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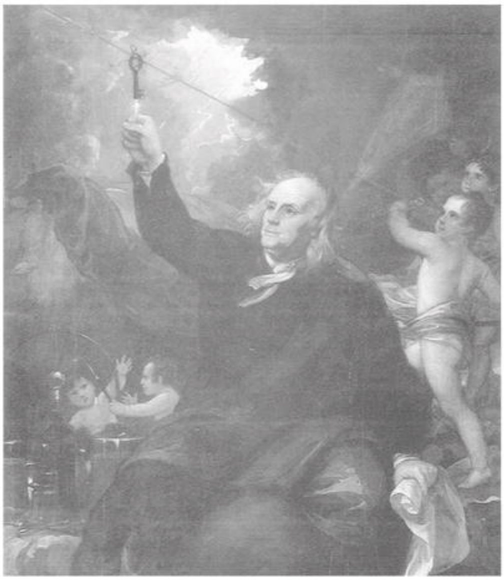

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On stabilisability of 2-D MIMO shift-invariant systems ☆

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Abstract

We concentrate on the linear spatially distributed time-invariant two-dimensional systems with multiple inputs and multiple outputs and with control action based on an array of sensors and actuators connected to the system. The system is described by the bivariate matrix polynomial fraction. Stabilisation of such systems is based on the relationship between stability of a bivariate polynomial and positiveness of a related polynomial matrix on the unit circle. Such matrices are not linear in the controller parameters, however, in simple cases, a linearising factorisation exists. It allows to describe the control design in the form of a linear matrix inequality. In more complicated cases, linear sufficient conditions are given. This concept is applied to a system with multiple outputs—a heat conduction in a long thin metal rod equipped with an array of temperature sensors and heaters, where heaters are placed in larger distances than sensors.

1. Introduction

Control of *spatially distributed systems* has always been a very active research topic with engineering applications in many areas. Such systems can be mathematically described by *partial differential equations* (PDEs). Control action can be based on a dense and regular array of sensors and actuators. Each sensor and actuator are connected to the system and each sensor and actuator or cells made from them are connected to a controller. There is an array of controllers interconnected by a *complex network*. Each controller collects data from that sensors and computes

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the control action for that actuators which are connected to it. This configuration is called *distributed control*.

Applications of distributed control can be found in various fields of the industry, see, e.g., [4,23,14,6,13,37]. Concerning two-dimensional (2-D) systems, one of the examples is a multi-zone crystal growth furnace, described in [1,7]. The furnace is depicted in Fig. 1(a). Block diagram of the closed-loop system is in Fig. 1(b). The problem consists in control of N zone temperatures by N energy inputs. The aim is to produce the temperature profile within the furnace, prescribed by N command signals $r_1, r_2, \ldots, r_N$.

In the process control, the approach to stabilisation presented in this paper can potentially be used if the process is described by a linear PDE and it is possible to obtain measurements by point sensors and to base the control action on point actuators. For instance, this is the case of the control of spatially distributed profiles. A particular example is the control of the temperature profile in the furnace or across the wafer in the rapid thermal chemical vapour deposition process. A heat conduction, diffusion, chemical reactions and other irreversible processes can be considered.

All systems and processes mentioned in the two last paragraphs are described by a parabolic PDE, which in the case of one temporal and one spatial variables has the form

$$d\frac{\partial v(t,x)}{\partial t} - c\frac{\partial^2 v(t,x)}{\partial x^2} - b\frac{\partial v(t,x)}{\partial x} - av(t,x) = f(t,x), \tag{1}$$

where a, b, c, d are the positive constants, v is a solution and f is the right-hand side. A particular example of Eq. (1) is a heat equation:

$$\frac{\partial v(t,x)}{\partial t} - \kappa\frac{\partial^2 v(t,x)}{\partial x^2} = f(t,x), \tag{2}$$

where v denotes temperature (°C), f the input heat (°C s^{-1}), t and x denote time (s) and a spatial coordinate (m), respectively, and κ is a constant (m^2 s^{-1}).

Assuming the parameters of the system do not depend on location, the *shift invariant* mathematical model can be derived. Then we talk about so-called *spatially invariant systems*. In such case, all controllers are identical, so, one can perform a design method once and obtain all the array of designed controllers. Shift invariance assumes the domain is infinite, i.e., the boundaries are at infinite distance, what is not realistic. However, this assumption is a reasonable simplification for the design. Simulations and verification of controllers should follow their design procedure.

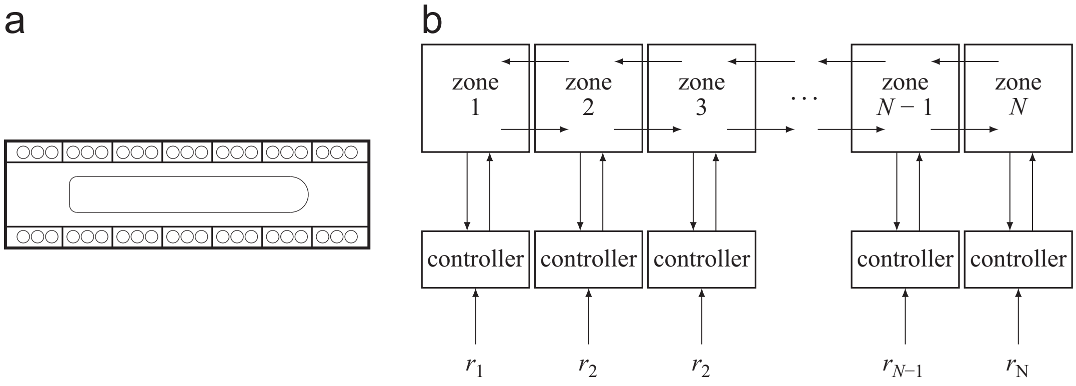

Fig. 1. (a) A sketch of a multizone crystal growth furnace. (b) Block diagram of the closed-loop system.

With the above assumption, Eq. (1) can be transformed to the description more suitable for the control design—the transfer function, whose coefficients are elements of a ring. It turned out in 1960s and 1970s that this type of systems can be studied within a class of systems whose coefficients are functions of parameters. The so-called *concept of linear systems over rings* can be used. Among the pioneer papers in this area were [34,19,36,20,22,5].

For stability analysis and stabilisation of shift-invariant systems with multiple inputs and multiple outputs (MIMO systems), the theory elaborated by Lin [25–29] and others [41,42,30] can also be adopted. A concept of so-called *generating polynomials* is established there and problems of stability analysis and stabilisation of multidimensional (n-D) systems are solved.

In the paper we shall concentrate on the linear spatially invariant time-invariant 2-D systems described by constant coefficient parabolic PDEs. They are modelled by a bivariate spatio-temporal transfer function in the form of the matrix polynomial fraction. Their stabilisation is based on the relationship between stability of a bivariate (matrix) polynomial and positiveness of a related polynomial matrix (e.g., Schur–Cohn matrix) on the unit circle. However, these conditions are not suitable for control system synthesis, since these matrices are usually bilinear in coefficients of original matrix polynomials (as well as in parameters of a controller). In the paper, this is fixed by linearising factorisation or deriving a new, linear, but no longer necessary condition. Conditions given in the paper lead to use the theory of and computation with *positive polynomials*, well described in [8,16].

The developed methods are demonstrated by means of a numerical example. A heat conduction in a long thin metal rod equipped with an array of temperature sensors and heaters is considered. It is supposed that heaters are placed in larger distances than sensors, in other words, there are some nodes without control action. The system is described by matrix polynomial fraction and a distributed stabilising controller is designed. Numerical simulations are given.

Let $\mathbb{Z}$ denote the set of integers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}(z,w)$ the set of rational functions in z,w over $\mathbb{R}$, $\mathbb{R}[z,w]$ the set of bivariate polynomials in z,w over $\mathbb{R}$, $\mathbb{R}[w]$ the ring of polynomials in w over $\mathbb{R}$, $\mathbb{R}^{m\times l}(z,w)$ the set of $m\times l$ matrices with entries in $\mathbb{R}(z,w)$, $\mathbb{R}^{m\times l}[z,w]$ the set of $m\times l$ matrices with entries in $\mathbb{R}[z,w]$ and $\mathbb{R}^{m\times l}[w]$ the set of $m\times l$ matrices with entries in $\mathbb{R}[w]$. Let a^* denote the complex conjugate of a and A^* denote the complex conjugate transpose of A.

2. Description of a system

A system description suitable for the control design will be proposed. As it was said, the aim is to base the control on action performed by an array of actuators and sensors. The presence of such an array naturally implies the discretisation of the original PDE in the spatial variables. Due to the digital implementation of control, the discretisation with respect to time is also performed. An explicit difference scheme [38,32] is considered for the discretisation of PDE. We obtain a discrete shift-invariant 2-D scalar input–output system with both the temporal and the spatial variables. It can be generally described by a convolution equation [15,21]

$$y[k_1,k_2]=\sum_{i_1=-\infty}^{k_1}\sum_{i_2=-\infty}^{\infty}h[k_1-i_1,k_2-i_2]u[i_1,i_2], \tag{3}$$

where k_1 is the discrete time variable, k_2 is the discrete spatial variable, u and y are doubly indexed input and output sequences with values in $\mathbb{R}$, respectively, and h is the impulse response of the system—a real-valued function defined on the Cartesian product $\mathbb{Z}\times\mathbb{Z}$, $h[k_1,k_2]=0$ for all $k_1<0$.

Since we assumed the infinite spatial domain, to obtain a transfer function we can perform a sequence of two z-transforms: one unilateral, corresponding to the temporal variable and the other bilateral, corresponding to the spatial variable. The same procedure was also done in [3]. The z-transform of each side of Eq. (3) gives $Y(z,w)=H(z,w)U(z,w)$, where $H(z,w)$ is the system transfer function defined as [15,21]

$$H(z,w)=\sum_{k_1=0}^{\infty}\sum_{k_2=-\infty}^{\infty}h[k_1,k_2]w^{-k_2}z^{-k_1}.$$

From a practical point of view, it is desirable to approximate $H(z,w)$ by a rational function, i.e., by the fraction of two polynomials in the variables z and w, w^{-1}. Let

$$H(z,w)=\frac{b(z,w)}{a(z,w)},\quad a,b\in\mathbb{R}[z,w]. \tag{4}$$

A MIMO system is described by a matrix, $P\in\mathbb{R}^{m\times l}(z,w)$,

$$P(z,w)=\begin{bmatrix} H_{1,1}(z,w) & \cdots & H_{1,l}(z,w)\\ \vdots & \cdots & \vdots\\ H_{m,1}(z,w) & \cdots & H_{m,l}(z,w)\end{bmatrix}, \tag{5}$$

whose every entry is the rational function of the form (4).

The bivariate rational matrix (5) can be written in the form of the (not necessarily minor coprime) left and right matrix fraction description (MFD), see, e.g., [25,42]. The left MFD is defined as

$$P(z,w)=A_{\mathrm{L}}^{-1}(z,w)B_{\mathrm{L}}(z,w), \tag{6}$$

where $A_{\mathrm{L}}\in\mathbb{R}^{m\times m}[z,w]$ and $B_{\mathrm{L}}\in\mathbb{R}^{m\times l}[z,w]$. Similarly, the right MFD is defined as

$$P(z,w)=B_{\mathrm{R}}(z,w)A_{\mathrm{R}}^{-1}(z,w). \tag{7}$$

Following the concept of systems over rings [36,20], the denominator matrix polynomial can be written in the form

$$A_{\mathrm{L}}[w](z)=A_n(w)z^n+A_{n-1}(w)z^{n-1}+\cdots+A_0(w), \tag{8}$$

where

$$A_k=\sum_{i=0}^{q}A_{k,i}w^i+A_{k,-i}w^{-i},\quad k=0,1,\ldots,n, \tag{9}$$

with $A_{k,i},A_{k,-i}\in\mathbb{R}^{m\times m}$. Furthermore, since we considered an explicit difference scheme for the discretisation, the matrix $A_{n,0}$ is regular and all entries of matrices $A_{n,i}$, $i=1,\ldots,q$, are equal to zero. Without loss of generality we consider $A_{n,0}=I_m$ and (8) in the form

$$A_{\mathrm{L}}[w](z)=I_mz^n+A_{n-1}(w)z^{n-1}+\cdots+A_0(w), \tag{10}$$

where I_m denotes the identity matrix of the dimension m. Similarly for A_{R}.

3. Stability

This section discusses the stability of systems described in the previous section. The classical definition of the bounded-input bounded-output (BIBO) stability follows, see, e.g., [15].

Definition 1 (*BIBO stability*). A system (3) is said to be BIBO stable if for all $r_1 > 0$ there exists $r_2 > 0$ such that any input sequence u satisfying $|u[k_1, k_2]| < r_1$ for all $[k_1, k_2]$, the corresponding output satisfies $|y[k_1, k_2]| < r_2$ for all $[k_1, k_2]$. A MIMO system is said to be BIBO stable if its every sub-component is BIBO stable.

In the n-D systems theory, it is convenient to define so-called *structural stability*, see, e.g, [15,25,42].

Definition 2 (*Structural stability*). A system described by Eq. (4), where a and b are relatively prime polynomials, is said to be structurally stable if $a(z, w) \neq 0$ for all $\{|z| \geq 1\} \cap \{|w| = 1\}$. A MIMO system is said to be structurally stable if every entry of Eq. (5) is structurally stable.

It is well known that the numerator of the transfer function of an n-D system can cause the BIBO stability in the case the relatively prime numerator and denominator share a common zero on the stability region boundary, i.e., the transfer function contains a non-essential singularity of the second kind, see [12,17] for the details. It follows from the above definitions and discussions in [18,12,17,15], that if a system is structurally stable, it is also BIBO stable.

Stability analysis can be based on so-called *generating polynomials* [25]. Let $F = [A_L \ B_L]$, $\eta = \binom{m+l}{l}$ and $\alpha_1, \alpha_2, \ldots, \alpha_\eta$ denote the η maximal order minors of F, with $\alpha_1 = \det(A_L)$. Similarly, the maximal order minors of a matrix $\begin{bmatrix} A_R \\ B_R \end{bmatrix}$ can be defined for the right MFD. The generating polynomials are defined as follows [25,42].

Definition 3 (*Generating polynomials*). Extracting the greatest common divisor $g(z)$ of $\alpha_1, \alpha_2, \ldots, \alpha_\eta$ gives $\alpha_i(z, w) = g(z, w)\beta_i(z, w)$, $i = 1, \ldots, \eta$. Then $\beta_1(z, w), \ldots, \beta_\eta(z, w)$ are called generating polynomials of $F(z, w)$, and also $P(z, w)$.

Note that an MFD is minor coprime if α_i's have no common factor, see [25] for the details. The following lemma expresses the stability in the manner of generating polynomials.

Lemma 1 (Lin *[25]*). *A system described by Eq.* (6) *or* (7) *is structurally stable if and only if* $\beta_1(z, w) \neq 0$ *for all* $\{|z| \geq 1\} \cap \{|w| = 1\}$.

The condition of Lemma 1 can be checked using a relationship between the stability of a 2-D polynomial and positiveness of Schur–Cohn matrix on the unit circle, see [35,3]. Due to the assumption (10), the polynomial $\beta_1(z, w)$ has the form

$$\beta_1(z, w) = \beta_{1n}(w)z^n + \beta_{1n-1}(w)z^{n-1} + \cdots + \beta_{10}(w), \tag{11}$$

where $\beta_{1k} = \sum_{i=0}^{q} \beta_{1k,i} w^i + \beta_{1k,-i} w^{-i}$, $k = 0, 1, \ldots, n$. The Schur–Cohn matrix corresponding to (11) reads

$$S_{\beta_1}(w) = S_1^* S_1 - S_2 S_2^*, \tag{12}$$

where

$$S_1 = \begin{pmatrix} \beta_{1n}(w) & \beta_{1n-1}(w) & \cdots & \beta_{11}(w) \\ 0 & \beta_{1n}(w) & \ddots & \beta_{12}(w) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \beta_{1n}(w) \end{pmatrix},$$

$$S_2 = \begin{pmatrix} \beta_{10}(w) & 0 & \cdots & 0 \\ \beta_{11}(w) & \beta_{10}(w) & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \beta_{1n-1}(w) & \beta_{1n-2} & \cdots & \beta_{10}(w) \end{pmatrix}.$$

See, e.g., [9,43] for the details. We now propose the following theorem.

Theorem 1. *A system described by Eq. (6) or (7) is structurally stable if and only if the Schur–Cohn matrix corresponding to $\beta_1(z,w)$ is positive definite on the unit circle, i.e., $S_{\beta_1}(w) \succ 0 \ \forall |w| = 1$.*

Proof. It follows from Lemma 1 and [35]. □

Test of positiveness of $S_{\beta_1}(w)$ on the unit circle can be performed using the advanced toolset of linear matrix inequalities (LMI), see, e.g., [40,10,11,8]. The matrix $S_{\beta_1}(w)$ is a pseudo-polynomial matrix with symmetric expansion in the form

$$S_{\beta_1}(w) = \sum_{i=-2q}^{2q} S_i w^i, \quad S_{-i} = S_i^*.$$

Hence, the result stated in [40,10] can be used: The matrix $S_{\beta_1}(w)$ is positive definite for all $|w| = 1$ if and only if there exists a symmetric matrix M of appropriate dimension such that

$$L(M) = \left(\begin{array}{c|ccc} S_0 & S_1 & \cdots & S_{2q} \\ \hline S_1^* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S_{2q}^* & 0 & \cdots & 0 \end{array}\right) + \left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I \\ 0 & \cdots & 0 \end{array}\right) M \left(\begin{array}{c|ccc} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right)$$
$$- \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \hline I & & \\ & \ddots & \\ & & I \end{array}\right) M \left(\begin{array}{c|ccc} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{array}\right) \succ 0. \tag{13}$$

An implementation of this test is easy with the available numerical solvers for LMIs, for instance, [39,31].

A disadvantage of the method based on Theorem 1 is that it requires computing the determinant of a matrix polynomial, which can be lengthy and numerically ill behaved. In what follows we show another method which is based on manipulating with coefficients of Eqs. (6) and (7) and does not require computing the determinant. The rest of this section holds for both left and right MFDs, therefore, the subscripts L and R are omitted.

Let $S_A(w) = (s_{i,j})$ be a matrix of the dimension $nm \times nm$ defined as

$$s_{i,j} = \sum_{k=1}^{i} A^*_{n+k-i} A_{n+k-j} - A^*_{j-k} A_{i-k} \tag{14}$$

for $i, j = 1, 2, \ldots, n$, where $A_{k,i}$'s are given by Eq. (9). We can formulate the following lemma based on [2, Theorem 1].

Lemma 2. *If* $S_A(w) \succ 0$ *for all* $|w| = 1$ *then* $\det(A) \neq 0$ *for all* $\{|w| = 1\} \cup \{|z| \geq 1\}$.

Proof. This lemma was proven for real matrix polynomials in [2]. Here, the situation is similar. Let

$$\sum_{i,j=1}^{n} z^{*j-1} s_{i,j} z^{i-1} = (|z|^2 - 1)^{-1} (A^*[w](z) A[w](z) - A_{\mathrm{I}}^*[w](z) A_{\mathrm{I}}[w](z)),$$

where $s_{i,j}$'s are given by Eq. (14) and $A_{\mathrm{I}}[w](z) = A_0(w) z^n + A_1 z^{n-1} + \cdots + I_m$. Let $\hat{z}$ be a root of $\det(A[w](z))$, i.e., $\det(A[w](\hat{z})) = 0$, and let a non-zero $m \times 1$ complex vector $x[w](\hat{z})$ be such that $A[w](\hat{z}) x[w](\hat{z}) = 0$. By hypothesis,

$$-x^*[w](\hat{z}) (|\hat{z}|^2 - 1)^{-1} (A_{\mathrm{I}}^*[w](\hat{z}) A_{\mathrm{I}}[w](\hat{z})) x[w](\hat{z}) > 0.$$

Hence, $(|\hat{z}|^2 - 1) < 0$. This implies that all the roots of $\det(A[w](z))$ lie in $\{|z| < 1\}$. □

Now, we can complete the following theorem.

Theorem 2. *A system described by (not necessarily coprime) Eq. (6) or (7) is (structurally) stable if* $S_A(w) \succ 0$ *for all* $|w| = 1$.

Proof. It follows from Definition 2 and Lemma 2. □

Like in Theorem 1, the positiveness of $S_A(w)$ on the unit circle can be checked using (13). In contrast to Theorem 1, the above condition is sufficient, not necessary, however, does not require to compute the determinant. A simple example follows.

Example 1. Consider a 2-D matrix polynomial

$$A = \begin{pmatrix} z + 0.1(w + w^{-1}) + 0.3 & 0.2(w + w^{-1}) \\ 0.2(w + w^{-1}) & z + 0.3(w + w^{-1}) \end{pmatrix}. \tag{15}$$

The matrix $S(w) = s_{i,j}$ is

$$s_{1,1} = -0.05(w^2 + w^{-2}) - 0.06(w + w^{-1}) + 0.81$$

$$s_{1,2} = s_{2,1} = -0.08(w^2 + w^{-2}) - 0.06(w + w^{-1}) - 0.16$$
$$s_{2,2} = -0.13(w^2 + w^{-2}) + 0.74$$

and Eq. (13) has the form

$$
\begin{aligned}
L(M) = & \begin{pmatrix} 0.81 & -0.16 & -0.06 & -0.06 & -0.05 & -0.08 \\ -0.16 & 0.74 & -0.06 & 0 & -0.08 & -0.13 \\ -0.06 & -0.06 & 0 & 0 & 0 & 0 \\ -0.06 & 0 & 0 & 0 & 0 & 0 \\ -0.05 & -0.08 & 0 & 0 & 0 & 0 \\ -0.08 & -0.13 & 0 & 0 & 0 & 0 \end{pmatrix} \\
& + \begin{pmatrix} I & & \\ \hline & \ddots & \\ & & I \\ 0 & \cdots & 0 \end{pmatrix} M \left(\begin{array}{c|ccc} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right) - \begin{pmatrix} 0 & \cdots & 0 \\ \hline I & & \\ & \ddots & \\ & & I \end{pmatrix} M \left(\begin{array}{c|ccc} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{array}\right) \succ 0.
\end{aligned}
$$

One can make sure that there exists a matrix M such that the above LMI holds. Hence, $\det(A) \neq 0$ for all $\{|w| = 1\} \cup \{|z| \geq 1\}$ and the system with the denominator (15) is structurally stable.

The sufficient and necessary condition which does not require to compute the determinant is more complicated. It could be formulated as an extension of [9, Theorem 1.1] to 2-D matrix polynomials. However, constructing of a reflection of a matrix polynomial, which is required there, is not simple in 1-D case more to 2-D case.

4. Stabilisation

The classical control scheme of Fig. 2 is considered. A system $P = A_{\mathrm{L}}^{-1} B_{\mathrm{L}} \in \mathbb{R}^{m \times l}(z, w)$ is feedback stabilisable if and only if there exists a controller $R = Y_{\mathrm{R}} X_{\mathrm{R}}^{-1} \in \mathbb{R}^{l \times m}(z, w)$ such that

$$A_{\mathrm{L}} X_{\mathrm{R}} + B_{\mathrm{L}} Y_{\mathrm{R}} = C_{\mathrm{L}} \tag{16}$$

is a stable matrix polynomial. A system $P = B_{\mathrm{R}} A_{\mathrm{R}}^{-1} \in \mathbb{R}^{m \times l}(z, w)$ is feedback stabilisable if and only if there exists a controller $R = X_{\mathrm{L}}^{-1} Y_{\mathrm{L}} \in \mathbb{R}^{l \times m}(z, w)$ such that

$$X_{\mathrm{L}} A_{\mathrm{R}} + Y_{\mathrm{L}} B_{\mathrm{R}} = C_{\mathrm{R}} \tag{17}$$

is a stable matrix polynomial. In the manner of generating polynomials, the stabilisation of P can be expressed by the following lemma, see [26, p. 158].

Lemma 3. *A system $P = A_{\mathrm{L}}^{-1} B_{\mathrm{L}}$ is feedback stabilisable if and only if there exists a controller R such that*

$$\sum_{i=1}^{\eta} \beta_{1i} \beta_{2i} = c \tag{18}$$

is stable polynomial, where $\beta_{21}, \ldots, \beta_{2\eta}$ denote the generating polynomials of $\left[\begin{smallmatrix} X_{\mathrm{R}} \\ Y_{\mathrm{R}} \end{smallmatrix}\right]$.

An analogous lemma holds for $P = B_{\mathrm{R}} A_{\mathrm{R}}^{-1}$. One make sure that all the polynomials β_{1i} are of the form (11). Without loss of generality, we can consider polynomials β_{2i} in the form (11) too.

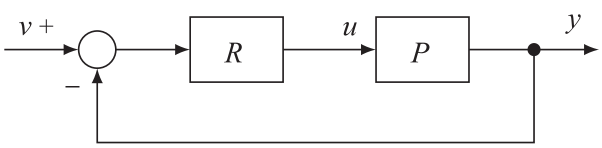

Fig. 2. Standard feedback configuration.

Then $c(z,w)$ is also in the form (11), i.e.,

$$c(z,w) = \sum_{k=0}^{\hat{n}} \sum_{i=0}^{\hat{q}} (c_{k,i} w^i + c_{k,-i} w^{-i}) z^k. \tag{19}$$

Since an explicit difference scheme was considered for discretisation, we further assume $c_{\hat{n},0} \neq 0$ and $c_{\hat{n},i} = 0,\ i = 1,\dots,\hat{q}$. Without loss of generality we consider $c_{\hat{n},0} = 1$ and Eq. (19) in the form

$$c(z,w) = z^{\hat{n}} + c_{\hat{n}-1}(w) z^{\hat{n}-1} + \cdots + c_0(w). \tag{20}$$

Now, we can formulate the following theorem.

Theorem 3. *A system P is feedback stabilisable if and only if there exists a controller R such that the Schur–Cohn matrix (12) corresponding to Eq. (18) is positive definite on the unit circle, i.e.,* $S_c(w) \succ 0\ \forall |w| = 1$.

Proof. It follows from Lemma 3 and Theorem 1.

Since $S_c(w)$ is a pseudo-polynomial matrix with symmetric expansion, its positiveness can be checked using Eq. (13).

A condition of stabilisability can also be based on Theorem 2. In what follows, the subscripts L and R are omitted in the case no matter whether the left or right fraction is used. We consider C in the form $C(z,w) = C_{\tilde{n}} z^{\tilde{n}} + C_{\tilde{n}-1}(w) z^{\tilde{n}-1} + \cdots + C_0(w)$ with

$$C_k = \sum_{l=0}^{\tilde{q}} C_{k,i} w^i + C_{k,-i} w^{-i}, \quad k = 0, 1, \dots, \tilde{n}-1, \tag{21}$$

where $C_{k,i}, C_{k,-i} \in \mathbb{R}^{\tilde{q} \times \tilde{q}}$. Since an explicit difference scheme was considered, we assume that $C_{\tilde{n},0}$ is regular and all entries of $C_{\tilde{n},i}, i = 1,\dots,\tilde{m}$, are equal to zero. Without loss of generality we consider $C_{\tilde{n},0} = I_{\tilde{m}}$ and

$$C(z,w) = I_{\tilde{m}} z^{\tilde{n}} + C_{\tilde{n}-1}(w) z^{\tilde{n}-1} + \cdots + C_0(w), \tag{22}$$

where $I_{\tilde{m}}$ denotes the identity matrix of the dimension $\tilde{m}$. Let $S_C(w) = (s_{i,j})$ be a matrix defined as

$$s_{i,j} = \sum_{k=1}^{i} C^*_{\tilde{n}+k-i} C_{\tilde{n}+k-j} - C^*_{j-k} C_{i-k}, \quad i,j = 1, 2, \dots, \tilde{n}, \tag{23}$$

where $C_{k,i}$'s are given by Eq. (21). The following theorem holds.

Theorem 4. *A system (6) is feedback stabilisable if there exists such a controller R that the matrix (23) corresponding to Eq. (16) is positive definite, i.e.,* $S_{C_L}(w) \succ 0\ \forall |w| = 1$. *A system (7) is feedback stabilisable if there exists such a controller R that the matrix (23) corresponding to Eq. (17) is positive definite, i.e.,* $S_{C_R}(w) \succ 0$, *for all* $|w| = 1$.

Proof. It follows immediately from Theorem 2. □

Since $S_{C_L}(w)$ and $S_{C_R}(w)$ are the pseudo-polynomial matrices with symmetric expansion, Eq. (13) can be used to check their positiveness.

5. Controller design

The controller design consists in finding such parameters that the condition given in Theorem 3 or Theorem 4 is satisfied. The coefficients of a (matrix) polynomials are unknown and are subject of design. Since neither the matrix $S_c(w)$ in Theorem 3 nor the matrices $S_{C_L}(w)$ and $S_{C_R}(w)$ in

Theorem 4 is linear in parameters of a controller, methods based on LMIs as presented in the previous sections cannot be used to design directly. We have to establish an equivalence of stability of a 2-D matrix polynomial and positiveness of a symmetric polynomial matrix whose all entries depend *linearly* on the coefficients of a polynomial matrix. We concentrate on scalar and matrix case separately.

5.1. Scalar case

The condition given in Theorem 3 is scalar. We will show latter that linearisation of $S_c(w)$ does not suffice to obtain a linear criterion in the controller parameters, because coefficients of the polynomial $c(w)$ are non-linear by themselves. However, what follows can be useful in the case when a controller has a special structure (which leads to $c(w)$ with linear coefficients). For systems of first order in the variable z (order in w can be arbitrary), the following condition was shown in [3].

Lemma 4 (*Augusta and Hurák [3, Theorem 3]*). *A polynomial (20) with $\hat{n}=1$ is stable if and only if*

$$\begin{pmatrix} 1 & c_0 \\ c_0^* & 1 \end{pmatrix} \succ 0. \tag{24}$$

The sufficient and necessary conditions for stabilisation of systems of a general order $\hat{n}$ cannot be derived in the form of an LMI due to a non-convexity of the set of parameters of stabilising controllers. One can deal with this by use of a non-convex optimisation or relaxation. Another possibility is to derive a condition in the form of an LMI, which is, however, no longer necessary. We do this, based on the famous *Schur conditions*, see, e.g., [43]. Let

$$\Phi_k(w) = \begin{pmatrix} c_{\tilde{n}}(w) & c_{\tilde{n}-1}(w) & \cdots & c_{\tilde{n}-k+1}(w) \\ 0 & c_{\tilde{n}}(w) & \ddots & c_{\tilde{n}-k+2}(w) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{\tilde{n}}(w) \end{pmatrix},$$

$$\Psi_k(w) = \begin{pmatrix} c_0(w) & 0 & \cdots & 0 \\ c_1(w) & c_0(w) & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ c_{k-1}(w) & c_{k-2} & \cdots & c_0(w) \end{pmatrix},$$

for $k=1,\dots,\tilde{n}$. Let $\delta_k(w) = \left(\begin{smallmatrix} \Phi_k & \Psi_k \\ \Psi_k^* & \Phi_k^* \end{smallmatrix}\right)$. The Schur conditions state that a polynomial c of the form (20) is stable if and only if $\det \delta_k > 0$, $k=1,\dots,\tilde{n}$, $\forall |w|=1$. More strict and not necessary, but linear in the coefficients of c, conditions are given in the following lemma.

Lemma 5. *A polynomial (20) is stable if $\delta_k(w) \succ 0$ $\forall |w|=1$, $k=1,\dots,\tilde{n}$.*

Proof. It follows from the Schur conditions and the fact that the determinant of a positive definite matrix is always positive. □

Note that Lemma 4 is a special case of Lemma 5 for $\tilde{n}=1$. The matrices δ_k are generally non-symmetric, so, we can use to check their positiveness the condition that $\delta_k \succ 0$ if and only if its

Hermitian part $\frac{1}{2}(\delta_k+\delta_k^*)\succ 0$. Their direct sum

$$\Delta(w)=\bigoplus_{k=1}^{\tilde{n}}\frac{1}{2}(\delta_k+\delta_k^*) \tag{25}$$

is the pseudo-polynomial matrix with symmetric expansion, hence, results of [40,10] can be used to control design as follows.

Theorem 5. *A system P is feedback stabilisable by a controller R if there exists a matrix M of an appropriate dimension that*

$$\begin{aligned} L(M,c_0,c_1,\ldots,c_{\tilde{n}-1})=&\left(\begin{array}{c|ccc}\Delta_0 & \Delta_1 & \cdots & \Delta_{2\tilde{q}}\\ \hline \Delta_1^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_{2\tilde{q}}^* & 0 & \cdots & 0\end{array}\right)+\left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I\\ 0 & \cdots & 0\end{array}\right)M\left(\begin{array}{c|ccc} I & & & 0\\ & \ddots & & \vdots\\ & & I & 0\end{array}\right)\\ &-\left(\begin{array}{ccc}0 & \cdots & 0\\ \hline I & & \\ & \ddots & \\ & & I\end{array}\right)M\left(\begin{array}{c|ccc}0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I\end{array}\right)\succ 0 \end{aligned} \tag{26}$$

holds, where Δ corresponds to Eq. (18) and is given by Eq. (25). For structure stabilisation with $\tilde{n}=1$ is the above condition sufficient and necessary.

Proof. It follows from Theorem 3 and Lemmata 4 and 5. □

5.2. Matrix case

The condition in Theorem 4 is in the form of a matrix inequality. Consider the case $\tilde{n}=1$. The matrix polynomial C has the form $C(z,w)=I_{\tilde{q}}z+C_0(w)$. The corresponding matrix S_C can be written as

$$S_C(w)=I_{\tilde{q}}-C_0^*C_0. \tag{27}$$

The following lemma holds.

Lemma 6. *The matrix polynomial (27) is positive definite for all $|w|=1$ if and only if*

$$\tilde{S}(w)=\begin{pmatrix} I_{\tilde{q}} & C_0\\ C_0^* & I_{\tilde{q}}\end{pmatrix} \tag{28}$$

is positive definite for all $|w|=1$.

Proof. $S(w)$ is the Schur complement of $\tilde{S}(w)$. Since $I_{\tilde{q}}\succ 0$, $\tilde{S}(w)\succ 0$ if and only if $S(w)\succ 0$. See, e.g., [43, Theorem 1.12] for the details. □

We obtained the condition as an LMI. The matrix (28) is a pseudo-polynomial matrix with symmetric expansion

$$\tilde{S}(w)=\sum_{i=-2\tilde{m}}^{2\tilde{m}}\tilde{S}_iw^i, \quad \tilde{S}_{-i}=\tilde{S}_i^*. \tag{29}$$

So, the result of [40,10] can be used to control design as follows. Let

$$L(M,C_0)=\left(\begin{array}{c|ccc}\tilde{S}_0 & \tilde{S}_1 & \cdots & \tilde{S}_{2\tilde{m}}\\ \hline \tilde{S}_1^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \tilde{S}_{2\tilde{m}}^* & 0 & \cdots & 0\end{array}\right)$$
$$+\left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I\\ 0 & \cdots & 0\end{array}\right)M\left(\begin{array}{c|ccc} I & & & 0\\ & \ddots & & \vdots\\ & & I & 0\end{array}\right)-\left(\begin{array}{ccc}0 & \cdots & 0\\ \hline I & & \\ & \ddots & \\ & & I\end{array}\right)M\left(\begin{array}{c|ccc}0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I\end{array}\right)\succ 0, \tag{30}$$

where $\tilde{S}(w)$ is given by Eqs. (28) and (29). The following theorem holds.

Theorem 6. *A plant of first order in the time variable described by $P(z,w)=A_L^{-1}B_L$ is stabilisable by a controller $R_R(z,w)=Y_RX_R^{-1}$ if $A_LX_R+B_LY_R=C_L$ is such a matrix polynomial that there exists a symmetric matrix M of an appropriate dimension such that Eq. (30) holds.*

A plant of first order in the time variable described by $P(z,w)=B_RA_R^{-1}$ is stabilisable by a controller $R_L(z,w)=X_L^{-1}Y_L$ if $X_LA_R+Y_LB_R=C_R$ is such a matrix polynomial that there exists a symmetric matrix M of an appropriate dimension such that Eq. (30) holds.

Proof. It follows from Lemma 6 and [40,10]. □

For $\tilde{n}>1$, it is possible to impose conditions of stability on the $\infty-$norm of a matrix constructed from the coefficient matrices. This technique was proposed for 1-D systems in [33]. One can make sure that it is not difficult to rewrite the below criterion to the form of an LMI.

Lemma 7 (*Ngo and Erickson [33, Theorem 2]*). *For the matrix polynomial (22) we define a matrix V as*

$$V(w)=(C_0\ \ C_1\ \ \cdots\ \ C_{\tilde{n}-1}). \tag{31}$$

If $\|V\|_\infty<1$ for all $|w|=1$, then $\det(C)\neq 0$ *for all* $\{|w|=1\}\cup\{|z|\geq 1\}$.

Proof. It follows from [33, Theorem 2]. □

5.3. An example

We consider four methods mentioned in the above text and plot the corresponding stability regions in terms of coefficients of matrix polynomials. It is the best way to see that some criteria presented in this paper are sufficient only.

Let a system be described by the left MFD (6) with

$$A_L=\begin{pmatrix} z+2 & 1\\ 1 & z+3\end{pmatrix},\quad B_L=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and a controller by $R=Y_{\mathrm{R}}X_{\mathrm{R}}^{-1}$, with

$$X_{\mathrm{R}}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad Y_{\mathrm{R}}=\begin{pmatrix}y_1(w) & y_2(w)\\ y_2(w) & y_1(w)+y_2(w)\end{pmatrix},$$

where $y_1(w)$ and $y_2(w)$ are scalar polynomials of w and $y_2^*(w)=y_2(w)$. The question is which values $y_1(w)$ and $y_2(w)$ can reach (for all $|w|=1$) so that closed-loop system is stable. We find these values by four different methods. The closed-loop matrix polynomial $C(w)$ is

$$C(w)=\begin{pmatrix}z+y_1(w)+2 & y_2(w)+1\\ y_2(w)+1 & z+y_1(w)+y_2(w)+3\end{pmatrix}.$$

1. *Computing determinant of C and its roots* gets stability criterion in the form of two non-linear inequalities—absolute values of the root has to be smaller than 1 for all $|w|=1$ for a system to be stable. The values of $y_1(w)$ and $y_2(w)$ satisfying this are depicted in Fig. 3a.
2. *Using Theorem 3 and Lemma 5 gets conditions*

$$\begin{pmatrix}1 & \xi\\ \xi & 1\end{pmatrix}\succ 0,\quad \begin{pmatrix}1 & 2y_1+y_2+5 & \xi & 0\\ 0 & 1 & 2y_1+y_2+5 & \xi\\ \xi & 2y_1+y_2+5 & 1 & 0\\ 0 & \xi & 2y_1+y_2+5 & 1\end{pmatrix}\succ 0,$$

   where $\xi=y_1^2+y_1y_2+5y_1-y_2^2+5$. The above conditions are bilinear, because coefficients of corresponding generating polynomials are also bilinear, in particular, $\beta_{24}=y_1^2+y_1y_2-y_2^2$. Values of $y_1(w)$ and $y_2(w)$ satisfying this are depicted in Fig. 3a.
3. *Using Theorem 4 and Lemma 6* gives the matrix (28) in the form

$$\begin{pmatrix}1 & 0 & y_1+2 & y_2+1\\ 0 & 1 & y_2+1 & y_1+y_2+3\\ y_1+2 & y_2+1 & 1 & 0\\ y_2+1 & y_1+y_2+3 & 0 & 1\end{pmatrix}.$$

   It is positive definite for values depicted in Fig. 3a. We obtained the same result as with the previous methods, however, this one is sufficient only and, in general, gives more strict conditions than previous ones.

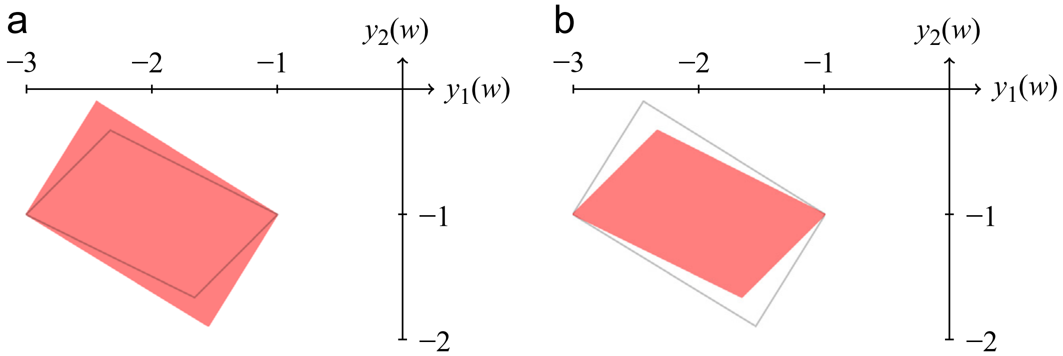

Fig. 3. An example: Real area of stability, given by methods 1—3 (a), stability area using the method 4 (b).

4. *Using Theorem 4 and Lemma 7* gives the matrix (31) in the form

$$V=\begin{pmatrix} y_1+2 & y_2+1 \\ y_2+1 & y_1+y_2+3 \end{pmatrix}$$

and leads to inequality $\max(|y_1+2|+|y_2+1|, |y_2+1|+|y_1+y_2+3|)<1$. It is satisfied for values depicted in Fig. 3b. Now, the result is different from the previous cases. Obtained conditions are more strict than previous ones.

6. An example: A heat conduction in a rod

In this section, the above described concept will be demonstrated by means of a numerical example. A distributed controller for a heat conduction in a long thin metal rod will be designed. The system is equipped with an array of temperature sensors and heaters and is sketched in Fig. 4.

6.1. Model of the system

The model was derived in [3]. The system is described by the heat equation (2). Using finite difference methods [38], partial derivatives are approximated by differences and Eq. (2) is transformed to the partial recurrence equation

$$v_{k+1,i}=\frac{T\kappa}{h^2}v_{k,i-1}+\left(1-2\frac{T\kappa}{h^2}\right)v_{k,i}+\frac{T\kappa}{h^2}v_{k,i+1}+q_{k,i}, \tag{32}$$

where $T>0$ is the sampling (time) period and $h>0$ denotes the distance between the nodes along the rod, k corresponds to discrete time and i to the coordinate of the node and, for simplicity, $q_{k,i}=Tf_{k,i}$.

The model (32) must give a sufficiently accurate approximation to the original model. A standard tool to analyse this is the von Neumann's analysis of the numerical stability [38], which gives a relation between T and h to guarantee the convergence, and which was performed on (32) in [3]. The two z-transforms of Eq. (32) give the transfer function

$$G(z,w)=\frac{1}{z+\left(2\frac{T\kappa}{h^2}-1\right)-\frac{T\kappa}{h^2}(w+w^{-1})}$$

with input heat as the input and temperature as the output. The variable z corresponds to time delay and the variable w corresponds to a shift along the spatial coordinate axes, $\kappa=\varkappa/\rho c_p$,

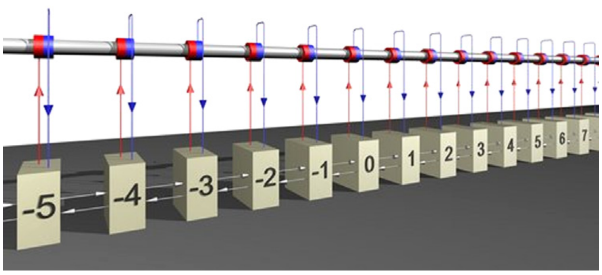

Fig. 4. Distributed control of a distributed parameter system: a rod with an array of heaters and temperature sensors and a distributed controller (an array of controllers) [3].

where $\varkappa$ is the thermal conductivity (W m^{-1} K^{-1}), ρ is the density (kg m^{-3}) and c_p is the heat capacity per unit mass (J K^{-1} kg^{-1}). Reasonable values are $\varkappa = 230$, $\rho = 2700$ and $c_p = 900$.

In contrast to [3], we suppose that heaters are placed in larger distances than sensors, i.e., there are some nodes without control action. Let the distance between heaters be *three* nodes, while sensors be placed at all nodes. In other words, let sensors and heaters make cells. Every cell contains three (neighbouring) sensors and one heater placed at the same node as the middle sensor. The system is then built from many cells placed side by side. Such the system can be described by Eq. (6) or (7) of size $m=3$. In particular, let $P(z,w) = A^{-1}B$, where

$$A = \begin{pmatrix} z + 2\frac{T\kappa}{h^2} - 1 & -\frac{T\kappa}{h^2} & -\frac{T\kappa}{h^2} w^{-1} \\ -\frac{T\kappa}{h^2} & z + 2\frac{T\kappa}{h^2} - 1 & -\frac{T\kappa}{h^2} \\ -\frac{T\kappa}{h^2} w & -\frac{T\kappa}{h^2} & z + 2\frac{T\kappa}{h^2} - 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The temperature (°C) and the input heat (°C s^{-1}) are considered to be the output and the input, respectively. For the numerical example we consider $h = 1/59$ m and $T = 10$ ms.

6.2. *A controller design*

Let a controller be described by $R(z,w) = YX^{-1}$. The closed-loop characteristic matrix polynomial is then $C(z,w) = AX + BY$. The plant is of order $n=1$. Extending the well-known results on solvability of a Diophantine equation in the 1-D setting [24], the closed-loop polynomial has to be of degree $2n-1$ or greater in the variable z to guarantee that a controller will be physically realisable. Hence, we put $\tilde{n} = 1$. Let the order of a controller be zero and

$$R(z,w) = YX^{-1}, \quad X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = (y_1 \;\; y_2 \;\; y_1), \tag{33}$$

where y_1 and y_2 are considered to be real constants for simplicity.

6.2.1. *Generating polynomials approach*

With Eq. (33) we get Eq. (18) in the form $c(w) = z^3 + (y_2 - 3)z^2 + (3 + 0.0066y_1 - 2y_2)z + 1 \times 10^{-5}(w + w^{-1})y_1 - 3.6 \times 10^{-8}(w + w^{-1}) + y_2 - 0.0065y_1 - 1$, where all coefficients are linear in controller parameters. Hence, it is possible to use Theorem 5. One can make sure that a matrix M in Eq. (26) exists, so, a stabilising controller (33) exists too. An algorithm implemented in Yalmip/SeDuMi returns $Y = (170 \; 2 \; 170)$.

6.2.2. *Positive polynomials approach*

With Eq. (33) we get closed-loop polynomial in the form

$$C = \begin{pmatrix} z + 2\frac{T\kappa}{h^2} - 1 & -\frac{T\kappa}{h^2} & -\frac{T\kappa}{h^2} w^{-1} \\ -\frac{T\kappa}{h^2} + y_1 & z + 2\frac{T\kappa}{h^2} - 1 + y_2 & -\frac{T\kappa}{h^2} + y_1 \\ -\frac{T\kappa}{h^2} w & -\frac{T\kappa}{h^2} & z + 2\frac{T\kappa}{h^2} - 1 \end{pmatrix}.$$

Since its leading coefficient is equal to identity matrix, it is possible to use Theorem 6 directly. We make sure that a matrix M in Eq. (30) exists, so, a stabilising controller (33) exists too. An algorithm implemented in Yalmip/SeDuMi returns $Y = (0.06 \; 0.93 \; 0.06)$.

6.3. Simulations

The aim of simulations is to make sure that closed-loop systems with controllers designed in the previous subsection are stable. We consider a rod of length 1 m with 59 nodes with zero boundary conditions and initial conditions given by Fig. 5a. Fig. 5b shows the response of the system with no controller. The responses of the closed-loop systems with controllers are shown in Fig. 6a, b. Both closed-loop systems are stable and the temperature goes to the origin. In Fig. 6b, one can see that temperature at nodes with control goes rapidly down, while the response is slower at nodes without control, but it is faster than response of the uncontrolled system.

7. Conclusions

The paper presented new results in the stability analysis and stabilisation of spatially invariant systems with multiple inputs and multiple outputs. Proposed methods are based on an LMI condition on positiveness of a polynomial matrix on the unit circle. The main restriction of this technique comes from a non-convexity of the set of matrix polynomial coefficients. In the paper

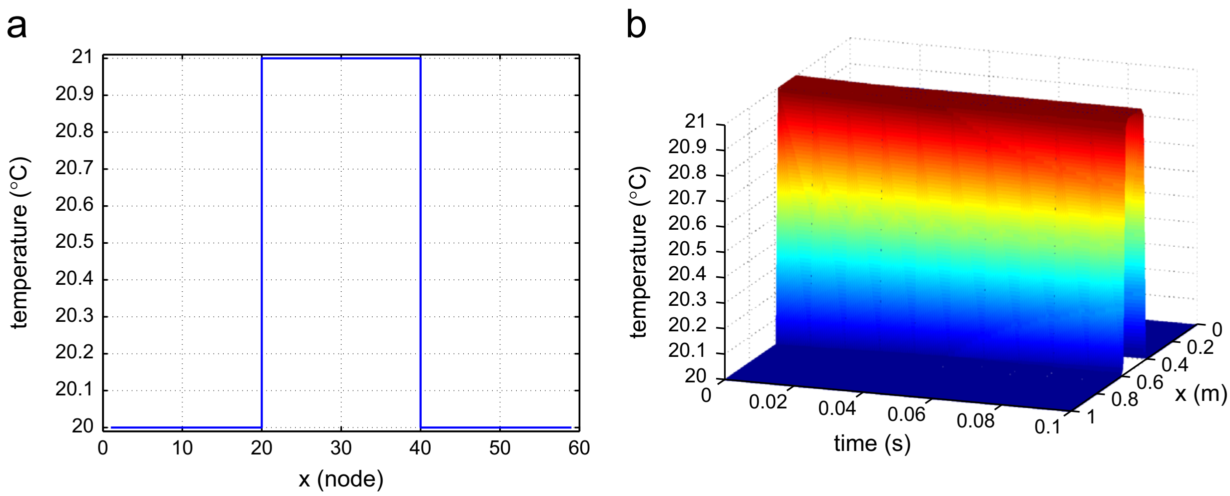

Fig. 5. Initial conditions (a), response to the initial conditions of the uncontrolled system (b).

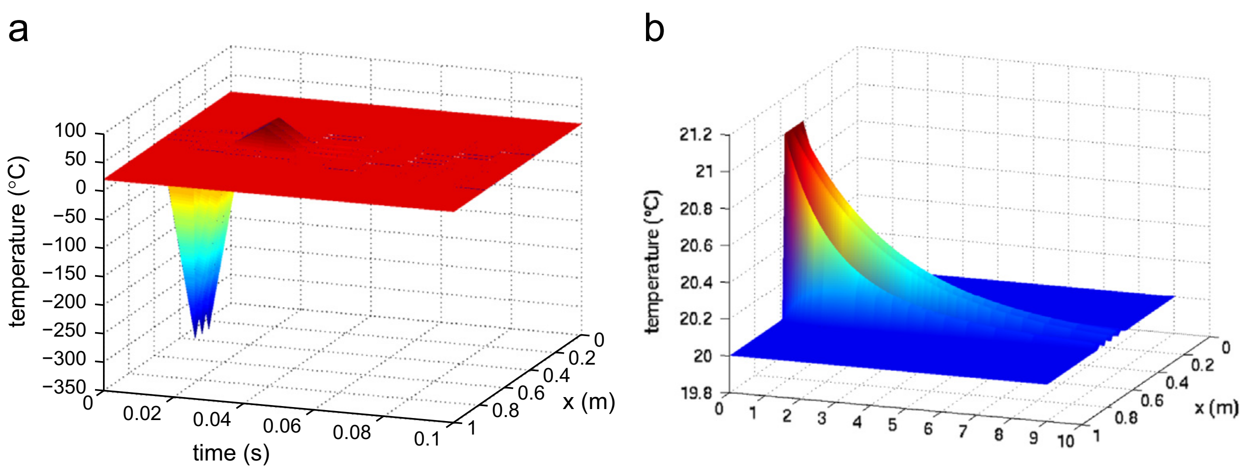

Fig. 6. Output of the closed-loop system with the controller obtained via generating polynomials (a), and via positive polynomials approach (b).

it was shown that there exists a linearising factorisation, which can remove this restriction, but it is available for systems of order one in the time variable only. However, the numerical example shows that a physically motivated problem can be solved realistically. For systems of higher order, other, linear in controller parameters, but more conservative, conditions were given.

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