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MODELLING OF WHEAT-FLOUR DOUGH MIXING AS AN OPEN-LOOP HYSTERETIC PROCESS

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Dedicated to the memory of Alexei Vadimovich Pokrovskii.

Abstract. Motivated by the fact that various experimental results yield strong confirmatory support for the hypothesis that *"the mixing of a wheat-flour dough is essentially a rate-independent process"*, this paper examines how the mixing can be modelled using the rigorous mathematical framework developed to model an incremental time evolving deformation of an elasto-plastic material. Initially, for the time evolution of a rate-independent elastic process, the concept is introduced of an *"energetic solution"* [24] as the characterization for the rate-independent deformations occurring. The framework in which it is defined is formulated in terms of a polyconvex stored energy density and a multiplicative decomposition of large deformations into elastic and nonelastic (plastic or viscous) components. The mixing of a dough to peak dough development is then modelled as a sequence of incremental elasto-nonelastic deformations. For such incremental processes, the existence of Sobolev solutions is guaranteed. Finally, the limit passage to vanishing time increment leads to the existence of an energetic solution to our problem.

1. **Introduction.**

In the breeding of new varieties of crops, such as wheat, the challenge is not only to maintain resistance against known pest and diseases but also to guarantee the product quality for which the crop has been grown. For wheat, the second aspect relates to guaranteeing that the new pest and disease resistant varieties include ones that make good breads, cakes or pastas.

As well as understanding the molecular differences between the wheat which make various types of breads, cakes or pastas, it is also important to study, in terms of the underlying molecular dynamics, the rheological (flow and deformation) behavior involved with the the various stages of their manufacture. For wheat, such insight is important from a number if independent perspective - the growing and

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ripening; the milling to make the flour; the mixing of the wheat-flour dough; the baking of the dough. In order to improve the energy efficiency of the processing and maintaining the quality of the end-product, such information feeds back iteratively on the breeding, the science and the technology.

It has already been hypothesized, on the basis of extensive experimental evidence [4, 12], that *"the mixing of a wheat-flour dough is essentially a rate-independent process"*. A detailed investigation and review of the possibility along with the results of a detailed experimental study are given in Anderssen *et al.* [4]. The fact that the rotational speed of the Mixograph slows as the elastic potential energy accumulates and correlates positively with the size of the bandwidth of the mixogram recording the dynamics of the mixing is highlighted in Anderssen *et al.* [3]. Such results were used as motivation by Anderssen *et al.* [2] in their examination of the global behavior of elastoplastic and viscoelastic materials with hysteresis-type state equations. Verification of the hysteretic nature of the extension, rupture and relax occurring during the mixing and recorded as a mixogram on a Mixograph was published by Gras *et al.* [12, 1].

In this paper, a theoretical basis is developed to explain the rheology of the apparent rate-independent dynamics of the mixing of a wheat-flour dough. The approach taken is based on utilizing the concept of Mielke and Theil [23, 24] of *"energetic solution"* as the characterization for the time evolution of a rate-independent deformation process. It is initially introduced for rate-independent elastic materials and then extended for the characterization of rate-independent elasto-nonelastic materials. The mixing of a dough is then modeled as a sequence of incremental elasto-nonelastic deformations. Regularity conditions are then established which guarantee the existence of energetic solutions for such incremental processes. The essence of the required regularity can then be viewed as an indirect characterization of the dynamics of the molecular interactions occurring within a wheat-flour dough during its mixing.

Here, the material properties of the dough are modelled using a polyconvex stored energy density while the mixing is modelled as a multiplicative decomposition of the large deformations into elastic and nonelastic components.

The role of hysteresis as the basis for the dynamics of physical processes has been studied by various authors including the study by Pokrovskii and colleagues [5] of rate-independent processes in terrestrial hydrology.

1.1. **The unifying thread and organization of the paper.** The strain hardening, which occurs in materials when subjected to repetitive (incremental) loading, is an open loop hysteretic process which tracks how the stress (or, alternatively, the strain) increases as a function of the repetitive loading. The associated mathematical modelling is quite different from that associated with closed loop hysteresis. When the open loop dynamics is rate-independent, which occurs in many materials, the associated mathematical modelling simplifies and is the focus of the current paper.

The theory of Sections 3, 3.1 and 4 is initially motivated with a discussion in Section 2 of the rate-independence of the mixing of a wheat-flour (bread) dough, where the historical and experimental evidence is reviewed.

Even though parts of the dough are rupturing during the mixing, at any stage, the dough is essentially experiencing a dominant extensional repetitive (incremental) loading like that applied to the strain hardening of steel. In both the steel and the dough situations, it is the molecular structural changes that the repetitive loading engenders that is the source, for both, of the observed strain hardening. For steel, the strain hardening occurs as a result of the formation of the dislocations through plastic deformation with the associated ductility (extent to which plastic deformation can occur) decreasing as the strain hardening increases. A similar phenomenology occurs within a dough during its (repetitive loading) mixing. The formation of the dislocations is replaced by the increasing alignment of the gluten protein chains within the dough, and the decreasing ductility by the increasing stress that must be applied in the extension of the dough as the mixing progresses (up to peak dough development).

Consequently, though the molecular mechanisms are different, the essential stress-strain dynamics are the same from a mathematical modelling perspective. The challenge becomes one of identifying how the structure of the mathematical modelling is formalized is see the difference. The natural choice is the stored energy density (functional), because it models how a system stores energy as repetitive loading is applied. The subsequent stress-strained dynamics is determined by its structure, as detailed in Sections 3, 3.1 and 4.

For the wheat-flour dough for hard wheats, various authors [9, 26] have concluded that, on the basis of rheological oscillatory shear experimentation, the development of the gluten network in the dough behaves in a hyperelastic manner and that an appropriate model for the associated stored energy density is Mooney-Rivlin (equation (1) in [9]). The Mooney-Rivlin stored energy density is a particular representative of the class of stored energy densities examined in Sections 3, 3.1 and 4.

2. The rate-independence of wheat-flour (bread) dough mixing.

Since the 1930, various instruments have been designed to measure the forces involved with the mixing of wheat-flour (bread) dough. They include the Mixogram and the Farinograph. Their initial importance related to them being able to assist with the categorize wheat on the basis of its intended use and, thereby, its economic value. Gradually, they played an increasingly important role in the development of cereal science by assisting with plant breeding through the discovery the role of wheat protein and starches in the making of good breads, cakes and pasta.

The first experimental proof of the dominant rate-independent behaviour of the mixing of a wheat-flour (bread) dough is contained in a paper by Kilborn and Tipples [14]. As shown in Fig. 6, 8 and 10 in their paper, the rate independence was established for a number of the properties of the breads made from them - loaf volume, crumb colour and crumb texture. Interestingly, in the bottom panels of their figures, the mixing time (MT), to reach the desired outcome, is plotted as a function of revolutions per minute (RPM). A plot of the number of revolutions, the product of MT and RPM, as a function of RPM would have produced a similar structure to that seen in the other panels of their figures. However, because the goal of the paper was a study of mixing intensity and work input to reach peak dough development, the fact that rate-independence had been established experimentally was overlooked.

On very early commercial mixers, there was a counter which counted the number of revolutions of the mixer and this was used by the operator to determine when a dough was fully developed.

The advantage of the Mixogram is that it is a pin mixer and has an action similar to that of commercial mixers. On a Mixogram, the force with which the dough being mixed resists its elongation between the moving and fixed pins is measured as a mixogram. The resulting rheological flow is dominantly elongational. The original scientific instruments, like the ones used by Kilborn and Tipples [14], were mechanical devices with the forces with which the dough resisted elongation recorded using a pen on moving paper device.

The invention of electronic versions of the Mixograph, recording the stress every millisecond, allowed the high resolution structure within a mixogram to be recorded electronically and examined in detail. Using such an instrument, the rate-independence of dough mixing to achieve peak dough development was confirmed scientifically and reported in papers by Anderssen, Gras and MacRitchie [3, 4]. It follows that, because of the rate-independence, dough is mixed once the right amount of stress has been applied independent of the speed of the mixing. This is easily visible from the positive one homogeneity of the dissipation potential. This establishes that once the appropriate amount of extensional energy has been accumulated by a dough, the molecular alignment of the gluten protein network associated with peak dough development will have been achieved.

A possible contributing molecular factor to the rate-independence of the mixing is that, in a hard (bread) wheat, there is strong cohesion between the gluten proteins and the starch granules. The cohesion is known to be the key genetic nuance that distinguishes a hard wheat from a soft one. Consequently, the molecular alignment process occurring during the mixing may be assisted by the presence of the starch granules since they make up about $60-70\%$ of a wheat-flour on a dry weight basis.

Much rheological wheat-flour dough research is not based on the direct recording of the dynamics of the mixing of a dough but on oscillatory shear [26] and extensional [25] experiments performed on samples of dough already mixed to peak dough development. Consequently, the conclusions thereby derived, since they are based on comparing different samples of the same wheat variety prepared on different mixing devices, are implicitly assuming that the mixing performed to produce the samples is rate-independent. To assume otherwise would required that comparisons could only be performed on samples which had been prepared using the same preparation and mixing protocols.

This contribution proposes a mathematical model of dough mixing. It is inspired by [4] where the authors observed that dough mixing to peak dough development can be viewed as essentially rate-independent; i.e., with respect to specified outcomes, the result of the dough mixing, which includes the quality of the bread made from the dough, is independent of the mixer speed. On the other hand, because the deformations occurring during the mixing are large, the associated mathematical modelling should reflect injectivity and orientation preservation of deformations as suggested in [10] as well as involving a multiplicative decomposition of the deformation into elastic and nonelastic contributions. If it is assumed that a dough behaves like a hyperelastic material (e.g. as in [9]), it follows that the stored energy density diverges to infinity whenever the deformation gradient tends to zero, which does not agree with convexity holding for the stored energy density. Consequently, the elasticity equations together with a flow rule governing the evolution of nonelastic deformations are not appropriate to describe equilibrium states of the material. To cope with this difficulty, we advantageously use the concept of *energetic solution* developed by Mielke and his collaborators [23, 24] to define a notion of a solution of rate-independent processes. In this way, we obtain a well-posed problem allowing for rigorous analysis. Future research will involve checking on the extent to which experimental data validate the appropriateness of this new approach to the modelling of the rate-independence of dough mixing.

This then represents justification for turning to the theory of *energetic solutions* to model rate-independent processes. Though this theory was developed to model elasto-plastic rate-independence process, like any mathematical theory, it can be applied to any practical situation where the assumptions of the theory are valid. This includes the modelling of rate-independence in a viscoelastic context. Here, the link concepts that matches the theory with the application are the form of the models for the stored energy density and the large deformations.

Comment. Though, through the viscosity, the dissipation of energy is involved, one cannot argue that rate-independence does occur. Rate-independence is measured with respect to a specified goal; namely, the goal that is achieved with the application of the same amount of energy independent of the speed with which the energy is accumulated. This is the situation for the mixing of a wheat flour dough to peak dough development. Here, the repetitive extensional elongation-and-rupture process occurring in a Mixograph performs an alignment of the various gluten proteins (and, in particular, the glutenins) which is recorded at an increasing extensional stress with which the dough resists the same extension. This is the essence of the open-loop hysteresis which characterizes the stress-strain process occurring in the mixing of a dough - the stress with which the dough is resisting elongation is increasing with the extensional strains remaining the same.

3. **Preliminaries and notation.** In what follows, $\Omega \subset \mathbb{R}^n$ is an open bounded Lipschitz domain representing the dough, $L^\beta(\Omega;\mathbb{R}^n)$, $1 \le \beta < +\infty$ denotes the usual Lebesgue space of mappings $\Omega \to \mathbb{R}^n$ whose modulus is integrable with the power β and $L^\infty(\Omega;\mathbb{R}^n)$ is the space of measurable and essentially bounded mappings $\Omega \to \mathbb{R}^n$. Further, $W^{1,\beta}(\Omega;\mathbb{R}^n)$ standardly represents the space of mappings which live in $L^\beta(\Omega;\mathbb{R}^n)$ and their gradients belong to $L^\beta(\Omega;\mathbb{R}^{n\times n})$. Finally, $W_0^{1,\beta}(\Omega;\mathbb{R}^n)$ is a subspace of $W^{1,\beta}(\Omega;\mathbb{R}^n)$ of maps with the zero trace on $\partial\Omega$. Finally, $C(\Omega)$ or $C(\mathbb{R}^{n\times n})$ denotes function spaces of functions continuous on Ω or $\mathbb{R}^{n\times n}$, respectively and $C^1(\Omega)$ denotes the spaces of continuously differentiable functions.

In the sequel, $\Omega \subset \mathbb{R}^n$ represents the so-called reference configuration of our material (dough), and $\partial\Omega \supset \Gamma_0, \Gamma_1$ which are disjoint. The overall deformation will be denoted $y : \Omega \to \mathbb{R}^n$. The evolution of the system will be controlled by external forces. Let $f(t) : \Omega \to \mathbb{R}^n$ be the (volume) density of external body forces and $g(t) : \Gamma_1 \subset \partial\Omega \to \mathbb{R}^n$ be the (surface) density of surface forces. The work of these forces done on the specimen models the action of mixing on the dough. The (hyper)elastic behavior of our specimen is influenced by a multidimensional internal variable z, which in the framework of elastoplasticity represents plastic deformation and hardening variables. Typically, its values live in some set Z to be specified later. In the present setting, we will also consider its gradient ∇z as e.g. in [7, 27] to capture nonlocal effects. From the mathematical point of view, it also provides us with additional compactness allowing for a passage to the limit, thus it is often called "regularization". Hence, the stored energy $\mathcal{W}$ of the material depends on the internal variable z. We are interested in the *rate-independent* evolution of the material. To this end, we assume the existence of a nonnegative convex potential $\delta = \delta(\dot z)$ of dissipative forces, where $\dot z$ denotes the time derivative of z. In order to ensure rate-independence, δ must be positively one-homogeneous, i.e., $\delta(\alpha \dot z) = \alpha\delta(\dot z)$ for all $\alpha \ge 0$.

The potential energy of our system can be written as ($\epsilon > 0$)

$$\mathcal{I}(t, y(t), z(t)) := \int_\Omega \mathcal{W}(\nabla y(t), z(t))\,\mathrm{d}x + \epsilon \int_\Omega |\nabla z(t)|^\omega\,\mathrm{d}x - L(t, y(t))\ , \tag{1}$$

where the work done by external forces is

$$L(t, y(t)) := \int_\Omega f(t) \cdot y(t)\,\mathrm{d}x + \int_{\Gamma_1} g(t) \cdot y(t)\,\mathrm{d}S \tag{2}$$

and the following energy balance is satisfied

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t, y(t), z(t)) = \dot{L}(t, y(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Diss}(z; [0, t])\ , \tag{3}$$

where

$$\mathrm{Diss}(z; [0, t]) := \int_0^t \int_\Omega \delta(\dot{z}(s))\,\mathrm{d}x\mathrm{d}s\ .$$

Hence, the integration with respect to time gives

$$\mathcal{I}(t, y(t), z(t)) + \mathrm{Diss}(z; [0, t]) = \mathcal{I}(0, y(0), z(0)) + \int_0^t \dot{L}(s, y(s))\,\mathrm{d}s\ .$$

We can also consider a more general form of δ which can also depend on (x, z), i.e. $\delta := \delta(x, z, \dot{z})$.

Typically, however, we do not have enough smoothness in the internal variable to compute the time derivative on the right-hand side of (3).

Following Mielke [19] we define a dissipation distance between two values of internal variables $z_0, z_1 \in Z$ as

$$D(x, z_0, z_1) := \inf_z \left\{ \int_0^1 \delta(x, z(s), \dot{z}(s))\,\mathrm{d}s;\ z(0) = z_0\ ,\ z(1) = z_1 \right\}\ , \tag{4}$$

where $z \in C^1([0, 1]; Z)$, and set

$$\mathcal{D}(z_1, z_2) = \int_\Omega D(x, z_1(x), z_2(x))\,\mathrm{d}x\ , \tag{5}$$

where $z_1, z_2 \in \mathbb{Z} := \{z : \Omega \to \mathbb{R}^M;\ z(x) \in Z \text{ a.e. in } \Omega\}$. We assume that $\mathbb{Z}$ is equipped with strong and weak topologies which define notions of convergence used below.

Following [11, 18] we impose the following assumptions on $\mathcal{D}$: (i) Weak lower semicontinuity:

$$\mathcal{D}(z, \tilde{z}) \le \liminf_{k \to \infty} \mathcal{D}(z_k, \tilde{z}_k)\ , \tag{6}$$

whenever $z_k \rightharpoonup z$ and $\tilde{z}_k \rightharpoonup \tilde{z}$.
(ii) Positivity: If $\{z_k\} \subset \mathbb{Z}$ is bounded and $\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0$ then

$$z_k \rightharpoonup z\ . \tag{7}$$

3.1. **Energetic solution.** Suppose that we look for the time evolution of $y(t) \in \mathbb{Y} \subset \{y : \Omega \to \mathbb{R}^n\}$ and $z(t) \in \mathbb{Z}$ during the time interval $[0, T]$. The following two properties are the key ingredients of the so-called energetic solution due to Mielke and Theil [23, 24].
(i) Stability inequality:
$\forall t \in [0, T]$, $\tilde{z} \in \mathbb{Z}$, $y \in \mathbb{Y}$:

$$\mathcal{I}(t, y(t), z(t)) \leq \mathcal{I}(t, \tilde{y}, \tilde{z}) + \mathcal{D}(z(t), \tilde{z}) \tag{8}$$

(ii) Energy balance: $\forall\ 0 \leq t \leq T$

$$\mathcal{I}(t, y(t), z(t)) + \mathrm{Var}(\mathcal{D}, z; [0, t]) = \mathcal{I}(s, y(0), z(0)) + \int_0^t \dot{L}(\xi, y(\xi))\, \mathrm{d}\xi \ , \tag{9}$$

where

$$\mathrm{Var}(\mathcal{D}, z; [s, t]) := \sup \left\{ \sum_{i=1}^{N} \mathcal{D}(z(t_i), z(t_{i-1}));\ \{t_i\} \text{ partition of } [s, t] \right\}$$

Definition 3.1. The mapping $t \mapsto (y(t), z(t)) \in \mathbb{Y} \times \mathbb{Z}$ is an energetic solution to the problem $(\mathcal{I}, \delta, L)$ if the stability inequality and the energy balance are satisfied.

Remark 1. For simplicity, we do not work with time-dependent Dirichlet boundary conditions here. However, the approach can be easily modified to include them.

It is convenient to put $\mathbb{Q} := \mathbb{Y} \times \mathbb{Z}$ and to set $q := (y, z)$. We define the set of stable states at time t as

$$\mathcal{S}(t) := \{q \in \mathbb{Q} : \forall \tilde{q} \in \mathbb{Q} :\ \mathcal{I}(t, q) \leq \mathcal{I}(t, \tilde{q}) + \mathcal{D}(q, \tilde{q})\} \tag{10}$$

4. **Applications to elastic-nonelastic processes.** Here, we discuss the application of the energetic solution approach to an elasto-plastic situation.

4.1. **Problem statement.** In what follows $y : \Omega \to \mathbb{R}^n$ will be a deformation of a body $\Omega \subset \mathbb{R}^n$ (in a fixed reference configuration) with the deformation gradient $F = \nabla y$. In particular, y covers both *elastic*, as well as *plastic* deformation. We define the multiplicative split, $F = F_{\mathrm{e}} F_{\mathrm{p}}$, into an elastic part F_{e} and an irreversible plastic part F_{p} which belongs to $\mathrm{SL}(n) := \{A \in \mathbb{R}^{n \times n};\ \det A = 1\}$. The so-called plastic strain F_{p} and the vector $p \in \mathbb{R}^m$ of hardening variables are internal variables influencing elasticity. In other words, $z(x) = (F_{\mathrm{p}}(x), p(x)) \in \mathrm{SL}(n) \times \mathbb{R}^m =: Z$ for almost all $x \in \Omega$.

The energy functional $\mathcal{I}$ now takes the form

$$\mathcal{I}(t, y(t), z(t) := \int_\Omega \mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)\, \mathrm{d}x - L(t, y(t)) \ , \tag{11}$$

with L given by (2).

In order to ease the notation we omit the dependence of $\mathcal{W}$ on x, however, all the statements in this paper may include nonhomogeneous $\mathcal{W}$, too.

In what follows, we suppose that

$$y \in \mathbb{Y} := \Big\{ y \in W^{1,d}(\Omega; \mathbb{R}^n);\ y = y_0 \text{ on } \Gamma_0\ ,\ \det\nabla y > 0 \text{ a.e. },$$
$$\int_\Omega \det\nabla y(x)\, \mathrm{d}x \leq |y(\Omega)| \Big\} \ , \tag{12}$$

where $\Gamma_0 \subset \partial\Omega$ with a positive surface measure. Moreover, we suppose that $\Gamma_0 \cap \Gamma_1 = \emptyset$. The inequality condition $\int_\Omega \det\nabla y(x)\,\mathrm{d}x \le |y(\Omega)|$ is the so-called injectivity condition [10] ensuring that y restricted to Ω is one-to-one. Here, $|y(\Omega)|$ denotes the three-dimensional Lebesgue measure of $y(\Omega)$, i.e., $\int_{y(\Omega)} 1\,\mathrm{d}x$. Further,

$$\mathbb{Z} := \{(F_p, p) \in W^{1,\beta}(\Omega;\mathbb{R}^{n\times n}) \times W^{1,\omega}(\Omega;\mathbb{R}^m) :\ F_p(x) \in \mathrm{SL}(n) \text{ for a.e. } x \in \Omega\}\ .$$

As $q = (y, z)$ it will be advantageous and will make no confusion to write $\mathcal{D}$ as dependent on q, i.e.,

$$\mathcal{D}(q_1, q_2) := \mathcal{D}(z_1, z_2)$$

if $q_1 = (y_1, z_1)$ and $q_2 = (y_2, z_2)$. Similarly, we may write $\mathcal{I}$ in terms of $q = (y, z)$ as

$$\mathcal{I}(t, q(t)) = \int_\Omega \mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)\,\mathrm{d}x - L(t, q(t))\ ,$$

where, obviously, $L(t, q(t) := L(t, y(t))$.

In this situation, $Q = (Q_1, Q_2)$ are *conjugate plastic stress* and *conjugate hardening forces*, respectively, defined as

$$Q_1 = \mathrm{div}\left(\frac{\partial\mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)}{\partial \nabla F_{\mathrm{p}}}\right) - \frac{\partial\mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)}{\partial F_{\mathrm{p}}} \tag{13}$$

and

$$Q_2 = \mathrm{div}\left(\frac{\partial\mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)}{\partial \nabla p}\right) - \frac{\partial\mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)}{\partial p}. \tag{14}$$

The elastic domain is defined as

$$\mathcal{Q}(x, z) = \partial^{\mathrm{sub}}_{\dot z}\delta(x, z, 0)\ , \tag{15}$$

where ∂^{sub} denotes the subdifferential in the sense of convex analysis.

4.2. **Assumptions on problem data.** As in [13] we will consider so-called *separable materials*, i.e., materials where the elasto-plastic energy density has the form

$$\mathcal{W}(F_{\mathrm{e}}, F_p, \nabla F_{\mathrm{p}}, p, \nabla p) := \mathcal{W}_1(F_{\mathrm{e}}) + \mathcal{W}_2(F_p, \nabla F_{\mathrm{p}}, p, \nabla p)\ , \tag{16}$$

which represents a standard constitutive assumption in plasticity. Nevertheless, our theory is not restricted only to separable materials. We start with assumptions on $\mathcal{W}$:

(i) $\mathcal{W}_1, \mathcal{W}_2 \ge 0$ are continuous in all their arguments. Moreover, $\mathcal{W}_1$ is polyconvex [6] which means that $\mathcal{W}_1(F) = h(F, \mathrm{cof}F, \det F)$ where $h : \mathbb{R}^{19} \to \mathbb{R} \cup \{+\infty\}$ is a convex function such that $\mathcal{W}_1(F) < +\infty$ if $\det F > 0$ and $\mathcal{W}_1(F) \to +\infty$ if $\det F \to 0_+$. Further, $+\infty > \mathcal{W}_1(F) \ge 0$ if $\det F > 0$ and

(ii) Suppose that there are two constants $C, c > 0$ so that the following assumptions hold for all arguments of involved functions: $\alpha, \beta > n$, $\omega > n$:

$$\mathcal{W}_1(F) \ge c|F|^\alpha\ ,$$

$$\begin{aligned} C(|F_p|^\beta + |G|^\beta + |p|^\omega + |\pi|^\omega) &\ge \mathcal{W}_2(F_{\mathrm{p}}, G, p, \pi) \\ &\ge c(|F_p|^\beta + |G|^\beta + |p|^\omega + |\pi|^\omega)\ , \end{aligned} \tag{17}$$

where $|\cdot|$ denotes the Euclidean norm;

(iii) $\mathcal{W}_2(F_{\mathrm{p}}, \cdot, p, \cdot)$ is convex for all $(F_{\mathrm{p}}, p) \in \mathbb{R}^{n\times n} \times \mathbb{R}^m$.

As an example we can consider the Mooney-Rivlin material of the form

$$\mathcal{W}_1(F) := \begin{cases} b|F|^\alpha + c|\mathrm{cof}F|^2 + \Gamma(\det F) & \text{if } \det F > 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{18}$$

Here $b, c > 0$ and $\Gamma(\delta) = d\delta^2 - e\log\delta$, $d, e > 0$ and we recall that $\mathrm{cof}F := \det F(F^{-\top})$ if F is invertible.

A simple example of $\mathcal{W}_2$, the density of energy stored in defects, is $(a_1, b_1 > 0)$

$$\mathcal{W}_2(F_\mathrm{p}, \nabla F_\mathrm{p}, p, \nabla p) := a_1|F_\mathrm{p}|^\beta + b_1|\nabla F_\mathrm{p}|^\beta \ . \tag{19}$$

We recall the following assumptions on $\mathcal{D}$:

(i) Lower semicontinuity:

$$\mathcal{D}(z, \tilde{z}) \le \liminf_{k\to\infty} \mathcal{D}(z_k, \tilde{z}_k) \ , \tag{20}$$

whenever $z_k \rightharpoonup z$ and $\tilde{z}_k \rightharpoonup \tilde{z}$.

(ii) Positivity: If $\{z_k\} \subset Z$ is bounded and $\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0$ then $z_k \rightharpoonup z$.

In order to prove the existence of a solution to (25) we must impose some data qualifications. In what follows, we assume that

$$f \in C^1\left([0,T]; L^{d^*}(\Omega; \mathbb{R}^n)\right) \ , \tag{21}$$

$$g \in C^1\left([0,T]; L^{d^\#}(\Gamma_1; \mathbb{R}^n)\right) \ , \tag{22}$$

where $d^* \ge nd/(n-d)$ if $1 \le d < n$ or $d^* \ge 1$ otherwise. Similarly, we suppose that $d^\# \ge (nd-d)/(n-d)$ if $d < n$ or $d^\# \ge 1$ otherwise.

If $\mathcal{D} : \mathbb{Q} \times \mathbb{Q} \to [0, +\infty)$, i.e., no irreversibility constraint is imposed on plastic processes as in the case of isotropic hardening, then it is sufficient if D from (4) satisfies

$$D(x, z_1, z_2) \le c(x) + C(|F_{\mathrm{p}_1}|^{\beta^*-\epsilon} + |F_{\mathrm{p}_2}|^{\beta^*-\epsilon} + |p_1|^{\omega^*-\epsilon} + |p_2|^{\omega^*-\epsilon}) \ , \tag{23}$$

where $\epsilon > 0$ is small enough and $\beta^* := n\beta/(n-\beta)$ if $n > \beta$ and $\beta^* < +\infty$ if $\beta \ge n$. Similarly, $\omega^* := n\omega/(n-\omega)$ if $n > \omega$ and $\omega^* < +\infty$ if $\omega \ge n$. Then the compact embedding ensures continuity of $\mathcal{D}$. We refer to [18, Ex. 3.4] for an example of isotropic-hardening dissipation. Another example might simply be $(a > 0)$

$$D(x, z_1, z_2) = a|F_{\mathrm{p}_1} - F_{\mathrm{p}_2}| \ , \tag{24}$$

which corresponds to $\delta(x, z, \dot{z}) = a|\dot{F}_\mathrm{p}|$.

Remark. The above analysis also applied to irreversible rate-independent processes, like that occurring in dough mixing, on assuming that $\mathcal{D}(x, z_1, z_2) = +\infty$, since this condition guarantees the irreversibility. Fuller details can be found in [18, 15].

In this way, through the introduction of a polyconvex stored energy functional and a multiplicative decomposition of the large deformations into elastic and nonelastic components, a rigorous mathematical framework has been constructed in which to model rate-independent processes such as the mixing of a dough to peak dough development and elasto-plastic strain hardening. The rigor is achieved through the rate-independent dynamics being characterized as an energetic solution.

Consequently, the final step, which validates the appropriateness of this framework, is to prove that energetic solutions exist.

Theorem 4.1. *Let $\alpha^{-1} + \beta^{-1} \leq d^{-1} < n^{-1} < 1$ and $\omega = \beta$. Let $q^0 \in \mathbb{Q}$ the initial condition be a stable state at $t = 0$. Let the assumptions on $\mathcal{W}$, $\mathcal{D}$, f and g from Section 4.2 hold. Let further (23) hold. Then there is a process $q : [0, T] \to \mathbb{Q}$ with $q(t) = (y(t), z(t))$ such that q is an energetic solution according to Definition 3.1.*

Proof. The proof is partially constructive as it it based on semidiscretization in time and a passage to the limit for a vanishing time increment. Thus, it also suggests an algorithm for a numerical solution. We consider a stable initial condition $q^0_\tau := q^0 \in \mathbb{Q}$.

Let us take $\tau > 0$, a time step, chosen in the way that $N = T/\tau \in \mathbb{N}$. For $1 \leq k \leq N$, $t_k := k\tau$, find $q^k_\tau \in \mathbb{Q}$ such that q^k_τ solves

$$\left.\begin{array}{ll} \text{minimize} & \mathcal{I}(t_k, q) + \mathcal{D}(q^{k-1}_\tau, q) \\ \text{subject to} & q^k_\tau \in \mathbb{Q}\ . \end{array}\right\} \tag{25}$$

Details of the proof can be found e.g. in [18]. □

5. **Conclusions.** The aim of this contribution has been to explain how the rate-independence of mixing a bread (wheat flour) dough to peak dough development can be modelled using the *energetic solution* formalism developed to model the strain hardening of elasto-plastic materials when subjected to repetitive loading. It is based on the mathematical theory of rate-independent processes developed by Mielke and his collaborators [17, 18, 20, 21, 22, 23, 24], which has already been successfully applied to elastoplasticity [8, 15, 16, 19], to mention a few papers in this direction. Dough is modeled as an elatic-nonelastic (e.g. elastoplastic like) material with isotropic hardening which allows for an open loop hysteresis loop observed in experiments. The elastic behavior of dough is described by a polyconvex energy density; e.g. (18) and [9]. Our model is restricted to injective and orientation-preserving deformations. Our approach easily allows for the incorporation of the volumetric constraint $\det F = 1$ (incompressibility of the dough) if required.

Future research will involve checking on the extent to which experimental data validate the appropriateness of this new approach to the modelling of the rate-independence of dough mixing. For instance, one can try to fit constants in the simple examples of energies and dissipations (18), (19), and (24), respectively. The proof of the existence of a solution uses semidiscretization in time which directly suggests a strategy for a possible numerical solution, using finite elements, for instance. The existence of a solution then justifies questions about existence of discrete approximates and their convergence as space-time mesh parameters tend to zero.

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