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Abstract Inspired by nonlinear quantile regression, the article introduces, investigates, discusses, and illustrates a new concept of generalized elliptical location quantiles. They may require less stringent moment assumptions, be less sensitive to outliers, be less rigid, employ more a priori information regarding the location of the distribution, and have higher potential for various regression generalizations than their common elliptical predecessor defined in the convex optimization framework by means of standard linear quantile regression. Furthermore, they still include an equivalent of their predecessor as a special case and inherit most of its favorable features such as the probability interpretation, natural equivariance properties, and good behavior for elliptical and symmetric distributions, which is demonstrated both by theoretical results and data examples with convincing graphical output. On the other hand, the new elliptical quantiles need not always be uniquely defined and they require somewhat different approach to their analysis and computation due to their intrinsically non-convex formulation.

Keywords Multivariate quantile · Elliptical quantile · Quantile regression · Multivariate statistical inference · Portfolio optimization

Mathematics Subject Classification 62H05 · 62J99 · 62G15

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1 Motivating introduction

Many practitioners investigate multivariate data and their distributions to understand not only the individual data coordinates but also their mutual relations. For a simple example of such data sets, let us consider the bivariate sample $(y_1, y_2)',\ i = 1, \ldots, n$, consisting of $n = 1{,}000$ daily stock log-returns of Intel (y_1) and General Electric (y_2) from 1996/01/03 to 1999/12/16. It was extracted from the datafile *rdj.tab*, included in the *copula* package for *R* (see, Hofert et al. 2012; R Core Team 2012), and already analyzed in Section 5.5 of (McNeil et al. 2005) by means of various copulas. Consequently, our sample can be expected to come from a roughly elliptical distribution with heavy tails, like many other financial data sets.

We propose a family of (elliptical) quantiles and index its members by their level $\tau \in (0, 1)$ and by the function g employed in their definition. They have many good properties: (B1) they are elliptical, (B2) they behave naturally under affine transformations, (B3) they are roughly nested for symmetric distributions, (B4) they preserve the centers and axes of symmetry, (B5) they can easily incorporate many types of a priori information regarding their shapes and centers, (B6) their levels directly correspond to their probability contents, (B7) their shapes and centers are τ-dependent, (B8) they can be made very robust and free of any moment assumptions by a suitable choice of g, (B9) they can work well even for complicated distributions, (B10) they are parameterized very naturally by means of their centers, shape matrices and inflation factors, and (B11) they coincide with the standard elliptical quantiles in the population case under very mild conditions if the distribution is truly elliptical. In the sample case, (B12) their border is not bound to contain a dimension-specific number of observations for most g's, and (B13) they can be computed even for very large and/or multidimensional data sets.

These quantiles can be used for investigating unconditional distributions where the elliptical shapes of all the quantiles seem appropriate. For example, if we compute them for the data sample described above and plot them in Fig. 1, we can check easily that the output is in good agreement with B1, B3, B4, B5, B6, and B7. Section 6 further illustrates their properties with simulated data examples; see e.g., Figs. 4, 3, 5, 2, and 6 for B5, B8, B9, B11, and B13, respectively. In our example, the quantiles identify rare/suspicious/profitable pairs of returns by linking all the pairs with probability and thus provide much useful information about them in a very distant future (under the assumption of stationarity), possibly helpful for trading the stocks together in long horizons. Then, the nearest history is not known and, therefore, the more sophisticated conditional models cannot be employed with usual benefits. See Sect. 4 for more details regarding the links to risk management and portfolio optimization.

Properties B7 and B10 imply that our concept generates quite a few natural processes indexed by τ and g that seem extremely promising not only for constructing goodness-of-fit tests, but also for testing various hypotheses regarding symmetry, ellipticity, and location/dispersion, respectively, thanks to B4, B11, and B5; see also Liu et al. (1999) for a way how they can be used. Not only are elliptical distributions very popular among practitioners, but also ellipsoids themselves have been already employed in various statistical contexts, e.g., for finding outliers, handling measurement errors, defining data depth, specifying copulas, discriminating data clusters,

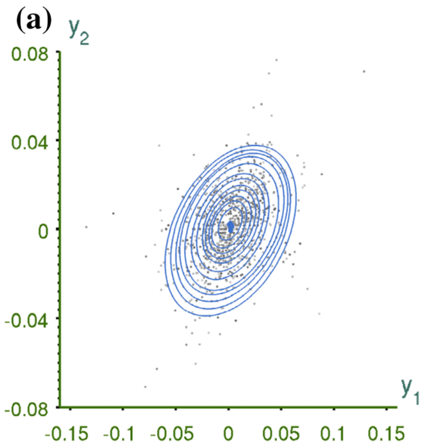


Fig. 1 Examples illustrating our elliptical quantile family. The *plots* show our elliptical g-quantile family in action. Both of them display some elliptical τ-g-quantiles, $\tau \in \{0.100, 0.200, \ldots, 0.800, 0.850, 0.875, 0.900, 0.925\}$, computed from $n = 1{,}000$ couples (y_1, y_2) of daily stock returns of Intel (y_1) and General Electric (y_2) from 1996/01/03 to 1999/12/16. The *left plot* contains elliptical g-quantiles for $g(z) = z/(1+z)$ that can be viewed as fully robustified elliptical quantiles of (Hlubinka and Šiman 2013), but this time parameterized directly by means of their centers. The *right plot* contains elliptical g-quantiles for $g(z) = \sqrt{z}$ using an a priori information about their centers, namely assuming their first coordinates as zero

constructing confidence regions, and for estimating scatter or location. This hints that our elliptical quantiles might once be used in many different statistical applications as well. Furthermore, possible regression generalizations of these elliptical quantiles will once be even much more useful than the location quantiles themselves and they will hopefully make all the time series models of conditional volatility somewhat redundant. Therefore, we consider the theory presented here as a very important cornerstone that may give rise to various interesting and important future achievements.

We can conclude that the elliptical quantiles presented here may outperform their predecessor of Hlubinka and Šiman (2013) in terms of B5, B8, B10, B12, and the possibility of regression generalizations. On the other hand, their interpretation may be less intuitive and their computation may become rather challenging, though it is still feasible, e.g., using the algorithms already developed for nonlinear single-response quantile regression. They are designed as direct competitors to the recent multivariate (regression) quantile concept arising from the directional approach to halfspace depth presented in Hallin et al. (2010a, b); Paindaveine and Šiman (2011); Kong and Mizera (2012); Hallin et al. (2014) implemented in Paindaveine and Šiman (2012a) and Paindaveine and Šiman (2012b), and employed, e.g., in McKeague et al. (2011); Šiman (2011) or Šiman (2014). They usually win, thanks to B5, B6, B10, and B13 if the elliptical shape of quantiles is really adequate, but they may lose the comparison owing to B1 and B3 for some highly asymmetric distributions. In fact, they win over any other depth-based multivariate quantile concept in the same contexts and for the same reasons (and sometimes also for B8); see Zuo and Serfling (2000) for a brief review of the most popular depth notions. This already implies that our concept is more than competitive because depth-based multivariate quantiles are generally considered superior to many others; see Serfling (2002) for this conclusion and a review of various multivariate quantiles (based on depth, norm minimization or M-estimation, inversion of mappings, gradients or generalized quantile processes) and their properties.

Consequently, we see as our main competitors the methods working with ellipsoids. Indeed, our approach seems similar to those based on minimum volume ellipsoids (see Rousseeuw 1985; Polonik 1997), minimum (weighted) covariance determinants (see Roelant et al. 2009), or other multivariate L1/S/CM/MM-estimators of location and scatter (see e.g., Kent and Tyler 1996; Tatsuoka and Tyler 2000; Roelant and Van Aelst 2007; Aelst and Willems 2007). Usually, they cannot also be used for direct quantile estimation, and when they can, such as in the case of minimum volume ellipsoids, then they have some unpleasant features including computational issues or slow rate of convergence of the resulting estimators. Roughly speaking, our method might be viewed as their quantile modification inspired by quantile regression.

Of course, one could also consider elliptical quantiles as only inflated copies of one another, defined by the same center and the same shape matrix. Any of the numerous location and scatter estimators could then be used for their definition and such quantiles might be desirable for elliptical distributions in the population case. Nevertheless, they would lead to much less inferential tools and possibilities and they would fail whenever the quantile shapes and/or centers depended on the quantile level.

Our elliptical g-quantiles are defined by means of non-convex optimization and therefore, unfortunately, they need not be uniquely defined in general. However, there is at least one important exception to this rule corresponding to $g(z) = z$ when we obtain (unique) elliptical quantiles virtually equivalent to those of Hlubinka and Šiman (2013), but better in terms of B5, B10, and the possibility of regression generalizations. Furthermore, such problems with ambiguity do not necessarily make a statistical concept useless, which is testified beyond any doubt by the widespread use of notoriously ambiguous but popular S-estimators or single-response nonlinear regression quantiles; see e.g., Tatsuoka and Tyler (2000); Koenker (2005) with references therein. In fact, the ambiguity might even be considered advantageous for some cases of multimodal distributions that may arise easily as mixtures of only two unimodal ones even in the univariate case; see Došlá (2009). Before the uniqueness questions are satisfactorily resolved, we nevertheless suggest to experiment with the choices of g cautiously, to look for the quantile centers only in the convex hull of the data cloud, to try various initial values for the computation, and to always use the elliptical quantiles for $g(z) = z$ as a benchmark.

The text proceeds as follows. Section 2 presents necessary notation and explains our definition of generalized elliptical quantiles, Sect. 3 studies their properties in the population case, Sect. 4 reveals their link to risk management, Sect. 5 comments on their computation in the sample case, Sect. 6 illustrates them with a few demo examples, and the last Sect. 7 discusses their use and impact on statistical inference.

2 Definitions and notation

Let us consider quite a general multivariate setup where an m-variate vector $\boldsymbol{Y} = (Y^{(1)}, \ldots, Y^{(m)})' \in \mathbb{R}^m$ has an absolutely continuous distribution and a probability density function differentiable almost everywhere.

Standard univariate location quantiles can be defined for any $\tau \in (0, 1)$ with the aid of the nonnegative convex real-valued check function $\rho_\tau(t) = t(\tau - \mathrm{I}(t < 0)) = \max\{(\tau-1)t, \tau t\}$ with a unique minimum; see e.g., Koenker (2005). This function was also used in Hlubinka and Šiman (2013) for defining multivariate location elliptical quantiles by means of convex optimization. Here, we choose a different approach, sacrifice simplicity to flexibility, and do not insist on the convexity of resulting optimization problems at any cost, which allows us to introduce a general class of (the whole processes of) elliptical location quantiles together with related parameters of location and scatter whose future impact on statistical inference seems hard to overestimate.

Definition 1 For any $\tau \in (0, 1)$, we define the generalized elliptical τ-quantile (or, τ-g-quantile) $\varepsilon_{g,\tau}(\boldsymbol{Y})$ of $\boldsymbol{Y}$, and corresponding lower and upper τ-g-quantile regions $\mathcal{E}^-_{g,\tau}(\boldsymbol{Y})$ and $\mathcal{E}^+_{g,\tau}(\boldsymbol{Y})$ as follows:

$$\begin{aligned}
\varepsilon_{g,\tau}(\boldsymbol{Y}) &= \{\boldsymbol{y} \in \mathbb{R}^m : g\big((\boldsymbol{y}-\boldsymbol{s})'\mathbb{A}_\tau(\boldsymbol{y}-\boldsymbol{s})\big) - c_\tau = 0\},\\
\mathcal{E}^-_{g,\tau}(\boldsymbol{Y}) &= \{\boldsymbol{y} \in \mathbb{R}^m : g\big((\boldsymbol{y}-\boldsymbol{s})'\mathbb{A}_\tau(\boldsymbol{y}-\boldsymbol{s})\big) - c_\tau < 0\},\\
\mathcal{E}^+_{g,\tau}(\boldsymbol{Y}) &= \{\boldsymbol{y} \in \mathbb{R}^m : g\big((\boldsymbol{y}-\boldsymbol{s})'\mathbb{A}_\tau(\boldsymbol{y}-\boldsymbol{s})\big) - c_\tau \geq 0\},
\end{aligned}$$

where $g(t) : [0, \infty) \mapsto [0, \infty)$ is a suitable strictly increasing smooth function with $g(0) = 0$ and $\mathbb{A}_\tau$, $\boldsymbol{s}_\tau$, and c_τ minimize the objective function

$$\Psi_\tau(\mathbb{A}, \boldsymbol{s}, c) := \mathrm{E}\,\rho_\tau(g\big((\boldsymbol{Y}-\boldsymbol{s})'\mathbb{A}(\boldsymbol{Y}-\boldsymbol{s})\big) - c) \tag{P$_1$}$$

subject to $\det(\mathbb{A}) = 1$ and $\mathbb{A} \in \mathrm{PSD}(m)$, i.e., $\mathbb{A} = \mathbb{A}_{m\times m}$ must be symmetric positive semidefinite. These two constraints together imply positive definiteness of $\mathbb{A}$.

We also tacitly assume that the expectation in (P_1) is finite and that its partial derivatives with respect to the parameters $\mathbb{A}$, $\boldsymbol{s}$, and c are exchangeable with the expectation sign.

As c_τ cannot be negative, any elliptical τ-g-quantile $\varepsilon_{g,\tau}$ may be described by the equation

$$(\boldsymbol{y}-\boldsymbol{s}_\tau)'\mathbb{A}_\tau(\boldsymbol{y}-\boldsymbol{s}_\tau) = g^{-1}(c_\tau),$$

and, therefore, its name is justified. Its definition is quite natural in view of the discussion after Definition 1 of Hlubinka and Šiman (2013). The assumption $\mathbb{A} \in \mathrm{PSD}(m)$ is necessary for the resulting generalized quantile regions to be really elliptical, the constraint on the determinant preserves their good equivariance properties while keeping them non-degenerate, properly scaled, and identical to those commonly considered for elliptically distributed $\boldsymbol{Y}$ if no ambiguity occurs, which will be proved below. This is why we focus on this determinant-based regularity constraint and leave the other possibilities mentioned in Hlubinka and Šiman (2013) aside. The parameter c remains unrestricted to guarantee a reasonable probabilistic interpretation of $\mathcal{E}^-_{g,\tau}(\boldsymbol{Y})$ as in Hlubinka and Šiman (2013). Since $\Psi_\tau(\mathbb{A}, \boldsymbol{s}, c)$ is not necessarily convex any more, there is no urgent need to keep the constraints convex, and we therefore directly assume $\det(\mathbb{A}) = 1$ contrary to $\sqrt[m]{\det(\mathbb{A})} \geq 1$ considered in Hlubinka and Šiman (2013). Other restrictions seem redundant unless we could expect or assume a special shape or position of the elliptical quantiles, when some special constraints and corresponding multipliers might indeed come on stage as is discussed and illustrated both in Hlubinka and Šiman (2013) and here in Sect. 6, but we will not pursue such a possibility below.

Clearly, we only get the alternative parameterization of the unique elliptical quantiles of Hlubinka and Šiman (2013) for $g(t) = t$. Consequently, the elliptical g-quantiles with $g(t) = t$ are uniquely defined for any $\tau \in (0, 1)$ if the distribution of $\boldsymbol{Y}$ has a connected support. They have much in common with the elliptical quantiles of Hlubinka and Šiman (2013), which is why we primarily focus on different choices of g hereinafter. Then, the elliptical τ-g-quantile, $\tau \in (0, 1)$, need not be uniquely defined as the objective function in (P_1) need not be quasiconvex. Hypothetically, it can have a unique global minimum together with several misleading local minima or it can attain its global minimum in a few different points. In the latter case, the generalized elliptical τ-g-quantile would not be uniquely defined. However, while $g(t) = t$ leads to second-order moment assumptions on $\boldsymbol{Y}$, $g(t) = \sqrt{t}$ requires only finite expectation $\mathrm{E}\,\boldsymbol{Y}$, and bounded g does not imply any moment restrictions on $\boldsymbol{Y}$ whatsoever, which probably indicates high robustness of the resulting elliptical g-quantile estimators. The alternative choices of $g(t) \neq t$ may also lead to quite flexible elliptical τ-g-quantiles not containing any boundary observation.

3 Population case

Unfortunately, the minimization problem involved in the definition of generalized elliptical quantiles need not be convex any more. Nevertheless, any optimal solution $\mathbb{A}_\tau$, $\boldsymbol{s}_\tau$, and c_τ, accompanied with the Lagrange multiplier L_τ corresponding to the constraint $-\det(\mathbb{A}_\tau) + 1 = 0$, still must lead to zero partial derivatives of the Lagrangian.

Consequently, the optimal solution must meet the following set of necessary conditions:

$$1 = \det(\mathbb{A}_\tau), \tag{1}$$

$$0 = P(\boldsymbol{Y} \in \mathcal{E}^-_{g,\tau}) - \tau, \tag{2}$$

$$\boldsymbol{0} = \frac{1}{1-\tau}\,\mathrm{E}\left[\gamma \boldsymbol{R}_\tau\, \mathrm{I}_{[\boldsymbol{Y} \in \mathcal{E}^+_{g,\tau}]}\right] - \frac{1}{\tau}\,\mathrm{E}\left[\gamma \boldsymbol{R}_\tau\, \mathrm{I}_{[\boldsymbol{Y} \in \mathcal{E}^-_{g,\tau}]}\right], \tag{3}$$

$$L_\tau \frac{\det(\mathbb{A}_\tau)}{\tau(1-\tau)}\mathbb{A}_\tau^{-1} = \frac{1}{1-\tau}\,\mathrm{E}\left[\gamma \boldsymbol{R}_\tau \boldsymbol{R}_\tau'\, \mathrm{I}_{[\boldsymbol{Y} \in \mathcal{E}^+_{g,\tau}]}\right] - \frac{1}{\tau}\,\mathrm{E}\left[\gamma \boldsymbol{R}_\tau \boldsymbol{R}_\tau'\, \mathrm{I}_{[\boldsymbol{Y} \in \mathcal{E}^-_{g,\tau}]}\right], \tag{4}$$

where $\boldsymbol{R}_\tau = \boldsymbol{Y} - \boldsymbol{s}_\tau$, $\gamma = \dot{g}(\boldsymbol{R}_\tau' \mathbb{A}_\tau \boldsymbol{R}_\tau)$, $\dot{g}(t) = \partial g(t)/\partial t$, and $\mathbb{A}_\tau$ is assumed symmetric positive semidefinite.

If we write $\bullet$ for the Hadamard (elementwise) product and realize that $\boldsymbol{1}'(\mathbb{A}_\tau \bullet (\mathbb{A}_\tau^{-1}))\boldsymbol{1} = m \det(\mathbb{A}_\tau) = m$, then we can multiply (2) and (4) elementwise, respectively, by c_τ and $\tau(1-\tau)\mathbb{A}_\tau$, sum up the results, and thus obtain an interesting expression for the multiplier L_τ:

$$L_\tau = \mathrm{E}\,\rho_\tau(\gamma \boldsymbol{R}_\tau' \mathbb{A}_\tau \boldsymbol{R}_\tau - c_\tau)/m \;\; (> 0). \tag{5}$$

In the special case of $g(t) = t$ and $\gamma = 1$, the Lagrange multiplier not only measures the impact of the boundary constraint, but also interprets the optimum value of the objective function and appears as a useful tool for statistical inference.

What can we say about the simplified gradient conditions (1)–(4) of the Lagrangian? The first Eq. (1) is easy to interpret as it only scales and regularizes the problem. The second Eq. (2) ensures that the resulting lower (elliptical) τ-g-quantile region $\mathcal{E}^-_{g,\tau}(\boldsymbol{Y})$ is non-empty, with its overall coverage probability naturally equal to τ. The third condition (3) can be rewritten as an equality of two (outer and inner) conditional weighted means, $E(\gamma \boldsymbol{R}_\tau | \boldsymbol{Y} \in \mathcal{E}^+_{g,\tau}) = \mathrm{E}(\gamma \boldsymbol{R}_\tau | \boldsymbol{Y} \in \mathcal{E}^-_{g,\tau})$. Finally, the fourth condition (4) makes $\mathbb{A}_\tau^{-1}$ proportionate to something like the difference of outer and inner conditional weighted variance matrices if $\mathrm{E}\sqrt{\gamma}\boldsymbol{R}_\tau \sim \boldsymbol{0}$. The interpretation of the gradient conditions remains complicated even if we further view $\gamma \boldsymbol{R}_\tau$ or $\sqrt{\gamma}\boldsymbol{R}_\tau$ like standardized residuals, which is more natural for some special choices of g such as $g(t) = \sqrt{t}$. Another simplification can be observed for $\tau \to 0$ when the conditional moments appearing in (3) and (4) either vanish or turn to the non-conditional ones.

Nevertheless, the generalized elliptical quantiles do have some welcome properties.

Theorem 1 *Let $\boldsymbol{Y} \in \mathbb{R}^m$ be a random vector with an absolutely continuous distribution, all required moments finite, and a density $f(\boldsymbol{y})$ differentiable almost everywhere. Then, its elliptical τ-g-quantile parameterized by $\mathbb{A}_\tau$, $\boldsymbol{s}_\tau$, and c_τ has the following properties for any possible g and any $\tau \in (0, 1)$:*

(1) it behaves naturally under affine transformations of $\boldsymbol{Y}$
(2) if $f(\boldsymbol{y}) = f(\mathbb{O}\boldsymbol{y})$ for an orthogonal matrix $\mathbb{O} = \mathbb{O}^{-1'}$, then there exists an elliptical τ-g-quantile, parameterized by $\mathbb{O}'\mathbb{A}_\tau\mathbb{O}$, $\mathbb{O}\boldsymbol{s}_\tau$, and c_τ

(A) If we further assume that the distribution of $\boldsymbol{Y}$ is elliptically symmetric with density $f(\boldsymbol{y}) \propto p\big((\boldsymbol{y} - \boldsymbol{s})'\mathbb{A}(\boldsymbol{y} - \boldsymbol{s})\big)$ where $\det(\mathbb{A}) = 1$ and p is a nonnegative real function, then

(3) if $\boldsymbol{s}_\tau \neq \boldsymbol{s}$, then there exist infinitely many τ-g-quantiles for $m \geq 2$
(4) if $\mathbb{A}_\tau \neq \mathbb{A}$, then there exist infinitely many τ-g-quantiles for $m \geq 2$
(5) the gradient conditions (1) to (4) are satisfied for $\boldsymbol{s}_\tau = \boldsymbol{s}$, $\mathbb{A}_\tau = \mathbb{A}$, the implied c_τ, and a multiplier $L_\tau > 0$

(B) If we alternatively assume that the elliptical τ-g-quantile is uniquely defined, then

(6) if $\boldsymbol{s}_\tau = (s_1, \ldots, s_m)'$, $\mathbb{A}_\tau = (a_{ij})_{i,j=1}^m$, and $f(\boldsymbol{y}) = f(\mathbb{J}\boldsymbol{y})$ for a sign-change matrix $\mathbb{J} = \mathbb{J}' = \mathbb{J}^{-1} = \mathrm{diag}(j_1, \ldots, j_m)$ with diagonal elements ± 1, then $s_i = 0$ whenever $j_i = -1$, $i \in \{1, \ldots, m\}$, and $a_{ij} = 0$ whenever $j_i j_j = -1$, $i, j \in \{1, \ldots, m\}$.
(7) if the distribution of $\boldsymbol{Y}$ is symmetric around a hyperplane, then $\boldsymbol{s}_\tau$ lies on the hyperplane
(8) if the distribution of $\boldsymbol{Y}$ is symmetric along an axis, then $\boldsymbol{s}_\tau$ lies on the axis
(9) if the distribution of $\boldsymbol{Y}$ is centrally symmetric, then $\boldsymbol{s}_\tau$ coincides with the center of symmetry

(10) *if the distribution of* $\boldsymbol{Y}$ *is elliptically symmetric, then its standard* τ*-quantile equals the elliptical* τ-g*-quantile*

Proof The proof is inspired by that of Theorem 2 in Hlubinka and Šiman (2013). Obviously, (1) follows directly from the definition, and both (3) and (4) are implied by (2) as they can be proved only for centered spherical distributions thanks to (1). Furthermore, (7), (8), and (9) result directly from (1) and (6). Consequently, we prove in detail only (2), (5), (6), and (10).

The assumption of (2) leads to

$$\Psi_\tau(\mathbb{A}, \boldsymbol{s}, c) = \Psi_\tau(\mathbb{O}'\mathbb{A}\mathbb{O}, \mathbb{O}\boldsymbol{s}, c)$$

for any $\mathbb{A}$, $\boldsymbol{s}$, and c. Consequently, $\mathbb{A}_\tau$, $\boldsymbol{s}_\tau$, and c_τ minimize Ψ_τ only when $\mathbb{O}'\mathbb{A}_\tau\mathbb{O}$, $\mathbb{O}\boldsymbol{s}_\tau$, and c_τ do it as well, which completes the proof of (2).

As for (5), we can simply verify the gradient conditions for $\boldsymbol{s}_\tau = 0$, $\mathbb{A}_\tau = \mathbb{I}_m$, the induced c_τ, and a centered spherical distribution, because of (1). The claim about L_τ follows from (5).

Furthermore, we can apply (2)–(6) as $\mathbb{J}$ is also an orthogonal matrix. Only the two parameterizations $\mathbb{A}_\tau$, $\boldsymbol{s}_\tau$, and c_τ and $\mathbb{J}\mathbb{A}_\tau\mathbb{J}$, $\mathbb{J}\boldsymbol{s}_\tau$, and c_τ must now lead to the same unique elliptical τ-g-quantile by assumption, which implies $s_i = 0$ whenever $j_i = -1$, and $a_{ij} = 0$ whenever $j_i j_j = -1$, $i, j \in \{1, \dots, m\}$.

Concerning (10), we can safely consider only $\boldsymbol{Y}$ spherically distributed around the origin thanks to (1). Then, $\mathbb{A}_\tau$ is diagonal and $\boldsymbol{s}_\tau = \boldsymbol{0}$ owing to (6). If we apply (2) to all (orthogonal) permutation matrices $\mathbb{P}$ and invoke the uniqueness assumption, we get $\mathbb{P}'\mathbb{A}_\tau\mathbb{P} = \mathbb{A}_\tau$. Therefore, the positive semidefinite $\mathbb{A}_\tau$ with unit determinant must be equal to the identity matrix, and the generalized elliptical quantile then coincides with the standard one because both have the same coverage probability. Note that the same could be said not only about any spherically symmetric $\boldsymbol{Y}$, but also about any $\boldsymbol{Y}$ with a marginally symmetric and exchangeable distribution.

Unfortunately, we have not been able to formulate a sensible criterion ruling out any ambiguity regarding the generalized elliptical quantiles even in the case of elliptical distributions. It seems not to be an easy task to unmask the complicated interplay between g and the density function of $\boldsymbol{Y}$ as the analogous results of Davies (1987) and Tatsuoka and Tyler (2000) for S-estimators are both quite involved and not directly applicable to our problem. This is why we put the choice of g and all its consequences aside for future research.

Finally, we should not forget to point out that the problem of finding our generalized elliptical quantiles and the auxiliary optimization problem

$$\min_{c, \boldsymbol{s}, \mathbb{A} \in \mathrm{PSD}(m)} -\det(\mathbb{A}) \equiv \max_{c, \boldsymbol{s}, \mathbb{A} \in \mathrm{PSD}(m)} \det(\mathbb{A})$$

subject to

$$\Psi_\tau(\mathbb{A}, \boldsymbol{s}, c) \le \Psi_\tau(\mathbb{A}_\tau, \boldsymbol{s}_\tau, c_\tau).$$

lead to virtually equivalent sets of gradient conditions, which indicates some remote similarity between our methodology and S-estimation (see Aelst and Willems 2007; Roelant et al. 2009). This might be used in further research to transfer some ideas from one concept to the other.

4 Links to risk management

Our elliptical quantile methodology also seems connected to risk management and portfolio optimization.

Let us interpret $\boldsymbol{Y} \in \mathbb{R}^m$ as a vector of asset returns and assume that all their extreme values are loss making. Then, the elliptical quantile $\varepsilon_{g,\tau}(\boldsymbol{Y})$ determines the most loss-making values of $\boldsymbol{Y}$ and, therefore, can be viewed as an extension of the well-known univariate value-at-risk concept to the multivariate case.

Portfolio risk behavior can be described not only by value-at-risk, but also by means of other risk measures such as tail conditional expectation TCE or shortfall s that can be defined for continuous scalar return Y and any $\tau \in (0, 1)$ by means of the τ-quantile $q_\tau(Y)$ as follows:

$$\mathrm{TCE}_\tau(Y) = -\,\mathrm{E}\big(Y|Y < q_t(Y)\big) \quad \text{and} \quad s_\tau(Y) = E(Y) + \mathrm{TCE}_\tau(Y);$$

(see e.g., Bertsimas et al. 2004). Interestingly, the shortfall can also be linked to our elliptical quantile concept.

Let us assume that $\boldsymbol{Y}$ always causes positive loss

$$Z = Z(\boldsymbol{Y}, \mathbb{A}, \boldsymbol{s}) = g\big((\boldsymbol{Y} - \boldsymbol{s})'\mathbb{A}(\boldsymbol{Y} - \boldsymbol{s})\big) > 0$$

for some $\boldsymbol{s} \in \mathbb{R}^m$ and $\mathbb{A} \in \mathrm{PSD}(m)$, $\det(\mathbb{A}) = 1$, and define

$$\Phi_\tau = \Phi_\tau(\mathbb{A}, \boldsymbol{s}) = \min_c \Psi_\tau(\mathbb{A}, \boldsymbol{s}, c)$$

and $C_\tau = C_\tau(\mathbb{A}, \boldsymbol{s})$ as a τ-quantile of Z so that $\Phi_\tau(\mathbb{A}, \boldsymbol{s}) = \Psi_\tau(\mathbb{A}, \boldsymbol{s}, C_\tau)$, $\tau \in (0, 1)$. Note that $\Psi_\tau(\mathbb{A}, \boldsymbol{s}, c)$ is convex in c and that C_τ would be uniquely defined under very mild additional conditions on the distribution of $\boldsymbol{Y}$, e.g., if $\boldsymbol{Y}$ were further required to have a positive density on a connected convex support. Then,

$$\Phi_\tau(\mathbb{A}, \boldsymbol{s}) = \mathrm{E}\, Z(\tau - \mathrm{I}(Z < C_\tau)) = -(1-\tau)\big(\mathrm{E}\, Z - \mathrm{E}(Z|Z \geq C_\tau)\big) > 0,$$

which implies

$$\Phi_\tau(\mathbb{A}, \boldsymbol{s})/(1-\tau) = s_{1-\tau}(-Z).$$

In other words, the elliptical quantiles described by $\mathbb{A}_\tau$, $\boldsymbol{b}_\tau$, and c_τ lead to $Z(\mathbb{A}_\tau, \boldsymbol{b}_\tau)$ with the minimal possible shortfall $s_{1-\tau}(-Z)$ under the assumptions on $\mathbb{A}$. This derivation is analogous to those in Paindaveine and Šiman (2011) and Bertsimas et al. (2004) and provides a way for transferring the results already obtained for shortfall to $\Phi_\tau(\mathbb{A}, \boldsymbol{s})$ or $\Psi_\tau(\mathbb{A}_\tau, \boldsymbol{s}_\tau, c_\tau)$, like in Paindaveine and Šiman (2011).

5 Computation

In the sample case with n observations $\boldsymbol{Y}_i$'s , $i = 1, \ldots, n$, from the population distribution assumed above, the sample generalized elliptical quantiles result from the definition if we take the expectation in (P_1) with respect to the empirical probability distribution.

Unfortunately, the problem of finding sample elliptical τ-g-quantiles is far from trivial as the corresponding sample objective function to be minimized is neither smooth nor convex. For the time being, we provide its preliminary solution by transforming it into an unconstrained nonlinear quantile regression task

$$\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} \rho_\tau \big(z_i - h(\boldsymbol{x}_i, \boldsymbol{\beta})\big)$$

for n responses z_i's, $i = 1, \ldots, n$, depending on corresponding regressors $\boldsymbol{x}_i$'s, $i = 1, \ldots, n$, in a nonlinear way described by a smooth function h and parameterized by a vector parameter $\boldsymbol{\beta}$. That is to say that the task is already a well-established research problem, see Koenker (2005), whose numerical solution has already been addressed successfully; see especially the algorithm of Koenker and Park (1996). Its MATLAB (2013) implementation in IPQR.M, available at http://sites.stat.psu.edu/~dhunter/code/qrmatlab, was employed, after a little improvement, as the basis for the computation of all the elliptical g-quantiles presented in the next section, using the transformation described below.

The constraints $\det(\mathbb{A}) = 1$ and $\mathbb{A} \in \mathrm{PSD}(m)$ can be made redundant by means of the Choleski decomposition. Indeed, any positive definite $\mathbb{A}$ can be represented as $\mathbb{C}'\mathbb{C}$ where $\mathbb{C} = (c_{i,j})_{i,j=1}^{m}$ is an upper triangular matrix. The assumption $\det(\mathbb{A}) = 1$ is then equivalent to writing $c_{m,m}$ as $1/(c_{1,1}c_{2,2}\ldots c_{m-1,m-1})$. The change to the unconstrained nonlinear quantile regression then lies in considering a constant zero vector of responses, treating observations as regressors, and using $-(g((\mathbb{C}\boldsymbol{Y}_i - \mathbb{C}\boldsymbol{s})'(\mathbb{C}\boldsymbol{Y}_i - \mathbb{C}\boldsymbol{s})) - c)$ as the nonlinear regression function where $\mathbb{C} = (c_{i,j})_{i,j=1}^{m}$ is the upper triangular matrix with the special $c_{m,m}$ described above. In other words, the metamorphosis lies in setting $z_i = 0$, $\boldsymbol{x}_i = \boldsymbol{Y}_i$, $\boldsymbol{\beta} = (\boldsymbol{v}', \boldsymbol{s}', c)'$ where $\boldsymbol{v} = \mathrm{vech}(\mathbb{C}')$ without the last $c_{m,m}$, and then $h(\boldsymbol{x}_i, \boldsymbol{\beta}) = -(g((\mathbb{C}\boldsymbol{Y}_i - \mathbb{C}\boldsymbol{s})'(\mathbb{C}\boldsymbol{Y}_i - \mathbb{C}\boldsymbol{s})) - c)$.

The algorithm of Koenker and Park (1996) requires some initial parameter estimates to start. We experimented with those derived from sample means and variance matrices but their robust counterparts might have led to even better performance. One should probably obtain the results for several wise choices of initial parameters and then compare them with one another before choosing the final solution. In any case, if there is some information about the parameters known or assumed in advance, it may be employed advantageously in the initial estimates as well.

We readily admit that our computational solution is suboptimal because it makes employing some a priori information about the non-diagonal elements of $\mathbb{A}_\tau$ virtually impossible. Nevertheless, we consider it as an important step forward towards a fully satisfactory algorithm.

6 Illustrations

At the end, we present some carefully designed pictures to illustrate the generalized elliptical g-quantiles and their properties. Their message should not be overemphasized as they are only included to show that some reasonable sample elliptical g-quantiles can be obtained at least for some choices of g and some data sets. For the sake of simplicity, we use only g in the form of g_1 to g_5 below:

$$g_1(z) = \sqrt[4]{z},\ \ g_2(z) = \sqrt{z},\ \ g_3(z) = z,\ \ g_4(z) = \frac{\sqrt{z}}{1+\sqrt{z}},\ \text{and}\ g_5(z) = \frac{z}{1+z}.$$

These five functions meet all the conditions of Definition 1 and allow a great deal of generality as they include both bounded and unbounded representatives, functions implicating different moment assumptions, and also the identity function g_3 leading to an alternative parameterization of the elliptical quantiles introduced in Hlubinka and Šiman (2013). We intentionally employ each of the functions to demonstrate that the family of reasonable elliptical g-quantiles is quite rich, although the optimal choice of g in specific contexts still remains debatable.

The cardinality of involved data sets ranges between $n = 250$ and $n = 500{,}000$ to show that the number of observations generally does not pose any problem. The elliptical quantile curves are always plotted for the same five quantile levels $\tau = 0.1, 0.3, 0.5, 0.7$, and 0.9 and always lighten with increasing τ.

Figure 2 demonstrates that elliptical g_i-quantiles, $i = 1, \ldots, 3$, computed from $n = 500{,}000$ bivariate normal observations $(Y_1, Y_2)' \sim N(0, 1) \times N(0, 4)$, closely match the nested standard population quantiles of multivariate normal distribution and virtually coincide at the same quantile levels.

On the other hand, Fig. 3 reveals that the choice of g matters in general and influences the robustness of the elliptical g-quantiles that seems improved for bounded

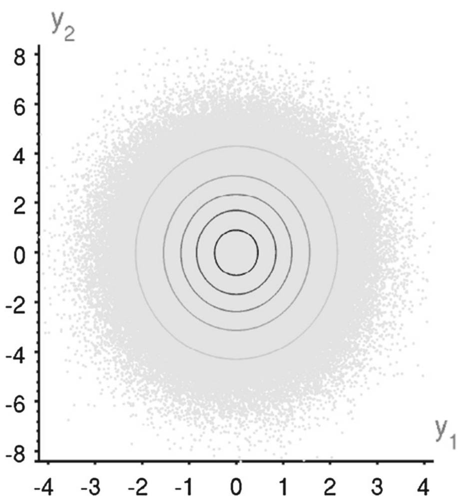
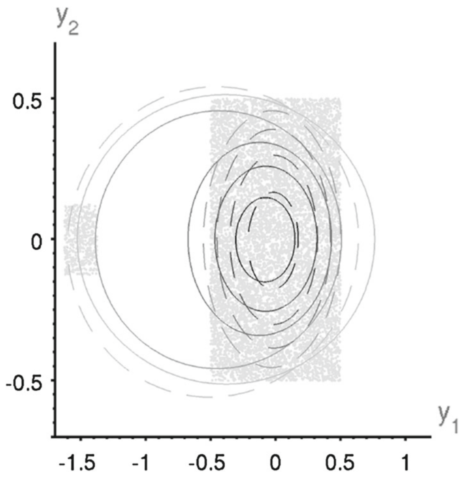


Fig. 2 Elliptical quantiles of a bivariate normal distribution. The *plot* shows elliptical τ-g_i-quantiles, $i \in \{1, 2, 3\}$, $\tau \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, from a random sample of $n = 500{,}000$ points from the bivariate normal distribution $N(0, 1) \times N(0, 4)$ where $g_1(z) = \sqrt[4]{z}$, $g_2(z) = \sqrt{z}$, and $g_3(z) = z$. The elliptical g_i-quantiles, $i = 1, \ldots, 3$, overlap and virtually coincide at the same quantile level. The quantile curves *lighten* with increasing τ

Fig. 3 Elliptical quantiles of a contaminated axially symmetric distribution. The *plot* displays elliptical τ-g_i-quantiles, $\tau \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, for $i = 3$ (*solid curves*) and $i = 5$ (*dashed curves*), i.e., for $g_3(z) = z$ and $g_5(z) = z/(1+z)$, computed from $n = 10{,}000$ observations driven by the model $0.95\boldsymbol{U}_1 + 0.05(\boldsymbol{U}_2 - (1.5, 0)')$ where the uniformly distributed bivariate vectors $\boldsymbol{U}_1 \sim U([-1/2, 1/2] \times [-1/2, 1/2])$ and $\boldsymbol{U}_2 \sim U([-1/8, 1/8] \times [-1/8, 1/8])$ are independent. The quantile curves *lighten* with increasing τ



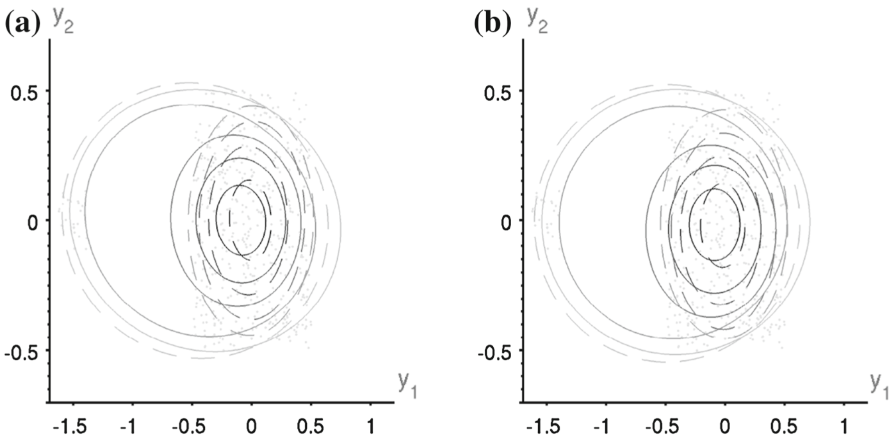


Fig. 4 Elliptical quantiles with a priori information. The plots show (*dark gray*) elliptical τ-g_i-quantiles, $\tau \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, for $i = 3$ (*solid curves*) and $i = 5$ (*dashed curves*), i.e., for $g_3(z) = z$ and $g_5(z) = z/(1+z)$, computed from $n = 500$ observations driven by the model used in Fig. 3 **a** with $s_2 = 0$, and **b** with $a_{12} = 0$, where $\mathbb{A} = (a_{ij})_{i,j=1}^2$ and $\boldsymbol{s} = (s_1, s_2)'$ stand for the quantile parameters. The quantile curves *lighten* with increasing τ

g's. It depicts both elliptical g_3-quantiles and g_5-quantiles computed from $n = 10{,}000$ observations driven by the model $0.95\boldsymbol{U}_1 + 0.05(\boldsymbol{U}_2 - (1.5, 0)')$ where the uniformly distributed bivariate vectors $\boldsymbol{U}_1 \sim U([-1/2, 1/2] \times [-1/2, 1/2])$ and $\boldsymbol{U}_2 \sim U([-1/8, 1/8] \times [-1/8, 1/8])$ are independent.

Figure 4 involves the same data model and the same functions g_3 and g_5 in g-quantiles as Fig. 3, but this time the elliptical g_3-quantiles and g_5-quantiles are computed only from $n = 500$ data points and with two types of available a priori information regarding $\mathbb{A}$ and $\boldsymbol{s}$: (a) with $s_2 = 0$, and (b) with $a_{12} = 0$, where $\mathbb{A} = (a_{ij})_{i,j=1}^2$

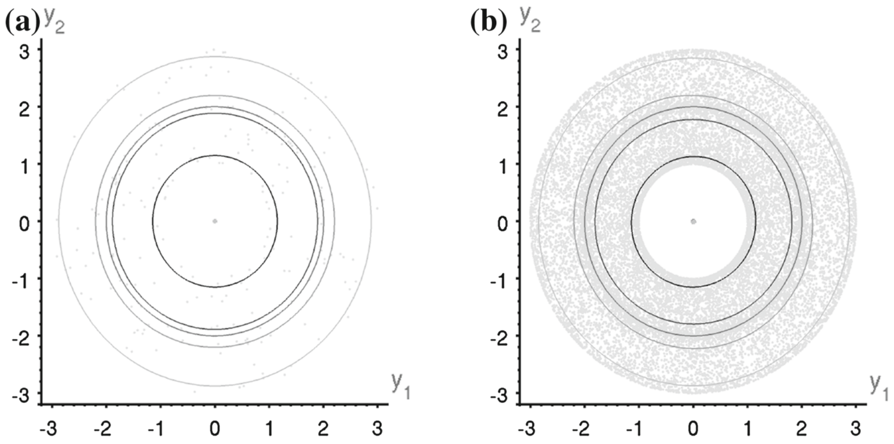


Fig. 5 Elliptical quantiles of a complicated elliptical distribution. The plots display (*dark gray*) elliptical τ-g_4-quantiles with their (*gray*) center points, $g_4(z) = \sqrt{z}/(1+\sqrt{z})$, $\tau \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, computed from (*light gray*) data points $(Y_1, Y_2)'$ following the model $(Y_1, Y_2)' = (2 + \sin^3(6\pi U_2))(\cos(2\pi U_1), \sin(2\pi U_1))'$ with independent $U_1, U_2 \sim U([0, 1])$, namely **a** from $n = 250$ observations with quantile parameters $\mathbb{A} = \text{diag}(1, 1)$ and $\boldsymbol{s} = (0, 0)'$ known or assumed in advance, and **b** from $n = 25{,}000$ observations without any a priori information. The quantile curves *lighten* with increasing τ

and $\boldsymbol{s} = (s_1, s_2)'$ stand for the quantile parameters. The output confirms that the elliptical g-quantiles easily incorporate a priori information about the symmetry of the underlying distribution.

Figure 5 shows elliptical g_4-quantiles obtained from bivariate observations $(Y_1, Y_2)' = (2 + \sin^3(6\pi U_2))(\cos(2\pi U_1), \sin(2\pi U_1))'$ where $U_1, U_2 \sim U([0, 1])$ are independent, namely (a) from $n = 250$ observations with both $\boldsymbol{s} = (0, 0)'$ and $\mathbb{A}$ equal to the identity matrix known or assumed in advance, and (b) from $n = 25{,}000$ observations without any a priori information available. This picture was included to demonstrate the facts that the concept is suitable even for complicated elliptical distributions and that a priori information regarding $\mathbb{A}$ and/or **s** can be employed easily, which improves the results, reduces the parametric space, and may be found useful for various statistical inference. In fact, if we set both $\mathbb{A}$ and **s** before the very computation, then the results answer the simple problem of how to inflate or deflate a given ellipsoid to cover the portion of observations determined by τ.

Finally, Fig. 6 confirms that the concept of elliptical g-quantiles is not restricted to the bivariate case and that it has no problem with affine equivariance. It depicts 3D elliptical g_2-quantiles computed from (a) $n = 1{,}000$ visible and (b) $n = 100{,}000$ invisible normally distributed points $(Y_1, Y_2, Y_3)' = (1, 1, 1)' + (N_1, N_2, N_3)\mathbb{V}^{1/2}$ with independent $N_1, N_2, N_3 \sim N(0, 1)$ and the symmetric matrix $\mathbb{V}$ with $\text{vech}(\mathbb{V}) = (1, 1/2, 0, 2, 1/2, 1)'$.

All the pictures included in this section testify that our research into elliptical quantiles proceeds in the right direction and that the concept presented here is worthy of serious consideration.

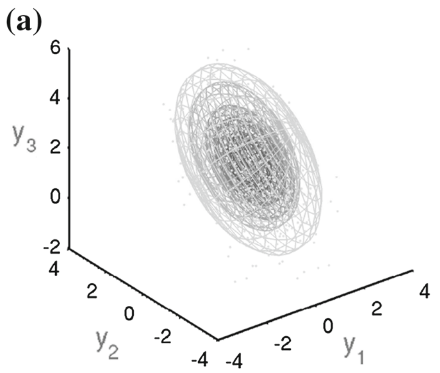


Fig. 6 Elliptical quantiles and equivariance in 3D. The plot shows (*dark gray*) 3D elliptical τ-g_2-quantiles, $g_2(z) = \sqrt{z}, \tau \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, computed from **a** $n = 1{,}000$ visible (*light gray*) and **b** $n = 100{,}000$ invisible data points $(Y_1, Y_2, Y_3)'$ following the model $(Y_1, Y_2, Y_3)' = (1, 1, 1)' + (N_1, N_2, N_3)\mathbb{V}^{1/2}$ with independent $N_1, N_2, N_3 \sim N(0, 1)$ and the symmetric matrix $\mathbb{V}$ with $\mathrm{vech}(\mathbb{V}) = (1, 1/2, 0, 2, 1/2, 1)'$. The quantile contours *lighten* with increasing τ

7 Discussion

We have introduced a class of location elliptical g-quantiles with an impressive collection of good properties (B1) to (B13) mentioned in the introductory section, including affine equivariance, preservation of symmetry, natural parameterization and probability coverage, location and shape dependence on the quantile level, easy incorporation of some types of apriori information, and possible resistance to outliers.

These quantiles have been designed (only) for situations when we have some information about the symmetry and/or location of the underlying data distribution without the full knowledge of its parametric family, particularly if we can also expect its heavy tails or level-dependent shape of its quantiles. Then, our elliptical quantiles excel above their competitors especially if the data are of medium to large size and of dimension three to seven or so.

Before uniqueness and asymptotic properties of these quantiles are sufficiently clarified, we suggest to use them mainly for exploratory analysis, to choose g cautiously, to search for the quantile centers only in the convex hull of the data cloud, to experiment with various initial values for the computation and to always use $g(z) = z$ as a benchmark as this particular choice of g leads to unique (but non-robust) elliptical quantiles, consistent in the sample case under very mild conditions.

The associated processes $\{c_\tau\}$, $\{\boldsymbol{b}_\tau\}$, and $\{\mathbb{A}_\tau\}$ of quantile parameters, indexed by the quantile level $\tau \in (0, 1)$, can also be used for statistical inference by means of Theorem 1. For example, $\{\boldsymbol{b}_\tau\}$ could be useful for testing central, axial, or half-space symmetry, $\{\mathbb{A}_\tau\}$ could be employed for testing ellipticity of centrally symmetric distributions, and $\{c_\tau\}$ could be used for identifying specific elliptical distributions. Currently, it is possible only heuristically but it will change when further results on uniqueness and asymptotics become available.

As quantiles belong to the cornerstones of statistics and stand behind many fundamental statistical tools, the same could be said about our elliptical quantiles as well.

They may likewise give rise to various control charts, diagnostic plots, L-statistics, descriptive functionals, regression generalizations and other useful statistical instruments, in the same way as many other multivariate quantiles. Only the applications of similarly defined standard univariate (regression) quantiles alone are numerous enough for quite a few book-length treatments and, as we see it, there is a good chance that our concept will once reach a similar degree of development and popularity as well, especially for its simplicity, intuitiveness, good properties, and various regression and shape extensions that we are currently investigating.

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