# Scenario Generation via $\mathcal{L}_1$ Norm

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**Abstract.** Optimization problems depending on a probability measure correspond to many economic and financial situations. It can be very complicated to solve these problems, especially when the "underlying" probability measure belongs to a continuous type. Consequently, the "underlying" continuous probability measure is often replaced by discrete one with finite number of atoms (scenario). The aim of the contribution is to deal with the above mentioned approximation in a special form of stochastic optimization problems with an operator of the mathematical expectation in the objective function.

The stability results determined by the help of the Wasserstein metric (based on the $\mathcal{L}_1$ norm) are employed to generate approximate distributions.

**Keywords:** One–stage stochastic programming problems, multistage stochastic problems, stability, Lipschitz property, $\mathcal{L}_1$ norm, Wasserstein metric, scenario generation, approximation error

**JEL classification:** C44
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## 1 Introduction

Let $(\Omega, \mathcal{S}, P)$ be a probability space, $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ an $s$–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$, $F := F_\xi(z)$ the distribution function of $\xi$, $P_F$ and $Z_F$ the probability measure and the support corresponding to $F$, respectively. Let, moreover, $g_0 := g_0(x, z)$ be a real–valued function defined on $R^n \times R^s$, $X_F \subset X \subset R^n$ a nonempty set generally depending on $F$ and $X \subset R^n$ a nonempty "deterministic" set. If $\mathsf{E}_F$ denotes the operator of mathematical expectation corresponding to $F$ and if for an $x \in X$ there exists finite $\mathsf{E}_F g_0(x, \xi)$, then rather general one-stage (static) "classical" stochastic optimization problem can be introduced in the form:

$$\text{to find} \quad \varphi(F, X_F) = \inf\big\{\mathsf{E}_F g_0(x, \xi) : \, x \in X_F\big\}. \tag{1}$$

The objective function in Problem (1) depends linearly on the probability measure $P_F$. We shall try to include in our consideration also a little relax problems. In particular, we consider problems that can be covered by the following type:

$$\text{to find} \quad \bar{\varphi}(F, X_F) = \inf\big\{\mathsf{E}_F \bar{g}_0(x, \xi, \mathsf{E}_F h(x, \xi)) : x \in X_F\big\}, \tag{2}$$

where $h := h(x, z)$ is an $m_1$–dimensional vector function defined on $R^n \times R^s$, $h = (h_1, \ldots, h_{m_1})$; $\bar{g}_0 := \bar{g}_0(x, z, y)$ is a real–valued function defined $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^{m_1}$.

**Remark 1.**
- The type of Problems (2) has begun recently to appear rather often in the literature (see, e. g., Ermoliev and Norkin [**?**]). Problem (2) covers Problem (1) with $\bar{g}_0(x, z, y) := g_0(x, z)$.
- Some problems from the class "Mean-Risk" can be covered by the type (2) (see, e. g., [**?**], [**?**], [**?**], [**?**]).

If $g_1 := g_1(x, z)$ is a real–valued function defined on $R^n \times R^s$, $g_2 := g_2(y, x, z)$ a real–valued function defined on $R^{n_1} \times R^n \times R^s$; $\mathcal{K}(x, z)$, for every $x \in X$, a measurable multifunction defined on $R^n \times R^s$, and

$$g_0(x, z) = g_1(x, z) + \inf\{g_2(y, x, z) : \, y \in \mathcal{K}(x, z)\}, \tag{3}$$

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then (1) is a two–stage stochastic programming problem (for more details see, e.g., [?] or [?]). However see also [?] to recognize that the well–known and often employed risk measure CVaR can be reformulated in the form of simple recourse problem and consequently in two–stage stochastic programming problem.

Two–stage stochastic programming problems correspond to applications in which it is necessary first to determine $x$ on a base of the knowledge $P_F$ only and, after the realization of the random element $\xi$, it is possible to correct the decision and to determine $y$.

The two–stage stochastic programming problems can be generalized to the multistage case. There are known a few types of different definitions of the multistage stochastic programming problems. We recall $(M+1)$–stage stochastic programming problem as the problem:

Find

$$\varphi_{\mathcal{F}}(M) = \inf \{\mathsf{E}_{F^{\xi^0}} g^0_{\mathcal{F}}(x^0,\, \xi^0) |\; x^0 \in \mathcal{K}^0\}, \tag{4}$$

where the function $g^0_{\mathcal{F}}(x^0,\, z^0)$ is defined recursively

$$\begin{aligned} g^k_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) \quad &= \quad \inf\{\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}\; g^{k+1}_{\mathcal{F}}(\bar{x}^{k+1},\, \bar{\xi}^{k+1})\, |x^{k+1} \in \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k)\}, \\ k = 0,\, 1,\, \ldots,\, M-1, & \\ g^M_{\mathcal{F}}(\bar{x}^M,\, \bar{z}^M) \quad &:= \quad g^M_0(\bar{x}^M,\, \bar{z}^M), \quad \mathcal{K}_0 := X^0. \end{aligned} \tag{5}$$

$\xi^j := \xi^j(\omega)$, $j = 0, 1, \ldots, M$ denotes an $s$–dimensional random vector defined on a probability space $(\Omega, \mathcal{S}, P)$; $F^{\xi^j}(z^j)$, $z^j \in R^s$, $j = 0, 1 \ldots, M$ the distribution function of the $\xi^j$ and $F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1})$, $z^k \in R^s$, $\bar{z}^{k-1} \in R^{(k-1)s}$, $k = 1, \ldots, M$ the conditional distribution function ($\xi^k$ conditioned by $\bar{\xi}^{k-1}$); $P_{F^{\xi^j}}$, $P_{F^{\xi^{k+1}|\bar{\xi}^k}}$, $j = 0, 1, \ldots, M$, $k = 0, 1, \ldots, M-1$ the corresponding probability measures; $Z^j := Z_{F^{\xi^j}} \subset R^s$, $j = 0, 1, \ldots, M$ the support of the probability measure $P_{F^{\xi^j}}$. Furthermore, the symbol $g^M_0 := g^M_0(\bar{x}^M, \bar{z}^M)$ denotes a continuous function defined on $R^{n(M+1)} \times R^{s(M+1)}$; $X^k \subset R^n$, $k = 0, 1, \ldots, M$ is a nonempty compact set; the symbol $\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) := \mathcal{K}^{k+1}_{F^{\xi^{k+1}|\bar{\xi}^k}}(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \ldots, M-1$ denotes a measurable multifunction defined on $R^{n(k+1)} \times R^{s(k+1)}$ with "values" subsets of $R^n$. $\bar{\xi}^k (:= \bar{\xi}^k(\omega)) = [\xi^0, \ldots, \xi^k]$; $\bar{z}^k = [z^0, \ldots, z^k]$, $z^j \in R^s$; $\bar{x}^k = [x^0, \ldots, x^k]$, $x^j \in R^n$; $\bar{X}^k = X^0 \times X^1 \ldots \times X^k$; $\bar{Z}^k := \bar{Z}^k_{\mathcal{F}} = Z_{F^{\xi^0}} \times Z_{F^{\xi^1}} \ldots \times Z_{F^{\xi^k}}$, $j = 0, 1, \ldots, k$, $k = 0, 1, \ldots, M$. Symbols $\mathsf{E}_{F^{\xi^0}}$, $\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}$, $k = 0, 1, \ldots, M-1$ denote the operators of mathematical expectation corresponding to $F^{\xi^0}$, $F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}$, $k = 0, \ldots, M-1$.

We have introduced three types of the stochastic optimization problems. The aim of the contribution is to suggest an approximate solution based on approximation of the continuous distributions by discrete one with finite number of atoms. To this end we employ the approach suggested in [?]. Furthermore, we generalize the former results in the case of "empirical" estimation approximation to the case of distributions with heavy tails. To this end we employ the stability results based on the Wasserstein metric with the "underlying" $\mathcal{L}_1$ norm.

## 2 Some Definitions and Some Assertions

First, we recall a few definitions and auxiliary assertions. To recall the first auxiliary assertion let $\mathcal{P}(R^s)$ denote the set of all (Borel) probability measures on $R^s$ and let the system $\mathcal{M}^1_1(R^s)$ be defined by the relation:

$$M^1_1(R^s) := \left\{\nu \in \mathcal{P}(R^s) : \int_{R^s} \|z\|_1 d\nu(z) < \infty\right\}, \quad \|\cdot\|_1 \quad \text{denotes} \quad \mathcal{L}_1 \quad \text{norm in} \quad R^s. \tag{6}$$

We introduce the system of the assumptions:

A.1
- $g_0(x, z)$ is either a uniformly continuous function on $X \times \mathbb{R}^s$, or $X$ is a bounded convex set and there exists $\varepsilon > 0$ such that $g_0(x, z)$ is a convex on $X(\varepsilon)$ and bounded on $X(\varepsilon) \times Z_F$ ($X(\varepsilon)$ denotes the $\varepsilon$–neighborhood of the set$X$);
- $g_0(x, z)$ is for $x \in X$ a Lipschitz function of $z \in \mathbb{R}^s$ with the Lipschitz constant $L$ (corresponding to the $\mathcal{L}_1$ norm) not depending on $x$;

B.1 $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$ and there exists $\varepsilon > 0$ such that

- $\bar{g}_0(x, z, y)$ is for $x \in X(\varepsilon), z \in R^s$ a Lipschitz function of $y \in Y(\varepsilon)$ with a Lipschitz constant $L_y$ where $Y(\varepsilon) = \{y \in R^{m_1} : y = h(x, z) \quad \text{for some} \quad x \in X(\varepsilon), \; z \in \mathbb{R}^s\}$ and $\mathsf{E}_F h(x, \xi)$, $\mathsf{E}_G h(x, \xi) \in Y(\varepsilon)$;
- for every $x \in X(\varepsilon), \; y \in Y(\varepsilon)$ there exist finite mathematical expectations $\mathsf{E}_F \bar{g}_0(x, \xi, \mathsf{E}_F h(x, \xi))$, $\mathsf{E}_F g_0^1(x, \xi, \mathsf{E}_G h(x, \xi))$, $\mathsf{E}_G g_0^1(x, \xi, \mathsf{E}_F h(x, \xi))$, and $\mathsf{E}_G g_0^1(x, \xi, \mathsf{E}_G h(x, \xi))$;
- $h_i(x, z), i = 1, \ldots, m_1$ are for every $x \in X(\varepsilon)$ Lipschitz functions of $z$ with the Lipschitz constants $L_h^i$ (corresponding to $\mathcal{L}_1$ norm),
- $\bar{g}_0(x, z, y)$ is for every $x \in X(\varepsilon), y \in Y(\varepsilon)$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant $L_z(x, \, y)$ (corresponding to $\mathcal{L}_1$ norm),
- $\bar{g}_0(x, \, z, \, y)$ is for every $x \in X, \; z \in R^s$ a Lipschitz function of $y \in Y$ with the Lipschitz constant $L^y(x, \, z)$ corresponding to $\mathcal{L}_1$ norm;

B.2 $\mathsf{E}_F \bar{g}_0(x, \xi, \mathsf{E}_F h(x, \xi))$, $\mathsf{E}_G \bar{g}_0(x, \xi, \mathsf{E}_G h(x, \xi))$ are continuous functions on $X$.

**Proposition 1** ([**?**], [**?**])**.** *Let $P_F, P_G \in \mathcal{M}_1^1(R^s)$ and let $X$ be a compact set. If*

1. *Assumption A.1 is fulfilled, then*

$$|\varphi(F, \, X) - \varphi(G, \, X)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i, \tag{7}$$

2. *Assumptions B.1, B.2 are fulfilled, then there exist $\hat{C} > 0$ such that*

$$|\bar{\varphi}(F, \, X) - \bar{\varphi}(G, \, X)| \le \hat{C} \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)| dz_i. \tag{8}$$

*The constant $\hat{C}$ can be estimated by the following relation*

$$\hat{C} \le \mathsf{E}_F[L^y(x, \, \xi)] \sum_{i=1}^{s} L_h^i + L^z(x, \, \mathsf{E}_G(h, \, \xi)).$$

Proposition 1 reduces (from the mathematical point of view) an $s$–dimensional case to one–dimensional. We employ this fact to define atoms of the discrete approximate distribution functions. Of course a stochastic dependence between components of $\xi$ is neglected by this approach. (The idea to reduce an $s$–dimensional case, $s > 1$ to one dimensional case is credited to G. Pflug [**?**] (see also Šmíd [**?**]).)

Evidently, if we approximate the continuous (w.r.t. Lebesque measure) probability measure $P_F$ by a discrete one with the finite number of atoms, we obtain mostly (from the numerical point of view) more "pleasant" problem.

# 3 Approximation

## 3.1 Deterministic Case

To construct first discrete approach we introduce the following assumption:

A.2 $P_{F_i}, i = 1, \ldots, s$ are absolutely continuous w. r. t. the Lebesgue measure on $R^1$,
($F_i, \; i = 1, \, \ldots, \, s$ are one–dimensional marginal distribution functions corresponding to $F$.)

Evidently, if A.2 is fulfilled, then for given $M_i, \, \bar{M}_i > 0, \, i = 1, \, \ldots, \, s$ there exist natural numbers $m_i, \, \bar{m}_i, \, i = 1, \, \ldots, \, s$ and points $z_{i, \, j}, \, \bar{z}_{i, \, k}, \in \bar{R}^1, \; j = 0, \, 1, \, \ldots, \, m_i, \, k = 1, \, \ldots, \, \bar{m}_i$ such that

$$-\infty = z_{i,\,0} < z_{i,\,1} < z_{i,\,2} < \ldots < z_{i,\,m_{i-1}} < z_{i,\,m_i} = \infty,$$

$$-\infty = \bar{z}_{i,\,0} < \bar{z}_{i,\,1} < \bar{z}_{i,\,2} < \ldots < \bar{z}_{i,\,\bar{m}_{i-1}} < \bar{z}_{i,\,\bar{m}_i} = \infty$$

and, simultaneously,

$$(L/s) \int\limits_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)| dz_i \,\leq\, M_i,\; i = 1,\, \ldots,\, s,$$
$$(\hat{C}/s) \int\limits_{-\infty}^{\infty} |F_i(z_i) - \bar{G}_i(z_i)| dz_i \,\leq\, \bar{M}_i,\; i = 1,\, \ldots,\, s,$$

where $G_i,\, \bar{G}_i,\, i = 1,\, \ldots,\, s$ are one dimensional discrete distribution functions with atoms in points $z_{i,\,j}, \bar{z}_{i,\,j},\; j = 1,\, \ldots,\, m_i,\, j = 1,\, \ldots,\, \bar{m}_i$, respectively.

Furthermore, it follows from the last relations that for every $M > 0,\, \bar{M} > 0$ there exist $s$–dimensional distribution functions $G,\, \bar{G}$ with marginals $G_i,\, \bar{G}_i,\, i = 1,\, \ldots,\, s$ such that

$$L \sum_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i \leq M, \tag{9}$$

$$\hat{C} \sum_{i=1}^{s} \int\limits_{-\infty}^{\infty} |F_i(z_i) - \bar{G}_i(z_i)| dz_i \leq \bar{M}. \tag{10}$$

We have proven the assertion

**Proposition 2.** *Let Assumption A.2 be fulfilled. Let moreover* $M, \bar{M} > 0$. *If*

1. *Assumption A.1 is fulfilled, then there exists a discrete distribution function G with discrete marginals* $G_i,\, i = 1,\, \ldots,\, s$ *such that*
$$|\varphi(F,\, X) - \varphi(G,\, X)| \leq M, \tag{11}$$
2. *Assumptions B.1, B.2 are fulfilled, then there exists a discrete distribution function* $\bar{G}$ *with discrete marginals* $\bar{G}_i,\, i = 1,\, \ldots,\, s$ *such that*
$$|\bar{\varphi}(F,\, X) - \bar{\varphi}(\bar{G},\, X)| \leq \bar{M}. \tag{12}$$

A possibility to employ the above mentioned approach (in the case of Problem (1)) it is necessary to assume that $g_0(x,\, z)$ is a Lipschits function of $z$ with the Lipschitz constant not depending on $x \in X$. It means, in the case of problem (3): if the function $g_1(x,\, z)$ fulfills this assumption, then a question arises if also

$$\inf\{g_2(y,\, x,\, z) :\, y \in \mathcal{K}(x,\, z)\} \tag{13}$$

fulfills this condition. To this end we consider two cases, separately. If (13) is a problem of linear programming, then the corresponding assertion can be found, e.g, in [**?**]. In the general nonlinear case we can find the corresponding assertion, e.g., in [**?**].

In the multistage case, we restrict to the case when the following assumption is fulfilled:

C.1 Random sequence $\{\xi^k\}_{k=-\infty}^{\infty}$ follows (generally) nonlinear autoregressive sequence

$$\xi^k = H(\xi^{k-1}\, \varepsilon^k),$$

where $\xi^0,\, \varepsilon^k,\, k = 1,\, 2,\, \ldots$ are stochastically independent $s$–dimensional random vectors defined on $(\Omega,\, \mathcal{S},\, P)$ and, moreover, $\varepsilon^k,\, k = 1,\, \ldots$ identically distributed. $H = (H_1,\, \ldots,\, H_s)$ is a Lipschitz vector function defined on $R^s$. We denote the distribution function corresponding to $\varepsilon^1 = (\varepsilon_1^1,\, \ldots,\, \varepsilon_s^1)$ by the symbol $F^{\varepsilon}$ and suppose the realization $\xi^0$ to be known.

Evidently, the multistage stochastic programming problem (4), (5) depends essentially on a system of (generally) conditional distribution functions

$$\mathcal{F} = \{F^{\xi^0}(z^0),\, F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\, k = 1,\, \ldots,\, M\}. \tag{14}$$

Consequently, if we replace $\mathcal{F}$ by another system $\mathcal{G}$

$$\mathcal{G} = \{G^{\xi^0}(z^0),\, G^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\, k = 1,\, \ldots,\, M\}, \tag{15}$$

we obtain another multistage stochastic programming problem with the optimal value denoted $\varphi_{\mathcal{G}}(M)$.

Under Assumption C.1 the system $\mathcal{F}$ is determined by $F^{\xi^0}$ and $F^{\varepsilon}$. Consequently, if we replace these two probability distribution functions by another $G^{\xi^0}$ and $G^{\varepsilon}$, we obtain another system $\mathcal{G}$.

Considering, furthermore, the constraint sets $\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k)$, $k = 0,\, \ldots,\, M-1$ not depending on the probability measure, then the assumptions under which

$$|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{G}}(M)| \le \sum_{i=1}^{s} C^i_W \int\limits_{R^1} |F^{\varepsilon}_i(z_i) - G^{\varepsilon}_i(z_i)| dz_i$$

can be found in [**?**]. Consequently, if we define discrete distributions $G^{\xi_0}, G^{\varepsilon}_i$ $i = 1,\, \ldots,\, s$ determined by the approach of Proposition 2, then we have an approximating system $\mathcal{G}$ given by discrete mostly conditional distributional functions.

Furthermore, it follows from results of the above mentioned work that this approach can be generalized to the case when constraints sets are given by the individual probability constraints.

### 3.2 Empirical Estimates Case

We introduce the next assumptions:

A.3
- $\{\xi^i\}_{i=1}^{\infty}$ is an independent random sequence corresponding to $F$,
- $F^N$ is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}, N = 1, 2, \ldots$.

**Proposition 3.** *[**?**], [**?**] Let $P_F \in \mathcal{M}^1_1(R^s)$, $X$ be a compact set. Let, moreover, Assumption A.3 be fulfilled. If*

1. *Assumption A.1 is fulfilled, then*

$$P\{\omega | |\varphi(F^N, X) - \varphi(F, X)| \xrightarrow[N\to\infty]{} 0\} = 1, \tag{16}$$

1. *Assumptions B.1, B.2 are fulfilled, then*

$$P\{\omega | |\bar{\varphi}(F^N, X) - \bar{\varphi}(F, X)| \xrightarrow[N\to\infty]{} 0\} = 1. \tag{17}$$

According to Proposition 3 we can see that $\varphi(F^N, X)$, $\bar{\varphi}(F^N, X)$ are, in the case of the "underlying" distributions $F$ with finite first moments (under some additional assumptions) consistent estimates of $\varphi(F, X)$, $\bar{\varphi}(F, X)$. It means that these estimates are consistent also in the case of the heavy tailed distributions (including the stable distributions, for the definition of stable distribution see, e.g. [**?**]) if there exists first absolute moments.

**Proposition 4.** *[**?**] Let $P_F \in \mathcal{M}^1_1(R^s)$, $t > 0$, $X$ be a compact set and for some $r > 2$ it holds that $\mathsf{E}_{F_i}|\xi_i|^r < +\infty, \quad i = 1, \ldots s$. Let, moreover, the constant $\gamma$ fulfil the inequalities $0 < \gamma < 1/2 - 1/r$. If*

1. *Assumptions A.1, A.2, A.3 are fulfilled, then*

$$P\{\omega |\, N^{\gamma} |\varphi(F,\, X) - \varphi(F^N, X)| > t\} \xrightarrow[N\to\infty]{} 0. \tag{18}$$

1. *Assumptions B.1, B.2, A.2, A.3 are fulfilled, then*

$$P\{\omega |\, N^{\gamma} |\bar{\varphi}(F,\, X) - \bar{\varphi}(F^N, X)| > t\} \xrightarrow[N\to\infty]{} 0. \tag{19}$$

## 4 Conclusion

The paper deals with optimization problems depending on a probability measure. Especially, it is considered the case when the "underlying" probability measure is absolutely continuous with respect to the Lebesque measure and the aim of the paper is to suggest approximate problems with a discrete probability measure. Moreover, this approximation can be defined with respect to required error. The case of one–stage, two–stage and multistage cases are considered separately. However, problems with an "underlying" autoregressive random sequence are considered in the multistage case only.

A results for empirical estimates are recalled in the second part of the contribution.

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