Model of Risk and Losses of a Multigeneration Mortgage Portfolio

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Abstract

During the last decades, Merton-Vasicek factor model (1987), later generalize by Frye at al. (2000), became standards in credit risk management. We present a generalization of these models allowing multiple sub-portfolios of loans possibly starting at different times and lasting more than one period. We show that, given this model, a one-to-one mapping between factors and the overall default rate and the charge-off rate exists, is differentiable and numerically computable.

Keywords

risk management, loan portfolio, default rate, charge off rate

1 Introduction

Consider a large portfolio of loans secured by colaterals. Assume that, at each period $1 \leq \tau \leq T$, a new generation of N^τ loans is added to the portfolio; precisely, τ is the time of the first repayment of the loans of the τ-th generation.

For simplicity, assume all the loans to have identical parameters, namely:

- the amounts of loans are unit (without loss of generality),
- all the loans last m periods
- their interest rate is ζ
- each loan is repaid annuity way, i.e., by identical installments

$$b = b(\zeta) = \begin{cases} \frac{\zeta}{1-v^m} & \zeta \neq 0 \\ \frac{1}{m} & \zeta = 0 \end{cases}, \qquad v = v(\zeta) = (1+\zeta)^{-1}.$$

Assume further that the wealth of the i-th debtor of the τ-th generation at time t fulfils

$$A_t^{\tau,i} = \exp\left\{ Y_t + Z_t^{\tau,i} \right\}, \qquad t \geq 1,$$

where

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- Y is a general stochastic common factor
- $Z^{\tau,i}$ is a stochastic indivudual factor such that
  - $Z_\tau^{\tau,i} = \sigma_1 U_1^{\tau,i}$, $\sigma_1 > 0$, where $U_1^i \sim \mathcal{N}(0,1)$
  - $Z_t^{\tau,i} = \phi Z_{t-1}^{\tau,i} + \sigma U_t^{\tau,i}$, $t > \tau$, for some constants $\phi \in \mathbb{R}$, $\sigma > 0$, where $U_s^{\tau,i} \sim \mathcal{N}(0,1)$ for any s.

Further, assume that the i-th loan belonging to the τ-th generation is secured by a collateral with price $P^{\tau,i}$ fulfilling

$$P_{\tau-1}^{\tau,i} = 1$$

(i.e., is equal to the size of the loan at the hypothetical time of its arrangement) and

$$P_t^{\tau,i} = \exp\left\{I_t + E_t^{\tau,i}\right\}, \qquad t > \tau,$$

where

- I is another common factor
- $E^{\tau,i}$ is a stochastic process fulfilling $E_t^{\tau,i} = \psi E_{t-1}^{\tau,i} + \rho V_t^{\tau,i}$, $t > \tau$, for some constants $\psi \in \mathbb{R}$ and $\rho > 0$ where $E_\tau^{\tau,i} = 0$ and $V_s^{\tau,i} \sim \mathcal{N}(0,1)$ for any s.

Finally, assume that

- $U_1^{1,1}, V_1^{1,1}, U_1^{1,2}, V_1^{1,2}, \dots U_1^{1,N^1}, V_1^{1,N^1}, U_1^{2,1}, V_1^{1,2}, \dots V_T^{1,N^1}, U_1^{2,1} \dots$ are mutually independent, independent of Y, I.

We say that the i-th loan of the τ-th generation defaults at t if

$$D_t^{\tau,i} = \mathbf{1}[B_t^{\tau,i} = 1, S_{t-1}^{\tau,i} = 1] = 1$$

where, for any θ,

$$B_\theta^{\tau,i} = \mathbf{1}\left[A_\theta^{\tau,i} < (\theta - \tau + 1)b\right] \tag{1}$$

is a variable indicating insufficiency of the wealth to cover the (accumulated) installments and

$$S_\theta^{\tau,i} = \begin{cases} 1 & \theta = \tau - 1 \\ \mathbf{1}[B_\tau^{\tau,i} = 0, B_{\tau+1}^{\tau,i} = 0, \dots, B_\theta^{\tau,i} = 0], & \tau \le \theta \le \tau + m - 1 \\ 0 & \text{otherwise} \end{cases}$$

is an indicator of "survival" up to θ

The percentage loss of the creditor from the τ-th generation associated with the i-th loan of the τ-th generation at time t may be then expressed as

$$G_t^{\tau,i} = \frac{D_t^{\tau,i} \max(0, h_t - P_t^{\tau,i})}{h_t^\tau}$$

where

$$h_t^\tau = h(t - \tau + 1, \zeta), \qquad h(\theta) = \begin{cases} b \sum_{\tau=\theta}^m v^{m-\tau+1} = bv\frac{1-v^{m-t+1}}{1-v} & \zeta \ne 0 \\ \frac{m-\theta+1}{m} & \zeta = 1 \end{cases}$$

is the principal outstanding at t (see [4] for a formula for h as well as that for b).

Remark 1. (i) If $T = 1, m = 1$ and $Y_1 \sim \mathcal{N}(0, \varsigma^2)$, $Z_1 \sim \mathcal{N}(0, \sigma^2), \varsigma^2 + \sigma^2 = 1$ then our setting replicates the Vasicek Model [6].
(ii) If $T = 1, m = 1$, the factors Y_1, Z_1, are as in (i), $I_1 = \alpha + \gamma Y_1$, $E_1 \sim \mathcal{N}(0, \varrho^2)$, for some $\alpha, \gamma > 0$ and $\varrho < 1$, $E_1 \perp\!\!\!\perp Z_1$, then our model coincides with [1] if their prices are regarded as log ones.

For each τ and t, denote

$$N_t^\tau = \sum_{i=1}^{N^\tau} S_{t-1}^{\tau,i} \qquad t > 0,$$

the number of debts having survived until τ and, for any $t > 0$, define

$$Q_t^{\tau,N^\tau} = \frac{\sum_{1 \le i \le N^\tau} D_t^{\tau,i}}{N_t^\tau}$$

the overall default rate of the τ-th generation and

$$G_t^{\tau,N^\tau} = \frac{\sum_{1 \le i \le N^\tau} G_t^{\tau,i}}{N_t^\tau}$$

its average chargeoff rate.

Finally, denote

$$Q_t^{N^1,N^2 \dots N^t} = \frac{\sum_{1 \le \tau \le t} \sum_{1 \le i \le N^\tau} D_t^{\tau,i}}{\sum_{1 \le \tau \le t} N_t^\tau}, \qquad t > 1,$$

$$G_t^{N^1,N^2 \dots N^t} = \frac{\sum_{1 \le \tau \le t} \sum_{1 \le i \le N^\tau} G_t^{\tau,i}}{\sum_{1 \le \tau \le t} N_t^\tau}, \qquad t > 1,$$

the overall default rate, chargeoff rate, respectively.

The goal of the present paper is to examine properties and computability of mapping Φ,

$$\Phi_t(Y_1, I_1, \dots Y_t, I_t; \theta) = (Q_t, L_t), \qquad t \ge 1, \tag{2}$$

and its inversion, where

$$\theta = (\zeta, \sigma_1, \sigma, \rho, \phi, \psi),$$

is the parameter vector and

$$Q_t = \lim_{\substack{N^1,N^2,\dots,N^t \to \infty \\ N^\theta/N^1 \to \omega_\theta, 2 \le \theta \le t}} Q_t^{N^1,N^2 \dots N^t}, \qquad G_t = \lim_{\substack{N^1,N^2,\dots,N^t \to \infty \\ N^\theta/N^1 \to \omega_\theta, 2 \le \theta \le t}} G_t^{N^1,N^2 \dots N^t}, \qquad t \ge 1$$

are the asymptotic versions of the overall rates given that N^θ/N^1 converges to $\omega_\theta > 0$ for each θ.

2 The Transformation

In the present Section, transformation Φ of the factors into overall rates is constructed by means of the functions transforming the factors into the generation-specific rates.

Proposition 2. *Let $1 \le \tau \le T$ and $\tau \le t \le T$. Then*

$$\mathbb{P}[D_t^\tau = 1 | S_{t-1}^\tau = 1, Y_1, Y_2, \dots, Y_t] = Q_t^\tau, \qquad Q_t^\tau = \lim_{N^\tau \to \infty} Q_t^{\tau, N^\tau} = q_t^\tau(Y_\tau, Y_{\tau+1}, \dots, Y_t; \theta),$$

$$\mathbb{E}[G_t^\tau | S_{t-1}^\tau = 1, Y_1, Y_2, \dots, Y_t, I_t] = G_t^\tau, \qquad G_t^\tau = \lim_{N^\tau \to \infty} G_t^{\tau, N^\tau} = g_t^\tau(Y_\tau, Y_{\tau+1}, \dots, Y_t, I_t; \theta)$$

where the functions q_t^τ and g_t^τ are strictly decreasing in Y_t, I_t, respectively, both continuously differentiable in all the factors and parameters.

Proof. See [5], Section 4. □

The following Theorem shows that properties of Φ and its inverse are analogous to the generation specific transformations.

Theorem 3. *Let $1 \le t \le T$. Then*
(i) Continuously differentiable mapping Φ fulfilling (2) exists and is given by

$$\Phi_t = \sum_{\tau=1}^{t} \pi_t^\tau \cdot (Q_t^\tau, G_t^\tau) \tag{3}$$

where

$$\pi_t^\tau = \begin{cases} \frac{\omega_\tau R_t^\tau}{\sum_{\theta=t-m+1}^{t} \omega_\theta R_t^\theta} & t-m+1 \le \tau \le t \\ 0 & \text{otherwise} \end{cases}, \qquad R_t^\tau = \prod_{\theta=\tau}^{t-1} (1 - Q_\theta^\tau).$$

(ii) A continuously differentiable inverse Ψ of Φ exists.

Proof. (i) Assume, for a while, that Y and I are deterministic (which makes $\mathbb{P}[\bullet | S_{t-1}, Y_1, \dots Y_t, I_t] = \mathbb{P}[\bullet | S_{t-1}]$ etc). Let $t > 0$, denote $N_t = \sum_{1 \le \tau \le t} N_t^\tau$. As

$$Q_t^{N^1, N^2 \dots N^t} = \sum_{1 \le \tau \le t} \left(\frac{\sum_{1 \le i \le N^\tau} D_t^{\tau, i}}{N_t} \right) = \sum_{1 \le \tau \le t} \left(Q^{\tau, N_t^\tau} \cdot \frac{N_t^\tau}{N_t} \right),$$

we may use the Strong Law of Large Numbers ([3] Theorem 4.23) together with [3] Corollary 4.5 to get that

$$Q_t = \sum_{1 \le \tau \le t} \left(\left(\lim_{N_\tau} Q^{\tau, N_t^\tau} \right) \cdot \left(\lim_{\substack{N^1, N^2, \dots, N^t \to \infty \\ N^i / N_1 \to \omega_\theta, 2 \le \theta \le t}} \frac{N_t^\tau}{N_t} \right) \right).$$

if the limits exist. Because, according to [5], Section 4, $\lim_{N_\tau} Q^{\tau, N_t^\tau} \to Q_t^\tau$ almost sure, it remains to show that

$$\lim_{\substack{N^1, N^2 \dots N^t \\ \dots}} \frac{N_t^\tau}{N_t} = \pi_t^\tau;$$

to do so, observe first that, by the Strong Law of Large Numbers,

$$\lim_{N^\tau} \frac{N_\theta^\tau}{N^\tau} = \mathbb{P}[S_{\theta-1}^\tau = 1] = \mathbb{P}[S_{\theta-1}^\tau = S_{\theta-2}^\tau = \dots = S_\tau^\tau = 1] = \prod_{s=\tau}^{\theta-1} \mathbb{P}[S_s = 1 | S_{s-1} = 1] = R_t^s$$

for any $\theta \ge \tau$ (the last equality holds due to Proposition 2 and because $\mathbb{P}[S_\theta = 1 | S_{\theta-1} = 1] = \mathbb{P}[D_\theta = 0 | S_{\theta-1} = 1]$). Thus, using [3] Corollary 4.5, we get, for any $t - m + 1 \le \tau \le t$, that

$$\lim_{\substack{N^1,N^2\dots N^t\\ \dots}} \frac{N_t^\tau}{N_t} = \lim_{\substack{N^1,N^2\dots N^t\\ \dots}} \frac{N_t^\tau/N^\tau}{\sum_{\theta=1}^t N_t^\theta/N^\theta N^\theta/N^1 N^1/N^\tau}$$
$$= \frac{\lim_{N^\tau} N_t^\tau/N^\tau}{\sum_{\theta=1}^t \lim_{N^\theta} N_t^\theta/N^\theta \lim_{N^\theta,N^1,N^\theta/N^1\to\omega_\theta} N^\theta/N^1 \lim_{N^\tau,N^1,N^\tau/N^1\to\omega_\theta} N^1/N^\tau} = \pi_t^\tau;$$

as $N_t^\tau = 0$ for the remaining $\tau's$, (3) is proved for deterministic Y and I. For stochastic ones, the formula holds by [2] 6.8.14. The differentiability and monotonicity follows from those of q^τ and g^τ.
Ad (ii). The assertion follows similarly as in [5], Section 4. □

Remark 4. Actual values Φ may be computed by means of a numerical technique, proposed in [5], applied to the generation specific mappings. The inverse Ψ of Φ, on the other hand, has to be computed numerically from Φ. A C++ software package computing both Φ nad Ψ, including its source code, may be found at https://github.com/cyberklezmer/phi.

3 Conclusions

We have proved existence a one-to-one computable transformation between factors and the overall default and charge-off rates in a multi-generation factor model of a loan portfolio and we have shown that properties of this transformation are analogous to the generation-specific transformations discussed in [5]. One of possible application or our result could be a realistic modeling a dynamics of a nation-wide default and charge-off rates assuming a hypothetical multi-generation nation-wide portfolio.

References

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