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A multilinear extension of a class of fuzzy bi-cooperative games[1]

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Abstract. In this paper, a new class of bi-cooperative games with fuzzy bi-coalitions is proposed in multilinear extension form. The extension is shown to be unique. The solution concept discussed in [3] is investigated and characterized for this class of games.

Keywords: Fuzzy sets, bi-cooperative games, bi-coalitions, LG value, fuzzy bi-coalitions

1. Introduction

In this paper, a new class of fuzzy bi-cooperative games (bi-cooperative games with fuzzy bi-coalitions) in multilinear extension form is proposed. The multilinear extension of a game [8] has been instrumental for easy computations of the Shapley like solutions for large games. In [7] a fuzzy cooperative game in multilinear extension form is defined. Here a similar approach to obtain a fuzzy bi-cooperative game is adopted however it breaks from the former due to the presence of bipolarity among players.

Bi-cooperative games á la Bilbao [2] consider problems arising from certain social and economic situations, where the players' set is divided into a partition of three groups viz., the group of the positive contributors, the negative contributors and the absentees. Each such partition can be uniquely represented by a pair of positive and negative contributors and is called a bi-coalition. A solution in this paradigm distributes the net payoff among the players that is accrued by the game. Labreuche and Grabisch [6] introduced the LG value for the class of crisp bi-cooperative games. Alternative solution concepts are found in [1, 4, 5]. However their difference is attributed to the two distinct orderings on the set of bi-coalitions namely, the product order and the order implied by monotonicity. Labreuche and Grabisch [6] highlighted this idea in details.

In a crisp bi-coalition the memberships of the players (rates of participation) can be considered as 1 for her full involvement as a positive or negative contributor and 0 for being indifferent or absentee.[2] Furthermore, it is possible to have players who participate partially in a bi-coalition, an idea similar to the notion of fuzzy cooperative games. In [3] a class of fuzzy bi-cooperative games and its solution concept (LG value) is defined. A particular class of bi-cooperative games namely the

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[2]A player in a crisp bi-coalition has three options : to join the group of positive contributors (denoted this by 1), the group of negative contributors (denoted by -1) or remain indifferent (denoted by 0). However if we only consider her rate of participation in the bi-coalition irrespective of her polarity, it ranges in [0, 1].

fuzzy bi-cooperative games in *Choquet integral* form is investigated and the solution is characterized with the LG axioms. A similar approach is adopted in this paper to obtain the LG value for the class of fuzzy bi-cooperative games in multilinear extension form.

The rest of the paper is organized as follows. Section 2 presents the notion of bi-cooperative games and a corresponding solution concept in both crisp and fuzzy environment. In Section 3, the notion of a fuzzy bi-cooperative game in multilinear extension form is introduced followed by an illustrative example. Finally, Section 4 brings some concluding remarks.

2. Model formulation

In this section, the basic definitions and results related to the development of the model is introduced. To a large extent, this section is based on [6] and [3]. Throughout the paper let $N = \{1, 2, 3, ..., n\}$ denote the players' set and

$$\mathcal{Q}(N) = \{(S, T) \mid S,\ T \subseteq N \text{ and } S \cap T = \emptyset\},$$

the set of all bi-coalitions of N. Let us consider the members of S in $(S, T) \in \mathcal{Q}(N)$ contribute positively and the members of T contribute negatively to the game. By what is called an abuse of notations we alternatively use i for the singleton set $\{i\}$, wherever there is no possibility of confusion.

2.1. Bi-cooperative games with crisp bi-coalitions

A bi-cooperative game is a pair (N, v) of which N is the players' set and $v : \mathcal{Q}(N) \to \mathbb{R}$, a real valued function such that $v(\emptyset, \emptyset) = 0$. Whenever N is fixed, (N, v) is replaced by v to simplify the notations. In [6] a value is defined on $\mathcal{BG}^N$, the class of all bi-cooperative games as a function $\Phi : \mathcal{BG}^N \to (\mathbb{R}^n)^{\mathcal{Q}(N)}$ which associates each bi-cooperative game v a vector $(\Phi_1(v), \Phi_2(v), ..., \Phi_n(v))$ representing a payoff distribution to the players in the game. There are two more definitions of a value found in the literature, see [1, 4, 5]. However, the definition given in [6] is seen to be more natural as it incorporates the bi-polar nature of the model. For a detailed discussion related to this idea, one can refer to [6].

Definition 2.1. Let $v \in \mathcal{BG}^N$. A player i is called *left monotone* with respect to v if

$$\forall (S, T) \in \mathcal{Q}(N \setminus i)\,, \quad v(S \cup i, T) \geq v(S, T).$$

A player i is *right monotone* with respect to v if

$$\forall (S, T) \in \mathcal{Q}(N \setminus i)\,, \quad v(S, T \cup i) \leq v(S, T).$$

The bi-cooperative game v is monotone if all players are left and right monotone with respect to v.

Remark 2.2. The expression $v(S \cup i, T) - v(S, T)$ (respectively $v(S, T) - v(S, T \cup i)$) is called the marginal contribution of player i with respect to $(S, T) \in \mathcal{Q}(N \setminus i)$ when she is a positive contributor (respectively a negative contributor).

Prior to the definition of the LG value of the class of crisp bi-cooperative games, define the following.

Definition 2.3. Let $(S, T) \in \mathcal{Q}(N)$ and $v \in \mathcal{BG}^N$. Player $i \in N$ is a null player for v, if it satisfies

$$v(S \cup i, T) = v(S, T) = v(S, T \cup i) \tag{1}$$

for every $(S, T) \in \mathcal{Q}(N \setminus i)$.

Now the LG value for the class $\mathcal{BG}^N$ is defined as follows.

Definition 2.4. A function $\Phi : \mathcal{BG}^N \to (\mathbb{R}^n)^{\mathcal{Q}(N)}$ defines the LG value if for every $(S, T) \in \mathcal{Q}(N)$ it satisfies the following axioms.

Axiom b1 (Efficiency) : If $v \in \mathcal{BG}^N$, it holds that,

$$\sum_{i \in N} \Phi_i(N, v)(S, T) = v(S, T)$$

Axiom b2 (Linearity) : For all $\alpha, \beta \in \mathbb{R}$, $b,\ v \in \mathcal{BG}^N$,

$$\Phi_i(N, \alpha b + \beta v)(S, T) = \alpha\ \Phi_i(N, b)(S, T) + \beta\ \Phi_i(N, v)(S, T).$$

Axiom b3 (Null Player Axiom) : If player i is null for $v \in \mathcal{BG}^N$, then $\Phi_i(N, v)(S, T) = 0$.

Axiom b4 (Intra-Coalition Symmetry) : If $v \in \mathcal{BG}^N$ and a permutation π is defined on N, such that $\pi S = S$ and $\pi T = T$, then it holds that, for all $i \in N$,

$$\Phi_{\pi i}(N, v \circ \pi^{-1})(S, T) = \Phi_i(N, v)(S, T)$$

where $\pi v(\pi L, \pi M) = v(L, M)$ and $\pi L = \{\pi i : i \in L\}$ for every $(L, M) \in \mathcal{Q}(N)$.

Axiom b5 (Inter-Coalition Symmetry) : Let $i \in S$ and $j \in T$, and $v_i,\ v_j$ be two bi-cooperative games such that for all $(S', T') \in \mathcal{Q}((S \cup T) \setminus \{i, j\})$,

$$v_i(S' \cup i, T') - v_i(S', T') = v_j(S', T') - v_j(S', T' \cup j)$$

$$v_i(S' \cup i, T' \cup j) - v_i(S', T' \cup j) = v_j(S' \cup i, T') - v_j(S' \cup i, T' \cup j)$$

Then,

$$\Phi_i(N, v_i)(S, T) = -\Phi_j(N, v_j)(S, T). \quad (2)$$

Axiom b6 (Monotonicity): Given $v,\ v' \in \mathcal{BG}^N$ such that $\exists i \in N$ with

$$v'(S', T') = v(S', T') \quad (3)$$

$$v'(S' \cup i, T') \geq v(S' \cup i, T') \quad (4)$$

$$v'(S', T' \cup i) \geq v(S', T' \cup i) \quad (5)$$

for all $(S', T') \in \mathcal{Q}(N \setminus i)$, then

$$\Phi_i(N, v')(S, T) \geq \Phi_i(N, v)(S, T).$$

In [6], an intuitive explanation about axioms *b1-b6* is given in details. In addition to the standard Shapley axioms viz., Efficiency (*b1*), Linearity (*b2*) and Null Player (*b3*), here we have the intra and inter-coalition symmetry axioms ((*b4*) and (*b5*) respectively) that take care of the anonymity of the players in two different ways. The intra-coalition symmetry axiom suggests that the role of the players of S, T and $N \setminus (S \cup T)$ is different. Thus symmetry holds only among players of S, players of T and players of $N \setminus (S \cup T)$. On the other hand the inter-coalition symmetry axiom tells that when the contribution of a player $i \in S$ to a game v_i is exactly the opposite of that of a player $j \in T$ to a game v_j, then the incentive payoff for i shall be exactly the opposite of the payoff for j. The monotonicity axiom (*b6*) says that the value to a player for a larger game can not be less than that of the smaller game.

The following theorem ensures existence and uniqueness of the LG value.

Theorem 2.5. *There exists a unique value $\Phi(N, v)(S, T)$ on $\mathcal{BG}^N$ for $(S, T) \in \mathcal{Q}(N)$ that satisfies Axiom (b1)- Axiom (b6) and is given by,*

$$\Phi_i(N, v)(S, T) = \sum_{K \subseteq (S \cup T) \setminus \{i\}} \frac{k!(s+t-k-1)!}{(s+t)!} [V(K \cup \{i\}) - V(K)] \quad (6)$$

for all $i \in N$ where for $K \subseteq S \cup T$, $V(K) := v(S \cap K, T \cap K)$. Moreover, if $i \in N \setminus (S \cup T)$, $\Phi_i(N, v)(S, T) = 0$.

An important corollary to Theorem 2.5 given in [6] is as follows.

Result 2.6. We have,

$$\forall i \in N \setminus (S \cup T), \quad \Phi_i(v)(S, T) = 0$$

$$\forall i \in S, \text{ with } i \text{ left monotone}, \Phi_i(v)(S, T) \geq 0 \quad (7)$$

$$\forall i \in T, \text{ with } i \text{ right monotone}, \Phi_i(v)(S, T) \leq 0$$

2.2. Bi-cooperative games with fuzzy bi-coalitions

Let $N = \{1, 2, ..., n\}$ be given. A fuzzy bi-coalition is an expression A on N given by,

$$A = \{< i, \mu_A^N(i), \nu_A^N(i) > \ | \ \min(\mu_A^N, \nu_A^N) = 0\}$$

where $\mu_A^N : N \to [0, 1]$, $\nu_A^N : N \to [0, 1]$ represent respectively, the membership functions over N of the fuzzy sets of positive and negative contributors of A. Note that the minimum condition in the above definition implies that the two roles (positive and negative contributions) are mutually exclusive so that one can not simultaneously put her partial participations in both of them, see [3].

Thus it follows that the functions μ_A^N and ν_A^N fully identify the fuzzy bi-coalition A of N. As N is fixed here, μ_A^N and ν_A^N can be replaced by μ_A and ν_A. Player i is a positive contributor in A if $\mu_A(i) > 0$ and a negative contributor if $\nu_A(i) > 0$. Let $\mathcal{FB}(N)$ denote the set of all fuzzy bi-coalitions on N. Every crisp bi-coalition can also be considered as a fuzzy bi-coalition with each player participating fully (with membership 1) or abstaining (with membership 0). Thus with an abuse of notations, write $\mathcal{Q}(N) \subseteq \mathcal{FB}(N)$.

For comparing the fuzzy bi-coalitions $A, B \in \mathcal{FB}(N)$, the following operations and relations are adopted.

$$A \preceq B \Leftrightarrow \mu_A(i) \leq \mu_B(i) \text{ and } \nu_A(i) \leq \nu_B(i)\ \forall i \in N.$$

$$A = B \Leftrightarrow \mu_A(i) = \mu_B(i) \text{ and } \nu_A(i) = \nu_B(i)\ \forall i \in N.$$

For any $A \in \mathcal{FB}(N)$, denote by $\mathcal{FB}(A)$, the set of all fuzzy bi-coalitions B such that $B \preceq A$.

The intersection of two fuzzy bi-coalitions A and B is obtained using the minimum operator '$\wedge$' as follows.

$$A \cap B = \{< i, \mu_A(i) \wedge \mu_B(i), \nu_A(i) \wedge \nu_B(i) >: i \in N\}$$

It follows that under the above mentioned ordering relation $\mathcal{FB}(N)$ is an inf-semilattice. However, the union is defined only on a restricted sub-domain of $\mathcal{FB}(N)$. For $A, B \in \mathcal{FB}(N)$ such that $\{\mu_A(i) \vee \mu_B(i)\} \bigwedge \{\nu_A(i) \vee \nu_B(i)\} = 0$, $\forall i \in N$, $A \cup B$ is defined as follows.

$$A \cup B = \{< i, \mu_A(i) \vee \mu_B(i), \nu_A(i) \vee \nu_B(i) >: i \in N\} \quad (8)$$

The Support of a fuzzy bi-coalition A, denoted by $Supp(A)$, is given by

$$Supp(A) = (\{i \in N | \mu_A(i) > 0\}, \{i \in N | \nu_A(i) > 0\}) \quad (9)$$

Note that, $Supp(A) \in \mathcal{Q}(N)$.

Definition 2.7. The null fuzzy bi-coalition $\emptyset_B$ is given by

$$\emptyset_B = \{< i, \mu_{\emptyset_B}(i), \nu_{\emptyset_B}(i) > | i \in N\}$$

where $\mu_{\emptyset_B}(i) = 0$, and $\nu_{\emptyset_B}(i) = 0 \ \ \forall i \in N$.

Thus a bi-cooperative game with fuzzy bi-coalitions is defined as follows.

Definition 2.8. A bi-cooperative game with fuzzy bi-coalitions (a fuzzy bi-cooperative game in short) is a function $w : \mathcal{FB}(N) \to \mathbb{R}$ with $w(\emptyset_B) = 0$. We call the value $w(A)$ the worth of A due to the fuzzy or partial contributions by the members of N.

The worth $w(A)$ for every $A \in \mathcal{FB}(N)$ is interpreted as the gain (whenever $w(A) > 0$) or loss (whenever $w(A) < 0$) that A can receive when the players participate in it in either of the three distinct capacities : positive, negative or absentees. Denote by $\mathcal{G}_{\mathcal{FB}}(N)$ the class of all fuzzy bi-cooperative games. It follows that the class $\mathcal{BG}^N$, of crisp bi-cooperative games is a subclass of the class $\mathcal{G}_{\mathcal{FB}}(N)$ of fuzzy bi-cooperative games.

Definition 2.9. Let $w \in \mathcal{G}_{\mathcal{FB}}(N)$. Player $i \in N$ is called left monotone in fuzzy sense with respect to w if for every $A, B \in \mathcal{FB}(N)$ such that $\mu_A(i) > \mu_B(i)$ with $\mu_A(j) = \mu_B(j)$ and $\nu_A(j) = \nu_B(j)$ for $i \neq j \in N$, it follows that $w(A) \geq w(B)$. Similarly, player i is right monotone in fuzzy sense with respect to w if for every $A, B \in \mathcal{FB}(N)$ such that $\nu_A(i) > \nu_B(i)$ with $\mu_A(j) = \mu_B(j)$ and $\nu_A(j) = \nu_B(j)$ for $i \neq j \in N$, it follows that $w(A) \leq w(B)$.

The game $w \in \mathcal{G}_{\mathcal{FB}}(N)$ is fuzzy monotone if every player is both left and right monotone in fuzzy sense.

Let us define the LG value for a fuzzy bi-cooperative game. Preparatory to this, following definitions due to [3] are given.

Definition 2.10. If $A \in \mathcal{FB}(N)$, and $w \in \mathcal{G}_{\mathcal{FB}}(N)$, the player $i \in N$ is said to be a null player for w in A if $w(B \cup I) = w(B)$ for all $B \in \mathcal{FB}(A)$ with $\mu_B(i) = \nu_B(i) = 0$ and all $I \in \mathcal{FB}(N)$ such that $\mu_I(j) = \nu_I(j) = 0$ when $j \neq i$, where the union $\cup$ is defined in (8).

Definition 2.11. Let $A \in \mathcal{FB}(N)$. For any permutation π on N, define the fuzzy bi-coalition πA by

$$\mu_{\pi A}(i) = \mu_A(\pi^{-1} i) \quad (10)$$

$$\nu_{\pi A}(i) = \nu_A(\pi^{-1} i) \quad (11)$$

Then πA is called a permutation of the fuzzy bi-coalition A.

The LG value for fuzzy bi-cooperative games is defined as follows.

Definition 2.1. A function $\Phi : \mathcal{G}_{\mathcal{FB}}(N) \to (\mathbb{R}^n)^{\mathcal{FB}(N)}$ is said to be the LG value on $\mathcal{G}_{\mathcal{FB}}(N)$ if it satisfies the following six axioms:

Axiom f1 (Efficiency): If $w \in \mathcal{G}_{\mathcal{FB}}(N)$ and $A \in \mathcal{FB}(N)$, then

$$\sum_{i \in N} \Phi_i(w)(A) = w(A).$$

Axiom f2 (Linearity): For $\alpha, \beta \in \mathbb{R}$ and $w, w' \in \mathcal{G}_{\mathcal{FB}}(N)$ it follows that $\Phi(\alpha w + \beta w') = \alpha \Phi(w) + \beta \Phi(w')$,

Axiom f3 (Null Player Axiom): If player $i \in N$ is a null player for $w \in \mathcal{G}_{\mathcal{FB}}(N)$, in $A \in \mathcal{FB}(N)$, then, $\Phi_i(w)(A) = 0$.

Axiom f4 (Intra coalition Symmetry): For any $w \in \mathcal{G}_{\mathcal{FB}}(N)$, and a permutation π, defined on N such that $\pi A = A$, it holds for all $i \in N$,

$$\Phi_i(w)(A) = \Phi_{\pi i}(\pi w)(A) \quad (12)$$

where $\pi w \in \mathcal{G}_{\mathcal{FB}}(N)$ is defined by $\pi w(\pi A) = w(A)$, with πA defined as in Definition 2.11.

Axiom f5 (Inter coalition Symmetry): Given $A \in \mathcal{FB}(N)$ and $i, j \in N$, if w_i and w_j are two bi-cooperative games with fuzzy bi-coalitions such that for every $B \in \mathcal{FB}(A)$ with $i, j \notin Supp(B)$ (i.e., $\mu_B(i) = 0 = \mu_B(j)$ and $\nu_B(i) = 0 = \nu_B(j)$), and every pair of $I, J \in \mathcal{FB}(N)$, such that $\mu_I(i) = \nu_J(j) > 0$ or $\mu_J(j) = \nu_I(i) > 0$ and $\mu_I(k) = \mu_J(k) = 0 = \nu_I(k) = \nu_J(k) \ \forall \ k \in N \setminus \{i, j\}$, it holds that,

$$\begin{aligned} & w_i(B \cup I) - w_i(B) \\ & \quad = w_j(B) - w_j(B \cup J) \\ & w_i(B \cup I \cup J) - w_i(B \cup J) \\ & \quad = w_j(B \cup I) - w_j(B \cup I \cup J) \end{aligned}$$

then, $\Phi_i(w_i)(A) = -\Phi_j(w_j)(A)$.

Axiom f6 (Monotonicity): Let w and w' be two bi-cooperative games with fuzzy bi-coalitions and $A \in \mathcal{FB}(N)$. Let further that there exists an $i \in N$ such that for every $I \in \mathcal{FB}(N)$ with $\mu_I(i) > 0$ or $\nu_I(i) > 0$, $\mu_I(j) = \nu_I(j) = 0$, $\forall j \neq i$, and for all $B \in \mathcal{FB}(A)$ such that $\mu_B(i) = \nu_B(i) = 0$, it holds that,

$$\begin{aligned} & w'(B) = w(B) \\ & w'(B \cup I) \geq w(B \cup I) \\ \text{then,} \quad & \Phi_i(w')(A) \geq \Phi_i(w)(A). \end{aligned}$$

In [3] it is remarked that it is easy to see that if Φ satisfies Axioms *f1-f6* then its restriction to the class of crisp bi-cooperative games satisfies Axioms *b1-b6*. Thus we can recover the crisp value from its fuzzy counterpart under restriction of its domain. As a matter of fact all the above axioms are obtained intuitively from their crisp analogues by generalizing the idea of participation of players. For example, *Axiom f5*(an analogue to *Axiom b5*), says that when contribution of a player i to a game w_i, is exactly opposite of that of player j, to a game w_j, (i and j having equal rates of participations) then i's payoff will be exactly opposite to the one for j. Similarly, *Axiom f6* implies that if i provides some positive contribution to $B \in \mathcal{FB}(A)$, and the added value for w' is greater than that for w or if i provides some negative contribution to $B \in \mathcal{FB}(A)$, and the negative added value for w' is lesser than that for w in absolute value, then its payoff due to w' can not be lesser than the one due to w. This establishes a well deserved link between the crisp and fuzzy frameworks pertaining to bi-cooperative games. Furthermore, the definition above can be adopted for any class of bi-cooperative games with fuzzy bi-coalitions. Following section deals with the notion of fuzzy bi-cooperative games in multilinear extension form and their properties.

3. Fuzzy bi-cooperative games in multilinear extension form

Given $v \in \mathcal{BG}^N$, define a fuzzy bi-cooperative game $w : \mathcal{FB}(N) \to \mathbb{R}$ in multilinear extension form as follows.

$$\begin{aligned} w(A) = & \prod_{i \in L} \mu_A(i) \prod_{j \in M} (1 - \nu_A(j)) v(L, \emptyset) \\ & + \prod_{i \in M} \nu_A(i) \prod_{j \in L} (1 - \mu_A(j)) v(\emptyset, M) \\ & + \sum_{(S,T) \in \mathcal{Q}^*(L,M)} \gamma(A) v(S, T) \end{aligned} \tag{13}$$

for every $A \in \mathcal{FB}(N)$ with $Supp(A) = (L, M) \in \mathcal{Q}(N)$ and $\mathcal{Q}^*(L, M) = \{(S, T) : S \subseteq L, T \subseteq M, (S, T) \notin \{(L, \emptyset), (\emptyset, M)\}\}$ and where,

$$\gamma(A) = \left\{ \prod_{i \in S} \mu_A(i) \prod_{j \in L \setminus S} (1 - \mu_A(j)) \right. \left. \prod_{i \in T} \nu_A(i) \prod_{j \in M \setminus T} (1 - \nu_A(j)) \right\} \tag{14}$$

Denote by $\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)$ the class of fuzzy bi-cooperative games in multilinear extension form. Observe that w restricted to $\mathcal{BG}^N$ gives the crisp bi-cooperative game v. More precisely we have the following.

For $(S, T) \in \mathcal{Q}(N)$, define a fuzzy bi-coalition $A^{(S,T)}$ as follows.

$$\mu_{A^{(S,T)}}(j) = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{if } j \notin S \end{cases} \tag{15}$$

$$\nu_{A^{(S,T)}}(j) = \begin{cases} 1 & \text{if } j \in T \\ 0 & \text{if } j \notin T \end{cases} \tag{16}$$

It is easy to see that $A^{(S,T)}$ represents the crisp bi-coalition (S, T). In the following uniqueness of the above multilinear extension is shown in the line of Owen [8].

Theorem 2.1. *The fuzzy bi-cooperative game* $w : \mathcal{FB}(N) \to \mathbb{R}$ *given by (13) is uniquely defined in the sense that if* v_1, v_2 *are two distict games then the corresponding fuzzy bi-cooperative games in multilinear extension form are also distinct .*

Proof. Observe that for a particular fuzzy bi-coalition A, we have exactly one of the following mutually exclusive cases.

Case (a): $L \neq \emptyset$, $M = \emptyset$; it follows from (13)

$$w(A) = \prod_{i \in L} \mu_A(i) \prod_{j \in M} (1 - \nu_A(j)) v(L, \emptyset)$$

Case (b): $L = \emptyset$, $M \neq \emptyset$; thus (13) becomes,

$$w(A) = \prod_{i \in M} \nu_A(i) \prod_{j \in L} (1 - \mu_A(j)) v(\emptyset, M)$$

Case (c): $L \neq \emptyset$, $M \neq \emptyset$; thus (13) becomes,

$$w(A) = \sum_{(S,T) \in Q^*(L,M)} \gamma(A) v(S, T)$$

where $\gamma(A)$ is given by (14). Let w have the form

$$w(A) = \sum_{(S,T) \in Q^*(L,M)} C_{(S,T)} \left\{ \prod_{i \in S} \mu_A(i) \prod_{i \in T} \nu_A(i) \right\}$$

where $C_{(S,T)}$ are constants. Then for every $(S, T) \in \mathcal{Q}(N)$, it follows from (15) and (16)

$$w(A^{(S,T)}) = \sum_{(S',T') \subseteq (S,T)} C_{(S',T')}$$

so that the condition $w(A^{(S,T)}) = v(S, T)$ reduces to

$$w(A^{(S,T)}) = \sum_{(S',T') \subseteq (S,T)} C_{(S',T')}$$

Now the proof follows exactly in the same way as of Owen ([8], pg-79). □

In what follows we define a function $\Phi : \mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N) \to (\mathbb{R}^n)^{\mathcal{FB}(N)}$ and show that it is the LG value for the class $\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)$. Define $\Phi : \mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N) \to (\mathbb{R}^n)^{\mathcal{FB}(N)}$ by

$$\begin{aligned}
\Phi_i(w)(A) = & \sum_{K \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} \times \prod_{t \in L} \mu_A(t) \prod_{r \in M} (1 - \nu_A(r)) \\
& [v(L \cap (K \cup i), \emptyset) - v(L \cap K, \emptyset)] \\
& + \sum_{K \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} \times \prod_{t \in M} \nu_A(t) \prod_{r \in L} (1 - \mu_A(r)) \\
& [v(\emptyset, M \cap (K \cup i)) - v(\emptyset, M \cap K)] \\
& + \sum_{K \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} \times \\
& \sum_{(S_0,T_0) \in Q_1^*} \left\{ \prod_{t \in S_0} \mu_A(t) \prod_{r \in L \setminus S_0} (1 - \mu_A(r)) \prod_{t \in T_0} \nu_A(t) \prod_{r \in M \setminus T_0} (1 - \nu_A(r)) \right\} v(S_0, T_0) \\
& - \sum_{(S_0',T_0') \in Q_2^*} \left\{ \prod_{t \in S_0'} \mu_A(t) \prod_{r \in L \setminus S_0'} (1 - \mu_A(r)) \prod_{t \in T_0'} \nu_A(t) \prod_{r \in M \setminus T_0'} (1 - \nu_A(r)) \right\} v(S_0', T_0')
\end{aligned} \tag{17}$$

where, $Q_1^* = Q^*(L \cap (K \cup i), M \cap (K \cup i))$ and $Q_2^* = Q^*(L \cap K, M \cap K)$. An equivalent expression of (17) is given below.

$$\begin{aligned}
\Phi_i(w)(A) = & \sum_{K \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} \times \prod_{t \in L} \mu_A(t) \prod_{r \in M} (1 - \nu_A(r)) \\
& [v(L \cap (K \cup i), \emptyset) - v(L \cap K, \emptyset)] \\
& + \sum_{K \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} \times \prod_{t \in M} \nu_A(t) \prod_{r \in L} (1 - \mu_A(r)) \\
& [v(\emptyset, M \cap (K \cup i)) - v(\emptyset, M \cap K)] \\
& + \sum_{(S,T) \in Q^*(L,M)} \{ \prod_{t \in S} \mu_A(t) \prod_{r \in L \setminus S} (1 - \mu_A(r)) \prod_{t \in T} \nu_A(t) \prod_{r \in M \setminus T} (1 - \nu_A(r)) \} \Phi_i(v)(S, T)
\end{aligned} \tag{18}$$

where $\Phi_i(v)$ on the right hand side of (18) is the i-th component of the LG value for the corresponding crisp bi-cooperative game v. Since for a particular fuzzy bi-coalition A, exactly one of the following mutually exclusive cases : Case (a): $L = \emptyset$, $M \neq \emptyset$; Case (b): $L \neq \emptyset$, $M = \emptyset$ and Case (c): $L \neq \emptyset$, $M \neq \emptyset$ occurs therefore exactly one of the three components on the right side of (17) is non-zero. In what follows, a second alternative expression of (17) is developed. Preparatory to this, define the following.

$$\mu_{A_K}(j) = \mu_A(j) \bigwedge \chi_K(j), \quad \forall j \in N \tag{19}$$

$$\nu_{A_K}(j) = \nu_A(j) \bigwedge \chi_K(j), \quad \forall j \in N \tag{20}$$

where χ_K is the characteristic (membership) function of K given by

$$\chi_K(j) = \begin{cases} 1 & \text{if } j \in K \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

It follows that

$$\begin{aligned} & Supp(A_K) && (22) \\ &= \big(\{j \in N\colon \mu_{A_K}(j) > 0\}, \{j \in N\colon \nu_{A_K}(j) > 0\}\big) \\ &= (L \cap K, M \cap K) && (23) \end{aligned}$$

Similarly, for $I \in \mathcal{FB}(N)$ such that

$$\mu_I(i) = \mu_A(i), \ \ \nu_I(i) = \nu_A(i) \tag{24}$$

$$\mu_I(j) = \nu_I(j) = 0, \ \text{if} \ \ i \neq j \tag{25}$$

we have *Supp* $(A_K \cup I) = (L \cap (K \cup i), M \cap (K \cup i))$ which follows from the facts that

$$\mu_{(A_K \cup I)}(j) = \mu_A(j) \bigwedge \chi_{K \cup \{i\}}(j) \tag{26}$$

$$\nu_{(A_K \cup I)}(j) = \nu_A(j) \bigwedge \chi_{K \cup \{i\}}(j) \tag{27}$$

Using (19) through (27),

$$\begin{aligned} & w(A_K \cup I) \\ &= \prod_{l \in L} \mu_A(l) \prod_{j \in M} (1 - \nu_A(j)) v(L \cap (K \cup i), \emptyset) \\ &\quad + \prod_{l \in M} \nu_A(l) \prod_{j \in L} (1 - \mu_A(j)) v(\emptyset, M \cap (K \cup i)) \\ &\quad + \sum_{(S_0, T_0) \in Q_1^*} \left\{ \prod_{l \in S_0} \mu_A(l) \prod_{j \in L \setminus S_0} (1 - \mu_A(j)) \right. \\ &\quad \left. \prod_{l \in T_0} \nu_A(l) \prod_{j \in M \setminus T_0} (1 - \nu_A(j)) \right\} v(S_0, T_0) && (28) \end{aligned}$$

$$\begin{aligned} w(A_K) &= \prod_{l \in L} \mu_A(l) \prod_{j \in M} (1 - \nu_A(j)) v(L \cap K, \emptyset) \\ &\quad + \prod_{l \in M} \nu_A(l) \prod_{j \in L} (1 - \mu_A(j)) \, v(\emptyset, M \cap K) \\ &\quad + \sum_{(S_0, T_0) \in Q_2^*} \left\{ \prod_{l \in S_0} \mu_A(l) \prod_{j \in L \setminus S_0} (1 - \mu_A(j)) \right. \\ &\quad \left. \prod_{l \in T_0} \nu_A(l) \prod_{j \in M \setminus T_0} (1 - \nu_A(j)) \right\} v(S_0, T_0) && (29) \end{aligned}$$

Therefore using (19) through (29), expression (17) can be simplified to the following.

$$\begin{aligned} & \Phi_i(w)(A) \\ &= \sum_{K \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l + m - k - 1)!}{(l + m)!} \\ &\quad \{w(A_K \cup I) - w(A_K)\} && (30) \end{aligned}$$

Our main theorem stands as follows.

Theorem 3.2. *The function $\Phi : \mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N) \to (\mathbb{R}^n)^{\mathcal{Q}(N)}$ given by (17) (or equivalently by (30)) is the unique LG-value for the class of fuzzy bi-cooperative games in multilinear extension form given by (13).*

Proof. The proof includes two parts: Φ given by (17) (or equivalently (30)) is an LG value for $\mathcal{G}^{\mathrm{m}}_{\mathcal{FB}}(N)$ and it is unique.

The uniqueness of Φ follows from the uniqueness of the LG value of the corresponding crisp bi-cooperative game and that of the multilinear extension followed by the axiom of linearity. Thus it is enough to show that Φ is an LG value i.e., Φ satisfies the LG-axioms given by *Axiom* $f1 - f6$. Let us proceed as follows.

Axiom f1 (Efficiency).

Let us consider $L \cup M = \{i_1, i_2, i_3, ... i_p\}$ in (**??**). It is worth mentioning here that the roles of the players (positive or negative) are not important in the formulation however it does not affect their bipolarity.

For every i_j $(j = 1, 2, ... p)$ and $K \subseteq (L \cup M) \setminus \{i_j\}$, in the following all the possible $(K \cup i_j)s$ (where $K \cup i_j = (L \cap (K \cup i_j)) \cup (M \cap (K \cup i_j)))$ have been listed as entries in a matrix P_k of p clusters with $k + 1$ repeated entries.

$$P_k = \left(A_k^1 \vdots A_k^2 \vdots \ldots \vdots A_k^p \right)$$

$$A_k^1 = \begin{pmatrix} \{i_2, \cdots i_{k+1}, i_1\} & \cdots \{i_{p-(k-1)}, \cdots i_p, i_1\} \\ \{i_2, \cdots i_{k+2}, i_1\} & \cdots \times \\ \vdots & \vdots \\ \{i_2, \cdots i_{p-1}, i_1\} & \times \\ \{i_2, \cdots i_p, i_1\} & \cdots \end{pmatrix}$$

$$A_k^2 = \begin{pmatrix} \{i_1, \cdots i_{k+1}, i_2\} & \cdots \{i_{p-(k-1)}, \cdots i_p, i_2\} \\ \{i_1, i_4 \cdots i_{k+2}, i_2\} & \{i_4, \cdots i_{k+4}, i_2\} \\ \vdots & \vdots \\ \{i_1, \cdots i_{p-1}, i_2\} & \\ \{i_1, \cdots i_p, i_2\} & \times \end{pmatrix}$$

$$\begin{matrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

$$A_k^p = \begin{pmatrix} \{i_1, \cdots i_k, i_p\} & \cdots \{i_{p-k}, \cdots i_p\} \\ \{i_1, \cdots i_{k+1}, i_p\} & \{i_3, \cdots i_p\} \\ \vdots & \vdots \\ \{i_1, \cdots i_{p-2}, i_p\} & \times \\ \{i_2, i_{p-(k-1)} \cdots i_{p-1}, i_p\} & \times \end{pmatrix}$$

In a similar way, for every $i_j (j = 1, 2, \ldots p)$ and $K \subseteq (L \cup M) \setminus i_j$, all the possible K s $(K = (L \cap K) \cup (M \cap K))$ have been listed as entries in a matrix Q_k of p clusters with $p - k$ repeated entries as follows.

$$Q_k = \left(B_k^1 \vdots B_k^2 \vdots \ldots \vdots B_k^p \right)$$

where the entries of $B_k^l (l = 1, 2, \ldots, p)$ are exactly same as those of the corresponding A_k^l with only the exception that they do not contain $i_l (l = 1, 2, \ldots, p)$.
Since the j^{th} cluster of Q_{k+1} corresponds to the j^{th} cluster of P_k, it follows from (17)

$$\sum_{i_j \in (L \cup M)} \Phi_{i_j}(w)(A)$$

$$= \sum_{k=0}^{p-1} \left(\sum_{i_j \in (L \cup M)} C_k^{i_j} \right) - \sum_{k=0}^{p-1} \left(\sum_{i_j \in (L \cup M)} D_k^{i_j} \right) \quad (31)$$

where

$$C_k^{i_j} = \sum_{K:|K|=k, i_j \notin K} \frac{k!(p-k-1)!}{p!}$$

$$\times \left[\prod_{i_t \in L} \mu_A(i_t) \prod_{i_r \in M} (1 - \nu_A(i_r)) v(L \cap (K \cup i_j), \emptyset) \right.$$

$$+ \prod_{i_t \in M} \nu_A(i_t) \prod_{i_r \in L} (1 - \mu_A(i_r)) v(\emptyset, M \cap (K \cup i_j))$$

$$+ \sum_{(S_0, T_0) \in Q^*(L \cap (K \cup \{i_j\}), M \cap (K \cup \{i_j\}))} \left\{ \prod_{i_t \in S_0} \mu_A(i_t) \right.$$

$$\prod_{i_r \in L \setminus S_0} (1 - \mu_A(i_r)) \prod_{i_t \in T_0} \nu_A(i_t)$$

$$\left. \left. \prod_{i_r \in M \setminus T_0} (1 - \nu_A(i_r)) \right\} v(S_0, T_0) \right]$$

$$D_k^{i_j} = \sum_{\substack{K:|K|=k \\ i_j \notin K}} \frac{k!(p-k-1)!}{p!}$$

$$\times \left[\prod_{i_t \in L} \mu_A(i_t) \prod_{i_r \in M} (1 - \nu_A(i_r)) v(L \cap K, \emptyset) \right.$$

$$+ \prod_{i_t \in M} \nu_A(i_t) \prod_{i_r \in L} (1 - \mu_A(i_r)) v(\emptyset, M \cap K)$$

$$+ \sum_{(S_0', T_0') \in Q^*(L \cap K,\ M \cap K)} \left\{ \prod_{i_t \in S_0'} \mu_A(i_t) \right.$$

$$\prod_{i_r \in L \setminus S_0'} (1 - \mu_A(i_r)) \prod_{i_t \in T_0'} \nu_A(i_t)$$

$$\left. \left. \prod_{i_r \in M \setminus T_0'} (1 - \nu_A(i_r)) \right\} v(S_0', T_0') \right]$$

Now

$$\sum_{i_j \in (L \cup M)} C_k^{i_j}$$

$$= \sum_{i_j \in (L \cup M)} \sum_{K:|K|=k, i_j \notin K} \frac{k!(p-k-1)!}{p!} \times$$

$$\left[\prod_{i_t \in L} \mu_A(i_t) \prod_{i_r \in M} (1 - \nu_A(i_r)) v(L \cap (K \cup i_j), \emptyset) \right.$$

$$+ \prod_{i_t \in M} \nu_A(i_t) \prod_{i_r \in L} (1 - \mu_A(i_r)) v(\emptyset, M \cap (K \cup i_j))$$

$$+ \sum_{(S_0, T_0) \in Q^*(L \cap (K \cup \{i_j\}), M \cap (K \cup \{i_j\}))} \left\{ \prod_{i_t \in S_0} \mu_A(i_t) \right.$$

$$\prod_{i_r \in L \setminus S_0} (1 - \mu_A(i_r)) \prod_{i_t \in T_0} \nu_A(i_t)$$

$$\left. \left. \prod_{i_r \in M \setminus T_0} (1 - \nu_A(i_r)) \right\} v(S_0, T_0) \right] \quad (32)$$

$$\sum_{i_j\in(L\cup M)} D_k^{i_j}$$

$$= \sum_{i_j\in(L\cup M)} \sum_{K:|K|=k, i_j\notin K} \frac{k!(p-k-1)!}{p!} \times$$

$$\Bigg[\prod_{i_t\in L} \mu_A(i_t) \prod_{i_r\in M} (1-\nu_A(i_r)) v(L\cap K, \emptyset)$$

$$+ \prod_{i_t\in M} \nu_A(i_t) \prod_{i_r\in L} (1-\mu_A(i_r)) v(\emptyset, M\cap K)$$

$$+ \sum_{(S_0', T_0')\in Q^*(L\cap K,\ M\cap K)} \Bigg\{ \prod_{i_t\in S_0'} \mu_A(i_t)$$

$$\prod_{i_r\in L\setminus S_0'} (1-\mu_A(i_r)) \prod_{i_t\in T_0'} \nu_A(i_t)$$

$$\prod_{i_r\in M\setminus T_0'} (1-\nu_A(i_r)) \Bigg\} v(S_0', T_0')\Bigg] \tag{33}$$

It follows from the matrix P_k and Q_k the entries of P_k and Q_{k+1} are identical whereas each entry of P_k is repeated $k+1$ times and each entry of Q_k is repeated $p-k$ times. In view of this along with (32) - (33) and noting that the expressions of $\sum_{i_j\in(L\cup M)} C_{k-1}^{i_j}$ and $\sum_{i_j\in(L\cup M)} D_k^{i_j}$ are identical for $k = 1, 2, \cdots p$, we see that in (31) the corresponding elements cancel each other. It follows that,

$$\sum_{i_j\in(L\cup M)} \Phi_{i_j}(w)(A) = \sum_{i_j\in(L\cup M)} C_{p-1}^{i_j}$$

$$= \prod_{i_t\in L} \mu_A(i_t) \prod_{i_r\in M} (1-\nu_A(i_r)) v(L\cap(K\cup i_j), \emptyset)$$

$$+ \prod_{i_t\in M} \nu_A(i_t) \prod_{i_r\in L} (1-\mu_A(i_r)) v(\emptyset, M\cap(K\cup i_j))$$

$$+ \sum_{(S_0,T_0)\in Q^*(L\cap(K\cup\{i_j\}), M\cap(K\cup\{i_j\}))} \Bigg\{ \prod_{i_t\in S_0} \mu_A(i_t)$$

$$\prod_{i_r\in L\setminus S_0} (1-\mu_A(i_r)) \prod_{i_t\in T_0} \nu_A(i_t)$$

$$\prod_{i_r\in M\setminus T_0} (1-\nu_A(i_r)) \Bigg\} v(S_0, T_0)$$

It follows from the fact that, when $|K| = p-1$ and $i_j \notin K$, $Q^*(L\cap(K\cup\{i_j\}), M\cap(K\cup\{i_j\})) = Q^*(L, M)$,

$$\sum_{i_j\in(L\cup M)} \Phi_{i_j}(w)(A)$$

$$= \prod_{i_t\in L} \mu_A(i_t) \prod_{i_r\in M} (1-\nu_A(i_r)) v(L, \emptyset)$$

$$+ \prod_{i_t\in M} \nu_A(i_t) \prod_{i_r\in L} (1-\mu_A(i_r)) v(\emptyset, M)$$

$$+ \sum_{(S_0,T_0)\in Q^*(L,M)} \Bigg\{ \prod_{i_t\in S_0} \mu_A(i_t)$$

$$\prod_{i_r\in L\setminus S_0} (1-\mu_A(i_r))$$

$$\prod_{i_t\in T_0} \nu_A(i_t) \prod_{i_r\in M\setminus T_0} (1-\nu_A(i_r)) \Bigg\} v(S_0, T_0)$$

$$= \quad w(A).$$

$N = \{i_1, i_2, ..., i_p\}$ being arbitrary the result holds for every N. This proves efficiency of Φ.

Axiom f2 (Linearity). This is obvious.

Axiom f3 (Null Player Axiom). Let $i \in N$ be a null player. Then by definition 2.10, for every $A \in \mathcal{FB}(N)$, we have $w(B\cup I) = w(B), \forall B \in \mathcal{FB}(A)$ with $\mu_B(i) = \nu_B(i) = 0$ and $\forall I \in \mathcal{FB}(N)$ such that $\mu_I(j) = \nu_I(j) = 0$, when $j \neq i$. It follows from (30),

$$\Phi_i(w)(A) = \sum_{K\subseteq(L\cup M)\setminus\{i\}} \frac{k!(l+m-k-1)!}{(l+m)!}$$

$$(w(A_K\cup I) - w(A_K)) = \ 0$$

Axiom f4 (Intra Coalition Symmetry). It follows from (13) that for any $A \in \mathcal{FB}(N)$, $w \in \mathcal{G}_{\mathcal{FB}}^{\mathrm{m}}(N)$ and a permutation π such that $\pi A = A$,

$$\Phi_{\pi i}(\pi w)(\pi A)$$

$$= \sum_{\pi K\subseteq\pi(L\cup M)\setminus\{\pi i\}} \frac{k!(l+m-k-1)!}{(l+m)!}$$

$$\{\pi w(\pi A_{\pi K}\cup(\pi I)) - \pi w(\pi A_{\pi K})\}$$

$$= \sum_{K'\subseteq(L\cup M)\setminus\{i\}} \frac{k!(l+m-k-1)!}{(l+m)!}$$

$$\{\pi w(\pi A_{\pi K}\cup(\pi I)) - \pi w(\pi A_{\pi K})\} \tag{34}$$

Following (19) and (20)

$$\begin{aligned}\mu_{\pi A_{\pi K}}(j) =& \mu_{\pi A}(j) \bigwedge \chi_{\pi K}(j), \quad \forall j \in N \\ =& \mu_A(\pi^{-1} j) \bigwedge \chi_K(\pi^{-1} j), \quad \forall j \in N \\ =& \mu_{A_K}(\pi^{-1} j) \\ =& \mu_{\pi A_K}(j)\end{aligned} \tag{35}$$

It follows that

$$\begin{aligned}\nu_{\pi A_{\pi K}}(j) =& \nu_{\pi A_K}(j), \quad \forall j \in N \\ \mu_{(\pi A_{\pi K} \cup (\pi I))}(j) =& \mu_{\pi(A_K \cup I)}(j), \quad \forall j \in N \\ \nu_{(\pi A_{\pi K} \cup (\pi I))}(j) =& \nu_{\pi(A_K \cup I)}(j), \quad \forall j \in N\end{aligned} \tag{36}$$

Putting (35) through (36) in (34) ,

$$\Phi_{\pi i}(\pi w)(\pi A) = \sum_{K' \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} \{\pi w(\pi(A_K \cup (I))) - \pi w(\pi(A_K))\} \tag{37}$$

Using the fact that $(\pi w)(\pi A) = w(A)$ and $\pi A = A$, (37) becomes

$$\begin{aligned}\Phi_{\pi i}(\pi w)(A) =& \sum_{K' \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} \{w(A_K \cup I) - w(A_K)\} \\ =& \Phi_i(w)(A)\end{aligned}$$

Axiom f5 (Inter coalition Symmetry). Let w^i and w^j be two bi-cooperative games with fuzzy bi-coalitions such that for every $B \in \mathcal{FB}(A)$ with $i, j \notin Supp\ B$ and every pair $I, J \in \mathcal{FB}(N)$ such that $\mu_I(i) = \nu_J(j) > 0$ or $\mu_J(j) = \nu_I(i) > 0$ and $\mu_I(k) = \mu_J(k) = 0 = \nu_I(k) = \nu_J(k), \quad \forall k \in N \setminus \{i, j\}$, it holds that,

1. $w^i(B \cup I) - w^i(B) = w^j(B) - w^j(B \cup J)$
2. $w^i(B \cup J \cup I) - w^i(B \cup J)$ $= w^j(B \cup I) - w^j(B \cup I \cup J)$

Given $K \subseteq N$ and $i, j, l \in N$, let us construct A_K^{ij} as follows.

$$\mu_{A_K^{ij}}(l) = \begin{cases} \mu_A(l) \bigwedge \chi_K(l), & \text{if } l \in N \setminus \{i, j\} \\ 0, & \text{otherwise} \end{cases} \tag{38}$$

$$\nu_{A_K^{ij}}(l) = \begin{cases} \nu_A(l) \bigwedge \chi_K(l), & \text{if } l \in N \setminus \{i, j\} \\ 0, & \text{otherwise} \end{cases} \tag{39}$$

where the function χ_K is given by (21).

Let $J \in \mathcal{FB}(N)$ be such that

$$\mu_J(j) = \mu_{A_K}(j), \nu_J(j) = \nu_{A_K}(j) \tag{40}$$

$$\mu_J(k) = \nu_J(k) = 0, if \ \ k \neq j \tag{41}$$

Therefore using the facts that $A_K = A_K^{ij} \cup J$ when $K \subseteq (L \cup M) \setminus \{i\}$ and $A_K = A_K^{ij} \cup I$ when $K \subseteq (L \cup M) \setminus \{j\}$, it follows from (17),

$$\begin{aligned}\Phi_i(w^i)(A) =& \sum_{K \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} \left(w^i(A_K \cup I) - w^i(A_K)\right), \quad i, j \notin K \\ =& \sum_{K \subseteq (L \cup M) \setminus \{i\}} \frac{k!(l+m-k-1)!}{(l+m)!} (w^i(A_K^{ij} \cup J \cup I) - w^i(A_K^{ij} \cup J))\end{aligned}$$

and

$$\begin{aligned}\Phi_j(w^j)(A) =& \sum_{K \subseteq (L \cup M) \setminus \{j\}} \frac{k!(l+m-k-1)!}{(l+m)!} \left(w^j(A_K \cup I) - w^j(A_K)\right) \\ =& \sum_{K \subseteq (L \cup M) \setminus \{j\}} \frac{k!(l+m-k-1)!}{(l+m)!} (w^j(A_K^{ij} \cup I \cup J) - w^j(A_K^{ij} \cup I))\end{aligned}$$

Using (1) and (2), upon further simplifications we obtain $\Phi_i(w^i)(A) = -\Phi_j(w^j)(A)$.

Axiom f6 (Monotonicity). Let w and w' be two bi-cooperative games with fuzzy bi-coalitions and $A \in \mathcal{FB}(N)$. Let further that $\exists$ an $i \in N$, such that for every $I \in \mathcal{FB}(N)$ with $\mu_I(i) > 0$ or $\nu_I(i) > 0$, $\mu_I(j) = \nu_I(j) = 0, \forall j \neq i$ and $\forall B \in \mathcal{FB}(A)$ such that $\mu_B(i) = \nu_B(i) = 0$, it holds that

1. $w'(B) = w(B)$
2. $w'(B \cup I) \geq w(B \cup I)$

The result follows from (30) along with (1) and (2).This completes the proof. □

Example 3.3. Let us take an illustrative example where the crisp bi-cooperative game $v : \mathcal{Q}(\{1, 2, 3\}) \to \mathbb{R}$ is given by $v(\emptyset, \emptyset) = 0,\ v(\{1\}, \emptyset) = 1,\ v(\{3\}, \emptyset) = 2,\ v(\emptyset, \{2\}) = 3,\ v(\{1\}, \{2\}) = 4,\ v(\{3\}, \{2\}) = 2,\ v(\{1, 3\}, \{2\}) = 5,\ v(\{1, 3\}, \emptyset) = 6.$ and $v(S, T) = 0$ for any other $(S, T) \in$

$\mathcal{Q}(\{1, 2, 3\})$. Let A be a fuzzy bi-coalition over $\{1, 2, 3\}$ given by

$$A = \{< 1, 0.1, 0 >, < 2, 0, 0.2 >, < 3, 0.3, 0 >\}$$

Thus using (13), $w(A) = .716$. After some computations, the LG value of w for A is found to be (0.0973, 0.4699, 0.1488). A close look at the values of the crisp bi-cooperative game reveals that player 2 is a negative contributor and both player 1 and 3 are positive contributors, however 3 is more influential than 1 in generating a value. Moreover, Player 3 has more membership as a positive contributor than 1 in the fuzzy bi-coalition A also. The LG value divides the value $w(A)$ among the three players accordingly.

4. Conclusion

This paper proposes a new class of fuzzy bi-cooperative games, namely the fuzzy bi-cooperative games in multilinear extension form. The LG value is obtained as a solution concept for these games. The class of games what is defined here generalizes Owen's multilinear extension. For the future research, the relationship between the defined value and the other payoff indices such as the fuzzy core, the fuzzy population monotonic allocation scheme etc., may be further considered.

References

[1] J.M. Bilbao, J.R. Fernández, N. Jiménez and J.J. López, The Shapley value for bi-cooperative games, *Annals of Operations Research* **158** (2008), 99–115.

[2] J.M. Bilbao, Cooperative Games on Combinatorial Structures, Kluwer Academic Publishers, Dordrecht, 2000.

[3] S. Borkotokey and P. Sarmah, Bi-cooperative games with fuzzy bicoalitions, *Fuzzy Sets and Systems, 198* (2012), 46–58.

[4] M. Grabisch and C. Labreuche, Bi-capacities, Part I: Definition, Mobius transform and interaction, *Fuzzy Sets and Systems* **151** (2005), 211–236.

[5] M. Grabisch and C. Labreuche, Bi-capacities, Part II: The Choquet integral, *Fuzzy Sets and Systems* **151** (2005), 237–259.

[6] C. Labreuche and M. Grabisch, A value for bi-cooperative games, *International Journal of Games Theory* **37**(3) (2008), 409–438.

[7] F. Meng and Q. Zhang, The Shapley function for fuzzy cooperative games with multilinear extension form, *Applied Mathematics Letters* **23** (2010), 644–650.

[8] G. Owen, Multilinear extensions of games, *Management Sciences* **18** (1972), 64–79.

[9] L. Zadeh, Fuzzy Sets, *Information and Control* **8** (1965), 338–353.