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Relationship between the concave integrals and the pan-integrals on finite spaces

Yao Ouyang [a,*], Jun Li [b], Radko Mesiar [c,d]

[a] Faculty of Science, Huzhou University, Huzhou, Zhejiang 313000, China
[b] School of Sciences, Communication University of China, Beijing 100024, China
[c] Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia
[d] UTIA CAS, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic

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ABSTRACT

This study discusses the relationship between the concave integrals and the pan-integrals on finite spaces. The *minimal atom* of a monotone measure is introduced and some properties are investigated. By means of two important structure characteristics related to minimal atoms: *minimal atoms disjointness property* and *subadditivity for minimal atoms*, a necessary and sufficient condition is given that the concave integral coincides with the pan-integral with respect to the standard arithmetic operations $+$ and $\cdot$ on finite spaces. Following this result, we have shown that these two integrals coincide if the underlying monotone measure is subadditive.

1. Introduction

In non-additive measure and integral theory, several prominent nonlinear integrals with respect to monotone measure (or capacity) have been defined and discussed in detail. Among them, we mention the Choquet integral [2] (see also [3,19]), the pan-integral introduced by Yang [27] (see also [25,24]) and the concave integral introduced by Lehrer [9] (see also [10]). Although all the three types of integrals coincide with the Lebesgue integral in the case where the monotone measure is σ-additive, they are significantly different from each other. The Choquet integral is based on chains of sets, while, similarly to the Lebesgue integral, the pan-integral deals with disjoint finite set systems. Finally, the concave integral deals with arbitrary finite set systems, see [16]. Note that all these integrals can be seen as particular instances of decomposition integrals recently introduced by Even and Lehrer [5], see also [16].

* Corresponding author.
E-mail addresses: oyy@hutc.zj.cn (Y. Ouyang), lijun@cuc.edu.cn (J. Li), mesiar@math.sk (R. Mesiar).

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Lehrer [9] showed that the Choquet integral is less than or equal to the concave integral and these two integrals coincide if and only if the underlying capacity ν is convex (also known as supermodular, see [3,19]). In [24] it was proved that the pan-integral with respect to the usual addition $+$ and usual multiplication $\cdot$ is less than or equal to the Choquet integral if the underlying monotone measure μ is superadditive, while the pan-integral is greater than or equal to the Choquet integral if μ is subadditive.

This paper will focus on the relationship between the concave integrals and the pan-integrals. We explore the conditions under which the concave integral coincides with the pan-integral w.r.t. the usual addition $+$ and usual multiplication $\cdot$.

We shall introduce the concept of *minimal atom* of a monotone measure and investigate its properties. As a special kind of atom of monotone measure, the minimal atom plays an essential role in our discussions. By means of minimal atoms we describe the relationship between the concave integrals and pan-integrals on finite spaces. We introduce two important concepts related to minimal atoms: *minimal atoms disjointness property* which is weaker than weak null-additivity, and *subadditivity w.r.t. minimal atoms* which is weaker than subadditivity. Our main results are in Section 4. We shall show that on finite spaces the concave integral coincides with the pan-integral w.r.t. the usual addition $+$ and usual multiplication $\cdot$ if and only if the underlying monotone measure μ satisfies both the minimal atoms disjointness property and subadditivity w.r.t. minimal atoms. As a direct corollary, we obtain that on finite spaces the above mentioned two integrals coincide if the monotone measure μ is subadditive.

2. Preliminaries

Let X be a nonempty set and $\mathcal{F}$ a σ-algebra of subsets of X. $\mathbf{F}$ denotes the class of all finite nonnegative real-valued measurable functions on $(X, \mathcal{F})$. Unless stated otherwise all the subsets mentioned are supposed to belong to $\mathcal{F}$, and all the functions mentioned are supposed to belong to $\mathbf{F}$.

We assume that μ is a monotone measure on $(X, \mathcal{F})$, i.e., $\mu : \mathcal{F} \to [0, +\infty]$ is an extended real-valued set function satisfying the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;
(2) $\mu(A) \leq \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{F}$. (monotonicity)

When μ is a monotone measure, the triple $(X, \mathcal{F}, \mu)$ is called a monotone measure space [19,24]. In some literature, the monotone measure constraint by $\mu(X) = 1$ is also known as capacity or fuzzy measure, or nonadditive probability, etc. (see [3,9,17,23,25]).

A monotone measure μ is said to be *weakly null-additive* [24], if $\mu(A \cup B) = 0$ whenever $\mu(A) = \mu(B) = 0$; *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any $A, B \in \mathcal{F}$.

The concept of a pan-integral [24,27] involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see also [15,19,20,25,24]). To be able to compare the concave and pan-integrals, in this paper we consider the pan-integrals based on the standard addition $+$ and multiplication $\cdot$ only. We recall the following definition.

Definition 2.1. Let $(X, \mathcal{F}, \mu)$ be a monotone measure space and $f \in \mathbf{F}$. The pan-integral of f on X w.r.t. the usual addition $+$ and usual multiplication $\cdot$ is given by

$$\int^{pan} f d\mu = \sup\left\{ \sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \leq f,\ \{A_i\}_{i=1}^{n} \subset \mathcal{F} \text{ is a partition of } X, \lambda_i \geq 0, n \in \mathbb{N} \right\}.$$

Lehrer [9] introduced a new integral known as concave integral (see also [10,22]), as follows:

Definition 2.2. (See [9].) Let $(X, \mathcal{F}, \mu)$ be a monotone measure space and $f \in \mathbf{F}$. The concave integral of f on X is defined by

$$\int^{cav} f d\mu = \sup\left\{\sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \le f,\ \{A_i\}_{i=1}^{n} \subset \mathcal{F}, \lambda_i \ge 0, n \in \mathbb{N}\right\}.$$

Note that the pan-integral is related to finite partitions of X, while the concave integral to any finite set systems of measurable subsets of X. Comparing these two integrals, it is obvious that for any $f \in \mathbf{F}$.

$$\int^{cav} f d\mu \ge \int^{pan} f d\mu.$$

In general,

$$\int^{cav} f d\mu \ne \int^{pan} f d\mu.$$

Example 2.3. Let $X = \{a, b\}$ and $\mu: 2^X \to [0, 1]$ be defined by

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ 0.4 & \text{if } E = \{a\}, \\ 0.5 & \text{if } E = \{b\}, \\ 1 & \text{if } E = X, \end{cases}$$

and put

$$f(x) = \begin{cases} 0.8 & \text{if } x = a, \\ 0.4 & \text{if } x = b. \end{cases}$$

Then we have $\int^{pan} f d\mu = 0.8 \times \mu(\{a\}) + 0.4 \times \mu(\{b\}) = 0.52$ and

$$\int^{cav} f d\mu = 0.4 \times \mu\big(\{a\}\big) + 0.4 \times \mu\big(\{a, b\}\big) = 0.56.$$

Thus, $\int^{cav} f d\mu > \int^{pan} f d\mu$.

3. Minimal atom of a monotone measure

The atom of a measure is an important concept in the classical measure theory. This concept was generalized in nonadditive measure theory, see [4,21] and further discussed, see [6,7,11,18,19,26].

Definition 3.1. (See [19,21].) Let μ be a monotone measure on $\mathcal{F}$. A set $A \in \mathcal{F}$ is called an atom of μ if $\mu(A) > 0$ and for every $B \subset A$ holds either

(i) $\mu(B) = 0$, or
(ii) $\mu(A) = \mu(B)$ and $\mu(A - B) = 0$.

To describe the relationship between the pan-integral and concave integral on finite space, we introduce the concept of *minimal atom* of a monotone measure.

Definition 3.2. Let μ be a monotone measure on $\mathcal{F}$. A set $A \in \mathcal{F}$ is called a *minimal atom* of μ if $\mu(A) > 0$ and for every $B \subset A$ holds either

(i) $\mu(B) = 0$, or
(ii) $A = B$.

Comparing Definition 3.1 and Definition 3.2, obviously, a minimal atom A of μ is a special atom of μ (it is also pseudo-atom of μ, see [7,26]). If A is a minimal atom of μ, then there is no proper subset B of A such that $\mu(B) > 0$. The following Example 3.3 shows that an atom of μ may not be a minimal atom of μ.

Example 3.3. Let $X = \{a, b\}$ and $\mu: 2^X \to [0, 1]$ be defined by

$$\mu(E) = \begin{cases} 1 & \text{if } E = \{a\}, \{a, b\}, \\ 0 & \text{else.} \end{cases}$$

Then X is an atom of μ, but it is not a minimal atom of μ. $\{a\}$ is an atom of μ and also a minimal atom of μ.

From the definition of minimal atom, we can easily obtain the following result. It will play a key role in the proof of Theorem 4.1 in the next section.

Proposition 3.4. *Let X be a finite set and μ be a monotone measure defined on $(X, \mathcal{F})$. Then every set $A \in \mathcal{F}$ with $\mu(A) > 0$ contains at least one minimal atom of μ.*

Proof. Suppose $A \in \mathcal{F}$ with $\mu(A) > 0$ and define

$$\mathcal{A} = \{E \in \mathcal{F} | E \subset A,\ \mu(E) > 0\}.$$

Then every minimal element of $\mathcal{A}$ is a minimal atom contained in A. □

We present some examples for minimal atoms in the following.

Example 3.5. Let $(X, \mathcal{F})$ be a measurable space and $\mu: \mathcal{F} \to [0, 1]$ be defined as

$$\mu(E) = \begin{cases} 0 & \text{if } E \neq X, \\ 1 & \text{if } E = X. \end{cases}$$

Then X is the only minimal atom of μ.

Example 3.6. Let $X = \{a, b, c, d\}$.

(1) Let $\mu: 2^X \to [0, 1]$ be defined by

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \\ \frac{3}{5} & \text{if } E = \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \\ \frac{2}{5} & \text{if } E = \{a, b\}, \{a, c\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then there are two minimal atoms of μ, namely $\{a, b\}$ and $\{a, c\}$.

(2) Let $\mu: 2^X \to [0,1]$ be defined by

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \\ \frac{2}{5} & \text{if } E \neq X, \\ 0 & \text{if } E = \emptyset. \end{cases}$$

Then every singleton is a minimal atom of μ.

In the following we introduce two concepts concerning minimal atoms of a monotone measure: *minimal atoms disjointness property* (Definition 3.7) and *subadditivity w.r.t. minimal atoms* (Definition 3.10). They play important role in our discussion.

Definition 3.7. A monotone measure μ on $(X, \mathcal{F})$ is said to have *the minimal atoms disjointness property*, if every two distinct minimal atoms of μ are disjoint, i.e., for every pair of minimal atoms A and B of μ, $A \neq B$ implies $A \cap B = \emptyset$.

Note that not every monotone measure possesses the minimal atoms disjointness property. For example, in Example 3.6(1), $\{a,b\}$ and $\{a,c\}$ are two minimal atoms of μ, but $\{a,b\} \cap \{a,c\} = \{a\} \neq \emptyset$. But, the following result reveals that the minimal atoms disjointness property is a rather weak condition.

Proposition 3.8. *Let $(X, \mathcal{F}, \mu)$ be a monotone measure space. If μ is weakly null-additive, then μ possesses the minimal atoms disjointness property.*

Proof. Let A, B be a pair of minimal atoms such that $A \cap B \neq \emptyset$ and $A \neq B$. Then, by the definition of minimal atom, we have $\mu(A \cap B) = 0$ and $\mu(A - (A \cap B)) = 0$. Since μ is weakly null-additive, $\mu(A) = \mu[(A - (A \cap B)) \cup (A \cap B)] = 0$, a contradiction. $\square$

The subadditivity of μ implies the weak null-additivity, so a subadditive monotone measure μ possesses the minimal atoms disjointness property.

Note that a monotone measure μ with the minimal atoms disjointness property may not be weakly null-additive. Therefore the minimal atoms disjointness property is a weaker condition than weak null-additivity.

Example 3.9. Let $X = \{a,b,c\}$ and $\mu: 2^X \to [0,1]$ be defined by

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset, \{b\}, \{c\}, \\ 1/2 & \text{if } E = \{a\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then μ has the minimal atoms disjointness property (there are only two minimal atoms of μ, namely $\{a\}$ and $\{b,c\}$). But μ is not weakly null-additive.

Definition 3.10. Let X be a finite set. A monotone measure μ on $(X, \mathcal{F})$ is said to be *subadditive w.r.t. minimal atoms*, if for every set $A \in \mathcal{F}$ with $\mu(A) > 0$, we have

$$\mu(A) \leq \sum_{i=1}^{n} \mu(A_i),$$

where $\{A_i\}_{i=1}^n$ is the set of all minimal atoms contained in A.

Proposition 3.11. *Let X be a finite set and μ be a monotone measure on $(X, \mathcal{F})$. If μ is subadditive, then it is subadditive w.r.t. minimal atoms.*

Proof. Let $A \in \mathcal{F}$ with $\mu(A) > 0$ and $\{A_i\}_{i=1}^n$ is the set of all minimal atoms contained in A.

Put

$$\widetilde{A}_0 = A \backslash (A_1 \cup A_2 \cup \cdots \cup A_n).$$

Then $\mu(\widetilde{A}_0) = 0$ (otherwise, from Proposition 3.4, $\widetilde{A}_0$ contains at least one minimal atom contained in A, a contradiction). Thus A is expressed as

$$A = A_1 \cup A_2 \cup \cdots \cup A_n \cup \widetilde{A}_0.$$

By the subadditivity of μ, then

$$\begin{aligned}\mu(A) &= \mu(A_1 \cup A_2 \cup \cdots \cup A_n \cup \widetilde{A}_0) \\ &\leq \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n).\end{aligned}$$

This shows that μ is subadditive w.r.t. minimal atoms. □

Note that a monotone measure μ with subadditivity w.r.t. minimal atoms may not be subadditive. For example, the monotone measure μ in Example 3.6(2) is subadditive w.r.t. minimal atoms, but $\mu(X) > \mu(\{a, b\}) + \mu(\{c, d\})$, thus it is not subadditive. Therefore, the subadditivity w.r.t. minimal atoms is really weaker than subadditivity.

Let μ be a monotone measure on a finite set X with the minimal atoms disjointness property. If μ is not subadditive w.r.t. minimal atoms, then there is at least one set A with $\mu(A) > 0$ such that

$$\mu(A) > \sum_{i=1}^{n} \mu(A_i),$$

where $\{A_i\}_{i=1}^n$ is the set of all minimal atoms contained in A. In this case we say that A is a *strictly superadditive set for minimal atoms* (shortly, *a strictly superadditive set*). We point out that for every strictly superadditive set A, there is a superadditive subset $B \subset A$ such that for any proper subset $C \subsetneq B$ with $\mu(C) > 0$, C is not a strictly superadditive set (we call B *a minimal strictly superadditive set* contained in A). In fact, let A be a strictly superadditive set, define

$$\mathcal{S}_A = \{B \mid B \subset A \text{ and } B \text{ is a strictly superadditive set}\}.$$

Every minimal element of $\mathcal{S}_A$ is a minimal strictly superadditive set contained in A.

Example 3.12. Let $X = \{a, b, c, d\}$. The monotone measure $\mu: 2^X \to [0, 1]$ is defined by

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \\ \frac{3}{5} & \text{if } E = \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \\ \frac{2}{5} & \text{if } E = \{a, b\}, \{c, d\}, \\ \frac{1}{5} & \text{if } E = \{a, c\}, \{b, c\}, \{c\}, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, X is a strictly superadditive set and

$$\mathcal{S}_X = \big\{X, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{c, d\}\big\}.$$

$\mathcal{S}_X$ contains three chains $X \supset \{a, b, d\}, X \supset \{a, c, d\} \supset \{c, d\}$ and $X \supset \{b, c, d\} \supset \{c, d\}$. There are two minimal strictly superadditive sets contained in X, namely $\{a, b, d\}$ and $\{c, d\}$.

4. The main results

Now we present our main results. We emphasize that in this section the universe X is a finite set and the monotone measure μ defined on $(X, \mathcal{F})$ is supposed to be finite, i.e., $\mu(X) < \infty$.

Theorem 4.1. *Let X be a finite space and μ be a finite monotone measure on $(X, \mathcal{F})$. Then, for all $f \in \mathbf{F}$*

$$\int^{pan} f d\mu = \int^{cav} f d\mu$$

if and only if the following two conditions hold:

(i) μ possesses the minimal atoms disjointness property;
(ii) μ is subadditive w.r.t. minimal atoms.

Proof. *Necessity*: Suppose that for all $f \in \mathbf{F}$, $\int^{pan} f d\mu = \int^{cav} f d\mu$. We use a proof by contradiction.

First of all, assume that the condition (i) is not true. Then there is a pair of minimal atoms A and B such that $A \neq B$, but $A \cap B \neq \emptyset$. We will show that $\int^{cav} f d\mu > \int^{pan} f d\mu$ for some f. Let k be a positive integer, define

$$f_k(x) = \begin{cases} 1, & \text{if } x \in A \backslash B; \\ k, & \text{if } x \in B \backslash A; \\ k+1, & \text{if } x \in A \cap B; \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_k \in \mathbf{F}$. Let $\sum_{i=1}^{n} \lambda_i \chi_{E_i}$ be an arbitrary summation such that $\lambda_i > 0$ for i from $\{1, .., n\}$, and $E_i \cap E_j = \emptyset, i \neq j$ and $\sum_{i=1}^{n} \lambda_i \chi_{E_i} \leq f_k$. Due to the fact that $f_k = 0$ on $X \backslash (A \cup B)$, evidently $E_i \subset A \cup B$ for any $i = 1, .., n$. Then there are two cases: (1) If there is some E_i, say E_1, such that $E_1 = B$, then $\lambda_1 \leq k$. Thus for any $i \neq 1$, $E_i \subset A \backslash B$, noting that A is a minimal atom of μ and $A \backslash B$ is a proper subset of A, then $\mu(E_i) = 0$. Therefore

$$\sum_{i=1}^{n} \lambda_i \mu(E_i) = \lambda_1 \mu(E_1) \leq k\mu(E_1) = k\mu(B). \tag{4.1}$$

(2) If for any i $(1 \leq i \leq n)$, $E_i \neq B$, then we have $\mu(E_i) = 0$ for any $E_i \subsetneqq B$ (since B is minimal atom of μ) and $\lambda_i \leq 1$ for any $E_i \cap (A - B) \neq \emptyset$. Thus

$$\sum_{i=1}^{n} \lambda_i \mu(E_i) \leq \sum_{i=1}^{n} \mu(E_i) \leq \mu_+(A \cup B), \tag{4.2}$$

where

$$\mu_+(A \cup B) = \sup\left\{ \sum_{i=1}^{n} \mu(F_i) \;\middle|\; \{F_i\} \text{ is a partition of } A \cup B \right\}$$

(for more details, see [12]).

Therefore, combining (4.1) and (4.2), we have

$$\begin{aligned} \int^{pan} f_k d\mu &= \max\left\{ \sum_{i=1}^{n} \lambda_i \mu(E_i) \;\middle|\; \sum_{i=1}^{n} \lambda_i \chi_{E_i} \leq f_k, E_i \subset A \cup B, E_i \cap E_j = \emptyset, i \neq j \right\} \\ &\leq \max\{k\mu(B), \mu_+(A \cup B)\}. \end{aligned}$$

Since $\mu_+(A \cup B) < \infty$ and $\mu(B) > 0$, there exists k_0 such that $k_0\mu(B) > \mu_+(A \cup B)$ and thus, on one hand, we have $\int^{pan} f_{k_0} d\mu \leq k_0\mu(B)$. On the other hand, by the definition of the concave integral and noting $\mu(A) > 0$, we have

$$\int^{cav} f_{k_0} d\mu \geq \mu(A) + k_0\mu(B) > k_0\mu(B).$$

We get a contradiction: $\int^{cav} f_{k_0} d\mu > \int^{pan} f_{k_0} d\mu$. So, the condition (i) is true.

Secondly, assume that the condition (ii) is not true (note that we have proved the condition (i) holds, i.e., μ has the minimal atoms disjointness property). Then there is a set $E \in \mathcal{F}$ such that $\mu(E) > \sum_{i=1}^{n} \mu(E_i)$, where $\{E_i\}_i$ is the set of all minimal atoms contained in E (E is called a strictly superadditive set, see Section 3). Noting the discussion to the subadditivity for minimal atoms in Section 3, there exists a strictly superadditive set $F \subset E$ such that for any proper subset $C \subsetneqq F$ with $\mu(C) > 0$, C is not a strictly superadditive set (F is called a minimal strictly superadditive set contained in E, see Section 3). Let $\{F_1, F_2, \cdots, F_n\}$ be the set of all minimal atoms contained in F (notice that Proposition 3.4 guarantees that the set $\{F_1, F_2, \cdots, F_n\}$ is nonempty).

Now, let the measurable function f be defined by

$$f(x) = \begin{cases} 2, & \text{if } x \in F_1; \\ 1, & \text{if } x \in F \backslash F_1; \\ 0, & \text{elsewhere.} \end{cases}$$

Let $\sum_{i=1}^{k} \lambda_i \chi_{A_i}$ be an arbitrary summation such that $A_i \cap A_j = \emptyset, i \neq j$ and $\sum_{i=1}^{k} \lambda_i \chi_{A_i} \leq f$. We can assume that $A_i \subset F$ and $\mu(A_i) > 0$ $(i = 1, 2, \ldots, k)$ without any loss of generality. Thus there are two cases:

Case I: There is some i_0 $(1 \leq i_0 \leq k)$ such that $A_{i_0} = F$. Then it follows from $A_i \subset F$ $(i = 1, 2, \ldots, k)$ and $A_i \cap A_j = \emptyset$ $(i \neq j)$ that $k = 1$, and hence $A_{i_0} = A_1 = F$, $\lambda_1 \leq 1$. Thus, we get

$$\sum_{i=1}^{k} \lambda_i \mu(A_i) = \lambda_1 \mu(A_1) \leq \mu(F). \tag{4.3}$$

Case II: For any i $(1 \leq i \leq k)$, $A_i \subsetneqq F$. Then there are two situations: (a) if there is some i_0 such that $A_{i_0} = F_1$, then $\lambda_{i_0} \leq 2$, and for every $i \neq i_0$, it follows from $A_i \subset F \backslash F_1$ that $\lambda_i \leq 1$; (b) if for any i $(1 \leq i \leq k)$, $A_i \neq F_1$, noting that F is minimal atom and $\mu(A_i) > 0$, then $A_i \cap (F \backslash F_1) \neq \emptyset$, and hence for any i $(1 \leq i \leq k)$, $\lambda_i \leq 1$. No matter what kind of situation, we have

$$\sum_{i=1}^{k} \lambda_i \mu(A_i) = \lambda_{i_0} \mu(A_{i_0}) + \sum_{i \neq i_0} \lambda_i \mu(A_i) \leq 2\mu(F_1) + \sum_{i \neq i_0} \mu(A_i). \tag{4.4}$$

Now let $\{A_{i,1}, A_{i,2}, \cdots, A_{i,s_i}\}$ be the set of all minimal atoms contained in A_i, $i = 1, 2, \ldots, k$. (Notice again that Proposition 3.4 guarantees that for every $i = 1, 2, \ldots, k$, the set $\{A_{i,1}, A_{i,2}, \cdots, A_{i,s_i}\}$ is nonempty.) Since F is a minimal strictly superadditive set contained in E and $A_i \subsetneqq F$, so A_i is not strictly superadditive set. Therefore we have

$$\mu(A_i) \leq \sum_{j=1}^{s_i} \mu(A_{i,j}) \quad (i = 1, 2, \ldots, k). \tag{4.5}$$

By combining (4.4) and (4.5), and noting that

$$\bigcup_{i=1}^{k} \{A_{i,1}, A_{i,2}, \cdots, A_{i,s_i}\} \subset \{F_1, F_2, \cdots, F_n\},$$

we obtain

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i\mu(A_i) &\le 2\mu(F_1)+\sum_{i\neq i_0}\mu(A_i)\\ &\le 2\mu(F_1)+\sum_{i\neq i_0}\lambda_i\left(\sum_{j=1}^{s_i}\mu(A_{i,j})\right)\\ &\le 2\mu(F_1)+\sum_{i=2}^{n}\mu(F_i)\\ &= \mu(F_1)+\sum_{i=1}^{n}\mu(F_i).\end{aligned} \tag{4.6}$$

Combining (4.3) and (4.6), and noting $\mu(F) > \sum_{i=1}^{n}\mu(F_i)$, it follows that

$$\begin{aligned}\int^{pan} f d\mu &= \max\left\{\sum_{i=1}^{k}\lambda_i\mu(A_i)\;\middle|\;\sum_{i=1}^{k}\lambda_i\chi_{A_i}\le f, A_i\subset F, A_i\cap A_j=\emptyset, i\neq j\right\}\\ &\le \max\left\{\mu(F),\mu(F_1)+\sum_{i=1}^{n}\mu(F_i)\right\}.\end{aligned}$$

But, by the definition of the concave integral and $\chi_{F_1}+\chi_F\le f$, and $\mu(F_1)>0$, then we have

$$\int^{cav} f d\mu \ge \mu(F_1)+\mu(F) > \max\left\{\mu(F),\mu(F_1)+\sum_{i=1}^{n}\mu(F_i)\right\}.$$

Thus, $\int^{cav} f d\mu > \int^{pan} f d\mu$. This is a contradiction. The condition (ii) is proved.

Sufficiency: Now assume that both conditions (i) and (ii) are true. Let $\{E_j\}_{j=1}^{n}$ be the set of all minimal atoms contained in X. For any function $f\in\mathbf{F}$ and an arbitrary summation $\sum_{i=1}^{k}\lambda_i\chi_{A_i}$ with $\sum_{i=1}^{k}\lambda_i\chi_{A_i}\le f$, define a sequence $\{l_j^{(i)}\}_{i,j}$ as

$$l_j^{(i)}=\begin{cases}1, & \text{if } E_j\subset A_i;\\ 0, & \text{otherwise,}\end{cases}$$

then we have

$$\begin{aligned}\sum_{j=1}^{n}\left(\sum_{i=1}^{k}\lambda_i l_j^{(i)}\right)\chi_{E_j} &= \sum_{i=1}^{k}\lambda_i\left(\sum_{j=1}^{n}l_j^{(i)}\chi_{E_j}\right)\\ &\le \sum_{i=1}^{k}\lambda_i\chi_{A_i}\\ &\le f,\end{aligned}$$

and by the condition (ii),

$$\sum_{i=1}^{k}\lambda_i\mu(A_i)\le\sum_{i=1}^{k}\lambda_i\left(\sum_{j=1}^{n}l_j^{(i)}\mu(E_j)\right)=\sum_{j=1}^{n}\left(\sum_{i=1}^{k}\lambda_i l_j^{(i)}\right)\mu(E_j).$$

Noting the condition (i), if we denote $\alpha_j = \sum_{i=1}^{k} \lambda_i l_j^{(i)}$ then for any summation $\sum_{i=1}^{k} \lambda_i \chi_{A_i} \leq f$ we have another summation $\sum_{j=1}^{n} \alpha_j \chi_{E_j} \leq f$ with $E_i \cap E_j = \emptyset, i \neq j$ and $\sum_{j=1}^{n} \alpha_j \mu(E_j) \geq \sum_{i=1}^{k} \lambda_i \mu(A_i)$. Therefore, we have

$$\begin{aligned}\int^{cav} f d\mu &= \sup\left\{ \sum_{i=1}^{k} \lambda_i \mu(A_i) \,\Big|\, \sum_{i=1}^{k} \lambda_i \chi_{A_i} \leq f \right\} \\ &\leq \sup\left\{ \sum_{j=1}^{n} \alpha_j \mu(E_j) \,\Big|\, \sum_{j=1}^{n} \alpha_j \chi_{E_j} \leq f, E_i \cap E_j = \emptyset, i \neq j \right\} \\ &\leq \int^{pan} f d\mu.\end{aligned}$$

From $\int^{cav} f d\mu \geq \int^{pan} f d\mu$, it follows that

$$\int^{cav} f d\mu = \int^{pan} f d\mu$$

for any function $f \in \mathbf{F}$.

The proof of the theorem is completed. □

Note that the conditions (i) and (ii) in Theorem 4.1 are independent, as shown in the following examples.

Example 4.2. Let $X = \{a, b\}$ and $\mathcal{F} = 2^X$. Let

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ \frac{1}{3} & \text{if } E = \{a\}, \{b\}, \\ 1 & \text{if } E = X. \end{cases}$$

Then $\{a\}$ and $\{b\}$ are all minimal atoms of μ, so μ possesses the minimal atoms disjointness property. But, it is not subadditive w.r.t. minimal atoms. In fact, $\mu(X) > \mu(\{a\}) + \mu(\{b\})$.

Example 4.3. Let $X = \{a, b, c, d\}$ and $\mathcal{F} = 2^X$. Let

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{a,b\}, \{a,c\}, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\{a, b\}$ and $\{a, c\}$ are all minimal atoms of μ, and μ is subadditive w.r.t. minimal atoms. But, $\{a, b\} \cap \{a, c\} = \{a\} \neq \emptyset$, so μ has not minimal atoms disjointness property.

Note 4.4. If we abandon the finiteness of μ, then both conditions (i) and (ii) in Theorem 4.1 are only sufficient, not necessary. That is, we can find an example which violates both conditions (i) and (ii) but the concave integral coincides with the pan-integral.

Example 4.5. Let $X = \{a, b, c\}$ and the monotone measure $\mu: 2^X \to [0, \infty]$ be defined as:

$$\mu(E) = \begin{cases} \infty & \text{if } E = X, \\ 1 & \text{if } E = \{a,b\} \text{ or } \{a,c\}, \\ 0 & \text{otherwise}. \end{cases}$$

Then there are two minimal atoms $\{a, b\}, \{a, c\}$. We note that $\{a, b\} \cap \{a, c\} = \{a\}$ and $\mu(X) > \mu(\{a, b\}) + \mu(\{a, c\})$, that is, both the conditions (i) and (ii) in Theorem 4.1 are not true. But for any function $f: X \to [0, \infty]$ we have $\int^{cav} f d\mu = \int^{pan} f d\mu$. In fact, if $\min\{f(x) \mid x \in X\} > 0$ then both the concave and pan-integrals are infinite. If $f(a) = 0$ or $f(b) = f(c) = 0$ then both the concave and pan-integrals are zero. If $f(a) > 0, f(b) > 0$ and $f(c) = 0$ then $\int^{cav} f d\mu = \int^{pan} f d\mu = \min\{f(a), f(b)\}$. Otherwise $\int^{cav} f d\mu = \int^{pan} f d\mu = \min\{f(a), f(c)\}$.

As a direct result of Proposition 3.8 and Theorem 4.1, we obtain the following corollary.

Corollary 4.6. *Let X be a finite set and μ be finite weakly null-additive monotone measure on $(X, \mathcal{F})$. Then, for all $f \in \mathbf{F}$*

$$\int^{pan} f d\mu = \int^{cav} f d\mu$$

if and only if μ is subadditive w.r.t. minimal atoms.

By Proposition 3.8 and Proposition 3.11, we know that if μ is subadditive then it satisfies the conditions (i) and (ii) in Theorem 4.1. Thus we obtain the next result.

Corollary 4.7. *Let X be a finite set. If μ is finite and subadditive, then for all $f \in \mathbf{F}$*

$$\int^{pan} f d\mu = \int^{cav} f d\mu.$$

Note 4.8. A monotone measure satisfying the conditions (i) and (ii) in Theorem 4.1 may not be subadditive. For example, the monotone measure μ in Example 3.6(2) satisfies the conditions (i) and (ii) in Theorem 4.1, but it is not subadditive. In fact, $\mu(X) > \mu(\{a, b\}) + \mu(\{c, d\})$.

The following example shows that the subadditivity in Corollary 4.7 is not a necessary condition.

Example 4.9. Let $X = \{1, 2, \cdots, n\}$ and $\mathcal{F} = \mathcal{P}(X)$ (the power set of X), μ be considered as in Example 3.5, i.e.,

$$\mu(E) = \begin{cases} 1 & \text{if } E = X, \\ 0 & \text{if } E \neq X. \end{cases}$$

Then, for all $f \in \mathbf{F}$

$$\int^{cav} f d\mu = \min\{f(x) \mid x \in X\} = \int^{pan} f d\mu.$$

But μ is not subadditive.

5. Concluding remarks

We have introduced the concept of minimal atom of a monotone measure (Definition 3.2) and, by means of characteristics of minimal atoms, obtained a necessary and sufficient condition that the concave integral coincides with the pan-integral w.r.t. arithmetic operations $+$ and $\cdot$ on finite spaces (Theorem 4.1).

In this study we have only considered the case that the underlying space X is finite. It should be pointed out that when X is an infinite space with finite σ-algebra $\mathcal{F}$ (i.e., $\mathcal{F}$ is finite as a set), then all of the results obtained in this paper remain true. However, when $\mathcal{F}$ is infinite, a monotone measure μ defined on $(X, \mathcal{F})$ may not have any minimal atom, as shown in the following examples. Thus, our approach based on minimal atoms does not apply to this situation.

Example 5.1. (1) Let $X = \mathbb{N}$ (the set of all positive integers) and $\mathcal{F} = 2^{\mathbb{N}}$. $\mu: \mathcal{F} \to [0,1]$ is defined by

$$\mu(E) = \begin{cases} 1 & \text{if } |E| = \infty \text{ and } 1 \in E, \\ 0 & \text{otherwise,} \end{cases}$$

where $|E|$ stands for the cardinality of E. Then each infinite subset E of $\mathbb{N}$ containing the element 1 is an atom of μ. However, since every infinite set has a proper infinite subset, there are no minimal atoms in this case.

(2) Let $X = [0,1]$ and $(X, \mathcal{B}, m)$ be the Borel measure space. If we define $\mu(A) = \sqrt{m(A)}$, then the monotone measure μ has no atom, thus it has no minimal atom.

In our further research, we will try to find some new methods in order to discuss the necessary and/or sufficient conditions such that the concave integral coincides with the pan-integral w.r.t. the usual addition $+$ and usual multiplication $\cdot$ on infinite spaces with infinite σ-algebra.

Finally, observe that the pan-integral [24,27] was established based on a special type of commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$. A related concept of generalizing Lebesgue integral based on a generalized ring $(\overline{R}_+, \oplus, \otimes)$ (the commutativity of $\otimes$ is not required), which is called *generalized Lebesgue integral*, was proposed and discussed in [28]. On the other hand, Mesiar et al. introduced *pseudo-concave integrals* [13] (see also [14]) and *pseudo-concave Benvenuti integrals* [8] by means of the pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$ of reals based on a generalized ring $(\overline{R}_+, \oplus, \otimes)$ (see also [1]). In next deeper study, we shall investigate the relationships among these three integrals based on a fixed generalized ring $(\overline{R}_+, \oplus, \otimes)$.

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