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**ELECTRONIC COMMUNICATIONS in PROBABILITY**

# On existence of progressively measurable modifications*

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**Abstract**

In this note we provide a short and simple proof that every adapted measurable stochastic process admits a progressively measurable modification.

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Existence of a progressively measurable modification of any adapted measurable process is frequently used in stochastic integration theory, however, the standard proof of this result (see e.g. [1, Théorème IV.30]) is far from being elementary. This proof was simplified and detailed by S. Kaden and J. Potthoff in [3], but their version remains rather lengthy. (In the paper [3], see also useful comments on older proofs.) We aim at showing that progressively measurable modifications may be constructed in a short and very straightforward way by defining them (almost) explicitly for a particular class of simple processes and then using standard approximation procedures, see Theorem 0.1 bellow. Having in mind applications to stochastic PDEs we prove Theorem 0.1 for Polish (i.e., complete separable metric) space-valued processes. If the state space has an additional linear structure we may consider conditional expectations and generalize the main result to them, see Corollary 0.2.

Let $(\Omega,\mathscr{F},\boldsymbol{P})$ be a probability space and $(\mathscr{F}_t)_{t\geq 0}$ a filtration in $\mathscr{F}$, no additional hypotheses on $(\mathscr{F}_t)$ being imposed. We shall denote by $\mathscr{B}(P)$ the Borel $\sigma$-algebra over a metric space $P$, by $L^1=L^1(\Omega,\mathscr{F},\boldsymbol{P})$ the Banach space of all (classes of equivalence of) integrable functions on $(\Omega,\mathscr{F},\boldsymbol{P})$ and by $\mathscr{M}$ the $\sigma$-algebra of all $(\mathscr{F}_t)$-progressively measurable sets, i.e.

$$\mathscr{M}=\bigcap_{T\geq 0}\bigl\{A\in\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F};\ A\cap\bigl([0,T]\times\Omega\bigr)\in\mathscr{B}([0,T])\otimes\mathscr{F}_T\bigr\}.$$

Both finite sets and infinite countable ones will be called countable in the sequel.

Now we may state our results.

**Theorem 0.1.** *Let $(D,\varrho)$ be a Polish space and $\alpha\colon\mathbb{R}_+\times\Omega\longrightarrow D$ an $(\mathscr{F}_t)$-adapted $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}$-measurable stochastic process. Then there exists an $\mathscr{M}$-measurable stochastic process $\beta\colon\mathbb{R}_+\times\Omega\longrightarrow D$ which is a modification of $\alpha$, that is,*

$$\boldsymbol{P}\bigl\{\alpha(t)=\beta(t)\bigr\}=1\quad\textit{for all } t\geq 0.$$

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**Corollary 0.2.** *Let $(X, \|\cdot\|)$ be a separable Banach space and $\alpha\colon \mathbb{R}_+ \times \Omega \longrightarrow X$ a $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}$-measurable stochastic process such that $\boldsymbol{E}\,\|\alpha(t)\| < \infty$ for every $t \geq 0$. Then there exists $\beta\colon \mathbb{R}_+ \times \Omega \longrightarrow X$ $\mathscr{M}$-measurable such that*

$$\boldsymbol{P}\big\{\boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t) \neq \beta(t)\big\} = 0 \quad \textit{for all } t \geq 0.$$

Note that conditional expectations are determined uniquely only as elements of $L^1(\Omega, \mathscr{F}, \boldsymbol{P}; X)$, so the corollary in fact says that there exists a progressively measurable $\beta$ such that $\beta(t)$ is a representative of the equivalence class $\boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t)$ for any $t \geq 0$. With a small abuse of terminology, we shall call $\beta$ a modification of the process $\big(\boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t)\big)$ in this case as well. Since it is not *a priori* obvious that the process $\big(\boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t)\big)$ has a measurable modification, Corollary 0.2 does not follow immediately from Theorem 0.1, but it follows easily from its proof. (Cf. also Remark 0.6 below.)

Corollary 0.2 remains valid in non-separable Banach spaces provided that $\alpha$ is separable-valued; in such a case, $\beta$ may be chosen separable-valued as well.

**Example 0.3.** *Let us show that an adapted measurable process need not be progressively measurable, i.e., passing to a modification cannot be avoided in general in Theorem 0.1. The following counterexample is anything but new (see e.g. [4, Example 1.17]), however, our argument does not use the nontrivial projection theorem.*

*Let $(\Omega, \mathscr{F}, \boldsymbol{P})$ be the unit interval $[0,1]$ equipped with its Borel $\sigma$-algebra and the Lebesgue measure. Let $\mathscr{A}$ be the $\sigma$-algebra generated by all finite subsets of $\Omega$; obviously, $\mathscr{A}$ consists of all subsets of $\Omega$ which are either countable or have a countable complement. Set $\mathscr{F}_t = \mathscr{A}$ for $t \in [0,1[$, $\mathscr{F}_t = \mathscr{F}$ for $t \geq 1$, $\Delta = \{(t,t) \in \mathbb{R}_+ \times \Omega;\ t \in [0, \frac{1}{2}]\}$ and define $\alpha = \mathbf{1}_\Delta$. The process $\alpha$ is plainly $(\mathscr{F}_t)$-adapted, $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}_1$-measurable, and $\beta \equiv 0$ is its progressively measurable modification. Striving after a contradiction, assume that $\Delta = \Delta \cap ([0, \frac{1}{2}] \times \Omega) \in \mathscr{B}([0, \frac{1}{2}]) \otimes \mathscr{F}_{1/2}$. Since $\mathbf{1}_\Delta$ is a pointwise limit of functions of the form $\sum_{j=1}^N c_j \mathbf{1}_{B_j \times C_j}$ with $c_j \in \mathbb{R}$, $B_j \in \mathscr{B}([0, \frac{1}{2}])$ and $C_j \in \mathscr{F}_{1/2}$, there exist sequences $\{A_k,\ k \geq 1\}$ of Borel subsets of $[0, \frac{1}{2}]$ and $\{D_k,\ k \geq 1\}$ of countable sets in $\Omega$ such that $\Delta \in \sigma(A_k,\ k \geq 1) \otimes \mathscr{D}$, where $\mathscr{D} = \sigma(D_k,\ k \geq 1)$. Set $D = \bigcup_{k=1}^\infty D_k$. The section $\{y \in \Omega;\ (t,y) \in \Delta\} = \{t\}$ belongs to $\mathscr{D}$ for any $t \in [0, \frac{1}{2}]$ and $\mathscr{D} \subseteq \mathscr{S} = \{S, \Omega \setminus S;\ S \subseteq D\}$, as $\mathscr{S}$ is a $\sigma$-algebra, so $\{t\} \subseteq D$ for all $t \in [0, \frac{1}{2}]$. Consequently, $[0, \frac{1}{2}] \subseteq D$, but $D$ is countable, this contradiction proves that $\alpha$ is not $(\mathscr{F}_t)$-progressively measurable.*

Before proceeding to the proof of Theorem 0.1 let us recall two well known results.

**Lemma 0.4.** *Let $(D, \varrho)$ be a separable metric space, $\psi\colon \mathbb{R}_+ \times \Omega \longrightarrow D$ a $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}$-measurable mapping and $\varepsilon > 0$. Then there exists $\tilde{\psi}\colon \mathbb{R}_+ \times \Omega \longrightarrow D$ $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}$-measurable with a countable range such that*

$$\sup_{\mathbb{R}_+ \times \Omega} \varrho(\psi, \tilde{\psi}) < \varepsilon.$$

*If $\psi$ is, in addition, $(\mathscr{F}_t)$-adapted then $\tilde{\psi}$ may be chosen $(\mathscr{F}_t)$-adapted as well.*

This is almost obvious; for the reader's convenience, we sketch a proof below.

**Lemma 0.5.** *Let $(D, \varrho)$ be a metric space and $f\colon \mathbb{R}_+ \longrightarrow D$ a regulated function, that is, the limits*

$$\lim_{s \to t+} f(s), \quad \lim_{s \to v-} f(s)$$

*exist for any $t \geq 0$ and $v > 0$. Then the sets*

$$M_+ = \big\{t \in \mathbb{R}_+;\ \lim_{s \to t+} f(s) \neq f(t)\big\}, \quad M_- = \big\{t \in (0, \infty);\ \lim_{s \to t-} f(s) \neq f(t)\big\}$$

*are countable.*

For real-valued functions, the result is well known, but since we do not know any suitable reference in the non-separable case, a proof of Lemma 0.5 is given at the end of the paper.

*Proof of Theorem 0.1.* The proof will be done in four steps.

$1^\circ$ Let $Y \in L^1$. By the martingale convergence theorem,

$$\lim_{s\to v+} \boldsymbol{E}\,(Y|\mathscr{F}_s) = \boldsymbol{E}\,(Y|\mathscr{F}_{v+}) \ \text{ and } \lim_{s\to w-} \boldsymbol{E}\,(Y|\mathscr{F}_s) = \boldsymbol{E}\,(Y|\mathscr{F}_{w-}) \quad \text{in } L^1$$

for all $v \geq 0$ and $w > 0$, and

$$\lim_{k\to\infty} \boldsymbol{E}\,(Y|\mathscr{F}_{s_k}) = \boldsymbol{E}\,(Y|\mathscr{F}_{w-}) \quad \boldsymbol{P}\text{-almost surely} \tag{0.1}$$

for any sequence $s_k \in [0,w)$, $s_k \nearrow w$, where we set as usual

$$\mathscr{F}_{t+} = \bigcap_{r>t} \mathscr{F}_r, \quad \mathscr{F}_{t-} = \bigvee_{r<t} \mathscr{F}_r.$$

We see that the function $\mathbb{R}_+ \longrightarrow L^1$, $t \longmapsto \boldsymbol{E}\,(Y|\mathscr{F}_t)$ is regulated, hence continuous on $\mathbb{R}_+ \setminus C$ for some countable set $C$ by Lemma 0.5. For any $t \geq 0$, fix an $\mathscr{F}_t$-measurable function $H_t\colon \Omega \longrightarrow \mathbb{R}$ such that $H_t = \boldsymbol{E}\,(Y|\mathscr{F}_t)$ $\boldsymbol{P}$-almost surely and define a sequence of progressively measurable processes

$$Y_n(t) = \mathbf{1}_{\{0\}}(t)H_0 + \sum_{k=0}^{\infty} \mathbf{1}_{(k2^{-n},(k+1)2^{-n}]\setminus C}(t) H_{k2^{-n}} + \sum_{s\in C\setminus\{0\}} \mathbf{1}_{\{s\}}(t) H_s,$$

$n \in \mathbb{N}$. Plainly $Y_n(t) = \boldsymbol{E}\,(Y|\mathscr{F}_t)$ $\boldsymbol{P}$-almost surely for $t \in \{0\} \cup C$ and all $n \geq 1$, and

$$\lim_{n\to\infty} Y_n(t) = \lim_{n\to\infty} \boldsymbol{E}\,(Y|\mathscr{F}_{(\lceil 2^n t\rceil -1)2^{-n}}) = \boldsymbol{E}\,(Y|\mathscr{F}_{t-}) = \boldsymbol{E}\,(Y|\mathscr{F}_t) \quad \boldsymbol{P}\text{-almost surely}$$

for $t \in (0,\infty) \setminus C$ by (0.1) and the definition of $C$, where by $\lceil r\rceil$ the upper integer part of $r \in \mathbb{R}$ is denoted. Set

$$\Gamma = \bigl\{(t,\omega) \in \mathbb{R}_+ \times \Omega;\ \exists \lim_{n\to\infty} Y_n(t,\omega)\bigr\}.$$

Then, due to completeness of $\mathbb{R}$ and progressive measurability of the processes $Y_n$, one gets

$$\Gamma = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k,r=m}^{\infty} \Bigl\{(t,\omega) \in \mathbb{R}_+ \times \Omega;\ |Y_k(t,\omega) - Y_r(t,\omega)| < \frac{1}{n}\Bigr\} \in \mathscr{M},$$

so the process

$$\Upsilon = \begin{cases} \lim_{n\to\infty} Y_n & \text{on } \Gamma, \\ 0 & \text{elsewhere} \end{cases}$$

is $\mathscr{M}$-measurable and clearly satisfies $\Upsilon_t = \boldsymbol{E}\,(Y|\mathscr{F}_t)$ $\boldsymbol{P}$-almost surely for every $t \geq 0$.

$2^\circ$ Let $U \subseteq \mathbb{R}_+ \times \Omega$ be a measurable rectangle, $U = I \times H$ for some interval $I \subseteq \mathbb{R}_+$ and $H \in \mathscr{F}$. Since

$$\boldsymbol{E}\,(\mathbf{1}_U(t)|\mathscr{F}_t) = \begin{cases} \boldsymbol{E}\,(\mathbf{1}_H|\mathscr{F}_t) & \text{for } t \in I, \\ 0 & \text{otherwise,} \end{cases}$$

we can check easily applying Step $1^\circ$ that the process $\bigl(\boldsymbol{E}\,(\mathbf{1}_U(t)|\mathscr{F}_t)\bigr)$ has an $\mathscr{M}$-measurable modification. Dynkin's $\pi/\lambda$ argument now implies that the system

$$\Lambda = \bigl\{B \in \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F};\ \bigl(\boldsymbol{E}\,(\mathbf{1}_B(t)|\mathscr{F}_t)\bigr) \text{ has an } \mathscr{M}\text{-measurable modification}\bigr\}$$

coincides with $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}$. The only point which may require a proof is closedness of $\Lambda$ under countable monotone unions. So take $A_k\in\Lambda$, $A_k\uparrow A$, let $\beta_n$ be an $\mathscr{M}$-measurable modification of $\big(\boldsymbol{E}\left(\mathbf{1}_{A_k}(t)|\mathscr{F}_t\right)\big)$. The sequence $\{\beta_k(t)\}$ is $\boldsymbol{P}$-almost surely nondecreasing for any $t\geq 0$, thus defining $\beta$ by $\lim_{k\to\infty}\beta_k$ whenever the limit exists in $\mathbb{R}$ and by $0$ otherwise we get the desired modification of $\big(\boldsymbol{E}\left(\mathbf{1}_A(t)|\mathscr{F}_t\right)\big)$.

3° Suppose that $\alpha$ satisfies hypotheses of Theorem 0.1 and moreover has a countable range. So there exist $N\in\mathbb{N}\cup\{\infty\}$, $x_j\in D$ for $j<N$, $x_i\neq x_j$ for $i\neq j$, and a $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}$-measurable partition $\{B_j,\,j<N\}$ of $\mathbb{R}\times\Omega$ into disjoint sets such that $\alpha=x_j$ on $B_j$. The process $\mathbf{1}_{B_j}$ is $(\mathscr{F}_t)$-adapted, since $\mathbf{1}_{B_j}(t)=\mathbf{1}_{\{x_j\}}(\alpha(t))$ and $\alpha$ is $(\mathscr{F}_t)$-adapted. From Step 2° we know that there exist $\mathscr{M}$-measurable processes $\xi_j$, $j<N$, satisfying $\mathbf{1}_{B_j}(t)=\xi_j(t)$ $\boldsymbol{P}$-almost surely for all $t\geq 0$. Set

$$C_j=\{\xi_j=1\},\quad \Gamma_j=C_j\setminus\bigcup_{i<j}C_i,\quad j<N,$$

choose an arbitrary $\tilde{x}\in D$ and define

$$\beta(t,\omega)=\begin{cases} x_j, & (t,\omega)\in\Gamma_j,\ j<N,\\ \tilde{x}, & (t,\omega)\notin\bigcup_{k<N}\Gamma_k.\end{cases}$$

The process $\beta$ is obviously $\mathscr{M}$-measurable and it is a modification of $\alpha$. Indeed, $\xi_j$ is a modification of $\mathbf{1}_{B_j}$, so $\mathbf{1}_{C_j}(t)=\mathbf{1}_{B_j}(t)$ $\boldsymbol{P}$-almost surely and disjointness of $B_j$'s yields $\mathbf{1}_{\Gamma_j}(t)=\mathbf{1}_{B_j}(t)$ $\boldsymbol{P}$-almost surely.

4° Let an arbitrary $\alpha$ satisfying the hypotheses of Theorem 0.1 be given. Using Lemma 0.4 we may find $(\mathscr{F}_t)$-adapted $\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{F}$-measurable processes $\alpha_n$ with countable ranges so that

$$\sup_{\mathbb{R}_+\times\Omega}\varrho(\alpha,\alpha_n)<\frac{1}{2^n},\quad n\geq 1.$$

Let $\beta_n$ be $\mathscr{M}$-measurable modifications of $\alpha_n$, $n\geq 1$, constructed in Step 3°. Since $\varrho(\alpha(t),\beta_n(t))<2^{-n}$ $\boldsymbol{P}$-almost surely holds for every $t\geq 0$, an $\mathscr{M}$-measurable modification of $\alpha$ may be defined by

$$\beta(t,\omega)=\begin{cases} \lim_{n\to\infty}\beta_n(t,\omega) & \text{if the limit exists,}\\ \hat{x} & \text{otherwise,}\end{cases}$$

where $\hat{x}\in D$ is an arbitrary (but fixed) point. Indeed, owing to completeness of $D$ it may be checked that $\{(t,\omega)\in\mathbb{R}_+\times\Omega;\ \exists\lim_{n\to\infty}\beta_n(t,\omega)\}\in\mathscr{M}$ as in Step 1°. □

**Remark 0.6.** *A. Irle [2] proved that the process $\big(\boldsymbol{E}\left(V_t|\mathscr{F}_t\right)\big)$ has a measurable modification whenever $V\geq 0$ is a measurable real-valued process using an idea loosely related to Step 1° of the above proof.*

**Remark 0.7.** *The proof of Theorem 0.1 simplifies further if additional continuity hypotheses are imposed on $\alpha$, for example, it is easy to show that an adapted measurable process continuous in probability has a progressively measurable (even predictable) modification (see e.g. [5, Proposition 3.21]).*

**Remark 0.8.** *Recall that a standard Borel space is a measurable space $(S,\Sigma)$ isomorphic to a measurable space of the form $(B,\mathscr{B}(B))$, where $B$ is a Borel subset of some Polish space. Let $\alpha$ be an adapted measurable process with values in an uncountable standard Borel space $(S,\Sigma)$. By the Borel isomorphism theorem (see e.g. [6, Theorem 3.3.13]) there exists a bijection $\iota\colon(S,\Sigma)\longrightarrow(\mathbb{R},\mathscr{B}(\mathbb{R}))$ such that both $\iota$ and $\iota^{-1}$ are measurable. The real-valued process $\iota\circ\alpha$ has an $\mathscr{M}$-measurable modification $\gamma$, the* *process $\beta = \iota^{-1} \circ \gamma$ is then an $\mathscr{M}$-measurable modification of $\alpha$. (For countable standard spaces, the argument may be modified in a straightforward way.)*

*Any Polish space endowed with its Borel $\sigma$-algebra is a standard Borel space, hence it would suffice to prove Theorem 0.1 for real-valued processes only and then use a Borel isomorphism, however, the proof does not simplify by choosing $D = \mathbb{R}$. Moreover, we prefer an elementary proof avoiding descriptive set theory.*

*Proof of Corollary 0.2.* The proof is analogous to that of Theorem 0.1. First, let us assume that $\alpha$ satisfies the assumptions of Corollary 0.2 and moreover has a countable range. Therefore,

$$\alpha = \sum_{j=1}^{\infty} x_j \mathbf{1}_{B_j}$$

for some $x_j \in X$ and a partition $\{B_j\}_{j=1}^{\infty}$ of $\mathbb{R}_+ \times \Omega$ into $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}$-measurable sets. By Step 2° in the proof of Theorem 0.1 we may find $\mathscr{M}$-measurable modifications $\zeta_j$, $j \geq 1$, of the processes $\big(\boldsymbol{E}\,(\mathbf{1}_{B_j}(t)|\mathscr{F}_t)\big)$. For any $t \geq 0$ fixed, $\zeta_j(t) \geq 0$ $\boldsymbol{P}$-almost surely and thus the integrability assumption on $\alpha$ implies

$$\boldsymbol{E} \sum_{j=1}^{\infty} \|x_j\|\, |\zeta_j(t)| = \sum_{j=1}^{\infty} \|x_j\| \boldsymbol{E}\, \mathbf{1}_{B_j}(t) = \boldsymbol{E}\, \|\alpha(t)\| < \infty,$$

whence we see that the series $\sum_{j=1}^{\infty} x_j \zeta_j(t)$ converges in $L^1$ and $\boldsymbol{P}$-almost surely; defining

$$\beta(t,\omega) = \begin{cases} \sum_{j=1}^{\infty} x_j \zeta_j(t,\omega) & \text{if the sum converges,} \\ 0 \in X & \text{otherwise,} \end{cases}$$

we obtain an $\mathscr{M}$-measurable process. Since $\boldsymbol{E}\,(\cdot|\mathscr{F}_t)$ is a continuous operator in $L^1$ for any $t \geq 0$, we get $\beta(t) = \boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t)$ $\boldsymbol{P}$-almost surely, that is, $\beta$ is an $\mathscr{M}$-measurable modification of $\big(\boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t)\big)$.

Finally, let $\alpha$ be an arbitrary process satisfying the hypotheses of Corollary 0.2. By Lemma 0.4 there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}$-measurable functions with a countable range such that

$$\sup_{\mathbb{R}_+ \times \Omega} \|\alpha - \alpha_n\| < \frac{1}{2^n}, \quad n \geq 1.$$

We have just proved that the processes $\big(\boldsymbol{E}\,(\alpha_n(t)|\mathscr{F}_t)\big)$ have $\mathscr{M}$-measurable modifications $\beta_n$, $n \geq 1$. Define

$$\beta(t,\omega) = \begin{cases} \lim_{n\to\infty} \beta_n(t,\omega) & \text{if the limit exists,} \\ 0 \in X & \text{otherwise.} \end{cases}$$

Then $\beta$ is an $\mathscr{M}$-measurable process and the estimate

$$\begin{aligned} \boldsymbol{E}\,\big\|\boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t) - \beta_n(t)\big\| &= \boldsymbol{E}\,\big\|\boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t) - \boldsymbol{E}\,(\alpha_n(t)|\mathscr{F}_t)\big\| \\ &\leq \boldsymbol{E}\,\|\alpha(t) - \alpha_n(t)\| < \frac{1}{2^n} \end{aligned}$$

implies that $\lim_{n\to\infty} \beta_n(t)$ exists and equals to $\boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t)$ $\boldsymbol{P}$-almost surely for any $t \geq 0$, and so $\beta(t) = \boldsymbol{E}\,(\alpha(t)|\mathscr{F}_t)$ $\boldsymbol{P}$-almost surely for all $t \geq 0$. □

*Proof of Lemma 0.4.* By $B(y,r)$ we shall denote an open ball in $D$ centered at $y$ with radius $r$. Let $\{y_j;\ j < N\}$ for some $N \in \mathbb{N} \cup \{\infty\}$ be a countable dense subset of $D$, then a mapping $\tilde{\psi}$ defined by

$$\tilde{\psi} = y_j \quad \text{on} \quad \Big\{(t,\omega);\ \psi(t,\omega) \in B(y_j,\varepsilon) \setminus \bigcup_{i<j} B(y_i,\varepsilon)\Big\}, \quad j < N,$$

has the desired properties. □

*Proof of Lemma 0.5.* We may assume that $D$ is separable, since the set $S_T = \{f(t);\ t \in [0,T]\}$ is totally bounded for any $T > 0$. Indeed, let $T > 0$ and $\varepsilon > 0$ be given. Define $f(r) = f(0)$ for $r < 0$, then $f$ is regulated on $\mathbb{R}$ and for any $t \geq 0$ there exists $\xi(t) > 0$ such that $d(f(x), f(x')) < \varepsilon$ whenever $x, x' \in (t - \xi(t), t)$ or $x, x' \in (t, t + \xi(t))$. By compactness of $[0,T]$, $t_1, \dots, t_k \in [0,T]$ may be found so that $[0,T] \subseteq \bigcup_{i=1}^k (t_i - \xi(t_i), t_i + \xi(t_i))$. Plainly,

$$S_T \subseteq \bigcup_{i=1}^{k} \Big( B(f(t_i - \frac{1}{2}\xi(t_i)), \varepsilon) \cup B(f(t_i), \varepsilon) \cup B(f(t_i + \frac{1}{2}\xi(t_i)), \varepsilon) \Big).$$

We shall prove that $M_-$ is countable, the proof for $M_+$ being almost the same. Let $\mathbf{B}$ be a countable base for the topology of $D$, then for any $t \in M_-$ there exists $U \in \mathbf{B}$ such that $f(t) \in U$ and $\lim_{s \to t-} f(s) \notin \bar{U}$. It suffices to show that the set

$$M_U = \{t \in M_-;\ f(t) \in U,\ \lim_{s \to t-} f(s) \notin \bar{U}\}$$

is countable for any $U \in \mathbf{B}$ fixed, as $M_- = \bigcup_{U \in \mathbf{B}} M_U$. For any $t \in M_U$ one may find $\delta(t) > 0$ such that $f(s) \notin U$ for all $s \in (t - \delta(t), t)$ and it may be checked easily that $(t_1 - \delta(t_1), t_1) \cap (t_2 - \delta(t_2), t_2) = \emptyset$, whenever $t_1, t_2 \in M_U$, $t_1 \neq t_2$. Therefore, we get a bijection between $M_U$ and a disjoint system $\{(t - \delta(t), t),\ t \in M_U\}$ of nonempty open intervals in $\mathbb{R}$, which is necessarily countable. ☐

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