# On the pricing of illiquid options with Black-Scholes formula

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**Abstract**

Detecting the fair, ie. no-arbitrage, price of an option is a very interesting and challenging task of quantitative finance. It results mostly from the fact that the option payoff is nonlinear and the price can be very sensitive to the changes of underlying factors (especially ATM options). From the other point of view, ATM vanilla options are often traded and liquid, while deep ITM and OTM options are mostly illiquid and it is difficult to estimate the model parameters. Another issue is how to obtain the market assumptions about riskless rate relevant for the option maturity and the future expected dividends. In this paper we focus on a particular problem of extracting parameters to value options on dividend paying stocks via BS model using real data from German option market.

**Keywords**

BS formula, German option market, illiquid option, implied parameters option valuation.

**JEL Classification:** G13, G14, C14, C52

## 1. Introduction

Options are a quite important type of financial derivatives since they allow to fit even very specific fears (hedging) and outlooks (speculation) about the future evolution. Due to the nonlinear payoff function and potential high sensitivity to changes in the input factors, such as volatility or even maturity, options are very challenging also for modeling purposes.

Obviously, since the standard option valuation model of Black and Scholes (and Merton) was based on the assumption of normally distributed returns, the presence of skewness and kurtosis at the market complicates the situation significantly. A common market practice is to use the market price as an exogenous variable to be put into the BS formula (Black and Scholes(1973), Merton (1973)). Thus, a so called *implied volatility* is obtained, ie. a number that assures that BS formula provides the right price. Such implied volatility can subsequently be used to value

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options, which are not traded at the market or are not sufficiently liquid, and even exotic options, which are not traded at the market at all.

However, an inexperienced user can be surprised by several consequences of such approach. First, since the illiquid options we wish to price have obviously at least some of the parameters different to the options used to get the implied volatility, we need to execute some kind of interpolation and subsequent smoothing to get nice resulting smile or smirk of the volatility curve or even surface. However, without respecting several important rules, the results might lead to arbitrage opportunity, ie. the option prices calculated with such volatilities might be mutually inconsistent, see eg. Benko et al. (2007). Second, the market mostly provides the maturity, moneyness, option price and implied volatility. But in the Black-Scholes formula there are two parameters more – the interest rate and dividend yield. Hence, the second issue the user is faced to is what are the market expectations about the interest rate and dividends over the option life?

In contrast to our previous research focused especially on the first issue, see eg. Kopa et al. (2014a,b), in this paper we study the second issue and show how to assure that option prices will be consistent with the market view on the interest rate as well as expected dividend yield so that illiquid options can be priced efficiently. We also address a specific problem related to the far OTM or ITM options.

We proceed as follows. In the subsequent section, basic theoretical foundations of options are provided. We proceed with the definition of the option pricing formula and the implied volatility approach. Finally, an efficient procedure to price illiquid options is suggested using real market data of German option market.

## 2. Option terminology

Options are nonlinear types of financial derivatives, which gives the holder the right (but not the obligation) to buy the underlying asset in the future (at maturity time) at prespecified exercise price. Simultaneously, the writer of the option has to deliver the underlying asset if the holder asks. Option analysis is subject of study of almost any book dealing with financial markets and especially those focusing on quantitative finance. In this section, we mostly follow Tichý (2011, 2012).

Options can be classified due to a whole range of criteria, such as counterparty position (short and long), maturity time, complexity of the payoff function, etc. The basic features are *the underlying asset* ($\mathcal{S}$), which should be specified as precisely as possible (it is important mainly for commodities),[4] *the exercise price* ($\mathcal{K}$), and *the maturity time* ($T$).

If the option can be exercised only at maturity time $T$, we call it *the European option.* By contrast, if it can be exercised also at any time prior the maturity day, ie. $t \in [0, T]$, we refer to it as *the American option.* A special type of options, possible to be classified somewhere between European and American options is *the Bermudan option*, which can be exercised at final number of times during the option life.

In dependency on the complexity of the payoff function, we usually distinguish simple *plain vanilla options* (PV) and *exotic options.* However, by a plain vanilla option we generally mean call and put options with the most simple payoff function.[5] Thus,

$$\Psi_{call}^{vanilla} = (\mathcal{S}_T - \mathcal{K})^+ \tag{1}$$

for vanilla call, and

$$\Psi_{put}^{vanilla} = (\mathcal{K} - \mathcal{S}_T)^+ \tag{2}$$

for vanilla put, where $(x)^+ \equiv \max(x; 0)$.

[4] It will be supposed that the underlying asset is a non-dividend stock if not stated otherwise.

[5] Sometimes, by plain vanilla options we mean any option which is regularly traded at the market.

Due to the definition of an option – it gives a right, but no obligation to make a particular trade – we can deduce basic differences between the short and the long position. While the payoff resulting from the long position is non-negative, either 0 or $\mathcal{S}_T - \mathcal{K}$, the payoff of the short position will never be positive, ie. it is either $\mathcal{K} - \mathcal{S}_T$ or 0. Moreover, it is obvious, that the long call payoff is not limited from above, but the short position payoff function goes only up to the exercise price (underlying asset price is zero). It should be noted that even if the value of long and short position is identical, their market prices mostly differ due to the transaction costs (a so called bid/ask spread).

The ratio of the current price of the underlying asset and the exercise price is commonly referred to as *moneyness*, which is equal to one for ATM options, higher than one for ITM options, ie. such options, which would lead to positive payoff if exercised immediately, and below one for OTM options. Sometimes, it is useful to work with log-moneyness, ie. zero for ATM, positive for ITM and negative for OTM options, or moneyness which uses forward prices instead of spot prices, ie. we take into account the time factor.

*Table 1: Spot price, exercise price and moneyness*

| **Relation of** | **Vanilla call** | | **Vanilla put** | |
|---|---|---|---|---|
| $\mathcal{S}_T$ and $\mathcal{K}$ | type | log/moneyness | type | log/moneyness |
| $\mathcal{S}_T > \mathcal{K}$ | ITM | $> 0/ > 1$ | OTM | $< 0/ < 1$ |
| $\mathcal{S}_T = \mathcal{K}$ | ATM | $= 0/ = 1$ | ATM | $= 0/ = 1$ |
| $\mathcal{S}_T < \mathcal{K}$ | OTM | $< 0/ < 1$ | ITM | $> 0/ > 1$ |

Any financial option with more complex payoff function than is the one of a standard European (American) call or put option is referred to as *the exotic option*. The majority of exotic options is traded outside organized exchanges, at so called OTC markets. However, several types of exotics are so popular that the major derivatives exchanges have listed them (e.g. some options with barriers).

By contrast, many exotic options are so unique that they are suitable only for investors for whom they were originally designed. Thus, within the payoff pattern special needs or believes and fears of corporate or institutional investors are respected. This fact decreases the liquidity significantly and also the pricing and hedging procedure can be substantially complicated.

## 3. Analytical valuation

Although the option valuation formula can be derived by various approaches, such as utilization of risk neutral expectations or solving of partial differential equations, it must – under the same conditions – always lead to the same result. In this section, valuation formulas assuming several kinds of available inputs will be considered.

**BS model.** Assuming the payoff function of plain vanilla call (1),

$$f_T = \Psi_{call}^{vanilla} = (\mathcal{S}_T - \mathcal{K})^+ ,$$

we get the valuation formula as follows (*BS model for vanilla call*):[6]

$$\mathcal{V}_{call}^{vanilla}(\tau; \mathcal{S}, \mathcal{K}, r, \sigma) = \mathcal{S}\mathcal{N}(d_+) - e^{-r\tau}\mathcal{K}\mathcal{N}(d_-). \tag{3}$$

Similarly, assuming the payoff function of vanilla put (2), we get (*BS model for vanilla put*):

$$\mathcal{V}_{put}^{vanilla}(\tau; \mathcal{S}, \mathcal{K}, r, \sigma) = e^{-r\tau}\mathcal{K}\mathcal{N}(-d_-) - \mathcal{S}\mathcal{N}(-d_+). \tag{4}$$

[6]Black and Scholes (1973); an alternative model for dividend paying stocks is due to Geske (1978), whereas the case of FX rates was analyzed by Garman and Kohlhagen (1983).

In both cases above:

$$d_{\pm} = \frac{\ln \frac{\mathcal{S}}{\mathcal{K}} + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{5}$$

Moreover, $\mathcal{S}$ is the underlying asset price at the valuation time ($t$) and it is supposed to follow log-normal distribution, $\tau$ is the time to maturity (ie. $\tau = T - t$), $r$ is the riskless rate valid over $\tau$, $\sigma$ is the volatility expected over the same period, both *per annum*, and $\mathcal{N}(x)$ is distribution function for standard normal distribution.

From the formulas above, a strict relation between prices of call and put options is obvious (*Put-Call Parity*):

$$\mathcal{V}_{call}^{vanilla}(\tau; \mathcal{S}, \mathcal{K}, r, \sigma) + Ke^{-r\tau} = \mathcal{V}_{put}^{vanilla}(\tau; \mathcal{S}, \mathcal{K}, r, \sigma) + S. \tag{6}$$

**BS model for call options with implied parameters.** Assume first that there are some options $\mathcal{V}_i$ traded at the market (eg. option exchange market) with sufficient liquidity. It follows that there is no need to apply the BS model since the correct prices can be obtained from the market, $\mathcal{V}_i^{mar}$.

Obviously, there still might be a need to price options written on the same underlying asset,[7] which are not traded at the market. For such purposes, we can use the so called implied volatility $\sigma_i^{imp}$, which can be obtained from the following equality:

$$\mathcal{V}_i^{mar} = \mathcal{S}\mathcal{N}\left(\frac{\ln \frac{\mathcal{S}}{\mathcal{K}} + \left(r + \frac{1}{2}\left(\sigma_i^{im}\right)^2\right)\tau}{\sigma_i^{imp}\sqrt{\tau}}\right) - e^{-r\tau}\mathcal{K}\mathcal{N}\left(\frac{\ln \frac{\mathcal{S}}{\mathcal{K}} + \left(r - \frac{1}{2}\left(\sigma_i^{imp}\right)^2\right)\tau}{\sigma_i^{imp}\sqrt{\tau}}\right). \tag{7}$$

Here $i$ in $f_i$ and $\sigma_i$ specifies the rest of option parameters – the maturity, log-moneyness and riskless rate $(\tau; \ln(\mathcal{S}/\mathcal{K}), r)$.

While both, the maturity and moneyness are unambiguously known, the preceding formula requires to estimate the riskless rate. In practice however, the market (ie. option exchange market organizers) already provides not only the traded price, but also the BS implied volatility, though, a potential user might be confused which riskless rate should be used to price the options with a given maturity.

The solution is obvious – since we know each data needed to apply formula (7), but the riskless rate, we can define a so called implied riskless rate, $r_i^{imp}$:

$$\mathcal{V}_i^{mar} = \mathcal{S}\mathcal{N}\left(\frac{\ln \frac{\mathcal{S}}{\mathcal{K}} + \left(r^{imp} + \frac{1}{2}\left(\sigma_i^{mar}\right)^2\right)\tau}{\sigma_i^{mar}\sqrt{\tau}}\right) - e^{-r^{imp}\tau}\mathcal{K}\mathcal{N}\left(\frac{\ln \frac{\mathcal{S}}{\mathcal{K}} + \left(r^{imp} - \frac{1}{2}\left(\sigma_i^{mar}\right)^2\right)\tau}{\sigma_i^{mar}\sqrt{\tau}}\right). \tag{8}$$

where once again $i$ states the maturity (the rate should be fixed for each moneyness) and $\sigma_i^{mar}$ is the volatility obtained from the market.

As concerns put options, a similar procedure can be derived. Alternatively, one can use the put-call parity (6).

**BS model for option on a dividend paying stock.** Applying the same ideas, which led to the derivation of standard BS model (see above), one can obtain also its extension for options written on stocks paying continuous-like dividend yield $q$ (*BS model for vanilla call on dividend paying stock*):

$$\mathcal{V}_{call}^{vanilla}(\tau; \mathcal{S}, \mathcal{K}, r, \sigma, q) = e^{-q\tau}\mathcal{S}\mathcal{N}(d_+) - e^{-r\tau}\mathcal{K}\mathcal{N}(d_-). \tag{9}$$

[7] If it is written on a different asset, it must be at least almost perfectly correlated.

and (*BS model for vanilla put on dividend paying stock*):

$$\mathcal{V}_{put}^{vanilla}(\tau; \mathcal{S}, \mathcal{K}, r, \sigma, q) = e^{-r\tau}\mathcal{K}\mathcal{N}(-d_-) - e^{-q\tau}\mathcal{S}\mathcal{N}(-d_+), \tag{10}$$

where

$$d_\pm = \frac{\ln\frac{\mathcal{S}}{\mathcal{K}} + (r - q \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{11}$$

**BS model for call option on a dividend paying stock with implied parameters.** If we would like to use the market prices of some option on a stock for which we assume some dividend payment, we need to estimate one parameter more – the implied dividend yield, ie. which level of dividends was expected by the market.

We proceed as follows. First, we obtain the market prices of available options on non-dividend stocks (eg. dividend free equity market index) and related market volatilities. Second step is to estimate the interest rates $r_i^{imp}$. In the next step we take the market prices $\mathcal{V}_I^{mar}$ and volatilities $\sigma_i^{mar}$ of available options on dividend-paying stocks and use them to estimate implied dividend yields $q_i^{imp}$ (together with the implied interest rates $r_i^{imp}$ from step two). Finally, we can price the target option $i$ with $\sigma_i^{mar}$, $r_i^{imp}$, $q_i^{imp}$:

$$\mathcal{V}_i = e^{-q_i^{imp}\tau}\mathcal{S}\mathcal{N}(d_+) - e^{-r_i^{imp}\tau}\mathcal{K}\mathcal{N}(d_-). \tag{12}$$

where

$$d_\pm = \frac{\ln\frac{\mathcal{S}}{\mathcal{K}} + \left(r_i^{imp} - q_i^{imp} \pm \frac{1}{2}\left(\sigma_i^{mar}\right)^2\right)\tau}{\sigma_i^{mar}\sqrt{\tau}}. \tag{13}$$

And similarly for put option.

## 4. Illustrative study

In the lines below we will show the procedure described in the previous section on real market data. First, we will consider a dividend-less equity market index and later one of its component stocks, which is supposed to pay some dividends. The option data were obtained from the Reuters terminal for September 15, 2011 and we considered all call and put options on DAX index maturing on June 15, 2012.

On a given day the spot price of the index option underlying asset (ie. the value of the index) was approximately 5508. First we calculate the log-moneyness for all available exercise prices between 500 to 10,000 (the interval length is sometimes 500, sometimes 100, or even 50). Since the log-moneyness of call is inversely related to the one of put options, we multiply the latter by minus one to get the adjusted monneyness for put.

In Figure 1 we show the market prices of call (black) and put options (gray) in dependency of such adjusted log-moneyness. It is apparent that most observations are located close to moneyness of zero (ie. close to ATM options) and the prices of put and call options are very close each to the other only for ATM options (moneyness is zero).

Similarly, in Figure 2, volatilities of call and put options provided by the market by using BS model are compared. This time, it is clear that the volatility is more or less similar for ATM as well as close to ATM options, while it diverges for deep ITM/OTM options. It is also apparent that the market volatilities of ITM/OTM calls are much lower than the volatilities of related OTM/ITM puts.

The most difficult step is to estimate the riskless rate, which should be possible unique for all options. As we can see in Figure 3, the value which matches the BS formula and market prices of calls, puts and volatilities oscillate slightly below 2% p.a. Unfortunately, it seems to be impossible to price deep ITM call options since the prices and volatilities provided by the

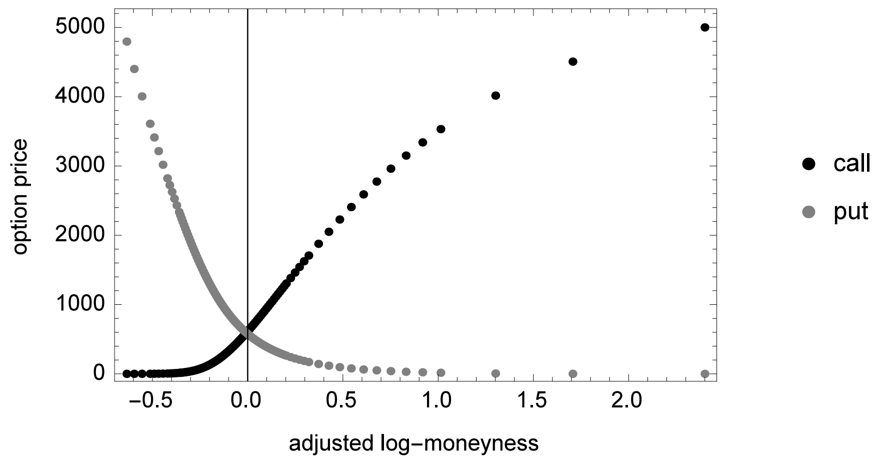


*Figure 1:* DAX call and put option market prices in dependency on absolute log-moneyness

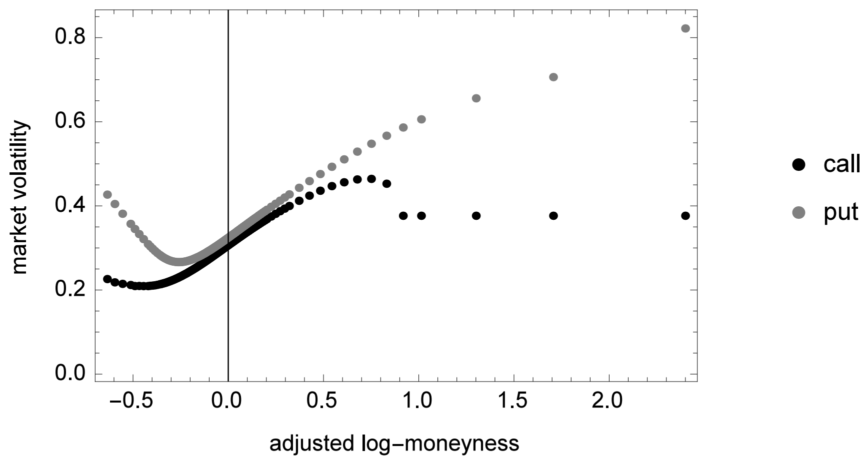


*Figure 2:* DAX call and put option market volatilities in dependency on absolute log-moneyness

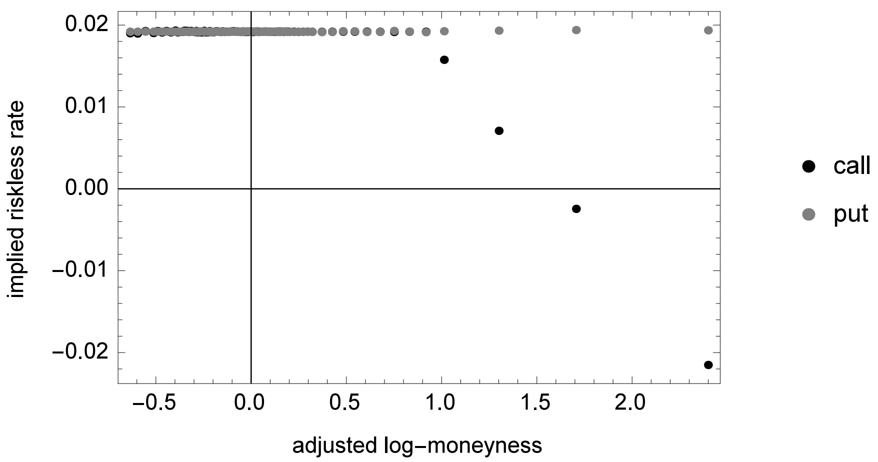


*Figure 3:* DAX call and put option implied riskless rates in dependency on absolute log-moneyness

market would require even negative interest rate to match the BS formula. This is especially the case of the call option with the lowest exercise price (500), which is about 10% of the spot price and which furthermore means that the option delta is in fact one, its gamma is zero and the option value is insensitive also to the interest rates (rho is zero as well).[8]

Applying some optimization steps we get riskless rate $r = 1.911\%$ p.a., a number which we will use further to obtain the implied dividend yield (ie. continuous dividend yield assumed by the market for a given stock).

This time we assume the options on Allianz, naturally with the same maturity as in the analysis above. From Figure 4 it is apparent that most of the call options are OTM (negative moneyness), while the puts are ITM (reverted moneyness is also negative).

If we focus on the volatility obtained from the market, see Figure 5, we can see significant differences for all levels of moneyness. The reason might be the dividend yield but also limited liquidity and resulting bid/ask spread.

Finally, in Figure 6 we provide the dividend yield estimated by the market. For most cases it is fixed just below 9.7% p.a. However, we can observe some instability for ITM puts.

The implication for pricing of OTC options on Allianz therefore is that the trader should ask for sufficient bid/ask spread even for close to ATM options to cover his or her cost of hedging by reverse position and should be very careful for deep ITM and OTM options.

## 5. Conclusion

This paper was focused on extracting of parameters for pricing of illiquid options by BS model, including market assumptions about the volatility, riskless rate and dividend yield.

Using empirical data (call and put option prices and implied volatilities) from German option market we have estimated the so called implied riskless interest rate using DAX index options in the first step. In the second step we have used the optimized interest rate (1.911%) to estimate the so called implied dividend yield for call and put options on Allianz, which is one of the components of DAX equity index. As a next step, the trader can use these data, ie. riskless interest rate, dividend yield and volatilities implied by the market price to valuate illiquid options on Allianz.

Within the analysis we have specifically stressed potential problems related to valuation of very deep ITM call options (unstable implied interest rate) and deep ITM put options (unstable implied dividend yield), as well as large differences in implied volatilities for related ITM/OTM call/put options. These real market observations should have an impact on bid/ask spread required by traders.

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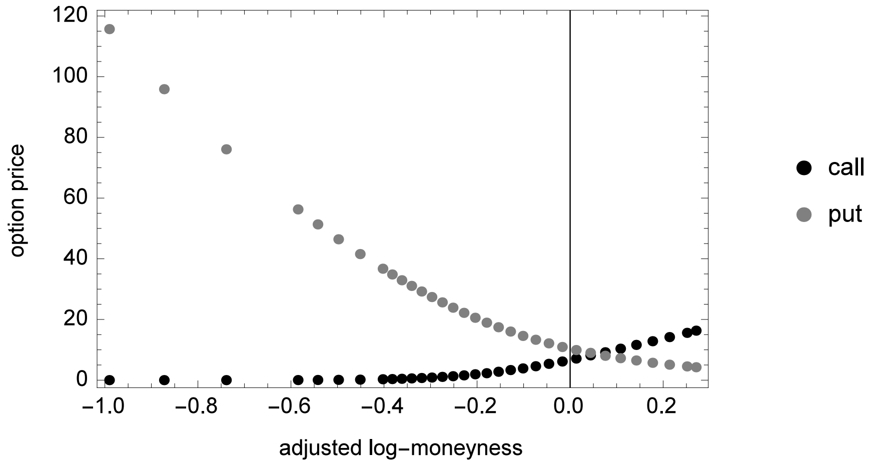


*Figure 4:* Allianz call and put option market prices on dependency in absolute log-moneyness

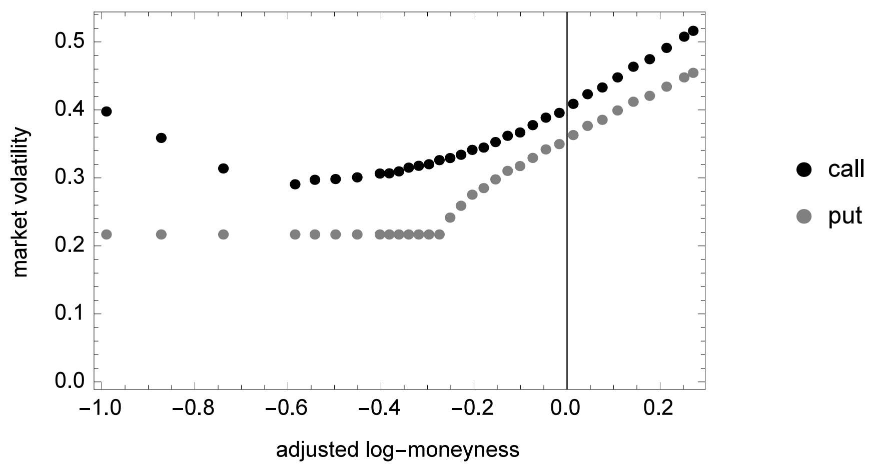


*Figure 5:* Allianz call and put option market volatilities in dependency on absolute log-moneyness

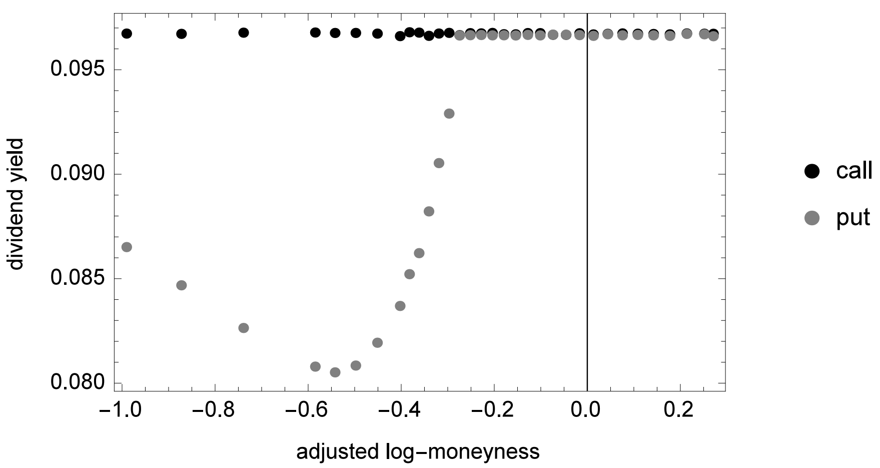


*Figure 6:* Allianz call and put option implied dividend rates in dependency on absolute log-moneyness