On the Tsallis Entropy for Gibbs Random Fields*

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Abstract. The Tsallis entropy, as a generalization of the standard Shannon-type entropy, was introduced by Constantino Tsallis (1988). Since that the concept has been extensively studied (see, e.g., Tsallis (2009)).
In the present paper we address the problem of generalizing the concept for infinite-dimensional systems, i.e., the random processes and fields. Apparently, rather well suited models are the Gibbs distributions (cf. e.g., Georgii (1988)).
We construct the appropriate Tsallis entropy rate either asymptotically by limit over a sequence of expanding volumes or by analogy with the exponential finite-dimensional distributions. Basic properties, taking into account the possible phase transitions, are also introduced.

Keywords: Tsallis entropy, Gibbs random field, phase transitions, Tsallis entropy rate

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1. Introduction

The Tsallis entropy is a generalization of the standard Shannon-type entropy. It was introduced by Constantino Tsallis (1988) as a basis for generalizing the standard statistical mechanics. In the scientific literature, the physical relevance of the Tsallis entropy was occasionally debated (see, e.g., Curado and Tsallis (1992)).

Recently, an increasingly wide spectrum of natural, artificial and social complex systems have been identified which confirm the predictions and consequences that are derived from this nonadditive entropy.

There are various fields and topics in finance and economics where entropy in general has brought interesting perspective. The list of relevant areas includes the general financial mathematics (see, e.g., Michael et al. (2002), Ramos et al. (1999), or Anteneodo et al. (2002)).

There are also many particular results and topics connected to the mathematical theory provided in the present paper, e.g., Bera and Park (2008) utilize entropy as an additional instrument for portfolio optimization, Eom et al. (2008) study relationship between predictability and efficiency using approximate entropy, Giglio et al. (2008) study complexity and relative efficiency of the financial markets, Kristoufek and Vosvrda (2014a,2014b) utilize entropy as a component of an efficiency

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measure, Maasoumi and Racine (2002) utilize entropy for series predictability detection, Ortiz-Cruz et al. (2012) study the crude oil efficiency using the entropy analysis, Qin et al. (2009) use cross-entropy for portfolio selection, Rompolis (2010) utilizes entropy for option pricing, or Shi and Shang (2013) use cross-entropy as an alternative to cross-correlations. Some of these results could be with an appropriate effort generalized for the depended data models treated here, but such task is behind the scope of this paper.

The Gibbs distributions, as defined within the area of statistical physics, represent natural models for dependent data. Moreover, the Gibbs random fields are well suited for generalizing the various concepts of entropy in the form of the appropriate rate. That means, first of all, the standard Shannon entropy, as well as the Rényi entropy (see Janzura(1999)). Tsallis entropy constitutes a rival to Rényi entropy (see, e.g., Nielsend and Noch (2011)) in the sense that it is also given as a parametric class which coincides with Shannon entropy at parameter point equal to 1.

The results on Gibbs random fields are partly from Künsch' paper (1982) and Georgii's book (1988), the statistical analysis results from Janžura (1997).

After recalling the definition of Tsallis entropy including the basic properties in Section 2, the following Sections 3,4, and 5 are devoted to the concept of Gibbs random fields. Finally Section 6 contains the new results concerning the Thallis entropy rate for Gibbs random fields, including again some fundamental properties. A short comment on the eventual performance is added in Section 7.

For the sake of simplicity, within this paper we consider only the finite state space. With an appropriate effort, the generalization is possible.

2. Tsallis entropy

Let $X_0 = \{x_1, \ldots, x_M\}$ be a finite set, and $\mathcal{P}_0$ the set of all probability measures on X_0. In particular, let $R(x) = \frac{1}{M}$ be the uniform distribution and δ_{x_0} the distribution concentrated to the single point $x_0 \in X_0$.

Then the *Tsallis entropy* for $P \in \mathcal{P}_0$ is defined as

$$S_q(P) = \frac{1}{q-1}\,[1 - \sum_{x \in X_0} P(x)^q]$$

where $q \neq 1$ is a real-valued parameter.

For pro $q = 1$ we obtain by limit the standard *Shannon type entropy*, namely

$$S_1(P) = \lim_{q \to 1} S_q(P) = \sum_{x \in X_0} [-\log P(x)] P(x)$$

Let us introduce the basic properties. We may easily verify that

$$S_q(P) \in [S_q(\delta_{x_0}), S_q(R)]$$

where

$$S_q(\delta_{x_0}) = 0$$

and

$$S_q(R) = \frac{1}{q-1}\,[1 - M^{1-q}]$$

for $q \neq 1$, resp.

$$S_1(R) = \log M.$$

Further, with the aid of an appropriate notation, we may also introduce a unifying approach. Namely, let us denote

$$\log^q z = \frac{1}{q-1}\,[1 - z^{1-q}].$$

Then we may write

$$S_q(P) = E_P[\log^q(\frac{1}{P(\bullet)})]$$

and, consequently,

$$S_q(P) \in [0, \log^q M].$$

Let us introduce the *exponential distribution* with a statistics $f : X_0 \to \mathcal{R}$, i.e.,

$$P^f(x) = \exp(f(x) - c(f))$$

where

$$c(f) = \log \sum_{x \in X_0} \exp(f(x))$$

is the normalizing constant.

Then, by direct substituting, we have

$$S_q(P^f) = \frac{1}{q-1}\,\{1 - \exp[\,c(q\,f) - q\,c(f)]\}$$

for $q \neq 1$ and

$$S_1(P^f) = c(f) - \int f dP^f.$$

3. Gibbs random fields

Let X_0 be again a finite set, and, for some $d \geq 1$, let $\mathbf{Z}^d$ be the d-dimensional integer lattice and $\mathcal{X} = X_0^{\mathbf{Z}^d}$ the corresponding product space. For $V \subset \mathbf{Z}^d$ we denote by $\mathcal{F}_V$ the σ-algebra generated by the projection $\mathrm{Proj}_V : \mathcal{X} \to X_0^V$, and by $\mathcal{B}_V$ the set of (bounded) $\mathcal{F}_V$-measurable functions. Further, we shall write $x_V = \mathrm{Proj}_V(x)$ for $x \in \mathcal{X}$, and we shall denote by $\mathcal{V} = \{W \subset \mathbf{Z}^d; 0 < |W| < \infty\}$ the system of finite non-void subsets of $\mathbf{Z}^d$.

We denote by $\mathcal{P}$ the set of all probability measures on $\mathcal{X}$ (with the product σ-algebra $\mathcal{F}$). Further, by $\mathcal{P}_S$ we denote the set of *stationary* (translation invariant) probability measures, i. e. $\mathcal{P}_S = \{P \in \mathcal{P}; P = P \circ \tau_j^{-1}$for every $j \in \mathbf{Z}^d\}$, where $\tau_j : \mathcal{X} \to \mathcal{X}$ is for every $j \in \mathbf{Z}^d$ the corresponding shift operator defined by $[\tau_j(x)]_t = x_{j+t}$

for every $t \in \mathbf{Z}^d$, $x \in \mathcal{X}$. Finally, by $\mathcal{P}_E$ we denote the set of *ergodic* probability measures, i. e. $\mathcal{P}_E = \{P \in \mathcal{P}_S; P(F) \in \{0,1\}$ for every $F \in \mathcal{E}\}$ where $\mathcal{E} = \{F \in \mathcal{F}; F = \tau_j^{-1}(F)$ for every $j \in \mathbf{Z}^d\}$.

The functions from $\mathcal{B} = \bigcup_{V \in \mathcal{V}} \mathcal{B}_V$ will be quoted as (finite range) *potentials*. For a potential $\Phi \in \mathcal{B}_V$, $V \in \mathcal{V}$, we define the *Gibbs specification* as a family of probability kernels

$$\Pi_\Lambda^\Phi(x_\Lambda | x_{\Lambda^c}) = [Z_\Lambda^\Phi(x_{\Lambda^c})]^{-1} \exp\left\{ \sum_{j \in \Lambda - V} \Phi \circ \tau_j(x) \right\}$$

with the normalizing constant

$$Z_\Lambda^\Phi(x_{\Lambda^c}) = \sum_{y_\Lambda \in X_0^\Lambda} \exp\left\{ \sum_{j \in \Lambda - V} \Phi \circ \tau_j(y_\Lambda, x_{\Lambda^c}) \right\}$$

for every $\Lambda \in \mathcal{V}$. Note that $\Lambda - V = \{j \in \mathbf{Z}^d; (j+V) \cap \Lambda \neq \emptyset\}$.

A probability measure $P \in \mathcal{P}$ is a *Gibbs distribution* (*Gibbs random field*) with the potential $\Phi \in \mathcal{B}$ if

$$P_{\Lambda|\Lambda^c}(x_\Lambda | x_{\Lambda^c}) = \Pi_\Lambda^\Phi(x_\Lambda | x_{\Lambda^c}) \quad \text{a. s.} \quad [P]$$

for every $\Lambda \in \mathcal{V}$. The set of such P's will be denoted by $G(\Phi)$, while the set of stationary (resp. ergodic) Gibbs distributions will be denoted by $G_S(\Phi) = G(\Phi) \cap \mathcal{P}_S$ (resp. $G_E(\Phi) = G(\Phi) \cap \mathcal{P}_E$). In general $G_E(\Phi) \neq \emptyset$. We may say that the (first-order) phase transition occurs if $|G(\Phi)| > 1$. Then, some elements are not ergodic, and some even may be not translation invariant (stationary) though the specification is so. For a detailed treatment and the examples see, e. g., Georgii (1988), Chapter 6.2. Unfortunately, the phase transitions, which are inherent for the infinite-dimensional models and cannot be easily avoided, mean the non-smoothness and non-regularity and make the treatment much more complicated.

Let us end this section with the observation that, since for $\Phi \in \mathcal{B}_V$ we have $\Pi_\Lambda^\Phi \in \mathcal{B}_{\Lambda+V-V}$ for every $\Lambda \in \mathcal{V}$, the above defined Gibbs random fields obey the (spatial) Markov property.

4. Equivalence of potentials

Besides the phase transitions, there is another non-uniqueness that can complicate the treatment, namely the possible equivalence of potentials. Two potentials Φ, $\Psi \in \mathcal{B}$ are *equivalent*, we write $\Phi \approx \Psi$, if $G(\Phi) = G(\Psi)$. There is a couple of equivalent characterizations (see, e. g., Georgii (1988) or Janžura (1994)), e. g. $\Phi \approx \Psi$ iff $\Pi_{\{0\}}^\Phi = \Pi_{\{0\}}^\Psi$. For our purposes there will be also important the following one:

$$\Phi \approx \Psi \quad \text{iff} \quad \int \Phi \, \mathrm{d}P = \int \Psi \, \mathrm{d}P + c \quad \text{for every } P \in \mathcal{P}_S \text{ with some fixed constant } c.$$

The equivalence can appear very easily, e. g.,

$$\Phi \approx \Phi + g - g \circ \tau_j + c \quad \text{for some } g \in \mathcal{B},\ j \in \mathbf{Z}^d, \text{ and a constant } c.$$

From the statistical analysis point of view the equivalence of potentials means breaking the basic identifiability condition, and, therefore, it seems better to avoid the phenomenon unambiguously.

A rather standard way consists in dealing with the equivalence classes instead of the particular potentials. But we prefer to restrict our considerations to a rich enough subclass of mutually nonequivalent potentials. Such subclass should contain representatives of all equivalence classes. That can be arranged by dealing with the so called *vacuum* potentials. Let us fix some state $v \in X_0$, quoted as the vacuum state. For any $V \in \mathcal{V}$, let us denote $\mathcal{B}_V^v = \{\Phi \in \mathcal{B}_V; \Phi(x_V) = 0 \text{ if } x_t = v \text{ for some } t \in V\}$. Further, in order to avoid equivalence by shifting, let us introduce $\mathcal{V}^0 = \{V \in \mathcal{V}; \min_{t\in V} t = 0\}$ where the minimum is with respect to some fixed complete (e. g. the lexicographical) ordering. Now, for a finite subsystem $\mathcal{A} \subset \mathcal{V}^0$ we set $\mathcal{B}_{\mathcal{A}}^v = \{\Phi \in \mathcal{B}; \Phi = \sum_{A\in\mathcal{A}} \Phi_A, \Phi_A \in \mathcal{B}_A^v \text{ for every } A \in \mathcal{A}\}$, and, consequently, $\mathcal{B}^v = \bigcup_{\mathcal{A}\subset\mathcal{V}^0, |\mathcal{A}|<\infty} \mathcal{B}_{\mathcal{A}}^v$ will be our class of vacuum potentials.

Proposition 1

i) For Φ, $\Psi \in \mathcal{B}^v$ it holds:

$$\Phi \approx \Psi \quad \text{iff} \quad \Phi = \Psi.$$

ii) For every $\Phi \in \mathcal{B}$ there exists $\Psi \in \mathcal{B}^v$ such that $\Phi \approx \Psi$.

Proof: i) We may observe that for Φ, $\Psi \in \mathcal{B}^v$ we have $\Phi - \Psi \in \mathcal{B}^v$, and $\Phi \approx \Psi$ iff $\Phi - \Psi \approx 0$.

That simplifies a bit the tedious calculation. For some $\Phi \in \mathcal{B}_{\mathcal{A}}^v$ with $\overline{A} = \bigcup_{A\in\mathcal{A}} A$ we must deduce $\Phi \equiv 0$ from the condition $\sum_{j\in\overline{A}} \Phi \circ \tau_j \in \mathcal{B}_{\overline{A}-\overline{A}\setminus\{0\}}$ (that is equivalent to $\Phi \approx 0$) by a proper sequence of substituting. Let $A^1 \in \mathcal{A}$ is minimal in the sense: $(A^1)^c \cap (A + t) \neq \emptyset$ for every $A \in \mathcal{A} \setminus \{A^1\}$ at $t \in \mathbf{Z}^d$. Then $\sum_{j\in-\overline{A}} \Phi \circ \tau_j(x_{A^1}, v_{(\overline{A}-\overline{A})\setminus A^1}) = \Phi_{A^1}(x_{A^1})$, and since by the assumption ϕ_{A^1} must not depend on x_0 we may always substitute $x_0 = v$ and obtain $\Phi_{A^1} \equiv 0$. Then we repeat the consideration with $\mathcal{A} \setminus \{A^1\}$, etc., and finally we obtain $\Phi_A \equiv 0$ for every $A \in \mathcal{A}$.

ii) The statement follows from Theorem 2.35 b) in Georgii (1988) or by direct calculations with the aid of Möbius formula for constructing the vacuum potentials. □

Thus, we can deal only with the *set of potentials* $\mathcal{B}^v$.

5. Limit results

For a fixed configuration $x \in \mathcal{X}$ and every $\Lambda \in \mathcal{V}$ we define a probability measure $\widehat{P}_x^\Lambda$ by

$$\int \Phi \, \mathrm{d}\widehat{P}_x^\Lambda = |\Lambda|^{-1} \sum_{t\in\Lambda} \Phi \circ \tau_t(x) \quad \text{for every } \Phi \in \mathcal{B}.$$

Such probability distributions will be called as *empirical random fields*. On the other hand, for fixed $\Phi \in \mathcal{B}_V$ we have $\int \Phi \, \mathrm{d}\widehat{P}^\Lambda \in \mathcal{B}_{V+\Lambda}$, which means that for specifying the marginal distribution $\widehat{P}_x^\Lambda/\mathcal{F}_V$ we actually need to have $x_{V+\Lambda} \in X_0^{V+\Lambda}$.

Now, let us consider a sequence of growing subsets $\{V_n\}_{n=1}^{\infty}$ in $\mathbf{Z}^d$, e.g., the cubes $V_n = [-n, n]^d$ for simplicity.

Then, for a fixed potential $\Phi \in \mathcal{B}^v$ let us denote by

$$c_\infty(\Phi) = \lim_{n\to\infty} |V_n|^{-1} \log Z_{V_n}^{\Phi}(x_{V_n{}^c})$$

the *pressure* corresponding to the potential Φ. The limit exists uniformly for every $x \in \mathcal{X}$ by, e.g., Theorem 15.30 in Georgii (1988).

In general, the pressure c_∞ is a convex continuous function on $\mathcal{B}$, strictly convex on $\mathcal{B}^v$, and even strongly convex on every compact subset of $\mathcal{B}^v$ (see Dobrushin and Nahapetian (1974)).

Further, for every $P \in \mathcal{P}_S$ we may set the *entropy rate*

$$S_1^\infty(P) = \lim_{n\to\infty} |V_n|^{-1} \int [-\log P_{V_n}(x_{V_n})] \,\mathrm{d}P(x)$$

where the limit exists by Theorem 15.12 in Georgii (1988).

Moreover, for every $Q \in \mathcal{P}_S$ and $P \in G_S(\Phi)$ we may set also the *I-divergence rate*

$$D^\infty(Q|P) = \lim_{n\to\infty} |V_n|^{-1} \int \left[\log \frac{Q_{V_n}(x_{V_n})}{P_{V_n}(x_{V_n})}\right] \mathrm{d}Q(x)$$

where the limit exists by Chapter 15.3 again in Georgii (1988).

Proposition 2

For $P \in \mathcal{P}_S$, the following statements are equivalent:

i) $P \in G_S(\Phi)$;

ii) $|V_n|^{-1} \log P_{V_n}(x_{V_n}) - \int \Phi \,\mathrm{d}\widehat{P}_x^{V_n} + c_\infty(\Phi) \to 0$ for $n \to \infty$;

iii) $S_1^\infty(P) = c_\infty(\Phi) - \int \Phi \,\mathrm{d}P$.

Proof: While i) $\Rightarrow$ ii) and ii) $\Rightarrow$ iii) are rather straightforward, the proof of iii) $\Rightarrow$ i) needs a rather sophisticated construction (see, e.g., Georgii (1988), Theorem 15.37 or Janžura (1999)). □

Proposition 3

i) For every $Q \in \mathcal{P}_S$ and $P \in G_S(\Phi)$ it holds:
$D^\infty(Q|P) = c_\infty(\Phi) - \int \Phi \,\mathrm{d}Q - S_1^\infty(Q) \geqq 0$.

ii) Every $P \in G_S(\Phi)$ creates a tangent functional to $c_\infty(\bullet)$ at Φ, i.e.,
$c_\infty(\Phi + \Psi) - c_\infty(\Phi) \geqq \int \Psi \,\mathrm{d}P$ for every $\Psi \in \mathcal{B}^v$.

iii) Function $q \rightarrowtail \int \Phi \, \mathrm{d}P^q$ where $P^q \in G_S(q\Phi)$ is growing. In particular, we have

$$\lim_{q \to 1+} \int \Phi \, \mathrm{d}P^q = \max_{P \in G_S(\Phi)} \int \Phi \, \mathrm{d}P$$

and

$$\lim_{q \to 1-} \int \Phi \, \mathrm{d}P^q = \min_{P \in G_S(\Phi)} \int \Phi \, \mathrm{d}P.$$

Proof: While i) is rather straightforward by substituting, the proof of ii) can be obtained from the formula for $D^\infty(P|Q) \geqq 0$ for $Q \in G_S(\Phi + \Psi)$. Finally, iii) follows from the convexity of function $c_\infty(\bullet)$.

□

6. Tsallis entropy for Gibbs random fields

The Tsallis entropy can be generalized for random processes or fields only asymptotically, i.e., as the appropriate rate. Therefore, for $P \in G_S(\Phi)$ with $\Phi \in \mathcal{B}^v$ let us denote

$$S_q^V(P) = \frac{1}{q-1} \, [1 - (\sum_{x_V \in X_0^V} P_V(x_V)^q)^{\frac{1}{|V|}}].$$

Then we obtain the main result.

Theorem 1

For $P \in G_S(\Phi)$ it holds

$$\lim_{n \to \infty} S_q^{V_n}(P) = S_q^\infty(P) = \frac{1}{q-1} \, \{1 - \exp[c_\infty(q\Phi) - qc_\infty(\Phi)]\}.$$

Proof.

We substitute for $P_{V_n}(x_{V_n})$ and calculate the limit with the aid of Proposition 2 ii).

□

We can obtain the same result by a direct *analogy* with the one-dimensional exponential distributions.

Remark 1

The Gibbs random fields can be understood as infinite dimensional exponential distributions. Then the function $c_\infty(\bullet)$ (within the statistical physics terminology known as pressure) is a straightforward counterpart of the "moment-generating function" $c(\bullet)$ as defined for the exponential distributions. In particular, for the i.i.d. case the notions coincide. Namely, for $\Phi \in \mathcal{B}_0^v$ we have exactly $c_\infty(\Phi) = c(\Phi)$.

Then the formula for $S_q^\infty(P)$ can be obtained as a direct analogy of the formula $S_q(P)$ for the exponential $P = P^f$.

□

Now, suppose $\Phi \in \mathcal{B}^v$, $P \in G_S(\Phi)$, and $Q^q \in G_S(q\Phi)$. Then, we observe

$$D^\infty(P|Q^q) = c_\infty(q\Phi) - c_\infty(\Phi) + (1-q)\int \Phi \,\mathrm{d}P \geqq 0.$$

At the same time we have also

$$D^\infty(Q^q|P) = c_\infty(\Phi) - c_\infty(q\Phi) - (1-q)\int \Phi \,\mathrm{d}Q^q \geqq 0.$$

Proposition 4

For $P \in G_S(\Phi)$ and $Q \in G_S(q\Phi)$ it holds

$$c_\infty(q\Phi) - qc_\infty(\Phi) = D^\infty(P|Q) + (1-q)\,(c_\infty(\Phi) - \int \Phi \,\mathrm{d}P),$$

and, simultaneously,

$$c_\infty(q\Phi) - qc_\infty(\Phi) = -D^\infty(Q|P) + (1-q)\,(c_\infty(\Phi) - \int \Phi \,\mathrm{d}Q).$$

Consequently, we have

$$(1-q)\,(c_\infty(\Phi) - \int \Phi \,\mathrm{d}P) \leqq c_\infty(q\Phi) - qc_\infty(\Phi) \leqq (1-q)\,(c_\infty(\Phi) - \int \Phi \,\mathrm{d}Q),$$

and

$$\frac{1}{q-1}\,\{1-\exp[(1-q)\,(c_\infty(\Phi)-\int \Phi \,\mathrm{d}Q)]\} \leqq S_q^\infty(P) \leqq \frac{1}{q-1}\,\{1-\exp[(1-q)\,(c_\infty(\Phi)-\int \Phi \,\mathrm{d}P)]\}$$

whenever $q > 1$, while

$$\frac{1}{q-1}\,\{1-\exp[(1-q)\,(c_\infty(\Phi)-\int \Phi \,\mathrm{d}P)]\} \leqq S_q^\infty(P) \leqq \frac{1}{q-1}\,\{1-\exp[(1-q)\,(c_\infty(\Phi)-\int \Phi \,\mathrm{d}Q)]\}$$

for $q < 1$.

Proof.

The results follow directly from the above formulas.

□

Theorem 2

For $P \in G_S(\Phi)$ it holds

$$\lim_{q\to 1+} S_q^\infty(P) = S_1^\infty(P^M)$$

and

$$\lim_{q\to 1-} S_q^\infty(P) = S_1^\infty(P^m)$$

where

$$P^M = \arg \max_{Q \in G_S(\Phi)} \int \Phi \, \mathrm{d}Q \quad \text{and} \quad P^m = \arg \min_{Q \in G_S(\Phi)} \int \Phi \, \mathrm{d}Q.$$

In case of no phase transitions, i.e., $G_S(\Phi) = \{P\}$, we have

$$\lim_{q \to 1} S_q^\infty(P) = S_1^\infty(P).$$

Proof.

The results follow directly from Proposition 3 iii) and Proposition 4.

□

7. A note on the performance

Unfortunately, for the Gibbs random fields, there is a lack of analytic formulas. While for calculating the expectations $\int \Psi \, \mathrm{d}P^\Phi$ we can substitute the theoretic terms by their simulated counterparts (cf., e. g., Younes (1989)), for the pressure $c_\infty(\Psi)$ the lack is absolute but we may apply the following approach.

Let us observe

$$c_\infty(\Phi) = c_\infty(\Psi) + \int (\Phi - \Psi) \, \mathrm{d}Q^{\tilde{\varepsilon}}$$

where $Q^{\tilde{\varepsilon}} \in G_S[\Psi + \tilde{\varepsilon}(\Phi - \Psi)]$ and $\tilde{\varepsilon} \in (0, 1)$.

Now, we may choose $\Psi \in \mathcal{B}^v_{\{0\}}$. Then, just as in Remark 1 we have $c_\infty(\Psi) = c(\Psi)$ which can be easily calculated.

For expressing $\int (\Phi - \Psi) \, \mathrm{d}Q^{\tilde{\varepsilon}}$ we can use the approximative Markov Chain Monte Carlo procedure (cf., e. g., Younes (1989)). Unfortunately, we do not know the exact value of $\tilde{\varepsilon}$. But, similarly as in Proposition 3 we observe

$$\int (\Phi - \Psi) \, \mathrm{d}Q^0 \leqq \int (\Phi - \Psi) \, \mathrm{d}Q^{\tilde{\varepsilon}} \leqq \int (\Phi - \Psi) \, \mathrm{d}Q^1.$$

There will be no problem if all the values coincide but that would mean $\Phi \approx \Psi$ which is in a non-trivial case hardly possible. Nevertheless, we may try at least to aim the eventuality, e.g., by taking Ψ in order to have $Q^0_{\{0\}} - Q^1_{\{0\}}$ which is easily feasible. Anyhow, there still will remain some gap, and, therefore we need an additional approximation, e.g., in the form

$$\frac{1}{N+1} \sum_{n=0}^{N} \int (\Phi - \Psi) \, \mathrm{d}Q^{\frac{n}{N}}$$

with N of a reasonable size, which also imitates the alternative expression

$$c_\infty(\Phi) = c_\infty(\Psi) + \int_0^1 [\int (\Phi - \Psi) \, \mathrm{d}Q^\varepsilon] \mathrm{d}\varepsilon.$$

Then the whole procedure can be in the same way repeated also for $c_\infty(q\Phi)$ in order to obtain both the terms needed for expressing the Tsallis entropy rate $S_q^\infty(P^\phi)$.

The method is rather complicated and tedious but it is caused by complexity of the concept.

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References

Anteneodo, C., Tsallis, C., and Martinez, A.S. (2002). Risk aversion in economic transactions, Europhys. Lett. 59, 635

Bera, A. and Park, S. (2008): Optimal portfolio diversification using the maximum entropy principle, Econometric Reviews 27(4-6), pp. 484-512

Curado E.M.F. and Tsallis C. (1991). Generalized statistical mechanics: connection with thermodynamics, J. Phys. A 24, L69-72.

Dobrushin, R. L. (1970). Prescribing a system of random variables by conditional distributions. *Theor. Probab. Appl.* 15, 458–486.

Dobrushin, R. L. and Nahapetian, B. S. (1974). Strong convexity of the pressure for lattice systems of classical physics (in Russian). *Teoret. Mat. Fiz.* 20, 223–234.

Eom, C., Oh, G. and Jung, W.-S. (2008): Relationship between efficiency and predictability in stock price change, Physica A 387, pp. 5511-5517

Georgii, H.-0. (1988). *Gibbs Measures and Phase transitions.* De Gruyter Studies in Mathematics 9, De Gruyter, Berlin.

Giglio, R., Matsushita, R. and Figueiredo, A. and Gleria, I. and Da Silva, S. (2008): Algorithmic complexity theory and the relative efficiency of financial markets, EPL 84, art. 48005

Janžura, M. (1994). Asymptotic behaviour of the error probabilities in the pseudo-likelihood ratio test for Gibbs–Markov distributions. In: *Asymptotic Statistics.* Proceedings, (P. Mandl, M. Hušková, eds.), 285–296, Physica–Verlag, Heidelberg.

Janžura M. (1997). Asymptotic results in parameter estimation for Gibbs random fields. *Kybernetika* , 33 , 2, 133-159.

Janžura M. (1999). On the concept of the asymptotic Rényi distances for random fields. *Kybernetika* , 35, 3, 353-366.

Kristoufek, L. and Vosvrda, M. (2014a): Commodity futures and market efficiency, Energy Economics 42, pp. 50-57

Kristoufek, L. and Vosvrda, M. (2014b): Measuring capital market efficiency long-term memory, fractal dimension and approximate entropy, European Physical Journal B 87, art. 162

Künsch, H. (1982). Decay of correlations under Dobrushin's uniqueness condition and its applications. *Comm. Math. Phys.*84, 207–222.

Maasoumi, E. and Racine, J. (2002): Entropy and predictability of stock returns, Journal of Econometrics 107, pp. 291-312

Michael F., Evans J., and Johnson M.D. (2002). Excess demand financial market model. Preprint.

Nielsen, F. and Noch R.(2011). On Rěnyi and Tsallis entropies and divergences for exponential families. IEEE Transactions on Information Theory 22(10), 17.

Ortiz-Cruz, A., Rodriguez, E. and Ibarra-Valdez, C. and Alvarez-Ramirez, J. (2012): Efficiency of crude oil markets: Evidences from informational entropy analysis, Energy Policy 41, pp. 365-373

Qin, Z., Li, X. and Ji, X. (2009): Portfolio selection based on fuzzy cross-entropy, Journal of Computational and Applied Mathematics 228, pp. 139-149

Ramos, F.M. Rosa, R.R., and Rodrigues Neto, C. (1999). Non-extensivity and turbulent-like behavior of financial markets. Preprint.

Rompolis, L. (2010): Retrieving risk neutral densities from European option prices based on the principle of maximum entropy, Journal of Empirical Finance 17(5), pp. 918-937

Shi, W. and Shang, P. (2013): Cross-sample entropy statistic as a measure of synchronism and cross-correlation of stock markets, Nonlinear Dynamics 71, pp. 539-554

Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics.*Journal of Statistical Physics* 52, 479–487.

Tsallis, C. (1995). Some comments on Boltzmann-Gibbs statistical mechanics, Chaos, Solitons and Fractals 6, 539.

Tsallis, C. (2009). Introduction to nonextensive statistical mechanics : approaching a complex world . New York, Springer.

Younes, L. (1989). Parameter inference for imperfectly observed Gibbsian fields. *Probab. Theory Rel. Fields* 82, 625–645.