\Delta N$

(iii) V_t *is non-decreasing both in* M_t *and* N_t
(iv) The constraint $\mathcal{M}(\tau)$ *is fulfilled with "=" for all* $\tau \geq t$

Proof. (i): Assume first that $T < \infty$. Then the problem (3) may be reformulated by means of Bellman equations

$$V(\xi_\tau) = \sup_{\delta_\tau,\sigma_\tau,C_\tau \text{ fulfilling } \mathcal{A}(\tau),\mathcal{M}(\tau),\mathcal{N}(\tau)} \left[C_\tau + e^{-\rho}\mathbb{E}(V_{\tau+1}(\xi_{\tau+1})|\xi_\tau)\right]$$

for $t \leq \tau < T$ and

$$V_T(\xi_T) = \mathbb{E}(M_T + e^{\pi_T}N_T|\xi_T) = M_T + \mathbb{E}(e^{\pi_T}|\xi_T)N_T.$$

We prove, by induction, that, for all $t \leq \theta \leq T$,

$$V_\theta(\xi_\theta) = \tilde{V}(M_\theta, N_\theta, h_\theta, v_{\eta,\theta}, T-\theta) = d_{T-\theta}M_\theta + e^{h_\theta}W(N_\theta, v_{\eta,\theta}, T-\theta) \tag{5}$$

for some functions $\tilde{V}$ and W such that W is non-decreasing in N and $d_\theta \in \mathbb{R}$ is deterministic.

If $\theta = T$ then (5) clearly holds with $d_0 = 1$ and

$$W(n,v,0) = n\mathbb{E}_{\eta\sim\mathcal{N}(0,v)}e^{-\eta} = ne^{v^2/2}.$$

(to see it, note that $\mathbb{E}(e^{\pi_T}|\xi_T)$ =$\mathbb{E}(e^{-\eta_T}e^{h_T}|\xi_T) = e^{h_T}\mathbb{E}(e^{-\eta_T}|\xi_T)$)

Now let $\tau < T$ and assume (5) to hold with $\theta = \tau + 1$. Put

$$x_\tau = e^{-h_\tau}(C_\tau - M_\tau).$$

Since

$$\Delta M_{\tau+1} = e^{h_\tau}(e^{\delta_\tau+\sigma_\tau}X_{\tau+1} - e^{\delta_\tau-\sigma_\tau}Y_{\tau+1})$$

we are getting

$$\begin{aligned}
V_\tau(\xi_t) = &\sup_{\delta_\tau,\sigma_\tau,x_\tau \text{ fulfilling } \mathcal{A}(\tau),\mathcal{M}(\tau),\mathcal{N}(\tau)} \\
&\left[M_\tau + e^{h_\tau}x_\tau + e^{-\rho}\mathbb{E}\left(d_{T-(\tau+1)}M_{\tau+1} + e^{h_{\tau+1}}W(N_{\tau+1}, v_{\eta,\tau+1}, T-(\tau+1))\right)\right] \\
&= (1 + e^{-\rho}d_{T-(\tau+1)})M_\tau \\
&+ e^{h_\tau}\sup_{\delta_t,\sigma_t,x_t,\ldots}[x_\tau + e^{-\rho}d_{T-(\tau+1)}e^{-h_\tau}\mathbb{E}(\Delta M_{\tau+1}|\xi_\tau) \\
&+ \mathbb{E}(e^{\Delta h_{\tau+1}}W(N_{\tau+1}, v_{\eta,\tau+1}, T-(\tau+1))|\xi_\tau)] \\
&= (1 + e^{-\rho}d_{T-(\tau+1)})M_\tau + e^{h_\tau}F(N_\tau, v_{\eta,\tau+1}, T-\tau)
\end{aligned}$$

where

$$F(N,v,\theta) = \sup_{\delta,\sigma}\{\phi(\delta,\sigma) + e^{-\rho}d_{\theta-1}R[e^{\delta+\sigma}(1-\frac{\delta+\sigma}{D}) - e^{\delta-\sigma}(\frac{\delta-\sigma}{D}-1)) \\ + \mathbb{E}(e^{-c_N(v,\sigma)\Delta N + c_e(v,\sigma)e}W(N+\Delta N, v_\eta(v,\sigma), \theta-1))] \quad (6)$$

s.t.

$$\sigma \geq 0 \qquad (\tilde{\mathcal{A}})$$

$$\mathbb{P}(\Delta N + N < 0) \leq \gamma \qquad (\tilde{\mathcal{N}})$$

where
$$\begin{bmatrix} \Delta N \\ e \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} k\delta \\ 0 \end{bmatrix}, \begin{bmatrix} k^2(v+v_\theta(v,\sigma)) & -kv \\ -kv & v+v_\pi+v_\zeta \end{bmatrix}\right)$$

Finally, since, by a textbook formula,

$$e|\Delta N \sim \mathcal{N}(-\frac{v}{v+v_\theta(v,\sigma)}(k^{-1}\Delta N_t - \delta_{t-1}), v+v_\pi+v_\zeta - \frac{k^2v^2}{v+v_\theta(v,\sigma)})$$

we have

$$\mathbb{E}(e^{-c_N(v,\sigma)\Delta N + c_e(v,\sigma)e}W) = \mathbb{E}(\mathbb{E}(e^{-c_N(v,\sigma)\Delta N + c_e(v,\sigma)e}W|\Delta N)) \\ = \mathbb{E}(\mathbb{E}(e^{c_e(v,\sigma)e}|\Delta N)e^{-c_N(v,\sigma)\Delta N}W) = \mathbb{E}(e^{w(\delta,\sigma,\Delta N)}W)$$

so

$$F(N,v,\theta) = \sup_{\delta,\sigma}\{\phi(\delta,\sigma) + e^{-\rho}d_{\theta-1}R[e^{\delta+\sigma}(1-\frac{\delta+\sigma}{D}) - e^{\delta-\sigma}(\frac{\delta-\sigma}{D}-1)) \\ + \mathbb{E}(e^{w(\sigma,\delta,\Delta N)}W(N+\Delta N, v_\eta(v,\sigma), \theta-1))]$$

Therefore, $V_\tau(\xi_\tau)$ and the optimal solutions depend, in fact, only on $N_\tau, M_\tau,\ h_\tau, v_{\eta,\tau}, T-\tau$ and, specially, δ_τ, σ_τ depends only on $N_\tau, v_{\eta,t}, T-\tau$.

As to the monotony: since W is non-decreasing in N and neigther v nor (X,Y,e) depend on N, we are getting that, given the same strategy, the value of (6) is greater given a greater N. Finally, as any strategy feasible given less N is feasible given any greater N, the monotony is proved.

Now, let $T = \infty$. We have

$$V(\xi_t) = \sup_{\delta_\tau,\sigma_\tau,C_\tau,\ \text{fulfilling } \mathcal{C}(\tau),\mathcal{A}(\tau),\mathcal{M}(\tau),\mathcal{N}(\tau),\tau\geq t} \mathbb{E}\left[\lim_{T\to\infty}\sum_{\tau=t}^{T} e^{-\rho(\tau-t)}C_\tau|\xi_t\right]$$

$$= \sup_{\substack{\delta_\tau,\sigma_\tau \text{ fulfilling } \mathcal{C}(\tau),\mathcal{A}(\tau),\mathcal{N}(\tau)\\ p_\tau\in[0,1],\varphi^{-1}(p_\tau)\leq(e^{h_\tau}\phi(\delta_\tau,\sigma_\tau)+M_\tau),\tau\geq t}} \mathbb{E}\left[\lim_{T\to\infty}\sum_{\tau=t}^{T} e^{-\rho(\tau-t)}\varphi^{-1}(p_\tau))|\xi_t\right]$$

$$= \lim_{T\to\infty} \sup_{\substack{\delta_\tau,\sigma_\tau \text{ fulfilling } \mathcal{C}(\tau),\mathcal{A}(\tau),\mathcal{N}(\tau)\\ p_\tau\in[0,1],\varphi^{-1}(p_\tau)\leq(e^{h_\tau}\phi(\delta_\tau,\sigma_\tau)+M_\tau),T\geq\tau\geq t}} \mathbb{E}\left[\sum_{\tau=t}^{T} e^{-\rho(\tau-t)}\varphi^{-1}(p_\tau))|\xi_t\right]$$

$$= \lim_{T\to\infty} \sup_{\delta_\tau,\sigma_\tau,C_\tau,\ \text{fulfilling } \mathcal{A}(\tau),\mathcal{M}(\tau),\mathcal{N}(\tau),T\geq\tau\geq t} \mathbb{E}\left[\sum_{\tau=t}^{T} e^{-\rho(\tau-t)}C_\tau|\xi_t\right]$$

$$= \lim_{T\to\infty} \tilde{V}(M_\tau, N_\tau, h_\tau, v_{\eta,\theta}, T-t)$$

$$= \lim_{T\to\infty} \sup_{\delta_t,\sigma_t,C_t,\ \text{fulfilling } \mathcal{A}(t),\mathcal{M}(t),\mathcal{N}(t)} \left[C_t + e^{-\rho}\mathbb{E}(\tilde{V}(M_\tau, N_\tau, h_\tau, v_{\eta,\theta}, T-(t+1))|\xi_\tau)\right]$$

$$= \sup_{\delta_t,\sigma_t,C_t,\ \text{fulfilling } \mathcal{A}(t),\mathcal{M}(t),\mathcal{N}(t)} \left[C_t + e^{-\rho}\mathbb{E}(\lim_{T\to\infty}\tilde{V}(M_\tau, N_\tau, h_\tau, v_{\eta,\theta}, T-(t+1))|\xi_\tau)\right]$$

$$= \sup_{\delta_t,\sigma_t,C_t,\ \text{fulfilling } \mathcal{A}(t),\mathcal{M}(t),\mathcal{N}(t)} \left[C_\tau + e^{-\rho}\mathbb{E}(V(\xi_{\tau+1})|\xi_\tau)\right] \quad (7)$$

The third equality follows from Fatou-Lesbeque theorem (allowing to change the order of limit and expectation) and variational analysis theory. Since $\varphi^{-1}(p_\tau)$ is a monotone and differentiable function for all τ almost surely, the objective functions: $f_T = \mathbb{E}(\sum_{\tau=t}^{T}\varphi^{-1}(p_\tau)|\xi_t)$ are monotone and differentiable, too. Moreover, the feasible solutions of the problem form a compact set. Monotonicity, differentiability and compactness imply the epi-convergence of f_T to $f_\infty = \mathbb{E}(\sum_{\tau=t}^{\infty}\varphi^{-1}(p_\tau)|\xi_t)$. Finally, having compact set of feasible solutions, the epi-convergence of objectives functions allows for interchange of supremum and limit. See [1](Theorem 1.10, Theorem 2.11) or [3](Th. 7.33) for more details.

It is clear from the third "=" of (7) that $V(\xi_t)$ depends only on $h_t, M_t, N_t, v_{\eta,t}, T-t$ and, as $\lim_T f(T-t)$ never depends on t whatever f is, it is clear that in fact $V(\xi_t,\infty)$ does not depend on t. The properties of a and b may be proved analogously to the case of finite horizon, using the equality of the first and last term in (7). The monotony of V follows from the fact that a limit of monotonous functions is monotonous.

(ii) Follows from the previous.

(iii) The monotony in N follows from the monotony of W, the monotony in M from (5).

(iv) Follows from the facts that the function on the LHS of constraint $(\tilde{\mathcal{M}})$ of problem (6) is non-decreasing in x and that the constraint $(\tilde{\mathcal{N}})$ does not depend on x. □

6 The Price and Volume

As it was already written above, our goal is to determine the joint dynamics of the midpoint price P_t and the the MM's inventory N_t. If we approximate

$$\delta(n, v) \doteq d_0 + d_1 n + d_2 v,$$

we get that, up to a constant, the price may be decomposed as follows

$$P_\tau = \underbrace{\pi_\tau}_{\text{fair price}} + \underbrace{\eta_\tau + d_2 v_{\eta,\tau-1}}_{\text{uncertainty}} + \underbrace{d_1 N_\tau}_{\text{inventory}}$$

which is analogous to a well known decomposition of spread, widely discussed in market macrostructure. If we further linearize

$$\sigma(n,v) = s_0 + s_1 n + s_2 v, \qquad v_\eta(v,\sigma) = w_0 + w_1\sigma + w_2 v$$

we get

$$v_{\eta,\tau} = w_0 + w_1\sigma_{\tau-1} + w_2 v_{\eta,\tau-1} = w_0 + w_1\sigma_{\tau-1} - \frac{w_2}{s_2}(s_0 + s_1 N_{\tau-1})$$

and, consequently,

$$\delta(N_\tau, v_{\eta,\tau}) \doteq d_0 + d_1 N_\tau + d_2(w_0 + w_1\sigma_{\tau-1} - \frac{w_2}{s_2}(s_0 + s_1 N_{\tau-1}))$$
$$= \beta_0 + \beta_\sigma\sigma_{\tau-1} + \beta_{\Delta N}\Delta N_\tau + \beta_N N_{\tau-1}$$

If we futher linearly approximate $c_N(v,\sigma) \equiv c_N$ and $c_e(v,\sigma) \equiv c_e$, we get

$$\Delta P_\tau = \Delta h_\tau + \Delta\delta(N_\tau, v_{\eta,\tau}) = c_N(k^{-1}\Delta N_\tau - \delta(N_\tau, v_{\eta,\tau})) + c_e(e_t - h_{t-1}) + \Delta\delta(N_\tau, v_{\eta,\tau})$$

$$\doteq \phi_0 + \phi_{\Delta\sigma}\Delta\sigma_{\tau-1} + \phi_\sigma\sigma_{\tau-2} + \phi_{\Delta N}\Delta N_\tau + \phi_{\Delta N_{-1}}\Delta N_{\tau-1} + \phi_N N_{\tau-2} + E_\tau$$

where E_τ are i.i.d. The following shows results of estimation of the latter equation based on 51412 observations of 10-second snapshots data of Exxon Mobile on ISE narket.

Model 3: OLS, using observations 3–51412 (T = 51410)
Dependent variable: d_P

	Coefficient	Std. Error	t-ratio	p-value
const	0,00101284	0,000413144	2,4515	0,0142
d_N	−2,78402e–05	7,70284e–07	−36,1428	0,0000
d_N_1	1,03080e–06	7,69970e–07	1,3388	0,1807
N_2	−8,38851e–10	4,55825e–09	−0,1840	0,8540
d_Sigma_1	−0,0231703	0,00350266	−6,6151	0,0000
Sigma_2	−0,0171499	0,00373884	−4,5870	0,0000

Mean dependent var	0,000019	S.D. dependent var	0,045355
Sum squared resid	103,0372	S.E. of regression	0,044771
R^2	0,025671	Adjusted R^2	0,025576
$F(5, 51404)$	270,8713	P-value(F)	6,6e–287
Log-likelihood	86744,62	Akaike criterion	−173477,2
Schwarz criterion	−173424,2	Hannan–Quinn	−173460,6
$\hat{\rho}$	−0,238393	Durbin–Watson	2,474708

7 Alternative models

Finally, let us discuss two alternative (sub) models and show that they are rejected in favour of our model.

7.1 Irrational liquidity takers

Assume, in the present Subsection, that the liquidity takers do not consider (their estimate of) the fair price but they buy and sell the stocks randomly, i.e.

$$X_\tau | \Xi_{\tau-1} \sim \mathrm{CP}(\lambda), \qquad Y_\tau | \Xi_{\tau-1} \sim \mathrm{CP}(\lambda).$$

This, however, means that X_τ, Y_τ, hence ΔN_τ, are independent of all the past, namely of $\Delta N_{\tau-1}$. This is, however, not true, as the correlation coeffitient of ΔN_τ and $\Delta M_{\tau-1}$, compoted from the data above, is significant on 0.0000 probability level.

7.2 Irrational market makers

Another possible violation of our model could be that the MM's do not (cannot) act rationally but they "take prices as they come". In particular, that once a market order arrives and causes a movement of its corresponding quote, the MM sets the new quote so that its jump (w.r.t. its value before the market order arrival) is proportional to the impact of the market order, which could be mathematically expressed as

$$\Delta M_\tau = \beta \Delta N_\tau + \epsilon_\tau.$$

Hosever, even this model is convincingly rejected by adding $\Delta M_{\tau-1}$ into the regression and observing that the probability level of the corresponding regressor is 0.0000.

8 Conclusion

A model of a rational behaviour of a risk averse partially informed market maker solving multistage decision problem was proposed, implying an easily tractable and estimable stochastic model of high frequency trade and quote data process, which was subsequently successfully tested by means of data from US electronic markets.

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