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FULL STABILITY OF LOCALLY OPTIMAL SOLUTIONS IN SECOND-ORDER CONE PROGRAMS*

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Abstract. The paper presents complete characterizations of Lipschitzian full stability of locally optimal solutions to second-order cone programs (SOCPs) expressed entirely in terms of their initial data. These characterizations are obtained via appropriate versions of the quadratic growth and strong second-order sufficient conditions under the corresponding constraint qualifications. We also establish close relationships between full stability of local minimizers for SOCPs and strong regularity of the associated generalized equations at nondegenerate points. Our approach is mainly based on advanced tools of second-order variational analysis and generalized differentiation.

Key words. variational analysis, second-order cone programming, full stability of local minimizers, nondegeneracy, strong regularity, quadratic growth, second-order subdifferentials, coderivatives

AMS subject classifications. 49J52, 90C30, 90C31

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1. Introduction.

This paper is devoted to the study of full Lipschitzian stability of locally optimal solutions to *second-order cone programs* (SOCPs), which are written as follows:

$$\text{minimize } \varphi_0(x) \text{ subject to } \Phi(x):=\big(\Phi^1(x),\ldots,\Phi^J(x)\big)\in\mathcal{Q}, \tag{1.1}$$

where the data $\varphi_0\colon\mathbb{R}^n\to\mathbb{R}$ and $\Phi^j\colon\mathbb{R}^n\to\mathbb{R}^{m_j+1}$, $j=1,\ldots,J$, are twice continuously differentiable ($\mathcal{C}^2$-smooth) around the reference points. The underlying set $\mathcal{Q}$ is given by

$$\mathcal{Q}:=\prod_{j=1}^{J}\mathcal{Q}_{m_j+1}\subset\mathbb{R}^l\quad\text{with }\ l:=\sum_{j=1}^{J}(m_j+1) \tag{1.2}$$

as the J-product of the *second-order, Lorentz, ice-cream cones*

$$\mathcal{Q}_{m_j+1}=\big\{(s_0,s_r)\in\mathbb{R}^{m_j+1}\,\big|\,s_0\ge\|s_r\|\big\},\quad j=1,\ldots,J. \tag{1.3}$$

It has been well recognized that the SOCP model (1.1) describes various classes of problems important for optimization theory and applications; see [1, 3] and the references therein. Note that, despite the $\mathcal{C}^2$-smoothness of the functions φ_0 and Φ in

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(1.1), this model belongs to nonsmooth optimization due to the nondifferentiability of the norm in (1.3) at the origin.

The concept of (Lipschitzian) *full stability* of local solutions to general optimization problems was introduced by Levy, Poliquin, and Rockafellar [11] who were largely motivated by the far-reaching extension of the *tilt stability* notion from Poliquin and Rockafellar [30] and the importance of these stability notions for the justification of numerical algorithms; see [11, 30] and also section 4 below for more details. In contrast to tilt stability for which significant progress has been achieved, especially during the recent years (see, e.g., [3, 6, 7, 8, 12, 20, 21, 24, 25, 26]), not much has been done for the study of full stability. The pioneering paper [11] contains a second-order subdifferential characterization of full stability in the unconstrained framework of finite-dimensional optimization with extended real-valued objectives. The next step was done in [27], where the authors obtained (mainly based on the general approach of [11] and the second-order calculus rules established in [26]) several second-order characterizations of full stability for finite-dimensional problems of nonlinear programming (NLP), extended nonlinear programming, and mathematical programs with polyhedral constraints (MPPC) under certain nondegeneracy conditions. Quite recently, Mordukhovich and Nghia [22] have developed a new approach to both Lipschitzian and Hölderian (introduced therein) full stability in finite and infinite dimensions and applied it to deriving constructive characterization of full stability in NLPs, infinite-dimensional problems with polyhedric constraints, and optimal control problems governed by semilinear elliptic equations without any nondegeneracy assumption.

The classes of constrained optimization problems for which full stability has been constructively characterized in [22, 27] entirely in terms of the initial data possess a certain *polyhedral* structure that has been significantly used in the proofs. Such a polyhedrality is not the case for SOCPs, where no results of this type have been known even for tilt stability. Note to this end that related coderivative characterizations of tilt stability for general conic programs has been recently obtained in [25] while not completely in terms of the initial program data. The major goal of this paper is to establish complete characterizations of full stability for SOCPs (1.1) at nondegenerate local minimizers via the initial data, where no difference arises between Lipschitzian and Hölderian versions. This will be achieved by using second-order generalized differential tools of variational analysis and their constructive realizations in the SOCP framework.

The rest of the paper is organized as follows. Section 2 recalls basic tools of generalized differentiation employed in *deriving* (while not in formulations) of the main characterizations of full stability. Section 3 makes a bridge between general second-order constructions and constraint qualifications developed in the variational approach to full stability from one side, and their effective realizations via the initial data in the SOCP setting from the other. In particular, employing the coderivative calculations for the metric projection onto the Lorentz cone by Outrata and Sun [29] as well as new calculations of the second-order subdifferential of the indicator function for this cone allow us to establish the equivalence between the *second-order constraint qualification* (SOCQ) developed in second-order variational calculus and the *nondegeneracy condition* well recognized in constrained optimization.

Section 4 contains second-order characterizations of full stability of locally optimal solutions to SOCPs. First we discuss characterizations given via the *partial strong metric regularity* of a first-order subgradient mapping and via an SOCP version of the *uniform second-order quadratic growth* under appropriate qualification conditions. Then we derive our main characterization of full stability of nondegenerate local solutions to SOCPs expressed via the *second-order strong sufficient condition* for SOCPs introduced in the nonparametric setting by Bonnans and Ramírez [2] from different perspectives. The proof of this result is based on the composite optimization approach to full stability with the usage of the aforementioned calculations of the second-order subdifferential together with recent second-order calculus rules developed in [28] and [19].

In section 5 we establish close relationships between full stability of local minimizers for SOCPs and Robinson's notion of *strong regularity* [31] of the associated KKT systems involving Lagrange multipliers. In fact we show that these two notions are *equivalent* under the *SOCQ/nondegeneracy* condition. The given proof of this result works for general problems of conic programming described by *reducible* cones. As a by-product of the obtained equivalence and the strong second-order sufficient optimality condition (SSOSC) characterization of full stability from section 4, we recover the characterization of strong regularity for SOCPs established by Bonnans and Ramírez [2] in a completely different way. Two illustrative examples are presented here showing, in particular, that full stability is a broader notion for SOCPs in the absence of nondegeneracy. The final section 6 contains some concluding remarks and formulates several open questions of further research.

Throughout the paper we use standard notation of variational analysis and second-order cone programming; see, e.g., [1, 17, 18, 33]. Everywhere $\mathbb{R}^n$ stands for the n-dimensional Euclidean space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. By $\mathbb{B}$ we denote the closed unit ball in the space in question and by $B_r(x) = B(x,r) := x + r\mathbb{B}$ the closed ball centered at x with radius $r > 0$. For a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ the symbol

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{ y\in\mathbb{R}^m \Big|\ \exists\, x_k \to \bar{x},\ \exists\, y_k \to y \ \text{ as } \ k\to\infty \ \text{ with } \ y_k \in F(x_k) \ \text{ for all } \ k\in\mathbb{N} := \{1,2,\ldots\}\Big\} \tag{1.4}$$

signifies the *Painlevé–Kuratowski outer limit* of F as $x \to \bar{x}$. Given $\Omega \subset \mathbb{R}^n$, the symbol $x \xrightarrow{\Omega} \bar{x}$ indicates that $x \to \bar{x}$ with $x \in \Omega$. For a linear operator/matrix A the notation A^* stands for the adjoint operator/matrix transposition; we also use the symbol * to indicate the duality/polarity correspondence. For a vector column $x \in \mathbb{R}^n$ the notation x^T stands for the corresponding vector row. Finally, by $\operatorname{cone}\Omega$ and $\operatorname{co}\Omega$ we denote the conic and convex hulls of Ω, respectively.

2. Tools of generalized differentiation. Here we recall and briefly discuss, mostly following the books [17, 33], some basic generalized differential constructions of variational analysis needed in the paper. Unless otherwise stated, all the sets $\Omega \subset \mathbb{R}^n$ under consideration are locally closed and the extended real-valued functions $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty,\infty]$ are lower semicontinuous around the reference points. Variational geometry is underlying in what follows, and so we first recall the original definition of the (limiting, Mordukhovich) *normal cone* to Ω at $\bar{x} \in \Omega$ given in [14] by

$$N_\Omega(\bar{x}) = \operatorname*{Lim\,sup}_{x\to\bar{x}} \Big[\operatorname{cone}\big(x - \Pi_\Omega(x)\big)\Big] \tag{2.1}$$

via the *projection operator* $\Pi_\Omega\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, which plays a crucial role in this paper. Observe that the normal cone (2.1) is often nonconvex while it can be represented via

(1.4) as

$$N_\Omega(\bar{x}) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}_\Omega(x) \quad \text{with} \quad \widehat{N}_\Omega(x;\Omega) := \left\{ v \in \mathbb{R}^n \,\middle|\, \limsup_{u \xrightarrow{\Omega} x} \frac{\langle v, u - x\rangle}{\|u - x\|} \le 0 \right\}, \tag{2.2}$$

where the convex cone $\widehat{N}_\Omega(x)$ in (2.2) is known as the collection of Fréchet or *regular normals* to Ω at $x \in \Omega$. There is the *duality/polarity* correspondence

$$\widehat{N}_\Omega(x) = T_\Omega(x)^* := \left\{ v \in \mathbb{R}^n \,\middle|\, \langle v, w\rangle \le 0 \ \text{ for all } \ w \in T_\Omega(x) \right\} \tag{2.3}$$

between $\widehat{N}_\Omega(x)$ in (2.2) and the (Bouligand–Severi) *tangent cone* to Ω at x defined by

$$T_\Omega(x) := \left\{ w \in \mathbb{R}^n \,\middle|\, \exists\, x_k \xrightarrow{\Omega} x,\ \alpha_k \ge 0 \ \text{ with } \ \alpha_k(x_k - x) \to w \ \text{ as } \ k \to \infty \right\}. \tag{2.4}$$

Note that for convex sets Ω the normal and tangent cone constructions in (2.1), (2.2), and (2.4) reduce to the corresponding constructions of convex analysis.

Given an extended real-valued function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ finite at $\bar{x}$, the only (first-order) *subdifferential* of φ at $\bar{x}$ we use in this paper is the one introduced in [14],

$$\partial\varphi(\bar{x}) := \left\{ v \in \mathbb{R}^n \,\middle|\, (v, -1) \in N_{\operatorname{epi}\varphi}(\bar{x}, \varphi(\bar{x})) \right\}, \tag{2.5}$$

via the normal cone (2.1) to the epigraph $\operatorname{epi}\varphi := \{(x,\alpha) \in \mathbb{R}^{n+1} |\ \alpha \ge \varphi(x)\}$. There are various analytic descriptions of the subdifferential (2.5), which are not used in what follows; see [17, 33]. It is easy to observe the normal cone representation

$$N_\Omega(\bar{x}) = \partial\delta_\Omega(\bar{x}), \quad \bar{x} \in \Omega, \tag{2.6}$$

via the subdifferential of the set indicator function $\delta_\Omega(x)$ equal to 0 for $x \in \Omega$ and ∞ otherwise. Note that, in spite of (or perhaps due to) the nonconvexity of the normal and subgradient sets (2.1) and (2.5), these constructions along with the associated coderivative of mappings enjoy comprehensive *calculus rules* based on *variational/extremal principles* of variational analysis.

Considering further a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with

$$\operatorname{dom} F := \left\{ x \in \mathbb{R}^n \,\middle|\, F(x) \ne \emptyset \right\} \quad \text{and} \quad \operatorname{gph} F := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \,\middle|\, y \in F(x) \right\},$$

define its *coderivative* [15] at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ by

$$D^*F(\bar{x},\bar{y})(v) := \left\{ u \in \mathbb{R}^n \,\middle|\, (u, -v) \in N_{\operatorname{gph} F}(\bar{x},\bar{y}) \right\}, \quad v \in \mathbb{R}^m \tag{2.7}$$

via the normal cone (2.2) to the graph $\operatorname{gph} F := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m |\ y \in F(x)\}$. The set-valued mapping $D^*F(\bar{x},\bar{y})\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is generally positive homogeneous, while in the case of smooth single-valued mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ (when we omit $\bar{y} = F(\bar{x})$ in the coderivative notation) it reduces to the *adjoint* derivative operator

$$D^*F(\bar{x})(v) = \left\{ \nabla F(\bar{x})^* v \right\}, \quad v \in \mathbb{R}^m.$$

Coming back to extended real-valued functions $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and following [16], define the *second-order subdifferential* (or generalized Hessian) of φ at $\bar{x}$ relative to $\bar{y} \in \partial\varphi(\bar{x})$ by

$$\partial^2\varphi(\bar{x},\bar{y})(u) := (D^*\partial\varphi)(\bar{x},\bar{y})(u), \quad u \in \mathbb{R}^n, \tag{2.8}$$

as the coderivative (2.7) of the first-order subdifferential mapping $\partial\varphi$. Observe that for $\varphi \in \mathcal{C}^2$ with the (symmetric) Hessian matrix $\nabla^2\varphi(\bar{x})$ we have

$$\partial^2\varphi(\bar{x})(u) = \big\{\nabla^2\varphi(\bar{x})u\big\} \text{ for all } u \in \mathbb{R}^n.$$

The second-order subdifferential construction (2.8) and its partial modification considered in section 4 play a crucial role in deriving the main results of this paper.

3. Basic calculations and relationships for SOCPs. The main goal of this section is to establish the equivalence *in the case of SOCPs* between two major *qualification conditions* introduced and developed from completely different viewpoints in general frameworks of variational analysis and optimization. The first condition appeared in [10] and then was used under some modifications in [23, 26] for deriving second-order chain rules of generalized differentiation. It has been recently applied in [26, 27] to characterize tilt and full stability in particular classes of optimization problems with a polyhedral structure and in [4] to derive necessary conditions in problems of optimal control of the sweeping process.

DEFINITION 3.1 (SOCQ for SOCPs). *Let $\bar{x}$ be a feasible solution to* (1.1) *with Q given by* (1.2)*, and $\bar{z} := \Phi(\bar{x}) \in \mathcal{Q}$. We say the* SOCQ *holds at $\bar{x}$ if*

$$\partial^2\delta_{\mathcal{Q}}\big(\bar{z}, \bar{y})(0) \cap \ker\nabla\Phi(\bar{x})^* = \{0\}, \tag{3.1}$$

where $\bar{y} \in N_{\mathcal{Q}}(\bar{z})$, and where $\ker A$ *stands for the kernel of the linear continuous operator A.*

Note that SOCQ (3.1) is automatic when the Jacobian matrix $\nabla\Phi(\bar{x})$ has full rank, which is not required here. It is shown in [27] that in the case of MPPCs the SOCQ condition is equivalent to the *polyhedral constraint qualification* (PCQ) introduced therein while for NLPs it reduces to the classical *linear independence constraint qualification* (LICQ), which amounts to the full rank of $\nabla\Phi(\bar{x})$ and is essentially stronger than the PCQ condition for MPPC.

Next we formulate the *nondegeneracy condition* taken from Bonnans and Shapiro [3, section 4.6] in the case of general conic programming that closely relates to the original one introduced by Robinson [32] under polyhedrality.

DEFINITION 3.2 (nondegeneracy condition for SOCPs). *Let $\bar{x}$ be a feasible solution to* (1.1)*. We say that $\bar{x}$ is a* nondegenerate point *of Φ with respect to $\mathcal{Q}$ if*

$$\nabla\Phi(\bar{x})\mathbb{R}^n + \operatorname{lin}\{T_{\mathcal{Q}}(\bar{z})\} = \prod_{j=1}^{J}\mathbb{R}^{m_j+1}, \tag{3.2}$$

where $T_{\mathcal{Q}}(\bar{z})$ is the tangent cone (2.4) *to $\mathcal{Q}$ at $\bar{z} = \Phi(\bar{x})$, and where* $\operatorname{lin}\{T_{\mathcal{Q}}(\bar{z})\}$ *stands for the largest linear subspace contained in $T_{\mathcal{Q}}(\bar{z})$.*

Taking the orthogonal complements on both sides of (3.2) allows us to rewrite (3.2) as

$$\operatorname{lin}\{T_{\mathcal{Q}}(\bar{z})\}^{\perp} \cap \ker\nabla\Phi(\bar{x})^* = 0. \tag{3.3}$$

Furthermore, since $\mathcal{Q}$ is a closed and convex set, the result of [3, Proposition 4.73] and the duality correspondence (2.3) applied to (3.3) tell us that $\operatorname{lin}\{T_{\mathcal{Q}}(\bar{z})\}^{\perp} = \operatorname{span}\{N_{\mathcal{Q}}(\bar{z})\}$, and so the nondegeneracy condition (3.2) is equivalent to

$$\operatorname{span}\{N_{\mathcal{Q}}(\bar{z})\} \cap \ker\nabla\Phi(\bar{x})^* = 0. \tag{3.4}$$

It follows from (3.1) and (3.4) that to show the equivalence between the SOCQ condition and the nondegeneracy condition for SOCPs we need to calculate the second-order subdifferential of the indicator function in (3.1) and relate it to the span of the normal cone to Ω in (3.4). This will be done in what follows by a series of assertions each of which is of its own independent interest.

To proceed, let us first recall some notions and properties from the *Jordan algebra* in connection with the second-order cone $\mathcal{Q}_{m+1}$; see [9]. Given any $u = (u_0, u_r)$ and $v = (v_0, v_r)$ in $\mathbb{R}^{m+1}$ with $u_0, v_0 \in \mathbb{R}$ and $u_r, v_r \in \mathbb{R}^m$ the *Jordan product* for $\mathcal{Q}_{m+1}$ is defined by

$$u \circ v := \left(\sum_{i=0}^{m} u_i v_i, u_0 v_r + v_0 u_r \right) = (\langle v, u \rangle, u_0 v_r + v_0 u_r).$$

It is well known that for any $u, v \in \mathcal{Q}_{m+1}$ we have $u \circ v = 0$ if and only if $\langle v, u \rangle = 0$, which is also equivalent to the following conditions:

$$\text{(3.5)} \qquad \begin{cases} \text{either } v = 0 \text{ or } u = 0, \\ \text{or there exists } \alpha \text{ such that } v_0 = \alpha u_0 \text{ and } v_r = -\alpha u_r. \end{cases}$$

Therefore for any $u, v \in \mathcal{Q}_{m+1}$ it follows from $\langle v, u \rangle = 0$ and $u, v \neq 0$ that

$$\text{(3.6)} \qquad v = (v_0, v_r) = t(u_0, -u_r),$$

where $t := v_0/u_0$. Given $u \in \mathbb{R}^{m+1}$, its *spectral decomposition* associated with $\mathcal{Q}_{m+1}$ is

$$\text{(3.7)} \qquad u = \lambda_1(u) c_1(u) + \lambda_2(u) c_2(u),$$

where $\lambda_1(u)$ and $\lambda_1(u)$ are the *spectral values* defined by

$$\text{(3.8)} \qquad \lambda_i(u) := u_0 + (-1)^i \|u_r\| \text{ for } i = 1, 2,$$

and where $c_1(u)$ and $c_2(u)$ are the *spectral vectors* of u given by

$$\text{(3.9)} \qquad c_i(u) := \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{u_r}{\|u_r\|}\right) & \text{if } u_r \neq 0, \\ \frac{1}{2}\Big(1, (-1)^i v_r\Big) & \text{if } u_r = 0, \quad i = 1, 2, \end{cases}$$

where v_r is a unit vector in $\mathbb{R}^m$. The decomposition in (3.8) is unique provided that $u_r \neq 0$.

Consider now the (single-valued) *metric projection* operator $\Pi_{\mathcal{Q}_{m+1}} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ onto the convex cone $\mathcal{Q}_{m+1}$. We know that $\Pi_{\mathcal{Q}_{m+1}}$ is continuously differentiable at any point $z \in \mathbb{R}^{m+1}$ with $\det(z) := z_0^2 - \|z_r\|^2 \neq 0$ and

$$\text{(3.10)} \qquad \nabla \Pi_{\mathcal{Q}_{m+1}}(z) = \begin{cases} 0 & \text{if } z_0 < -\|z_r\|, \\ I_{m+1} & \text{if } z_0 > \|z_r\|, \\ \frac{1}{2}\begin{pmatrix} 1 & w_r^T \\ w_r & H \end{pmatrix} & \text{if } -\|z_r\| < z_0 < \|z_r\|, \end{cases}$$

where $w_r := \frac{z_r}{\|z_r\|}$ and $H := (1 + \frac{z_0}{\|z_r\|}) I_m - \frac{z_0}{\|z_r\|} w_r w_r^T$. The following major result taken from Outrata and Sun [29, Theorems 3, 4] gives us an explicit calculation of the coderivative (2.7) of the projection operator $\Pi_{\mathcal{Q}_{m+1}}$ onto the Lorentz cone (1.3).

LEMMA 3.3 (coderivative of the metric projection onto the Lorentz cone). *For an arbitrary vector $z \in \mathbb{R}^{m+1}$ consider its spectral decomposition* (3.7) *and take any $u \in \mathbb{R}^{m+1}$. Then the coderivative of the projection operator $\Pi_{\mathcal{Q}_{m+1}}$ is calculated as follows.*

(i) *If* $\det(z) \neq 0$, *then* $D^*\Pi_{\mathcal{Q}_{m+1}}(z)(u) = \nabla\Pi_{\mathcal{Q}_{m+1}}(z)u$.

(ii) *If* $\det(z) = 0$ *but* $\lambda_2(z) \neq 0$, *i.e.*, $z \in \mathrm{bd}\,\mathcal{Q}_{m+1}\backslash\{0\}$, *then*

$$D^*\Pi_{\mathcal{Q}_{m+1}}(z)(u) = \begin{cases} \mathrm{co}\,\{u, A(z)u\} & \text{if } \langle u, c_1(z)\rangle \geq 0, \\ \{u, A(z)u\} & \text{otherwise,} \end{cases} \tag{3.11}$$

where the matrix $A(z)$ *is defined by* $A(z) := I_{m+1} + \frac{1}{2}\begin{pmatrix} -1 & \frac{z_r^T}{\|z_r\|} \\ \frac{z_r}{\|z_r\|} & -\frac{z_r z_r^T}{\|z_r\|^2} \end{pmatrix}$.

(iii) *If* $\det(z) = 0$ *but* $\lambda_1(z) \neq 0$, *i.e.*, $z \in \mathrm{bd}\,(-\mathcal{Q}_{m+1})\backslash\{0\}$, *then*

$$D^*\Pi_{\mathcal{Q}_{m+1}}(z)(u) = \begin{cases} \mathrm{co}\,\{0, B(z)u\} & \text{if } \langle u, c_1(z)\rangle \geq 0, \\ \{0, B(z)u\} & \text{otherwise,} \end{cases} \tag{3.12}$$

where the matrix $B(z)$ *is defined by* $B(z) := \frac{1}{2}\begin{pmatrix} 1 & \frac{z_r^T}{\|z_r\|} \\ \frac{z_r}{\|z_r\|} & \frac{z_r z_r^T}{\|z_r\|^2} \end{pmatrix}$.

(iv) *If* $z = 0$, *then we have*

$$\begin{aligned} D^*\Pi_{\mathcal{Q}_{m+1}}(0)(u) &= \mathcal{D}u \cup \Big(\mathcal{Q}_{m+1} \cap (u - \mathcal{Q}_{m+1})\Big) \\ &\quad \cup \bigcup_{A \in \mathcal{A}} \mathrm{co}\,\{u, Au\} \cup \bigcup_{B \in \mathcal{B}} \mathrm{co}\,\{0, Bu\}, \end{aligned} \tag{3.13}$$

where the sets $\mathcal{D}$, $\mathcal{A}$, *and* $\mathcal{B}$ *are defined, respectively, by*

$$\begin{aligned} \mathcal{D} &:= \Big\{0, I_{m+1}\Big\} \\ &\quad \bigcup \left\{ \frac{1}{2}\begin{pmatrix} 1 & w^T \\ w & 2\alpha I_m + (1-2\alpha)ww^T \end{pmatrix} \,\middle|\, w \in \mathbb{R}^m,\ \|w\| = 1,\ \alpha \in [0,1] \right\}, \\ \mathcal{A} &:= \left\{ I_{m+1} + \frac{1}{2}\begin{pmatrix} -1 & w^T \\ w & -ww^T \end{pmatrix} \,\middle|\, w \in \mathbb{R}^m,\ \|w\| = 1,\ u_0 - \langle u_r, w\rangle \geq 0 \right\}, \\ \mathcal{B} &:= \left\{ \frac{1}{2}\begin{pmatrix} 1 & w^T \\ w & ww^T \end{pmatrix} \,\middle|\, w \in \mathbb{R}^m,\ \|w\| = 1,\ u_0 + \langle u_r, w\rangle \geq 0 \right\}. \end{aligned}$$

The next lemma presents two coderivative properties for the projection operator onto the Lorentz cone that are important for deriving the full stability characterizations in section 4.

LEMMA 3.4 (coderivative properties for the metric projection). *Let* $\mathcal{C}$ *be a convex subset of* $\mathbb{R}^n$. *Then for any* $z \in \mathcal{C}$, $u \in \mathbb{R}^n$, *and* $q \in D^*\Pi_{\mathcal{C}}(z)(u)$, *the following properties hold:*

(i) $\langle q, u\rangle \geq 0$ *and*

(ii) $\langle q, u - q\rangle \geq 0$.

Proof. Fix $q \in D^*\Pi_{\mathcal{C}}(z)(u)$ and recall the well-known relationship

$$\mathrm{co}\,D^*\Pi_{\mathcal{C}}(z)(u) = \Big\{ A^*u \,\Big|\, A \in \mathcal{J}\Pi_{\mathcal{C}}(z) \Big\}$$

between the coderivative and (Clarke's) *generalized Jacobian* of any single-valued locally Lipschitzian mapping; see, e.g., [15, 33]. Thus we find $A \in \mathcal{J}\Pi_{\mathcal{C}}(z)$ with $q = A^*u$. It follows from [13, Proposition 1(i)] that $A^* = A$, and so we have $q = Au$. Employing finally the assertions from [13, Propositions 1(ii), (iii)] that ensure properties (i) and (ii) of this lemma for the generalized Jacobian of $\Pi_{\mathcal{C}}$ completes the proof of the claimed coderivative properties. ∎

The next step is to calculate the normal cone $N_{\mathcal{Q}_{m+1}}(z)$ at $z \in \mathcal{Q}_{m+1}$, which is the crucial ingredient of the nondegeneracy condition (3.4). We have the following result.

LEMMA 3.5 (calculating normals to the Lorentz cone). *For any $z := (z_0, z_r) \in \mathcal{Q}_{m+1}$ the normal cone to $\mathcal{Q}_{m+1}$ at z is calculated by*

$$N_{\mathcal{Q}_{m+1}}(z) = \begin{cases} -\mathcal{Q}_{m+1} & \text{if } z = 0, \\ \{0\} & \text{if } z \in \operatorname{int} \mathcal{Q}_{m+1}, \\ \left\{(-tz_0, tz_r) \,\middle|\, t \geq 0\right\} & \text{if } z \in \operatorname{bd} \mathcal{Q}_{m+1} \setminus \{0\}. \end{cases} \tag{3.14}$$

Proof. The proof follows from the calculation of the tangent cone to $\mathcal{Q}_{m+1}$ given in [2, Lemma 25] and the duality between the normal and tangent cones under consideration. $\square$

In the sequel we often use the *self-dual property* $\mathcal{Q}_{m+1} = -\mathcal{Q}^*_{m+1}$ of the Lorentz cone $\mathcal{Q}_{m+1}$, which implies the equivalence

$$q \in N_{\mathcal{Q}_{m+1}}(z) \Longleftrightarrow \begin{cases} -q \in \mathcal{Q}_{m+1}, \\ \langle q, z \rangle = 0, \\ z \in \mathcal{Q}_{m+1}. \end{cases} \tag{3.15}$$

Furthermore, Lemma 3.5 immediately implies the span representation

$$\operatorname{span}\{N_{\mathcal{Q}_{m+1}}(z)\} = \begin{cases} \operatorname{span}\{-\mathcal{Q}_{m+1}\} = \mathbb{R}^{m+1} & \text{if } z = 0, \\ \{0\} & \text{if } z \in \operatorname{int} \mathcal{Q}_{m+1}, \\ \left\{(-tz_0, tz_r) \,\middle|\, t \in \mathbb{R}\right\} & \text{if } z \in \operatorname{bd} \mathcal{Q}_{m+1} \setminus \{0\}. \end{cases} \tag{3.16}$$

It is easy to extend Lemma 3.5 to the product cone $\mathcal{Q} = \prod_{j=1}^{J} \mathcal{Q}_{m_j+1}$ by the well-known product formula for the normal cone:

$$N_{\mathcal{Q}}(z) = \prod_{j=1}^{J} N_{\mathcal{Q}_{m_j+1}}(z_j), \tag{3.17}$$

where $z\colon = (z_1, \ldots, z_J) \in \mathcal{Q}$. We can also observe that

$$\operatorname{span}\left\{\prod_{j=1}^{J} A_j\right\} = \prod_{j=1}^{J} \operatorname{span}\{A_j\}$$

provided that the sets A_j, $j = 1, \ldots, J$, are cones. Thus plugging $A_j = N_{\mathcal{Q}_{m_j+1}}(z_j)$ into this formula gives us the representation

$$\operatorname{span}\{N_{\mathcal{Q}}(z)\} = \prod_{j=1}^{J} \operatorname{span}\{N_{\mathcal{Q}_{m_j+1}}(z_j)\}. \tag{3.18}$$

To proceed further, consider first the case of $\mathcal{Q} = \mathcal{Q}_{m+1}$ in (1.1) when (3.1) reduces to

$$\partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, \bar{y})(0) \cap \ker \nabla\Phi(\bar{x})^* = \{0\}, \quad \bar{y} \in N_{\mathcal{Q}_{m+1}}(\bar{z}). \tag{3.19}$$

It follows from [28, Lemma 19] that

$$q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}\big(\bar{z}, \bar{y}\big)(p) \Longleftrightarrow -p \in D^*\Pi_{\mathcal{Q}_{m+1}}(\bar{z}+\bar{y}, \bar{z})(-q-p) \tag{3.20}$$

when $\bar{z} \in \mathcal{Q}_{m+1}$ and $\bar{y} \in N_{\mathcal{Q}_{m+1}}(\bar{z})$. Employing now the lemmas above and relationship (3.20) allow us to calculate the second-order subdifferential of $\delta_{\mathcal{Q}_{m+1}}$ at the origin $p = 0$, which leads consequently to establishing the equivalence between the SOCQ and nondegeneracy conditions.

THEOREM 3.6 (calculating the second-order subdifferential for the indicator function of the Lorentz cone). *Let* $z\colon = (z_0, z_r) \in \mathcal{Q}_{m+1}$*, and let* $-y \in N_{\mathcal{Q}_{m+1}}(z)$*. Then we have*

$$\partial^2 \delta_{\mathcal{Q}_{m+1}}(z, -y)(0) = \operatorname{span}\big\{N_{\mathcal{Q}_{m+1}}(z)\big\}. \tag{3.21}$$

Proof. It follows from the property (3.15) that the inclusion $-y \in N_{\mathcal{Q}_{m+1}}(z)$ yields

$$y \in \mathcal{Q}_{m+1} \ \text{ and } \ \langle z, y\rangle = 0. \tag{3.22}$$

We split the proof of (3.21) into the following six cases according to the position of $z, y \in \mathcal{Q}_{m+1}$.

Case 1: $z = 0$, $y \in \operatorname{int}\mathcal{Q}_{m+1}$. Take $q \in \partial^2\delta_{\mathcal{Q}_{m+1}}(0,-y)(0)$, which by (3.20) amounts to $0 \in D^*\Pi_{\mathcal{Q}_{m+1}}(-y, 0)(-q)$. Invoking Lemma 3.3(i) together with (3.10), we have

$$D^*\Pi_{\mathcal{Q}_{m+1}}(-y, 0)(-q) = \nabla\Pi_{\mathcal{Q}_{m+1}}(-y)(-q) = 0(-q) = 0$$

due to $-y_0 < -\|y_r\|$. This tells us that $\partial^2\delta_{\mathcal{Q}_{m+1}}\big(0,-y\big)(0) = \mathbb{R}^{m+1}$. Employing now Lemma 3.5 concludes the proof of (3.21) in this case.

Case 2: $z \in \operatorname{int}\mathcal{Q}_{m+1}$. This gives us $y = 0$. Take $q \in \partial^2\delta_{\mathcal{Q}_{m+1}}(z, 0)(0)$, which by (3.20) is equivalent to $0 \in D^*\Pi_{\mathcal{Q}_{m+1}}(z,z)(-q)$. Since $z \in \operatorname{int}\mathcal{Q}_{m+1}$, it says that $z_0 > \|z_r\|$, and thus by Lemma 3.3(i) together with (3.10) we get

$$0 \in D^*\Pi_{\mathcal{Q}_{m+1}}(z,z)(-q) = \nabla\Pi_{\mathcal{Q}_{m+1}}(z)(-q) = I_{m+1}(-q) = -q \Longleftrightarrow q = 0.$$

This shows that $\partial^2\delta_{\mathcal{Q}_{m+1}}\big(z,0\big)(0) = \{0\}$. Observing now that $\operatorname{span}\{N_{\mathcal{Q}_{m+1}}(z)\} = \{0\}$ due to $z \in \operatorname{int}\mathcal{Q}_{m+1}$ completes the proof of (3.21) in this case.

Case 3: $z, y \in \operatorname{bd}\mathcal{Q}_{m+1}\backslash\{0\}$. Denote $c := z - y$ and then deduce from (3.6) and (3.22) that

$$c = (c_0, c_r) = (z_0 - y_0, z_r - y_r) = \left(z_0 - y_0, -\frac{z_0+y_0}{y_0}y_r\right) = \left(z_0 - y_0, \frac{z_0+y_0}{z_0}z_r\right), \tag{3.23}$$

which gives us therefore that

$$\|c_r\| = \left|\frac{z_0+y_0}{y_0}\right| \cdot \|y_r\| = |z_0+y_0| = z_0 + y_0. \tag{3.24}$$

Pick $q \in \partial^2\delta_{\mathcal{Q}_{m+1}}(z,-y)(0)$ and write down, by (3.20), that

$$0 \in D^*\Pi_{\mathcal{Q}_{m+1}}(z-y, z)(-q). \tag{3.25}$$

Taking into account that $-\|c_r\| < c_0 < \|c_r\|$ and employing Lemma 3.3(i), we arrive at

$$D^*\Pi_{\mathcal{Q}_{m+1}}(z-y,z)(-q) = \frac{1}{2}\begin{pmatrix} 1 & \frac{c_r^T}{\|c_r\|} \\ \frac{c_r}{\|c_r\|} & H \end{pmatrix}\begin{pmatrix} -q_0 \\ -q_r \end{pmatrix}$$
$$\text{with } H := \left(1 + \frac{c_0}{\|c_r\|}\right) I_m - \frac{c_0}{\|c_r\|}\frac{c_r c_r^T}{\|c_r\|^2}.$$

This allows us to restate condition (3.25) in the form

$$\begin{pmatrix} 1 & \frac{c_r^T}{\|c_r\|} \\ \frac{c_r}{\|c_r\|} & H \end{pmatrix}\begin{pmatrix} -q_0 \\ -q_r \end{pmatrix} = 0$$

with $q = (q_0, q_r)$, which can be equivalently rewritten as

$$\text{(3.26)} \qquad \begin{cases} q_0 + \dfrac{1}{\|c_r\|}\langle c_r, q_r\rangle = 0, \\ \dfrac{q_0}{\|c_r\|}c_r + \dfrac{2z_0}{z_0+y_0}q_r - \dfrac{z_0-y_0}{z_0+y_0}\dfrac{c_r}{\|c_r\|^2}\langle c_r, q_r\rangle = 0. \end{cases}$$

Involving the last equality in (3.23), the first equation in (3.26) can be reformulated as

$$q_0 + \left\langle \frac{z_r}{z_0}, q_r \right\rangle = 0,$$

which, by taking into account that $z_0 \neq 0$, shows that

$$\text{(3.27)} \qquad \langle q, z\rangle = q_0 z_0 + \langle z_r, q_r\rangle = 0.$$

Furthermore, it follows that $\langle c_r, q_r\rangle = -q_0\|c_r\|$ due to the first equation in (3.26), and thus

$$\text{(3.28)} \qquad \begin{aligned} 0 &= \frac{q_0}{\|c_r\|}c_r + \frac{2z_0}{z_0+y_0}q_r - \frac{z_0-y_0}{z_0+y_0}\frac{c_r}{\|c_r\|^2}\langle c_r, q_r\rangle \\ &= \frac{q_0}{\|c_r\|}c_r + \frac{2z_0}{z_0+y_0}q_r - \frac{z_0-y_0}{z_0+y_0}\frac{c_r}{\|c_r\|^2}(-q_0\|c_r\|) \\ &= \frac{2z_0}{z_0+y_0}q_r + \frac{q_0}{\|c_r\|}\frac{2z_0}{z_0+y_0}c_r, \end{aligned}$$

which yields $\|c_r\|q_r = -q_0 c_r$ and so $\|q_r\| = \frac{|q_0|}{\|c_r\|}\|c_r\| = |q_0|$. This tells us that either $q \in \mathcal{Q}_{m+1}$ or $-q \in \mathcal{Q}_{m+1}$. Using it together with (3.27) and (3.15), observe that $q = (-tz_0, tz_r)$ for some $t \in \mathbb{R}$, which verifies the inclusion $\partial^2\delta_{\mathcal{Q}_{m+1}}(z,-y)(0) \subset \text{span}\{N_{\mathcal{Q}_{m+1}}(z)\}$ by Lemma 3.5.

To justify the converse inclusion in this case, take $q := (-tz_0, tz_r) \in \text{span}\{N_{\mathcal{Q}_{m+1}}(z)\}$ for some $t \in \mathbb{R}$. It suffices to show that the relationships in (3.26) hold for the chosen vector q, which is surely equivalent to $q \in \partial^2\delta_{\mathcal{Q}_{m+1}}(z,-y)(0)$. Indeed, by $z_0 = \|z_r\|$ we have

$$-tz_0 + \frac{1}{\|c_r\|}\langle c_r, tz_r\rangle = t\left(-z_0 + \frac{\|z_r\|^2}{z_0}\right) = 0$$

implying the first equation in (3.26). The other one therein means that, by the calculation in (3.28) with $q=(-tz_0,tz_r)$ and $c_r=\frac{z_0+y_0}{z_0}z_r$, that

$$\frac{2z_0}{z_0+y_0}q_r+\frac{q_0}{\|c_r\|}\frac{2z_0}{z_0+y_0}c_r=0,$$

which is definitely true and thus verifies (3.21) in this case.

Case 4: $z=0$, $y\in \text{bd}\,\mathcal{Q}_{m+1}\backslash\{0\}$. As shown in (3.16), we have $\text{span}\,\{N_{\mathcal{Q}_{m+1}}(z)\}=\mathbb{R}^{m+1}$. Take $q\in\partial^2\delta_{\mathcal{Q}_{m+1}}(0,-y)(0)$, which means by (3.20) that $0\in D^*\Pi_{\mathcal{Q}_{m+1}}(-y,0)(-q)$. Since $-y\in\text{bd}\,(-\mathcal{Q}_{m+1})\backslash\{0\}$, it follows from Lemma 3.3(iii) that $0\in D^*\Pi_{\mathcal{Q}_{m+1}}(-y,0)(-q)$ for all $q\in\mathbb{R}^{m+1}$. This verifies that $\partial^2\delta_{\mathcal{Q}_{m+1}}(0,-y)(0)=\mathbb{R}^{m+1}$ and thus justifies (3.21) in this case.

Case 5: $z\in\text{bd}\,\mathcal{Q}_{m+1}\backslash\{0\}$, $y=0$. By (3.16) we have that

$$\text{span}\,\{N_{\mathcal{Q}_{m+1}}(z)\}=\Big\{(-tz_0,tz_r)\Big|\ t\in\mathbb{R}\Big\}.$$

Take $q:=(q_0,q_r)\in\partial^2\delta_{\mathcal{Q}_{m+1}}(z,0)(0)$, which is equivalent by (3.20) to $0\in D^*\Pi_{\mathcal{Q}_{m+1}}(z,z)(-q)$. Employing Lemma 3.3(ii) with $z\in\text{bd}\,\mathcal{Q}_{m+1}\backslash\{0\}$ tells us that the latter inclusion amounts to

$$G_sq=0\ \text{ for }\ G_s:=(1-s)I_{m+1}+sA(z),\ s\in[0,1],\tag{3.29}$$

where $A(z)$ is defined in Lemma 3.3(ii). We claim that the solution set $\Lambda(s)$ to (3.29) is

$$\Lambda(s)=\begin{cases}\{0\} & \text{if }\ s\in[0,1),\\ \text{span}\,\{N_{\mathcal{Q}_{m+1}}(z)\} & \text{if }\ s=1.\end{cases}\tag{3.30}$$

To verify (3.30), observe that for $s=0$ (3.29) reduces to $I_{m+1}q=0$, which clearly has only the trivial solution $q=0$. For $s=1$ (3.29) is equivalent to

$$A(z)q=q+\frac{1}{2}\begin{pmatrix}-1 & \frac{z_r^T}{\|z_r\|}\\ \frac{z_r}{\|z_r\|} & -\frac{z_rz_r^T}{\|z_r\|^2}\end{pmatrix}\begin{pmatrix}q_0\\ q_r\end{pmatrix}=0,$$

which reduces by $\|z_r\|=z_0$ to the system of equations

$$\begin{cases}q_0+\dfrac{1}{z_0}\langle z_r,q_r\rangle=0,\\[2mm] q_r+\dfrac{q_0}{2z_0}z_r-\dfrac{\langle z_r,q_r\rangle}{2z_0^2}z_r=0.\end{cases}\tag{3.31}$$

Take now $q\in\text{span}\,\{N_{\mathcal{Q}_{m+1}}(z)\}$ meaning that $q=(-tz_0,tz_r)$ for some $t\in\mathbb{R}$. Then q surely satisfies (3.31), and thus $q\in\Lambda(1)$. Conversely, let $q\in\Lambda(1)$ and show that $q\in\text{span}\,\{N_{\mathcal{Q}_{m+1}}(z)\}$. Indeed, reformulating the first equation in (3.31) gives us

$$\langle q,z\rangle=q_0z_0+\langle z_r,q_r\rangle=0,\tag{3.32}$$

and we get from the second equation in (3.31) that $q_r=-\frac{q_0}{z_0}z_r$; hence $\|q_r\|=|q_0|$. It allows us to deduce that either $q\in\mathcal{Q}_{m+1}$ or $-q\in\mathcal{Q}_{m+1}$. Using this together with (3.32) and self-dual property (3.15), we observe that $q=(-tz_0,tz_r)$ for some $t\in\mathbb{R}$, which verifies (3.30) for $s=1$. Considering finally the case of $s\in(0,1)$, we can show similarly to the previous calculation that the equation $G_s q = 0$ is equivalent to the following system:

$$\left\{\begin{array}{l}\left(1-\dfrac{s}{2}\right)q_0+\dfrac{s}{2z_0}\langle z_r,q_r\rangle=0,\\[2mm] q_r+\dfrac{sq_0}{2z_0}z_r-\dfrac{s\langle z_r,q_r\rangle}{2z_0^2}z_r=0.\end{array}\right. \tag{3.33}$$

Let us verify that $q\colon = (q_0, q_r) = 0$. Indeed, assuming the contrary tells us that $q_0 \neq 0$, since otherwise we get $q_r = 0$ by the equations in (3.33). Substituting now the expression $\langle z_r, q_r\rangle = -(\frac{2}{s}-1)z_0q_0$ from the first equation into the second equation in (3.33), we arrive at

$$0=q_r+\frac{sq_0}{2z_0}z_r+\frac{s(\frac{2}{s}-1)z_0q_0}{2z_0^2}z_r=q_r+\frac{q_0}{z_0}z_r.$$

Multiplying the latter by z_r and taking into account that $\|z_r\| = z_0$ lead us to

$$\left(1-\frac{2}{s}\right)z_0q_0+q_0z_0=0,$$

which yields $s = 1$ due to $z_0q_0 \neq 0$. This contradicts the assumption on $s \in (0, 1)$ and hence shows that $q = 0$. Invoking (3.30) justifies the inclusion $\partial^2\delta_{\mathcal{Q}_{m+1}}(0,-y)(0) \subset \operatorname{span}\{N_{\mathcal{Q}_{m+1}}(z)\}$.

To verify the converse inclusion, take $q \in \operatorname{span}\{N_{\mathcal{Q}_{m+1}}(z)\}$ and get from the above that it is a solution to (3.29) for $s = 1$. This means that $0 \in D^*\Pi_{\mathcal{Q}_{m+1}}(z,z)(-q)$, which implies by (3.20) that $q \in \partial^2\delta_{\mathcal{Q}_{m+1}}(z,0)(0)$ and completes the proof of (3.21) in this case.

Case 6: $y, z = 0$. Since $\operatorname{span}\{N_{\mathcal{Q}_{m+1}}(z)\} = \mathbb{R}^{m+1}$ in this case and since $0 \in D^*\Pi_{\mathcal{Q}_{m+1}}(0,0)(q)$ for any $q \in \mathbb{R}^{m+1}$ by Lemma 3.3(iv), we get similarly to Case 4 that $\partial^2\delta_{\mathcal{Q}_{m+1}}(z,0)(0) = \mathbb{R}^{m+1}$, which completes the proof of the theorem. ∎

This theorem immediately implies the equivalence between the second-order qualification and nondegeneracy conditions for SOCPs with only one Lorentz cone $\mathcal{Q} = \mathcal{Q}_{m+1}$ in (1.1).

COROLLARY 3.7 (equivalence between SOCQ and nondegeneracy for SOCPs with one Lorentz cone). *Consider SOCP* (1.1) *with one Lorentz cone* $\mathcal{Q} = \mathcal{Q}_{m+1}$ *therein. Then for any* $\bar{x} \in \mathbb{R}^n$ *with* $\bar{z} = \Phi(\bar{x}) \in \mathcal{Q}_{m+1}$ *and any* $\bar{y} \in -N_{\mathcal{Q}_{m+1}}(\bar{z})$ *we have that SOCQ* (3.19) *amounts to saying that* $\bar{x}$ *is an SOCP nondegenerate point* (3.2) *with* $\mathcal{Q} = \mathcal{Q}_{m+1}$ *and* $J = 1$.

Proof. As mentioned above, the nondegeneracy condition (3.2) in this case can be rewritten in the dual form (3.4). The latter reduces to SOCQ (3.19) by Theorem 3.6. ∎

To extend the equivalence in Corollary 3.7 to the general case of the product cone $\mathcal{Q}$ in (1.1), we need the following product lemma for the second-order subdifferential of set indicators. Although we present this result for products of two sets, it can be easily obtained by induction for the case of finitely many sets in the product.

LEMMA 3.8 (second-order subdifferential of the indicator function for product sets). *Let* $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^m$*, let* $x = (x_1, x_2) \in \Omega$*, and let* $y = (y_1, y_2) \in N_\Omega(x)$*. Then for any* $u = (u_1, u_2) \in \mathbb{R}^n \times \mathbb{R}^m$ *we have the equality*

$$\partial^2\delta_\Omega(x,y)(u)=\partial^2\delta_{\Omega_1}(x_1,y_1)(u_1)\times\partial^2\delta_{\Omega_2}(x_2,y_2)(u_2). \tag{3.34}$$

Proof. The well-known product formula for the normal cone (2.1) tells us that

$$N_{\Omega_1\times\Omega_2}(x_1,x_2) = N_{\Omega_1}(x_1)\times N_{\Omega_2}(x_2), \tag{3.35}$$

and thus $y_i \in N_{\Omega_i}(x_i)$ for $i = 1, 2$ in the notation above. By definition (2.8) of the second-order subdifferential in the case of $\varphi = \delta_\Omega$ and by formula (2.6) we have

$$\partial^2\delta_\Omega(x,y)(u) = (D^*\partial\delta_{\Omega_1\times\Omega_2})(x,y)(u) = (D^*N_{\Omega_1\times\Omega_2})(x,y)(u).$$

Furthermore, it is easy to deduce from the coderivative definition (2.7) and formula (3.35) that

$$(D^*N_{\Omega_1\times\Omega_2})(x,y)(u) = (D^*N_{\Omega_1})(x_1,y_1)(u_1)\times(D^*N_{\Omega_2})(x_2,y_2)(u_2).$$

Combining all the above verifies the claimed result (3.34). □

Now we are ready to establish the main result of this section.

THEOREM 3.9 (equivalence between SOCQ and nondegeneracy conditions for general SOCPs). *Consider the general SOCP* (1.1) *with the product cone* (1.2) *and take any* $\bar{x} \in \mathbb{R}^n$ *with* $\bar{z} = (\bar{z}^1,\ldots,\bar{z}^J) = \Phi(\bar{x}) \in \mathcal{Q}$ *and* $\bar{y} = (\bar{y}^1,\ldots,\bar{y}^J) \in -N_{\mathcal{Q}}(\bar{z})$. *Then SOCQ* (3.1) *is equivalent to the nondegeneracy condition* (3.2).

Proof. Letting $u = 0$ in Lemma 3.8 in the case of $\Omega = \mathcal{Q}$ therein, we get that

$$v := (v^1,\ldots,v^J) \in \partial^2\delta_{\mathcal{Q}}(\bar{z},-\bar{y})(0) \Longleftrightarrow v^j \in \partial^2\delta_{\mathcal{Q}_{m_j+1}}(\bar{z}^j,-\bar{y}^j)(0), \quad j = 1,\ldots,J.$$

This gives us by Corollary 3.7 that

$$\partial^2\delta_{\mathcal{Q}}(\bar{z},-\bar{y})(0) = \prod_{j=1}^{J}\partial^2\delta_{\mathcal{Q}_{m_j+1}}(\bar{z}_j,-\bar{y}_j)(0) = \prod_{j=1}^{J}\operatorname{span}\left\{N_{\mathcal{Q}_{m_j+1}}(\bar{z}_j)\right\}.$$

Now using the product relationship (3.18), we arrive at

$$\partial^2\delta_{\mathcal{Q}}(\bar{z},-\bar{y})(0) = \operatorname{span}\left\{\prod_{j=1}^{J}N_{\mathcal{Q}_{m_j+1}}(z_j)\right\} = \operatorname{span}\{N_{\mathcal{Q}}(z)\}, \tag{3.36}$$

which completes the proof of the theorem due to the equivalence between forms (3.2) and (3.4) of the nondegeneracy condition for SOCPs. □

4. Characterizations of full stability for SOCPs. In this section we derive complete second-order characterizations of full stability of local optimal solutions to SOCPs (1.1) entirely in terms of their initial data. As mentioned in section 1, the notion of full stability was introduced in [11] in the unconstrained framework of optimization with extended-real-valued objectives. To adopt this approach for the case of SOCPs under consideration, we consider the two-parametric version $\mathcal{P}(w,v)$ of (1.1) written as

$$\text{minimize } \ \varphi_0(x,w) + \delta_{\mathcal{Q}}(\Phi(x,w)) - \langle v,x\rangle \ \text{ over } \ x \in \mathbb{R}^n \tag{4.1}$$

with $\Phi(x,w) = (\Phi^1(x,w),\ldots,\Phi^J(x,w))$ and the product cone $\mathcal{Q}$ taken from (1.2). Recall that the mappings φ_0 and Φ^j, $j = 1,\ldots,J$, are assumed to be twice continuously differentiable around the reference points. In accordance with [11], the vector

$w \in \mathbb{R}^d$ signifies *basic* perturbations while the linear parametric shift of the objective with $v \in \mathbb{R}^n$ represents *tilt* perturbations. Define the function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^d \to \overline{\mathbb{R}}$ by

$$\varphi(x,w) := \varphi_0(x,w) + \delta_{\mathcal{Q}}(\Phi(x,w)) \ \text{ with } \ x \in \mathbb{R}^n,\ w \in \mathbb{R}^d \tag{4.2}$$

and fix in what follows the triple $(\bar{x},\bar{w},\bar{v})$ such that $\Phi(\bar{x},\bar{w}) \in \mathcal{Q}$ and $\bar{v} \in \partial_x\varphi(\bar{x},\bar{w})$. Given a number $\gamma > 0$, consider the (local) optimal value function

$$m_\gamma(w,v) := \inf_{\|x-\bar{x}\|\le\gamma} \Big\{\varphi(x,w) - \langle v,x\rangle\Big\}, \quad (w,v) \in \mathbb{R}^d \times \mathbb{R}^n, \tag{4.3}$$

and the corresponding parametric family of optimal solution sets in (4.1) defined by

$$M_\gamma(w,v) := \mathrm{argmin}_{\|x-\bar{x}\|\le\gamma}\Big\{\varphi(x,w) - \langle v,x\rangle\Big\}, \quad (w,v) \in \mathbb{R}^d \times \mathbb{R}^n, \tag{4.4}$$

with the convention that argmin:=$\emptyset$ when the expression under minimization is ∞. In these terms, $\bar{x}$ is a *locally optimal solution* to $\mathcal{P}(\bar{w},\bar{v})$ if $\bar{x} \in M_\gamma(\bar{w},\bar{v})$ for some $\gamma > 0$ sufficiently small.

DEFINITION 4.1 (full stability). *A point $\bar{x}$ is a fully stable locally optimal solution to problem $\mathcal{P}(\bar{w},\bar{v})$ if there exist a number $\gamma > 0$ and neighborhoods W of $\bar{w}$ and V of $\bar{v}$ such that the mapping $(w,v) \mapsto M_\gamma(w,v)$ is single valued and Lipschitz continuous with $M_\gamma(\bar{w},\bar{v}) = \bar{x}$ and the function $(w,v) \mapsto m_\gamma(w,v)$ is likewise Lipschitz continuous on $W \times V$.*

When the basic perturbation vector w is absent, Definition 4.1 corresponds to *tilt stability* of local minimizers introduced in [30] in the general extended real-valued framework of unconstrained optimization. In this case the value function (4.3) is automatically locally Lipschitzian.

We proceed with characterizing full stability (as well as tilt stability as a particular case) for SOCPs under appropriate constraint qualifications. Recall (cf. [31] and [3, Definition 2.86]) that the *Robinson constraint qualification* (RCQ) with respect to x holds for (1.1) at $(\bar{x},\bar{w})$ if

$$0 \in \mathrm{int}\Big\{\Phi(\bar{x},\bar{w}) + \nabla_x\Phi(\bar{x},\bar{w})\mathbb{R}^n - \mathcal{Q}\Big\}. \tag{4.5}$$

It is well known that this condition can be equivalently described as

$$N_{\mathcal{Q}}(\Phi(\bar{x},\bar{w})) \cap \ker \nabla_x\Phi(\bar{x},\bar{w})^* = \{0\}, \tag{4.6}$$

which obviously reduces to the classical *Mangasarian–Fromovitz constraint qualification* with respect to x in the case of NLP.

Let us first present a characterization of full stability of locally optimal solutions to SOCPs under RCQ (4.5). Consider the partial first-order subdifferential mapping $\partial_x\varphi\colon \mathbb{R}^n \times \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ of the function φ from (4.2) and define the *partial inverse* of $\partial_x\varphi$ by

$$S_\varphi(w,v) := \Big\{x \in \mathbb{R}^n \,\Big|\, v \in \partial_x\varphi(x,w)\Big\}. \tag{4.7}$$

Observe that under the validity of RCQ at $(\bar{x},\bar{w})$, and hence around this point, we have the partial subdifferential representation

$$\partial_x\varphi(x,w) = \nabla_x\varphi_0(x,w) + \nabla_x\Phi(x,w)^* N_{\mathcal{Q}}(\Phi(x,w)) \ \text{ near } \ (\bar{x},\bar{w}). \tag{4.8}$$

It follows by employing the elementary subdifferential sum rule in (4.2) and then the chain rule from [33, Exercise 10.26(b)] for the amenable composition $\delta_{\mathcal{Q}}(\Phi(x,w))$ valid under RCQ in the equivalent form (4.6) due to the convexity of the second-order product cone $\mathcal{Q}$. This implies the local representation of the partial inverse (4.7) in terms of the initial data of (1.1) by

$$S_\varphi(w,v)=\Big\{x\in\mathbb{R}^n\Big|\ v\in\nabla_x\varphi_0(x,w)+\nabla_x\Phi(x,w)^*N_{\mathcal{Q}}(\Phi(x,w))\Big\}. \tag{4.9}$$

Following [27, Definition 3.3], we say that the mapping $\partial_x\varphi\colon\mathbb{R}^n\times\mathbb{R}^d\rightrightarrows\mathbb{R}^n$ is *partially strongly metrically regular* (PSMR) at $(\bar{x},\bar{w},\bar{v})$ if its partial inverse (4.7) admits a Lipschitzian single-valued localization around this point. In the parameter-independent case of $\varphi(x,w)=\varphi(x)$ this notion reduces to strong metric regularity [5] of the subgradient mapping $\partial\varphi$ defined as an abstract version of Robinson's strong regularity [31] for generalized equations; see section 5.

The next result is actually a constructive implementation of the general full stability characterization from [27, Theorem 3.4] to the case of SOCPs. Its tilt stability versions in various optimization settings were given in [6, 12, 24, 27].

THEOREM 4.2 (PSMR characterization of full stability in SOCPs under RCQ). *Let $\Phi(\bar{x},\bar{w})\in\mathcal{Q}$ for SOCP* (1.1)*, and let RCQ* (4.5) *hold at $(\bar{x},\bar{w})$. Then $\bar{x}$ is a fully stable locally optimal solution to $\mathcal{P}(\bar{w},\bar{v})$ in* (4.1) *with $\bar{v}$ satisfying the condition*

$$\bar{v}\in\nabla_x\varphi_0(\bar{x},\bar{w})+\nabla_x\Phi(\bar{x},\bar{w})^*N_{\mathcal{Q}}(\Phi(\bar{x},\bar{w})) \tag{4.10}$$

if and only if $\bar{x}\in M_\gamma(\bar{w},\bar{v})$ for some $\gamma>0$ and the mapping $\partial_x\varphi$ from (4.8) *is PSMR at $(\bar{x},\bar{w},\bar{v})$.*

Proof. We apply [27, Theorem 3.4] to problem $\mathcal{P}(\bar{w},\bar{v})$ in (4.1). It follows from the subdifferential formula (4.8) verified above that the partial inverse mapping (4.7) is locally represented by (4.9) and that the stationary condition $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$ reduces to (4.10). Furthermore, we deduce from [11, Proposition 2.2] that the assumed RCQ implies the validity of the basic constraint qualification as well as the parametric continuous prox-regularity of the function φ at $(\bar{x},\bar{w},\bar{v})$ imposed in [27, Theorem 3.4]. Then the "only if" part of this theorem follows from [27, Theorem 3.4(ii)]. Likewise, the "if" part of the theorem is a direct consequence of [27, Theorem 3.4(i)] due to the aforementioned calculus rules. ∎

To proceed further, consider the *KKT system*

$$\bar{v}=\nabla_xL(\bar{x},\bar{w},\bar{\lambda}),\ \ -\bar{\lambda}\in N_{\mathcal{Q}}(\Phi(\bar{x},\bar{w}))\subset\mathbb{R}^l\ \text{ with }\ l=\sum_{j=1}^J(m_j+1), \tag{4.11}$$

associated with the local minimizer $\bar{x}$ in $\mathcal{P}(\bar{w},\bar{v})$, where L is the *Lagrangian* for (1.1) given by

$$L(x,w,\lambda):=\varphi_0(x,w)-\sum_{j=1}^J\langle\lambda^j,\Phi^j(x,w)\rangle,\quad \lambda^j\in\mathbb{R}^{m_j+1}. \tag{4.12}$$

It is well known in conic programming (see, e.g., [3, Theorem 3.9]) that RCQ (4.5) ensures the existence a Lagrange multiplier vector $\bar{\lambda}=(\bar{\lambda}^1,\ldots,\bar{\lambda}^J)$ satisfying the KKT system (4.11). On the other hand, it is easy to see that the *uniqueness* of the Lagrange multiplier $\bar{\lambda}\in\mathbb{R}^l$ in (4.11) is guaranteed by the *partial* in x SOCQ

$$\partial^2\delta_{\mathcal{Q}}\big(\bar{z},\bar{y}\big)(0)\cap\ker\nabla_x\Phi(\bar{x},\bar{w})^*=\{0\}\ \text{ with }\ \bar{z}:=\Phi(\bar{x},\bar{w}),\ \bar{y}\in N_{\mathcal{Q}}(\bar{z}), \tag{4.13}$$

as well as by the *partial* in x *nondegeneracy condition*

$$\nabla_x\Phi(\bar{x},\bar{w})\mathbb{R}^n+\operatorname{lin}\{T_{\mathcal{Q}}(\bar{z})\}=\prod_{j=1}^{J}\mathbb{R}^{m_j+1} \tag{4.14}$$

equivalent to (4.13) by Theorem 3.9. In what follows we use only these versions of SOCQ and nondegeneracy conditions and thus omit the adjective "partial" when no confusion arises.

In the rest of this section we derive two second-order characterizations of full stability in SOCPs under the *equivalent SOCQ/nondegeneracy conditions* (4.13) and (4.14). The proofs of both of them strongly involve the recent chain rule from [19, Theorem 4.3] for the partial second-order subdifferential for functions of two variables defined by scheme (2.8) as follows; see [11]. Given a function $\psi\colon\mathbb{R}^n\times\mathbb{R}^d\to\overline{\mathbb{R}}$ of the variables $(x,w)\in\mathbb{R}^n\times\mathbb{R}^d$, its *partial second-order subdifferential* at $(\bar{x},\bar{w})$ relative to some $\bar{y}\in\partial_x\psi(\bar{x},\bar{w})$ is

$$\partial_x^2\psi(\bar{x},\bar{w},\bar{y})(u):=(D^*\partial_x\psi)(\bar{x},\bar{w},\bar{y})(u),\quad u\in\mathbb{R}^n. \tag{4.15}$$

To avoid confusion, note that in [26, 27] the second-order construction (4.15) is labeled as the "extended partial second-order subdifferential" when using the symbol $\widetilde{\partial}_x^2\psi$.

The following second-order chain rule is a particular case of [19, Theorem 4.3] while its nonparametric counterpart with $\Phi=\Phi(x)$ was first obtained in [28, Theorem 7]. Both results were proved under the nondegeneracy condition. Their SOCQ versions hold due to Theorem 3.9.

LEMMA 4.3 (second-order chain rule for parametric compositions in SOCPs). *Let* $\bar{v}\in\partial_x(\delta_{\mathcal{Q}}\circ\Phi)(\bar{x},\bar{w})$ *in the SOCP setting* (1.1) *with* $\bar{z}:=\Phi(\bar{x},\bar{w})\in\mathcal{Q}$ *under the validity of the SOCQ/nondegeneracy condition at* $(\bar{x},\bar{w})$*, and let* $\bar{y}\in\mathbb{R}^J$ *be the* (*unique*) *vector in* $\partial\delta_{\mathcal{Q}}(\bar{z})$ *satisfying* $\bar{v}=\nabla_x\Phi(\bar{x},\bar{w})^*\bar{y}$*. Then for all* $u\in\mathbb{R}^J$ *we have the equality*

$$\begin{aligned}\partial_x^2(\delta_{\mathcal{Q}}\circ\Phi)(\bar{x},\bar{w},\bar{v})(u)=&\Big(\nabla_{xx}^2\langle\bar{y},\Phi\rangle(\bar{x},\bar{w})u,\nabla_{xw}^2\langle\bar{y},\Phi\rangle(\bar{x},\bar{w})u\Big)\\&+\Big(\nabla_x\Phi(\bar{x},\bar{w}),\nabla_w\Phi(\bar{x},\bar{w})\Big)^*\partial^2\delta_{\mathcal{Q}}(\bar{z},\bar{y})(\nabla_x\Phi(\bar{x},\bar{w})u).\end{aligned} \tag{4.16}$$

Proof. The point follows from [19, Theorem 4.3] by the convexity and pointedness of the cone $\mathcal{Q}$ and by the automatic validity in finite dimensions of the sequentially normally compact assumptions imposed therein. ∎

Our next goal is to characterize full stability of local minimizers for SOCPs via the following SOCP version [2] of the uniform second-order growth condition from [3, Definition 5.16]. Given $\varphi\colon\mathbb{R}^n\times\mathbb{R}^d\to\overline{\mathbb{R}}$ in (4.2) with $\Phi(\bar{x},\bar{w})\in\mathcal{Q}$ and given $\bar{v}\in\partial_x\varphi(\bar{x},\bar{w})$, we say that the *SOCP uniform second-order growth condition* (*USOGC*) holds at $(\bar{x},\bar{w},\bar{v})$ if there exist $\eta>0$ as well as neighborhoods U of $\bar{x}$, W of $\bar{w}$, and V of $\bar{v}$ such that for any $(w,v)\in W\times V$ and any stationary point $x_{wv}\in U$ satisfying $v\in\partial_x\varphi(x_{wv},w)$ we have

$$\varphi_0(x,w)\ge\varphi_0(x_{wv},w)+\langle v,x-x_{wv}\rangle+\eta\|x-x_{wv}\|^2\ \text{ for }\ x\in U,\ \Phi(x,w)\in\mathcal{Q}. \tag{4.17}$$

THEOREM 4.4 (characterizing full stability in SOCPs via USOGC). *Let* $(\bar{x},\bar{w})$ *be such that* $\Phi(\bar{x},\bar{w})\in\mathcal{Q}$ *in the SOCP framework* (1.1) *under the SOCQ/nondegeneracy condition at* $(\bar{x},\bar{w})$*, and let* $\bar{v}$ *be taken from* (4.10)*. Then* $\bar{x}$ *is a fully stable local* *minimizer of* $\mathcal{P}(\bar{w},\bar{v})$ *in* (4.1) *if and only if* $\bar{x} \in M_\gamma(\bar{w},\bar{v})$ *for some* $\gamma > 0$ *and USOGC* (4.17) *is satisfied at* $(\bar{x},\bar{w},\bar{v})$.

Proof. The necessity part of the theorem follows from [27, Theorem 3.7(i)] specified in the SOCP setting. To justify the sufficiency part, find numbers $\gamma,\sigma > 0$ such that USOGC (4.17) holds with $U := \operatorname{int}\mathbb{B}_\gamma(\bar{x})$, $W := \operatorname{int}\mathbb{B}_\gamma(\bar{w})$, $V := \operatorname{int}\mathbb{B}_\gamma(\bar{v})$, and $\eta := \frac{1}{2\sigma}$. It easily follows from (4.17) that $M_\gamma(\bar{w},\bar{v}) = \{\bar{x}\}$. Using this together with [11, Proposition 3.5] shows that for all $(w,v) \in W \times V$ we have $\emptyset \neq M_\gamma(w,v) \subset \operatorname{int}\mathbb{B}_{\frac{\gamma}{2}}(\bar{x})$. This guarantees the existence of the unique minimizer $x_{wv} \in \operatorname{int}\mathbb{B}_{\frac{\gamma}{2}}(\bar{x})$ of (4.1) for any $(w,v) \in W \times V$. It is worth mentioning that the basic constraint qualification in [11, Proposition 3.5] holds under the validity of the SOCQ/nondegeneracy condition at $(\bar{x},\bar{w})$. Let us now prove that the mapping $(w,v) \mapsto x_{wv}$ is a Lipschitzian single-valued localization of (4.9) around $(\bar{w},\bar{v})$ and thus Theorem 4.2 ensures that $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w},\bar{v})$.

To proceed, define the set-valued mapping $\Theta : \mathbb{R}^d \rightrightarrows \mathbb{R}^n \times \mathbb{R}^n$ by

(4.18)
$$\Theta(w) := \operatorname{gph}\partial_x\varphi(\cdot,w) = \Big\{(x, \nabla_x\varphi_0(x,w) - \nabla_x\Phi(x,w)^*\lambda)\Big|\ -\lambda \in N_{\mathcal{Q}}(\Phi(x,w))\Big\}$$

with φ taken from (4.2). We claim Θ has the Lipschitz-like/Aubin property around $(\bar{w},\bar{x},\bar{v})$, which means that there exists a number $\ell > 0$ such that

$$\Theta(w) \cap (U \times V) \subset \Theta(w') + \ell\|w - w'\|\mathbb{B}_1(0) \quad \text{for all } w, w' \in W. \tag{4.19}$$

Indeed, it follows from [11, Propositions 3.2 and 4.3] via the Mordukhovich criterion of [33, Theorem 9.40] that the latter property holds for (4.18) if and only if

$$(0,q) \in \partial_x^2\varphi(\bar{x},\bar{w},\bar{v})(0) \Longrightarrow q = 0. \tag{4.20}$$

Employing now in φ the second-order sum rule from [17, Proposition 1.121] and then the second-order chain rule from Lemma 4.3 above, we get that (4.20) is equivalent to the implication

$$\Big[\nabla_x\Phi(\bar{x},\bar{w})^*p = 0,\ \nabla_w\Phi(\bar{x},\bar{w})^*p = q,\ p \in \partial^2\delta_{\mathcal{Q}}(\bar{z},-\bar{\lambda})(0)\Big] \Longrightarrow q = 0,$$

where $-\bar{\lambda}$ is the unique vector in $N_{\mathcal{Q}}(\Phi(\bar{x},\bar{w}))$ satisfying (4.11). But this is automatic under the imposed SOCQ (4.13), which thus verifies (4.19).

Let us finally show that the mapping $(w,v) \mapsto x_{wv}$ is Lipschitz continuous around $(\bar{w},\bar{v})$. To furnish this, take $w_1, w_2 \in W$ and $v_1, v_2 \in V$ and observe that USOGC (4.17) implies the existence and uniqueness of minimizers $x_{w_1v_1}, x_{w_2v_2}$ for $\mathcal{P}(\bar{w},\bar{v})$ on $\operatorname{int}\mathbb{B}_{\frac{\gamma}{2}}(\bar{x})$. This yields by (4.18) that $(x_{w_1v_1}, v_1) \in \Theta(w_1)$ and $(x_{w_2v_2}, v_2) \in \Theta(w_2)$. Since $(x_{w_2v_2}, v_2) \in U \times V$, we get from the Lipschitzian condition (4.19) that

$$\|x_{w_2v_2} - \widetilde{x}\| + \|v_2 - \widetilde{v}\| \le \ell\|w_1 - w_2\| \quad \text{for some } (\widetilde{x},\widetilde{v}) \in \Theta(w_1). \tag{4.21}$$

Suppose without loss of generality that $(\widetilde{x},\widetilde{v}) \in U \times V$ and employing USOGC (4.17) shows that $M_\gamma(w_1,\widetilde{v}) = \{\widetilde{x}\}$. Implementing (4.17) again with the chosen number $\eta := \frac{1}{2\sigma}$ tells us that

$$\varphi_0(x_{w_1v_1}, w_1) \ge \varphi_0(\widetilde{x}, w_1) + \langle\widetilde{v}, x_{w_1v_1} - \widetilde{x}\rangle + \frac{1}{2\sigma}\|x_{w_1v_1} - \widetilde{x}\|^2,$$
$$\varphi_0(\widetilde{x}, w_1) \ge \varphi_0(x_{w_1v_1}, w_1) + \langle v_1, \widetilde{x} - x_{w_1v_1}\rangle + \frac{1}{2\sigma}\|x_{w_1v_1} - \widetilde{x}\|^2$$

from which we readily deduce the estimates

$$\begin{aligned}\|x_{w_1v_1}-\widetilde{x}\| &\le \sigma\|\widetilde{v}-v_1\| \le \sigma\|\widetilde{v}-v_2\|+\sigma\|v_2-v_1\|\\ &\le \sigma\ell\|w_1-w_2\|+\sigma\|v_2-v_1\|.\end{aligned}$$

Combining this together with (4.21) shows that

$$\begin{aligned}\|x_{w_1v_1}-x_{w_2v_2}\| &\le \|\widetilde{x}-x_{w_2v_2}\|+\|x_{w_1v_1}-\widetilde{x}\|\\ &\le \ell(\sigma+1)(\|w_1-w_2\|+\|v_1-v_2\|),\end{aligned}$$

which verifies the claimed local Lipschitz continuity and thus completes the proof. ∎

The last and most important topic of this section is deriving second-order subdifferential descriptions of full stability of locally optimal solutions to SOCPs, which eventually leads us to complete characterizations of this notion entirely in terms of the initial data of (1.1). The next lemma revisits the result of [27, Theorem 5.1] on the second-order subdifferential characterization of full stability replacing, for the case of SOCPs, the full rank assumption on $\nabla_x\Phi(\bar{x},\bar{w})$ by the milder SOCQ/nondegeneracy condition in (4.13) and (4.14).

LEMMA 4.5 (second-order subdifferential characterization of fully stable local minimizers for SOCPs at nondegenerate solutions). *Let $\bar{x}$ be a feasible solution to the unperturbed problem $\mathcal{P}(\bar{w},\bar{v})$ in* (4.1) *with some $\bar{w}\in\mathbb{R}^d$ and $\bar{v}\in\mathbb{R}^n$ taken from* (4.10) *under the SOCQ/nondegeneracy condition at $(\bar{x},\bar{w})$, and let $\bar{\lambda}$ be the unique vector satisfying*

$$\nabla_x\varphi_0(\bar{x},\bar{w})-\nabla_x\Phi(\bar{x},\bar{w})^*\bar{\lambda}=\bar{v} \quad \text{and} \quad -\bar{\lambda}\in N_{\mathcal{Q}}(\bar{z}). \tag{4.22}$$

Then $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w},\bar{v})$ if and only if we have the implication

$$[(p,q)\in\mathcal{T}(\bar{x},\bar{w},\bar{v})(u),\ u\ne 0]\Longrightarrow\langle p,u\rangle>0 \tag{4.23}$$

for the set-valued mapping $\mathcal{T}(\bar{x},\bar{w},\bar{v})\colon\mathbb{R}^n\rightrightarrows\mathbb{R}^n\times\mathbb{R}^d$ defined by

$$\begin{aligned}\mathcal{T}(\bar{x},\bar{w},\bar{v})(u):=&\Big(\nabla^2_{xx}\varphi_0(\bar{x},\bar{w})u,\nabla^2_{xw}\varphi_0(\bar{x},\bar{w})u\Big)\\ &-\Big(\nabla^2_{xx}\langle\bar{\lambda},\Phi\rangle(\bar{x},\bar{w})u,\nabla^2_{xw}\langle\bar{\lambda},\Phi\rangle(\bar{x},\bar{w})u\Big)\\ &+\Big(\nabla_x\Phi(\bar{x},\bar{w}),\nabla_w\Phi(\bar{x},\bar{w})\Big)^*\partial^2\delta_{\mathcal{Q}}(\bar{z},-\bar{\lambda})(\nabla_x\Phi(\bar{x},\bar{w})u),\quad u\in\mathbb{R}^n.\end{aligned}$$

Proof. The proof follows the lines in the proof of [27, Theorem 5.1] with replacement of the second-order chain rule for (4.15) from [26, Theorem 3.1] obtained under the full rank assumption by that of Lemma 4.3 established under the more subtle SOCQ
(4.13). ∎

It is worth mentioning that Lemma 4.5 *cannot* be derived from [27, Theorem 5.2] under the SOCQ/nondegeneracy condition, since the outer function $\delta_{\mathcal{Q}}$ for SOCPs is neither piecewise linear nor piecewise linear quadratic of the type considered therein.

Now we proceed with expressing the second-order subdifferential characterization of full stability in Lemma 4.5 *entirely in terms of the initial data* of the SOCP (1.1) under consideration. To accomplish it, we need to *calculate the second-order subdifferential* $\partial^2\delta_{\mathcal{Q}}$ in condition (4.23) at *nonzero* points, which is of its own interest. This calculation is partly similar to but generally different from that at zero derived in Theorem 3.6 while it also strongly involves the results of Lemma 3.3 on the coderivative description for the metric projection onto the Lorentz cone $\mathcal{Q}_{m+1}$. The key ingredients are given in the following lemma.

LEMMA 4.6 (domain of the second-order subdifferential for the indicator function of the Lorentz cone). *Let $z = (z_0, z_r) \in \mathcal{Q}_{m+1}$, and let $-y \in N_{\mathcal{Q}_{m+1}}(z)$ for the Lorentz cone $\mathcal{Q}_{m+1}$ in* (1.3). *Then we have the representation*

$$\text{(4.24)}\quad \operatorname{dom} \partial^2 \delta_{\mathcal{Q}_{m+1}}(z, -y) = \left\{ u \in \mathbb{R}^{m+1} \;\middle|\; \begin{array}{lll} u = 0 & \textit{if} & y \in \operatorname{int} \mathcal{Q}_{m+1}, \\ u \in \mathbb{R}(y_0, -y_r) & \textit{if} & y \in \operatorname{bd} \mathcal{Q}_{m+1} \backslash \{0\},\ z = 0, \\ \langle u, y \rangle = 0 & \textit{if} & y, z \in \operatorname{bd} \mathcal{Q}_{m+1} \backslash \{0\} \end{array} \right\}.$$

Proof. We split the proof into the following six cases according to the position of $z, y \in \mathcal{Q}_{m+1}$.

Case 1: $z = 0,\ y \in \operatorname{int} \mathcal{Q}_{m+1}$. Take $q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(0, -y)(u)$, which by (3.20) amounts to $-u \in D^* \Pi_{\mathcal{Q}_{m+1}}(-y, 0)(-q - u)$. Arguing as in Case 1 from the proof of Theorem 3.6 gives us $D^* \Pi_{\mathcal{Q}_{m+1}}(-y, 0)(-q - u) = 0$, which yields $u = 0$ and thus $\operatorname{dom} \partial^2 \delta_{\mathcal{Q}_{m+1}}(0, -y)(0) = \{0\}$.

Case 2: $z \in \operatorname{int} \mathcal{Q}_{m+1}$. This implies that $y = 0$. Take $q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(z, 0)(u)$, which is equivalent by (3.20) to $-u \in D^* \Pi_{\mathcal{Q}_{m+1}}(z, z)(-q - u)$. Since $z \in \operatorname{int} \mathcal{Q}_{m+1}$, it says that $z_0 > \|z_r\|$ and thus Lemma 3.3(i) together with (3.10) tells us that

$$\begin{aligned} -u \in D^* \Pi_{\mathcal{Q}_{m+1}}(z, z)(-q - u) &= \nabla \Pi_{\mathcal{Q}_{m+1}}(z)(-q - u) \\ &= I_{m+1}(-q - u) = -q - u \Longleftrightarrow q = 0. \end{aligned}$$

Therefore we have $\operatorname{dom} \partial^2 \delta_{\mathcal{Q}_{m+1}}(z, 0) = \mathbb{R}^{m+1}$ in this case.

Case 3: $z, y \in \operatorname{bd} \mathcal{Q}_{m+1} \backslash \{0\}$. Taking $q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(z, -y)(u)$ and using (3.20) give us

$$\text{(4.25)}\qquad -u \in D^* \Pi_{\mathcal{Q}_{m+1}}(z - y, z)(-q - u).$$

First we claim that $\langle u, y \rangle = 0$. To show this, observe that $-y \in N_{\mathcal{Q}_{m+1}}(z)$ yields $\langle y, z \rangle = 0$ due to the property (3.15). Put $c = (c_0, c_r) := z - y$ and deduce from (3.6) the validity of equalities (3.23) and (3.24) for these vectors. Similarly to Case 3 in the proof of Theorem 3.6 with $q = (q_0, q_r)$ and $u = (u_0, u_r)$ we arrive at the coderivative representation

$$D^* \Pi_{\mathcal{Q}_{m+1}}(z - y, z)(-q - u) = \frac{1}{2} \begin{pmatrix} 1 & \frac{c_r^T}{\|c_r\|} \\ \frac{c_r}{\|c_r\|} & H \end{pmatrix} \begin{pmatrix} -q_0 - u_0 \\ -q_r - u_r \end{pmatrix},$$

where H is defined in the proof of Theorem 3.6. This allows us to reduce (4.25) to the system

$$\text{(4.26)}\quad \begin{cases} \dfrac{1}{2}(q_0 + u_0) + \dfrac{1}{2\|c_r\|} \langle c_r, q_r + u_r \rangle = u_0, \\ \dfrac{1}{2}(q_0 + u_0) \dfrac{c_r}{\|c_r\|} + \dfrac{z_0}{z_0 + y_0}(q_r + u_r) - \dfrac{z_0 - y_0}{2(z_0 + y_0)} \dfrac{c_r}{\|c_r\|^2} \langle c_r, q_r + u_r \rangle = u_r. \end{cases}$$

Arguing similarly to Theorem 3.6 in this case, we deduce from (4.26) that

$$u_0 \|c_r\| = \langle c_r, u_r \rangle = \left\langle -\frac{\|c_r\|}{y_0} y_r, u_r \right\rangle$$

and thus verify that $\langle u, y \rangle = 0$. Hence for $\langle u, y \rangle \neq 0$ we have $\partial^2 \delta_{\mathcal{Q}_{m+1}}(z, -y)(u) = \emptyset$.

It remains to consider the situation when $\langle u, y\rangle = 0$ and show that $\mathrm{dom}\,\partial^2\delta_{\mathcal{Q}_{m+1}}(z,-y)\neq\emptyset$. To proceed, take $\bar q := -\frac{y_0}{z_0}(u_0,-u_r)$ and verify that $\bar q\in\partial^2\delta_{\mathcal{Q}_{m+1}}(z,-y)(u)$, which amounts to saying that $\bar q$ satisfies the system of equations in (4.26). Indeed, it follows from (3.23) that

$$\langle c_r,u_r\rangle = -\frac{y_0+z_0}{y_0}\langle y_r,u_r\rangle = \frac{y_0+z_0}{y_0}y_0u_0 = u_0(z_0+y_0) = u_0\|c_r\|$$

due to $\langle y,u\rangle = 0$. This implies the equalities

$$\begin{aligned}\frac{1}{2}(\bar q_0+u_0)+\frac{1}{2\|c_r\|}\langle c_r,\bar q_r+u_r\rangle &= \frac{1}{2}\left(-\frac{y_0}{z_0}+1\right)u_0+\frac{1}{2\|c_r\|}\left\langle c_r,\left(\frac{y_0}{z_0}+1\right)u_r\right\rangle\\ &= \frac{1}{2}\left[\frac{(z_0-y_0)u_0}{z_0}+\frac{(y_0+z_0)u_0}{z_0}\right] = u_0,\end{aligned}\tag{4.27}$$

and thus the first equation in (4.26) holds. Also it follows from (4.27) that

$$\begin{aligned}&\frac{1}{2}\Big(\bar q_0+u_0\Big)\frac{c_r}{\|c_r\|}+\frac{z_0}{z_0+y_0}\Big(\bar q_r+u_r\Big)-\frac{z_0-y_0}{2(z_0+y_0)}\frac{c_r}{\|c_r\|^2}\Big\langle c_r,\bar q_r+u_r\Big\rangle\\ &= \frac{1}{2}\Big(-\frac{y_0}{z_0}+1\Big)\frac{c_r}{\|c_r\|}u_0+\frac{z_0}{z_0+y_0}\Big(\frac{y_0}{z_0}+1\Big)u_r-\frac{z_0-y_0}{2(z_0+y_0)}\frac{c_r}{\|c_r\|^2}\frac{\|c_r\|^2}{z_0}u_0\\ &= \frac{1}{2}\frac{c_r}{\|c_r\|}u_0\Big(\frac{y_0-z_0}{z_0}-\frac{y_0-z_0}{z_0}\Big)+\frac{z_0}{z_0+y_0}\Big(\frac{y_0+z_0}{z_0}\Big)u_r = u_r,\end{aligned}$$

and therefore the second equation in (4.26) is satisfied as well. This verifies that $\bar q\in\partial^2\delta_{\mathcal{Q}_{m+1}}(z,-y)(u)$ and hence completes the proof in this case.

Case 4: $z=0$, $y\in\mathrm{bd}\,\mathcal{Q}_{m+1}\backslash\{0\}$. Take $q\in\partial^2\delta_{\mathcal{Q}_{m+1}}(0,-y)(u)$ and get by (3.20) that

$$-u\in D^*\Pi_{\mathcal{Q}_{m+1}}(-y,0)(-q-u).\tag{4.28}$$

Let us first check that $u\in\{t(y_0,-y_r)|\ t\in\mathbb{R}\}$. Indeed, it follows from Lemma 3.3(iii) that

$$\begin{aligned}&D^*\Pi_{\mathcal{Q}_{m+1}}(-y,0)(-q-u)\\ &\quad\in\left\{\frac{t}{2}B(-y)(-q-u)\Big|\ B(-y)=\begin{pmatrix}1 & -\frac{y_r^T}{\|y_r\|}\\ -\frac{y_r}{\|y_r\|} & \frac{y_ry_r^T}{\|y_r\|^2}\end{pmatrix},\ \ t\in[0,1]\right\}.\end{aligned}$$

This together with (4.28) ensures the existence of $t_0\in[0,1]$ such that

$$\begin{pmatrix}u_0\\ u_r\end{pmatrix}=\frac{t_0}{2}\begin{pmatrix}1 & -\frac{y_r^T}{\|y_r\|}\\ -\frac{y_r}{\|y_r\|} & \frac{y_ry_r^T}{\|y_r\|^2}\end{pmatrix}\begin{pmatrix}q_0+u_0\\ q_r+u_r\end{pmatrix}.\tag{4.29}$$

For $t_0=0$ we have $u=0$ and so $u\in\{t(y_0,-y_r)|\ t\in\mathbb{R}\}$. If $t_0\neq 0$, we rephrase (4.29) as

$$\begin{cases}\dfrac{t_0}{2}\left[q_0+u_0-\dfrac{\langle y_r,q_r+u_r\rangle}{\|y_r\|}\right]=u_0,\\[2ex] \dfrac{t_0}{2}\left[\dfrac{-(q_0+u_0)}{\|y_r\|}y_r+\dfrac{\langle y_r,q_r+u_r\rangle}{\|y_r\|^2}y_r\right]=u_r.\end{cases}\tag{4.30}$$

The first equation in (4.30) gives us the relationship

$$\langle y_r, q_r + u_r\rangle = \|y_r\|\left(-\frac{2}{t_0}u_0 + u_0 + q_0\right). \tag{4.31}$$

Multiplying both sides of the second equation in (4.30) by y_r and using (4.31) yield

$$\begin{aligned}\langle y_r, u_r\rangle &= \frac{t_0}{2}\Big[-(q_0+u_0)\|y_r\| + \langle y_r, q_r+u_r\rangle\Big]\\ &= \frac{t_0}{2}\left[-(q_0+u_0)\|y_r\| + \|y_r\|\left(-\frac{2}{t_0}u_0 + u_0 + q_0\right)\right]\\ &= -u_0\|y_r\| = -u_0 y_0,\end{aligned}$$

which implies that $\langle y, u\rangle = 0$. Furthermore, substituting (4.31) into the second equation in (4.30) brings us to the expression

$$u_r = \frac{t_0}{2}\left[\frac{-(q_0+u_0)}{\|y_r\|}y_r + \frac{-\frac{2}{t_0}u_0 + u_0 + q_0}{\|y_r\|}y_r\right] = \frac{-u_0}{\|y_r\|}y_r,$$

and thus $\|u_r\| = |u_0|$. If $u_0 = 0$, we have $u = 0$ and $u \in \{t(y_0, -y_r)|\ t \in \mathbb{R}\}$. Otherwise, it follows that either $u \in \mathcal{Q}_{m+1}$ or $-u \in \mathcal{Q}_{m+1}$. Using this together with $\langle y, u\rangle = 0$ and (3.5), observe that $u = (-ty_0, ty_r)$ for some $t \in \mathbb{R}$ and so verify $u \in \mathbb{R}(y_0, -y_r)$. It remains to show that if $(u_0, u_r) = t_0(y_0, -y_r)$ for some $t_0 \in \mathbb{R}$, then $(u_0, u_r) \in \operatorname{dom}\partial^2\delta_{\mathcal{Q}_{m+1}}(0, -y)$. Indeed, note that the vector $q = (q_0, q_r) = (0, 0)$ satisfies the system

$$\begin{pmatrix} u_0 \\ u_r \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\frac{y_r^T}{\|y_r\|} \\ -\frac{y_r}{\|y_r\|} & \frac{y_r y_r^T}{\|y_r\|^2}\end{pmatrix}\begin{pmatrix} q_0 + u_0 \\ q_r + u_r\end{pmatrix} \tag{4.32}$$

and that by Lemma 3.3(iii) we have the inclusion

$$\frac{1}{2}\begin{pmatrix} 1 & -\frac{y_r^T}{\|y_r\|} \\ -\frac{y_r}{\|y_r\|} & \frac{y_r y_r^T}{\|y_r\|^2}\end{pmatrix}(-q-u) \in D^*\Pi_{\mathcal{Q}_{m+1}}(-y, 0)(-q-u).$$

This together with (4.32) leads us to

$$-u \in D^*\Pi_{\mathcal{Q}_{m+1}}(-y, 0)(-q-u) \ \text{ and hence }\ 0 = q \in \partial^2\delta_{\mathcal{Q}_{m+1}}(0, -y)(u),$$

which completes the proof of (4.24) in this case.

Case 5: $z \in \operatorname{bd}\mathcal{Q}_{m+1}\backslash\{0\}$, $y = 0$. Employing Lemma 3.3(ii), we have

$$-u \in D^*\Pi_{\mathcal{Q}_{m+1}}(z, z)(-u), \ \ u \in \mathbb{R}^{m+1},$$

which amounts to $0 \in \partial^2\delta_{\mathcal{Q}_{m+1}}(z, 0)(u)$ by the equivalence in (3.20). This tells us $\operatorname{dom}\partial^2\delta_{\mathcal{Q}_{m+1}}(z, 0) = \mathbb{R}^{m+1}$ and ends the proof of (4.24) in this case.

Case 6: $z = 0$, $y = 0$. Similarly to Case 5 we have by employing Lemma 3.3(iv) that

$$-u \in D^*\Pi_{\mathcal{Q}_{m+1}}(0, 0)(-u), \ \ u \in \mathbb{R}^{m+1},$$

which amounts to $0 \in \partial^2\delta_{\mathcal{Q}_{m+1}}(0,0)(u)$ by (3.20). This shows that $\mathrm{dom}\,\partial^2\delta_{\mathcal{Q}_{m+1}}(0,0) = \mathbb{R}^{m+1}$ in this case and thus ends the proof of the lemma. ∎

To derive our main characterization of full stability of locally optimal solutions to SOCP (1.1) in terms of the initial data, we need one more result about properties of the second-order subdifferential $\partial^2\delta_{\mathcal{Q}_{m+1}}$. For formulating this result and for the subsequent developments below, let us fix a pair $(\bar{x},\bar{w})$ with $\Phi(\bar{x},\bar{w}) \in \mathcal{Q}$ and a Lagrange multiplier with $-\bar{\lambda} \in N_{\mathcal{Q}}(\Phi(\bar{x},\bar{w}))$. Following [2], introduce the function

$$\mathcal{H}(\bar{x},\bar{w},\bar{\lambda}) := \sum_{j=1}^{J} \mathcal{H}^j(\bar{x},\bar{w},\bar{\lambda}), \tag{4.33}$$

where each $n \times n$ matrix $\mathcal{H}^j(\bar{x},\bar{w},\bar{\lambda})$ is defined by

$$\mathcal{H}^j(\bar{x},\bar{w},\bar{\lambda}) := \begin{cases} -\frac{\bar{\lambda}_0^j}{\bar{z}_0^j}\nabla_x\Phi^j(\bar{x},\bar{w})^* \begin{pmatrix} 1 & 0^T \\ 0 & -I_{m_j} \end{pmatrix} \nabla_x\Phi^j(\bar{x},\bar{w}) & \text{if } \bar{z}^j \in \mathrm{bd}\,\mathcal{Q}_{m_j+1}\backslash\{0\}, \\ 0 & \text{otherwise} \end{cases} \tag{4.34}$$

with $\bar{\lambda}^j = (\bar{\lambda}_0^j, \bar{\lambda}_r^j)$ and $\bar{z}^j = \Phi^j(\bar{x},\bar{w}) = (\bar{z}_0^j, \bar{z}_r^j)$ for $j = 1,\ldots,J$. In the case of one Lorentz cone $\mathcal{Q} = \mathcal{Q}_{m+1}$ in (4.33), i.e., when $J = 1$, we drop the index j in (4.33) and (4.34) for convenience. To unify consideration of similar cases regarding the position of the vectors $\bar{\lambda}$ and $\bar{z} = \Phi(\bar{x},\bar{w})$ in $\mathcal{Q}_{m+1}$, we write $(\bar{z},\bar{\lambda}) \in I_1$ if $\bar{z},\bar{\lambda} \in \mathrm{bd}\,\mathcal{Q}_{m+1}\backslash\{0\}$ and $(\bar{z},\bar{\lambda}) \in I_2$ otherwise.

LEMMA 4.7 (properties of the second-order subdifferential of $\delta_{\mathcal{Q}_{m+1}}$). *Given the vectors $\bar{z} := (\bar{z}_0,\bar{z}_r) \in \mathcal{Q}_{m+1}$ and $-\bar{\lambda} \in N_{\mathcal{Q}_{m+1}}(\bar{z})$, the following assertions hold.*

(i) *If $(\bar{z},\bar{\lambda}) \in I_1$ and $\nabla_x\Phi(\bar{x},\bar{w})u \in \mathrm{dom}\,\partial^2\delta_{\mathcal{Q}_{m+1}}(\bar{z},-\bar{\lambda})$ with $u \in \mathbb{R}^n$, then for any vector $q \in \partial^2\delta_{\mathcal{Q}_{m+1}}(\bar{z},-\bar{\lambda})(\nabla_x\Phi(\bar{x},\bar{w})u)$ we have*

$$\langle \mathcal{H}(\bar{x},\bar{w},\bar{\lambda})u,u\rangle = \langle q, \nabla_x\Phi(\bar{x},\bar{w})u\rangle. \tag{4.35}$$

(ii) *If $(\bar{z},\bar{\lambda}) \in I_2$ and $\nabla_x\Phi(\bar{x},\bar{w})u \in \mathrm{dom}\,\partial^2\delta_{\mathcal{Q}_{m+1}}(\bar{z},-\bar{\lambda})$, then*

$$0 \in \partial^2\delta_{\mathcal{Q}_{m+1}}(\bar{z},-\bar{\lambda})(\nabla_x\Phi(\bar{x},\bar{w})u). \tag{4.36}$$

Proof. To verify (i) first, we see that if $\nabla_x\Phi(\bar{x},\bar{w})u = 0$, then (4.35) clearly holds. Suppose that $\beta := (\beta_0,\beta_r) = \nabla_x\Phi(\bar{x},\bar{w})u \neq 0$. Taking into account that $(\beta_0,\beta_r) \in \mathrm{dom}\,\partial^2\delta_{\mathcal{Q}_{m+1}}(\bar{z},-\bar{\lambda})$ and using Lemma 4.6 give us that $\langle\bar{\lambda},\beta\rangle = 0$, which implies in turn that $\beta_r \neq 0$. Fix now $q \in \partial^2\delta_{\mathcal{Q}_{m+1}}(\bar{z},-\bar{\lambda})(\beta)$ and get by (3.20) that

$$-\beta \in D^*\Pi_{\mathcal{Q}_{m+1}}(\bar{z}-\bar{\lambda},\bar{z})(-q-\beta).$$

It follows from the inclusion $-\bar{\lambda} \in N_{\mathcal{Q}_{m+1}}(\bar{z})$ due to the property (3.15) that $\langle\bar{\lambda},\bar{z}\rangle = 0$. Denote $c = (c_0,c_r) := \bar{z}-\bar{\lambda}$ and derive, similarly to Case 3 in the proof of Lemma 4.6, that these elements with $q = (q_0,q_r)$ and $\beta = (\beta_0,\beta_r)$ satisfy system (4.26) therein and thus the relationships

$$\langle c_r,\beta_r\rangle = \beta_0\|c_r\|, \quad \langle c_r,q_r\rangle = -q_0\|c_r\|. \tag{4.37}$$

Multiplying both sides of the second equation in (4.26) by u_r and unifying it with (4.37) yield

$$\begin{aligned}\|\beta_r\|^2 &= \frac{1}{2}(q_0+\beta_0)\frac{\langle c_r,\beta_r\rangle}{\|c_r\|} + \frac{\bar{z}_0}{\bar{z}_0+\bar{\lambda}_0}\langle q_r+\beta_r,\beta_r\rangle - \frac{\bar{z}_0-\bar{\lambda}_0}{2(\bar{z}_0+\bar{\lambda}_0)}\frac{\langle c_r,\beta_r\rangle}{\|c_r\|^2}\langle c_r,q_r+\beta_r\rangle \\ &= \frac{1}{2}(q_0+\beta_0)\beta_0 + \frac{\bar{z}_0}{\bar{z}_0+\bar{\lambda}_0}(\langle q_r,\beta_r\rangle + \|\beta_r\|^2) - \frac{\bar{z}_0-\bar{\lambda}_0}{2(\bar{z}_0+\bar{\lambda}_0)}\beta_0(\beta_0-q_0),\end{aligned}$$

which implies in turn the equality

$$\langle q_r, \beta_r\rangle = \frac{\bar\lambda_0}{\bar z_0}\|\beta_r\|^2 + \frac{\bar z_0 - \bar\lambda_0}{2\bar z_0}\beta_0(\beta_0 - q_0) - \frac{\bar z_0 + \bar\lambda_0}{2\bar z_0}(q_0+\beta_0)\beta_0.$$

Using this and the definition of $\mathcal{H}$ in (4.34), we arrive at the expression

$$\begin{aligned}\langle q, \beta\rangle &= \beta_0 q_0 + \langle q_r, \beta_r\rangle = \beta_0 q_0 + \frac{\bar\lambda_0}{\bar z_0}\|\beta_r\|^2 + \frac{\bar z_0 - \bar\lambda_0}{2\bar z_0}\beta_0(\beta_0 - q_0) - \frac{\bar z_0 + \bar\lambda_0}{2\bar z_0}(q_0+\beta_0)\beta_0\\ &= -\frac{\bar\lambda_0}{\bar z_0}(\beta_0^2 - \|\beta_r\|^2) = \langle \mathcal{H}(\bar x, \bar w, \bar\lambda)u, u\rangle,\end{aligned}$$

which verifies (4.35) and thus completes the proof of assertion (i).

To justify assertion (ii), we proceed as in the corresponding proof of Lemma 4.6 and show that in the case of $(\bar z, \bar\lambda) \in I_2$ inclusion (4.36) is satisfied. ∎

Considering now the general SOCP setting (1.1), define the *critical cone*

$$\mathcal{C}(\bar x, \bar w) := \Big\{u \in \mathbb{R}^n \Big|\ \nabla_x\varphi_0(\bar x, \bar w)u \le 0,\ \nabla_x\Phi(\bar x, \bar w)u \in T_{\mathcal{Q}}(\Phi(\bar x, \bar w))\Big\} \tag{4.38}$$

at $(\bar x, \bar w)$ with $\Phi(\bar x, \bar w) \in \mathcal{Q}$. It is well known [3, Proposition 3.10] that under the nondegeneracy condition (4.14) the critical cone (4.38) admits the representation

$$\mathcal{C}(\bar x, \bar w) = \Big\{u \in \mathbb{R}^n \Big|\ \nabla_x\Phi(\bar x, \bar w)u \in T_{\mathcal{Q}}(\Phi(\bar x, \bar w)) \cap \bar\lambda^{\perp}\Big\}, \tag{4.39}$$

where $\bar\lambda$ is a unique solution of the KKT system (4.11) with $\bar v$ taken from (4.10). Furthermore, it is proved in [2, Theorem 30] that the nondegeneracy condition (4.14) ensures that

$$\begin{aligned}&\operatorname{span}\{\mathcal{C}(\bar x, \bar w)\}\\ &= \left\{u \in \mathbb{R}^n \left|\ \begin{array}{lll} \nabla_x\Phi^j(\bar x, \bar w)u = 0 & \text{if} & \bar\lambda^j \in \operatorname{int}\mathcal{Q}_{m_j+1},\\ \nabla_x\Phi^j(\bar x, \bar w)u \in \mathbb{R}(\bar\lambda_0^j, -\bar\lambda_r^j) & \text{if} & \bar\lambda^j \in \operatorname{bd}\mathcal{Q}_{m_j+1}\backslash\{0\},\ \bar z^j = 0,\\ \langle\nabla_x\Phi^j(\bar x, \bar w)u, \bar\lambda^j\rangle = 0 & \text{if} & \bar\lambda^j, \bar z^j \in \operatorname{bd}\mathcal{Q}_{m_j+1}\backslash\{0\}\end{array}\right.\right\}.\end{aligned}$$

Assuming further the validity of the nondegeneracy condition (4.14) and following Bonnans and Ramírez [2] who considered the case when the basic perturbation parameter $\bar w$ is absent, we say that the SSOSC holds at $(\bar x, \bar w)$ if

$$\langle u, \nabla^2_{xx}L(\bar x, \bar w, \bar\lambda)u\rangle + \langle\mathcal{H}(\bar x, \bar w, \bar\lambda)u, u\rangle > 0 \ \text{ for all }\ u \in \operatorname{span}\{\mathcal{C}(\bar x, \bar w)\}\backslash\{0\} \tag{4.40}$$

with $\mathcal{H}$ defined in (4.33). Note that condition (4.40) is an SOCP counterpart of the SSOSC condition first introduced by Robinson [31] for problems of nonlinear programming.

Now we are ready to establish the main result of this paper characterizing full stability of locally optimal solutions to SOCPs in terms of their initial data.

THEOREM 4.8 (SSOSC characterization of full stability for SOCPs). *Let $\bar x$ be a feasible solution to problem $\mathcal{P}(\bar w, \bar v)$ in* (4.1) *for some $\bar w \in \mathbb{R}^d$ and $\bar v$ satisfying* (4.10) *under the validity of the SOCQ/nondegeneracy condition at $(\bar x, \bar w)$, and let $\bar\lambda \in \mathbb{R}^l$ be the corresponding unique Lagrange multiplier satisfying the KKT system* (4.11). *Then $\bar x$ is a fully stable locally optimal solution to* (4.1) *if and only if SSOSC* (4.40) *holds at $(\bar x, \bar w)$.*

Proof. First we consider the case of just one Lorentz cone $\mathcal{Q} = \mathcal{Q}_{m+1}$ in (1.1) and define the set-valued mapping $\mathcal{T}(\bar{x}, \bar{w}, \bar{v}) = (\mathcal{T}_1(\bar{x}, \bar{w}, \bar{v}), \mathcal{T}_2(\bar{x}, \bar{w}, \bar{v}))\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^d$ by

(4.41)
$$\begin{cases} \mathcal{T}_1(\bar{x}, \bar{w}, \bar{v})(u) := \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u + \nabla_x \Phi(\bar{x}, \bar{w})^* \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u), \\ \mathcal{T}_2(\bar{x}, \bar{w}, \bar{v})(u) := \nabla^2_{xw} L(\bar{x}, \bar{w}, \bar{\lambda})u + \nabla_w \Phi(\bar{x}, \bar{w})^* \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u) \end{cases}$$

for all $u \in \mathbb{R}^n$, where $\bar{z} = \Phi(\bar{x}, \bar{w})$ as usual. Let us start with justifying the "if" part of the theorem and suppose that SSOSC (4.40) holds. According to Lemma 4.5 we need to verify implication (4.23) for the mapping $\mathcal{T}(\bar{x}, \bar{w}, \bar{v})$ in (4.41). This amounts to saying that

$$\langle p, u \rangle > 0 \ \text{ whenever } \ p \in \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u + \nabla_x \Phi(\bar{x}, \bar{w})^* \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u), \quad u \neq 0,$$

which can be rewritten as follows:

$$\langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u \rangle + \langle q, \nabla_x \Phi(\bar{x}, \bar{w})u \rangle > 0 \tag{4.42}$$

for all $q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u)$ with $u \neq 0$.

To proceed, take any $q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\beta_0, \beta_r)$ with $u \neq 0$, where we use the notation $\beta = (\beta_0, \beta_r) =: \nabla_x \Phi(\bar{x}, \bar{w})u$. This implies $\beta \in \operatorname{dom} \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})$, and thus Lemma 4.6 and the above description of $\operatorname{span}\{\mathcal{C}(\bar{x}, \bar{w})\}$ tell us that $u \in \operatorname{span}\{\mathcal{C}(\bar{x}, \bar{w})\} \setminus \{0\}$. Using the notation of that lemma, consider first the case of $(\bar{z}, \bar{\lambda}) \in I_2$, which gives us $\mathcal{H}(\bar{x}, \bar{w}, \bar{\lambda}) = 0$. Thus the assumed SSOSC (4.40) reduces to $\langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u \rangle > 0$ in this case. Using now Lemma 3.4(ii) and the second-order subdifferential representation (3.20) bring us to

$$\langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u \rangle + \langle q, \nabla_x \Phi(\bar{x}, \bar{w})u \rangle \geq \langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u \rangle + 0 > 0,$$

which shows that condition (4.42) is satisfied and thus $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w}, \bar{v})$ by Lemma 4.5. Considering further the remaining case of $(\bar{z}, \bar{\lambda}) \in I_1$ in Lemma 4.7, we deduce from it that $\langle \mathcal{H}(\bar{x}, \bar{w}, \bar{\lambda})u, u \rangle = \langle q, \nabla_x \Phi(\bar{x}, \bar{w})u \rangle$, and so

$$\langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u \rangle + \langle q, \nabla_x \Phi(\bar{x}, \bar{w})u \rangle = \langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u \rangle + \langle \mathcal{H}(\bar{x}, \bar{w}, \bar{\lambda})u, u \rangle > 0$$

due to SSOSC (4.40), which brings us to (4.42) and hence justifies the "if" part of the theorem.

To prove the "only if" part, suppose that $\bar{x}$ is a fully stable locally optimal solution to $\mathcal{P}(\bar{w}, \bar{v})$, which is obviously a usual local minimizer for this problem. Thus the well-known first-order necessary optimality conditions for $\mathcal{P}(\bar{w}, \bar{v})$ under the assumed SOCQ/nondegeneracy condition (4.13) tells us that there is a unique vector $-\bar{\lambda} \in N_{\mathcal{Q}_{m+1}}(\Phi(\bar{x}, \bar{w}))$ satisfying (4.11). It follows from Lemma 4.5 that condition (4.42) is satisfied for all $q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u)$ with $u \neq 0$. Let us show that it implies the validity of SSOSC (4.40). To proceed, fix any $u \in \operatorname{span}\{\mathcal{C}(\bar{x}, \bar{w})\} \setminus \{0\}$ and observe that $\beta = (\beta_0, \beta_r) \in \operatorname{dom} \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda}) \neq \emptyset$ in the notation above. Considering the case of $(\bar{z}, \bar{\lambda}) \in I_2$, we deduce from Lemma 4.7(ii) that $0 \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u)$. Letting $q = 0$ in (4.42) and taking into account that $\mathcal{H}(\bar{x}, \bar{w}, \bar{\lambda}) = 0$ allow us to conclude that (4.40) holds in this case. In the other case of $(\bar{z}, \bar{\lambda}) \in I_1$, consider the vector $q := -\frac{\bar{\lambda}_0}{\bar{z}_0}(\beta_0, -\beta_r)$ and deduce from Lemma 4.7(i) that

$$\langle q, \beta \rangle = -\frac{\bar{\lambda}_0}{\bar{z}_0}\Big(\beta_0^2 - \|\beta_r\|^2\Big) = \langle \mathcal{H}(\bar{x}, \bar{w}, \bar{\lambda})u, u\rangle.$$

Similarly to the proof of Case 3 in Lemma 4.6, we get that $q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\beta_0, \beta_r)$ and, substituting this vector into (4.42), arrive at SSOSC (4.40). This completes the proof of the theorem in the case of the single Lorentz cone $\mathcal{Q} = \mathcal{Q}_{m+1}$ in (1.1).

It remains to consider the general case of the product cone (1.2) in (1.1). Define the mapping $\mathcal{T}(\bar{x}, \bar{w}, \bar{v}) = (\mathcal{T}_1(\bar{x}, \bar{w}, \bar{v}), \mathcal{T}_2(\bar{x}, \bar{w}, \bar{v}))\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^d$ by (4.41) with replacing $\mathcal{Q}_{m+1}$ by $\mathcal{Q}$ therein. Then using the results above for one cone $\mathcal{Q}_{m+1}$ in (1.1) and the product formula for the second-order subdifferential in Lemma 3.8. This completes the proof of the theorem. $\square$

To conclude this section, observe that the calculations and results presented above are new also in the study of *tilt stability* for SOCPs when the basic perturbation parameter w is absent in (4.1) and thus its nominal value $\bar{w}$ does not appear in the obtained characterizations. In this case the partial versions of SOQC (4.13) and (4.14) reduce to their full counterparts in (3.1) and (3.2), respectively, and the partial SSOSC (4.40) agrees with its nonparametric version from [2]. Thus the latter provides a characterization of tilt stability for SOCPs in accordance with Theorem 4.8 when the basic perturbation parameter is dropped.

5. Relationships with strong stability of generalized equations.

In this section we address the relationship between full stability of local minimizers for $\mathcal{P}(\bar{w}, \bar{v})$ in (4.1) and Robinson's notion of strong regularity for the associated parametric *KKT system* (4.11). We show that these notions are in fact *equivalent* under the imposed SOCQ/nondegeneracy condition. As a by-product of this result, we recover the SSOSC characterization of strong regularity obtained, via a completely different approach, by Bonnans and Ramírez [2].

Considering two closed and convex sets $Z \subset \mathbb{R}^n$ and $C \subset \mathbb{R}^p$, recall [3, Definition 3.135] that $Z \subset \mathbb{R}^n$ is $\mathcal{C}^2$-*reducible* at $\bar{z} \in Z$ to C if there is a neighborhood U of $\bar{z}$ and a $\mathcal{C}^2$-smooth mapping $h\colon \mathbb{R}^n \to \mathbb{R}^p$ such that $\delta_Z(z) = \delta_C(h(z))$ for all $z \in U \cap Z$, and the derivative operator $\nabla h(\bar{z})\colon \mathbb{R}^n \to \mathbb{R}^s$ is surjective. If this holds for all $z \in Z$, then we say that Z is $\mathcal{C}^2$-reducible to C. It is proved in [2, Lemma 15] that the Lorentz cone $\mathcal{Q}_{m+1}$ is $\mathcal{C}^2$-reducible at $\bar{z} \in \mathcal{Q}_{m+1}$ to

$$K_m := \begin{cases} \mathcal{Q}_{m+1} & \text{if } \bar{z} = 0, \\ \{0\} & \text{if } \bar{z} \in \operatorname{int} \mathcal{Q}_{m+1}\backslash\{0\}, \\ \mathbb{R}_- & \text{if } \bar{z} \in \operatorname{bd} \mathcal{Q}_{m+1}\backslash\{0\}. \end{cases}$$

It readily implies that the product cone (1.2) is also $\mathcal{C}^2$-reducible at $\bar{z} \in \mathcal{Q}$ to some closed and convex cone $K \subset \mathbb{R}^s$. This allows us to find a neighborhood U of $\bar{z}$ and a $\mathcal{C}^2$-smooth mapping $h\colon \prod_{j=1}^J \mathbb{R}^{j+1} \to \mathbb{R}^s$ with $0 \le s \le \sum_{j=1}^J (m_j + 1)$, such that

$$\begin{cases} \delta_{\mathcal{Q}}(z) = \delta_K(h(z)) \text{ for all } z \in U \cap \mathcal{Q} \ \text{ and} \\ \nabla h(\bar{z})\colon \prod_{j=1}^J \mathbb{R}^{j+1} \to \mathbb{R}^s \ \text{ is surjective.} \end{cases} \tag{5.1}$$

The following theorem is of its own interest while playing a crucial role in establishing the subsequent equivalence between full stability of SOCP (1.1) and Robinson's strong regularity of the associated KKT system with respect to canonical perturbations.

THEOREM 5.1 (equivalence between full stability of SOCP and Lipschitz continuity of the single-valued solution map to the associated KKT system). *Let $\bar{x}$ be a constraint nondegenerate point of SOCP in* (4.1). *Then it is a fully stable locally optimal solution to problem $\mathcal{P}(\bar{w},\bar{v})$ in* (4.1) *with $\bar{v}$ from* (4.10) *if and only if we have $\bar{x} \in M_\gamma(\bar{w},\bar{v})$ for some $\gamma > 0$ and the solution map $S_{KKT}\colon (w,v) \mapsto (x,\lambda)$ for the KKT system* (4.11) *is single valued and Lipschitz continuous around $(\bar{w},\bar{v},\bar{x},\bar{\lambda})$.*

Proof. To justify the "if" part of the theorem, observe first that RCQ (4.5) is obviously fulfilled in this setting. Then Theorem 4.2 tells us that $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w},\bar{v})$ provided that the subgradient mapping $\partial_x\varphi$ in (4.8) is PSMR at $(\bar{x},\bar{w},\bar{v})$ for $\bar{v}$ from (4.10). To verify this property, note that the local single valuedness and Lipschitz continuity of the KKT solution map S_{KKT} around $(\bar{w},\bar{v},\bar{x},\bar{\lambda})$ implies, by using the calculus rules in (4.11) similarly to deriving (4.8), that the partial inverse mapping (4.9) admits a single-valued Lipschitzian localization around $(\bar{x},\bar{w},\bar{v})$. But this exactly means the claimed PSMR of $\partial_x\varphi$.

To prove the "only if" part of the theorem, suppose that $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w},\bar{v})$. Then $\bar{x} \in M_\gamma(\bar{w},\bar{v})$ and it follows from the nondegeneracy condition that the mapping S_{KKT} is single valued on some neighborhood $W \times V$ of $(\bar{w},\bar{v})$, i.e., $S_{KKT}(w,v) = \{(x_{wv},\lambda_{wv})\}$ therein. Indeed, the full stability of $\bar{x}$ ensures that the mapping $\partial_x\varphi$ is PSMR at $(\bar{x},\bar{w},\bar{v})$, which implies the uniqueness of the critical point x_{wv} for any $(w,v) \in W \times V$ while the uniqueness of the Lagrange multiplier λ_{wv} is due to the nondegeneracy. Since the Lipschitz continuity of the mapping $(w,v) \mapsto x_{wv}$ around $(\bar{x},\bar{w})$ follows directly from the full stability of $\bar{x}$, it remains to verify that the mapping $(w,v) \mapsto \lambda_{wv}$ is Lipschitz continuous around $(\bar{w},\bar{v})$ as well.

To proceed, observe that due to the validity of RCQ (4.5) in this setting there is a positive number $\rho < \infty$ such that $\|\lambda_{wv}\| \le \rho$ for all $(w,v) \in W \times V$. Let $\ell > 0$ be a common Lipschitz constant for the mappings $\nabla_x\varphi_0$, $\nabla_x\Phi$, and $(w,v) \mapsto x_{wv}$ on $W \times V$. Due to reducibility (5.1) of the cone $\mathcal{Q}$ at $\bar{z} = \Phi(\bar{x},\bar{w})$ we have

$$N_{\mathcal{Q}}(z) = \nabla h(z)^* N_K(h(z)) \quad \text{whenever} \quad z \in U \cap \mathcal{Q} \tag{5.2}$$

with $\operatorname{rank} \nabla h(\bar{z}) = s$. This easily implies (see [3, p. 315]) that the Jacobian matrix $\nabla_x\psi(\bar{x},\bar{w})$ for the composition $\psi := h \circ \Phi$ is of full rank s.

Pick now any $w_1, w_2 \in W$ and $v_1, v_2 \in V$ and denote $z_i := \Phi(x_{w_iv_i}, w_i)$ for $i = 1,2$. It follows from (5.2) and the composite representation $\delta_{\mathcal{Q}} \circ \Phi = \delta_K \circ \psi$ that there are unique normals $-\mu_{w_iv_i} \in N_K(h(z_i))$ for $i = 1,2$ satisfying the conditions

$$\lambda_{w_iv_i} = \nabla h(z_i)^*\mu_{w_iv_i} \quad \text{and} \quad \nabla_x\Phi(x_{w_iv_i},w_i)^*\lambda_{w_iv_i} = \nabla_x\psi(x_{w_iv_i},w_i)^*\mu_{w_iv_i}, \quad i = 1,2, \tag{5.3}$$

which readily ensure the equations

$$\left\{\begin{aligned} v_2 &= \nabla_x\varphi_0(x_{w_2v_2},w_2) + \nabla_x\Phi(x_{w_2v_2},w_2)^*\lambda_{w_2v_2} \\ &= \nabla_x\varphi_0(x_{w_2v_2},w_2) + \nabla_x\psi(x_{w_2v_2},w_2)^*\mu_{w_2v_2}, \\ v_1 &= \nabla_x\varphi_0(x_{w_1v_1},w_1) + \nabla_x\Phi(x_{w_1v_1},w_1)^*\lambda_{w_1v_1} \\ &= \nabla_x\varphi_0(x_{w_1v_1},w_1) + \nabla_x\psi(x_{w_1v_1},w_1)^*\mu_{w_1v_1}. \end{aligned}\right.$$

Combining the above relationships, we arrive at

$$\begin{aligned} &\nabla_x\psi(x_{w_2v_2},w_2)^*(\mu_{w_2v_2} - \mu_{w_1v_1}) \\ &\quad = \Big(\nabla_x\psi(x_{w_1v_1},w_1) - \nabla_x\psi(x_{w_2v_2},w_2)\Big)^*\mu_{w_1v_1} \\ &\qquad + \nabla_x\varphi_0(x_{w_1v_1},w_1) - \nabla_x\varphi_0(x_{w_2v_2},w_2) + v_2 - v_1. \end{aligned} \tag{5.4}$$

Having rank $\nabla_x\psi(\bar{x},\bar{w}) = s \le n$, we can always reduce the situation to the square case of $s = n$. Indeed, when $s < n$, construct a linear mapping $\widetilde{\psi}\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^{n-s}$ such that the matrix

$$\overline{\psi}(x,w) := (\psi(x,w), \widetilde{\psi}(x,w))\colon \mathbb{R}^n \times \mathbb{R}^d \longrightarrow \mathbb{R}^n \tag{5.5}$$

has full rank for any fixed (x,w) close to $(\bar{x},\bar{w})$. We do it by taking an orthogonal basis $\{b_1,\ldots,b_{n-s}\}$ in the $(n-s)$-dimensional space $\{u \in R^n|\ \nabla_x\psi(\bar{x},\bar{w})u = 0\}$ and then defining $\widetilde{\psi}(x,w) := (\langle b_1,x\rangle,\ldots,\langle b_{n-s},x\rangle)$. Denote further $\overline{\delta}_K(z,q) := \delta_K(z)$ for all $z \in \mathbb{R}^s$ and $q \in \mathbb{R}^{n-s}$. Employing now an elementary first-order chain rule for "full rank" compositions yields

$$\begin{aligned}\partial_x(\overline{\delta}_K \circ \overline{\psi})(x,w) &= \nabla_x\overline{\psi}(x,w)^*\partial\overline{\delta}_K(\overline{\psi}(x,w))\\ &= \Big(\nabla_x\psi(x,w)^*, b_1,\ldots,b_{n-s}\Big)\begin{pmatrix}\partial\delta_K(z)\\ 0^{n-s}\end{pmatrix}\\ &= \nabla_x\psi(x,w)^*N_K(z)\end{aligned}$$

with $z = \psi(x,w)$. This ensures the existence of the unique subgradients $\zeta_1 \in \partial\overline{\delta}_K(z_1')$ and $\zeta_2 \in \partial\overline{\delta}_K(z_2')$ such that $z_1' = \overline{\psi}(x_{w_1v_1},w_1)$, $z_2' = \overline{\psi}(x_{w_2v_2},w_2)$, and

$$\nabla_x\psi(x_{w_iv_i},w_i)^*\mu_{w_iv_i} = \nabla_x\overline{\psi}(x_{w_iv_i},w_i)^*\zeta_i \ \text{ with } \ \zeta_i = (\mu_{w_iv_i},0^{n-s}),\quad i=1,2. \tag{5.6}$$

Combining the above relationships with (5.4), we deduce that

$$\begin{aligned}&\nabla_x\overline{\psi}(x_{w_2v_2},w_2)^*(\zeta_2-\zeta_1)\\ &\quad= \Big(\nabla_x\overline{\psi}(x_{w_1v_1},w_1) - \nabla_x\overline{\psi}(x_{w_2v_2},w_2)\Big)^*\zeta_1\\ &\qquad+ \nabla_x\varphi_0(x_{w_1v_1},w_1) - \nabla_x\varphi_0(x_{w_2v_2},w_2) + v_2 - v_1.\end{aligned} \tag{5.7}$$

Since the mapping $\overline{\psi}$ in (5.5) is invertible in x, we apply to it the standard inverse function theorem and arrive at

$$\begin{aligned}&\|\zeta_2-\zeta_1\|\\ &\quad\le \Big\|\Big(\nabla_x\overline{\psi}(x_{w_2v_2},w_2)^*\Big)^{-1}\Big\|\Big(\|\nabla_x\overline{\psi}(x_{w_1v_1},w_1) - \nabla_x\overline{\psi}(x_{w_2v_2},w_2)\|\cdot\|\mu_{w_1v_1}\|\\ &\qquad\qquad+ \|\nabla_x\varphi_0(x_{w_1v_1},w_1) - \nabla_x\varphi_0(x_{w_2v_2},w_2)\| + \|v_2-v_1\|\Big)\\ &\quad\le \gamma\Big[\|\mu_{w_1v_1}\|\ell\Big(\|x_{w_2v_2}-x_{w_1v_1}\| + \|w_2-w_1\|\Big)\\ &\qquad+ \ell\Big(\|x_{w_2v_2}-x_{w_1v_1}\| + \|w_2-w_1\|\Big) + \|v_2-v_1\|\Big]\end{aligned} \tag{5.8}$$

by substituting (5.6) into (5.4), where $\gamma > 0$ is an upper bound of $\|(\nabla_x\overline{\psi}(x,w)^*)^{-1}\|$ over (x,w) near $(\bar{x},\bar{w})$. Since $\|\lambda_{wv}\| \le \rho$ for $(w,v) \in W \times V$, $\lambda_{w_iv_i} = \nabla h(z_i)^*\mu_{w_iv_i}$ for $i = 1,2$, and $\nabla h(\bar{z})$ is of full rank, we find arguing as above that there is $\breve{h}\colon \prod_{j=1}^J \mathbb{R}^{j+1} \to \prod_{j=1}^J \mathbb{R}^{j+1}$ such that

$$\partial(\breve{\delta}_K \circ \breve{h})(z) = \nabla\breve{h}(z)^*\partial\breve{\delta}_K(\breve{h}(z)) = \nabla h(z)^*N_K(h(z)) \tag{5.9}$$

with $\breve{\delta}_K(z,q) := \delta_K(z)$ for all $z \in \mathbb{R}^s$ and $q \in \mathbb{R}^{\sum_{j=1}^J(m_j+1)-s}$. This gives us

$$\lambda_{w_iv_i} = \nabla h(z_i)^*\mu_{w_iv_i} = \nabla\breve{h}(z_i)^*\breve{\mu}_{w_iv_i} \tag{5.10}$$

with $\breve{\mu}_{w_iv_i} \in \partial\breve{\delta}_K(\breve{h}(z)) = (N_K(h(z)), 0^{\sum_{j=1}^J (m_j+1)-s})$, which ensures in turn that

$$\breve{\mu}_{w_iv_i} = \left(\mu_{w_iv_i}, 0^{\sum_{j=1}^J (m_j+1)-s}\right). \tag{5.11}$$

Using now (5.10) and (5.11) together with the classical inverse function theorem for the mapping $\breve{h}$ invertible in x tells us that the family of $\{\mu_{wv}\}$ is uniformly bounded when the pairs (w, v) are close enough to $(\bar{w}, \bar{v})$. Without loss of generality, suppose that

$$\|\mu_{wv}\| \le \rho \text{ for all } (w, v) \in W \times V. \tag{5.12}$$

Also the equalities in (5.3) imply the relationships

$$\begin{aligned}
\|\lambda_{w_2v_2} - \lambda_{w_1v_1}\| &= \|\nabla h(z_2)^*\mu_{w_2v_2} - \nabla h(z_1)^*\mu_{w_1v_1}\| \\
&\le \|\nabla h(z_2)^*\| \cdot \|\mu_{w_2v_2} - \mu_{w_1v_1}\| + \|\nabla h(z_2) - \nabla h(z_1)\| \cdot \|\mu_{w_1v_1}\| \\
&\le \tau\|\mu_{w_2v_2} - \mu_{w_1v_1}\| + \rho\ell'\Big(\|x_{w_2v_2} - x_{w_1v_1}\| + \|w_2 - w_1\|\Big) \\
&\le \tau\|\zeta_2 - \zeta_1\| + \rho\ell'\Big(\|x_{w_2v_2} - x_{w_1v_1}\| + \|w_2 - w_1\|\Big),
\end{aligned} \tag{5.13}$$

where $\tau > 0$ is an upper bound of $\|\nabla h(z)^*\|$ for all z sufficiently close to $\bar{z}$, and where $\ell' > 0$ is a Lipschitz constant for the mapping ∇h around $\bar{z}$.

Combining finally the estimates in (5.8), (5.12), and (5.13) with the local Lipschitz continuity of the mapping $(w, v) \mapsto x_{wv}$ allows us to conclude that $(w, v) \mapsto \lambda_{wv}$ is Lipschitz continuous around $(\bar{w}, \bar{v})$, which thus completes the proof of the theorem. ∎

It is easy to see that the given proof of Theorem 5.1 does not exploit specific futures of SOCPs in (1.1) and holds true for any $\mathcal{C}^2$-*reducible* cones generating problems of conic programming.

Consider next the *canonically* perturbed version of (1.1) denoted by $\widetilde{\mathcal{P}}_{\bar{w}}(v_1, v_2)$:

$$\begin{cases} \text{minimize } \varphi_0(x, \bar{w}) - \langle v_1, x\rangle \text{ subject to } x \in \mathbb{R}^n, \\ \Phi(x, \bar{w}) + v_2 \in \mathcal{Q} \end{cases} \tag{5.14}$$

for any $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^m$. The following important lemma relates the full stability of $\bar{x}$ in $\mathcal{P}(\bar{w}, \bar{v})$ to this property of $\bar{x}$ in the canonically perturbed problem $\widetilde{\mathcal{P}}_{\bar{w}}(\bar{v}, 0)$.

Lemma 5.2 (full stability with respect to canonical perturbations). *Let $\bar{x}$ be a feasible solution to problem $\mathcal{P}(\bar{w}, \bar{v})$ in (4.1) for some $\bar{w} \in \mathbb{R}^d$ and $\bar{v}$ from (4.10) under the validity of the SOCQ/nondegeneracy condition at $(\bar{x}, \bar{w})$. Then $\bar{x}$ is a fully stable local minimizer of $\mathcal{P}(\bar{w}, \bar{v})$ if and only if it is a fully stable local minimizer of $\widetilde{\mathcal{P}}_{\bar{w}}(\bar{v}, 0)$.*

Proof. Observe first that the SOCQ/nondegeneracy condition for $\mathcal{P}(\bar{w}, \bar{v})$ at $(\bar{x}, \bar{w})$ is equivalent to this property for the $\widetilde{\mathcal{P}}_{\bar{w}}(\bar{v}, 0)$ at $(\bar{x}, \bar{w})$. It follows from Lemma 4.5 that the full stability of $\bar{x}$ for problem $\widetilde{\mathcal{P}}_{\bar{w}}(\bar{v}, 0)$, in the sense of Definition 4.1 applied to this problem, is equivalent to

$$[(p, q) \in \widetilde{\mathcal{T}}_{\bar{w}}(\bar{x}, 0, \bar{v})(u),\ u \ne 0] \Longrightarrow \langle p, u\rangle > 0, \tag{5.15}$$

where the set-valued mapping $\widetilde{\mathcal{T}}_{\bar{w}}(\bar{x}, 0, \bar{v})\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ is defined by

$$\begin{aligned}
\widetilde{\mathcal{T}}_{\bar{w}}(\bar{x}, 0, \bar{v})(u) := &\Big(\nabla^2_{xx}\varphi_0(\bar{x}, \bar{w})u, 0\Big) + \Big(\nabla^2_{xx}\langle\bar{\lambda}, \Phi\rangle(\bar{x}, \bar{w})u, 0\Big) \\
&+ \Big(\nabla_x\Phi(\bar{x}, \bar{w}), I_m\Big)^* \partial^2\delta_\Theta(\bar{z}, \bar{\lambda})(\nabla_x\Phi(\bar{x}, \bar{w})u), \quad u \in \mathbb{R}^n.
\end{aligned}$$

It is not hard to check that (5.15) can be written equivalently as

$$\langle u, \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})u\rangle + \langle q, \nabla_x \Phi(\bar{x}, \bar{w})u\rangle > 0$$

for all $q \in \partial^2 \delta_{\mathcal{Q}_{m+1}}(\bar{z}, -\bar{\lambda})(\nabla_x \Phi(\bar{x}, \bar{w})u)$ with $u \neq 0$, which reduces to condition (4.42) characterizing the full stability of $\bar{x}$ in $\mathcal{P}(\bar{w}, \bar{v})$ and thus completes the proof. □

In the rest of this section, suppose for convenience and without loss of generality that $\bar{v} = 0$. Following Robinson [31], we say that the pair $(\bar{x}, \bar{\lambda})$ is *strongly regular* for the KKT system (4.11) if the solution map to the *linearized* generalized equation

$$\text{(5.16)} \qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \begin{bmatrix} \nabla^2_{xx} L(\bar{x}, \bar{w}, \bar{\lambda})(x - \bar{x}) - \nabla_x \Phi(\bar{x}, \bar{w})^*(\lambda - \bar{\lambda}) \\ -\Phi(\bar{x}, \bar{w}) - \nabla_x \Phi(\bar{x}, \bar{w})(x - \bar{x}) \end{bmatrix} + \begin{bmatrix} 0 \\ N_{\mathcal{Q}}^{-1}(-\lambda) \end{bmatrix}$$

has a Lipschitz continuous single-valued localization around $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^s$.

THEOREM 5.3 (equivalence between full stability and strong regularity for SOCPs under the nondegeneracy condition). *Let* $(\bar{x}, \bar{w})$ *be a feasible solution to* $\mathcal{P}(\bar{w}, \bar{v})$ *with* $\bar{v} = 0$ *satisfying* (4.10)*, and let* $\bar{\lambda} \in \mathbb{R}^s$ *be a unique Lagrange multiplier of the KKT system* (4.11)*. Imposing the nondegeneracy condition* (4.14)*, or the SOCQ condition* (4.13)*, we have that* $\bar{x}$ *is a fully stable locally optimal solution to* $\mathcal{P}(\bar{w}, \bar{v})$ *if and only if* $\bar{x} \in M_\gamma(\bar{w}, \bar{v})$ *for some* $\gamma > 0$ *and* $(\bar{x}, \bar{\lambda})$ *is a strongly regular solution to* (4.11)*. Furthermore, this strong regularity implies the validity of the SOCQ/nondegeneracy condition* (4.14) *for SOCPs.*

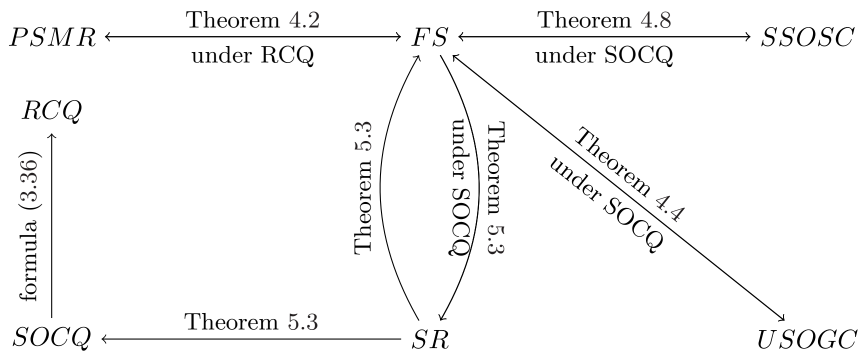
Proof. Note that the generalized equation (5.16) can be consider as a linearization of the *canonically perturbed* KKT system (4.11) as follows:

$$\text{(5.17)} \qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \begin{bmatrix} \nabla_x L(x, \bar{w}, \lambda) \\ -\Phi(x, \bar{w}) \end{bmatrix} + \begin{bmatrix} 0 \\ N_{\mathcal{Q}}^{-1}(-\lambda) \end{bmatrix}$$

with the perturbation pair (v_1, v_2) varied around $(\bar{v}, 0) \in \mathbb{R}^n \times \mathbb{R}^s$ with $\bar{v} = 0$. It has been well recognized in optimization theory (see, e.g., [5, Theorem 2B.10]) that the single valuedness and Lipschitz continuity of the canonically perturbed generalized equation/KKT system is equivalent to these properties of its linearization (5.16). Taking now a fully stable locally optimal solution $\bar{x}$ to $\mathcal{P}(\bar{w}, 0)$ and applying Lemma 5.2, we get that $\bar{x}$ is a fully stable locally optimal solution to problem $\widetilde{\mathcal{P}}_{\bar{w}}(0, 0)$ in (5.14). Then Theorem 5.1 tells us that the solution map $S_{KKT} \colon (v_1, v_2) \mapsto (x, \lambda)$ of the KKT system (5.17) is single valued and Lipschitz continuous around $(0, 0, \bar{x}, \bar{\lambda})$. By [5, Theorem 2B.10] these properties are satisfied for linearization (5.16) around $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^s$. This says that the pair $(\bar{x}, \bar{\lambda})$ is strongly regular for the KKT system (4.11).

To justify the converse implication, suppose that $(\bar{x}, \bar{\lambda})$ is a strongly regular solution to (4.11). It follows from [5, Theorem 2B.10] that the solution map $S_{KKT} \colon (v_1, v_2) \mapsto (x, \lambda)$ of the KKT system (5.17) is single valued and Lipschitz continuous around $(0, 0, \bar{x}, \bar{\lambda})$. Recalling that $\bar{x} \in M\gamma(\bar{w}, \bar{v})$ for some $\gamma > 0$ and combining it with Theorem 5.1 tell us that $\bar{x}$ is a fully stable locally optimal solution to $\widetilde{\mathcal{P}}_{\bar{w}}(0, 0)$ and therefore to $\mathcal{P}(\bar{w}, 0)$. Since strong regularity implies nondegeneracy for conic programming (see, e.g., [3, Theorem 5.24]), this ensures the validity of SOCQ in the SOCP setting by Theorem 3.9. □

For the reader's convenience we present now the diagram on relationships between the main properties considered in this paper. Here the symbols FS and SR stand for full stability and strong regularity, respectively, while the other abbreviations have been defined earlier.



Finally in this section, we give two examples illustrating some important features of full stability for SOCPs and its relationships with strong regularity. The first example shows that, for the same SOCP, one local minimizer is fully stable while another is not under the validity of the nondegeneracy condition for both of them.

EXAMPLE 5.4 (full stability for different local minimizers under nondegeneracy). For the three-dimensional SOCP (1.1) consider the corresponding problem $\mathcal{P}(w,v)$ in (4.1) given by

$$\mathcal{P}(w,v)\begin{cases}\text{minimize} & \frac{1}{2}\|u-a(w)\|^2+\frac{1}{2}\|z-b(w)\|^2+wu_1z_1-\langle v,(u,z)\rangle\\ \text{subject to} & \\ & u\in\mathcal{Q}_3,\, z\in\mathcal{Q}_3,\end{cases}$$

where $x:=(u,z)\in\mathbb{R}^3\times\mathbb{R}^3$, $w\in\mathbb{R}$, and

$$a(w):=\begin{bmatrix} w\\ \frac{w}{1+w}\\ 0\end{bmatrix},\quad b(w):=\begin{bmatrix} w\\ 0\\ \frac{w}{1+w}\end{bmatrix}.$$

Taking the reference perturbation pair $(\bar{w},\bar{v})=(0.5,0^{\mathbb{R}^6})$, it is easy to verify that $\bar{x}=(\bar{u},\bar{z})=(\frac{1}{3},\frac{1}{3},0,\frac{1}{3},0,\frac{1}{3})$ is a nondegenerate local minimizer of $\mathcal{P}(\bar{w},\bar{v})$ and that $\bar{\lambda}=0$ is the corresponding (unique) Lagrange multiplier of the KKT system (4.11). As shown in [28, Table 4.1], the given point $(\bar{x},\bar{\lambda})$ is strongly regular for (4.11). Thus it follows from Theorem 5.3 that $\bar{x}$ is a fully stable locally optimal solution to the problem $\mathcal{P}(\bar{w},\bar{v})$ under consideration.

Consider now the perturbation pair $(\widetilde{w},\widetilde{v})=(1,0^{\mathbb{R}^6})$. In this case we check that $\widetilde{x}=(\widetilde{u},\widetilde{z})=(0.5,0.5,0,0.5,0,0.5)$ is a nondegenerate minimizer of $\mathcal{P}(\widetilde{w},\widetilde{v})$ and, again, $\bar{\lambda}=0$ is the corresponding (unique) Lagrange multiplier of the associated KKT system (4.11). Applying [2, Theorem 30] allows us to conclude that the pair $(\widetilde{x},\bar{\lambda})$ is not strongly regular for (4.11), and thus Theorem 5.3 shows that $\widetilde{x}$ is not a fully stable locally optimal solution to $\mathcal{P}(\widetilde{w},\widetilde{v})$.

The second example demonstrates that full stability is indeed a broader concept than strong regularity for SOCPs in the *absence of nondegeneracy*.

EXAMPLE 5.5 (full stability without strong regularity). In the full stability framework, consider the following problem $\mathcal{P}(w,v)$ for the two-dimensional SOCP:

$$\mathcal{P}(w,v)\begin{cases}\text{minimize} & -x-vx\\ \text{subject to} & \\ & \begin{bmatrix}1-x\\1-w\end{bmatrix}\in\mathcal{Q}_2\end{cases}$$

with $x,w,v\in\mathbb{R}$. Picking the reference perturbation pair $(\bar{w},\bar{v})=(1,0)$, it is easy to see that $\bar{x}=1$ is the unique minimizer for $\mathcal{P}(\bar{w},\bar{v})$, which is not nondegenerate but satisfies RCQ (4.5). To examine the full stability in this case, we invoke Theorem 4.2 and check the PSMR property of the corresponding partial subgradient mapping

$$\partial_x\varphi(x,w) := -1-[-1,0]N_{\mathcal{Q}_2}\left(\begin{bmatrix}1-x\\1-w\end{bmatrix}\right). \tag{5.18}$$

Denote by $G\colon\mathbb{R}\times\mathbb{R}\rightrightarrows\mathbb{R}$ the partial inverse (4.7) of the mapping $\partial_x\varphi$ in (5.18). It readily follows from (5.18) that there are neighborhoods $\mathcal{W}$ of $\bar{w}$ and $\mathcal{V}$ of $\bar{v}$ on which the mapping G is single valued and admits the representation

$$G(w,v)=\left\langle\begin{array}{lll} w & \text{for} & w\le 1,\\ 1-w & \text{for} & w>1,\end{array}\right.$$

which shows that the mapping $\partial_x\varphi$ is PSMR at $(\bar{x},\bar{w},\bar{v})$. Thus we deduce from Theorem 4.2 that $\bar{x}=1$ is a fully stable locally optimal solution to the problem $\mathcal{P}(\bar{w},\bar{v})$ under consideration. Observe further that the corresponding KKT system (4.11) attains the form

$$v=-1-[-1,0]\lambda,\quad -\lambda\in N_{\mathcal{Q}_2}\left(\begin{bmatrix}1-x\\1-w\end{bmatrix}\right). \tag{5.19}$$

It is easy to see that the triple $(\bar{x},\bar{w},\bar{v})$ satisfies (5.19) along with *any* vector $\lambda\in\mathcal{Q}_2$ whose first component amounts to 1, and thus the solution map to the linearized system (5.16) is not single valued. This shows that strong regularity is violated in this case, and thus full stability is a broader concept than strong regularity for SOCPs in the absence of nondegeneracy.

6. Concluding remarks. This paper presents several second-order characterizations of the important notion of full stability for locally optimal solutions to the general problems of SOCP. The main result, Theorem 4.8, describes necessary and sufficient conditions for full stability entirely in terms of the initial data of SOCPs under the constraint nondegeneracy, which happens to be equivalent to the SOCQ which appeared in the core theory of second-order variational calculus. The interplay between general forms of second-order variational analysis and generalized differentiation from one side and their specific implementations in the SOCP framework is one of the major themes of this paper. It is worth mentioning that most of the results obtained in this paper including the main Theorem 4.8 are also new for tilt stability of locally optimal solutions to SOCPs, which is a more narrow notion than full stability.

Among the most challenging issues of further research is to study full and tilt stability for SOCPs without the *nondegeneracy* assumption, which reduces to the classical (and rather restrictive) LICQ condition for NLPs. In the latter framework the LICQ condition has been recently essentially relaxed in the study of tilt and full stability by exploiting some specific features of NLPs; see [21, 22, 24]. It seems to be highly important to relax the SOCQ/nondegeneracy assumption for tilt and full stability of SOCPs and other remarkable classes of problems in conic programming.

On the other hand, the established equivalence between full/tilt stability and strong regularity under the nondegeneracy condition opens the gate to the broad usage of the former in numerical applications. Furthermore, it has been well recognized in optimization that nondegeneracy is in fact *necessary* for strong regularity while is not the case for tilt and full stability. Thus the latter stability notions promise to be an efficient tool of qualitative and numerical analysis of optimization problems even in the absence of nondegeneracy and strong regularity.

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