Wavelet Coefficients Energy Redistribution and Heisenberg Principle of Uncertainty

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Abstract. This paper introduces the relation of Heisenberg Principle of Uncertainty to maximum Wavelet level of decomposition where the Wavelet coefficients typically provide sparse representation. In Quantum physics, Heisenberg Principle of Uncertainty states that we cannot exactly know the position and momentum of a particle simultaneously. In time frequency domain, Heisenberg Principle states that we cannot exactly know information about time and frequency simultaneously. Time delta multiplied by frequency delta is greater than some arbitrary constant. This restriction has different consequences in Fourier transformation and Wavelet transformation, which is more suitable for non-stationary times series analysis. Wavelets fits into this principle because a basic Wavelet is characterized by short time and high frequency. However, when the Wavelet is stretched then it has longer time anhd lower frequency. This principle is inherent in the nature of things and has nothing to do with numerical precision of the Wavelet analysis. First part of the paper summarizes Heisenberg Principle of Uncertainty, Wavelet transformation and signal energy. Second part presents Wavelet analysis of Apple Inc. stock daily closing price showing energy redistribution depending on the Wavelet decomposition level based on the choice of the Wavelet used for decomposition and the level of decomposition.

Keywords: Heisenberg Principle of Uncertainty, Wavelet Transformation, signal energy, signal entropy.

JEL classification: C44
AMS classification: 90C15

1 Heisenberg Principle of Uncertainity definition in time frequency domain

Theoretical part of this paper sumarises various definitions of Heisenberg Principle of Uncertainty from literature in references and define basic signal properties and also Distrete Wavelet Transformation which is used in numerical part. We start with definiton by Gabor atoms and their relation to Heisenberg Uncertainty and continue with Windowed Fourier transformation to Continuous Wavelet transformation.

1.1 Gabor atoms

Heisenberg Uncertainty as a analogy was first time used in freqeuncy domain by Gabor during time frequency dictionary definition constructed from waveforms of unit energy $||\phi_\gamma|| = 1$. Following paragraph is based on pioneer work of Mallat in Wavelet theory [1]. Lets start with definition of time localisation u of ϕ_γ and spread of ϕ_γ around u.

$$u = \int t.|\phi_\gamma(t)|^2 dt \text{ and } \sigma^2_{t,\gamma} = \int |t-u|^2.|\phi_\gamma(t)|^2 dt \tag{1}$$

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For frequency localisation and spread we have similar equations

$$\xi = (2\pi)^{-1} \int \omega.|\hat{\phi}_\gamma(\omega)|^2 d\omega \text{ and } \sigma^2_{\omega,\gamma} = (2\pi)^{-1} \int |\omega - \xi|^2.|\hat{\phi}_\gamma(\omega)|^2 d\omega \tag{2}$$

The Fourier Parseval formula shows dependency of $\langle f, \phi_\gamma \rangle$ on the values of $f(t)$ and $\hat{f}(\omega)$. Rectangle of size $\sigma_{t,\gamma} \times \sigma_{\omega,\gamma}$ is centerd at (u, ξ) for (t, ω).

$$\langle f, \phi_\gamma \rangle = \int_{-\infty}^{+\infty} f(t).\phi^\star_\gamma(t)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega).\hat{\phi^\star}_\gamma(\omega)d\omega \tag{3}$$

Heisenberg box represents Gabor atom ϕ_γ as is depicted on following figure 1. This box may be understand as "quantum of information over an elementary resolution cell." [1]. This rectangle has minimum surface defined

$$\sigma_{t,\gamma}.\sigma_{\omega,\gamma} \geq \frac{1}{2} \tag{4}$$

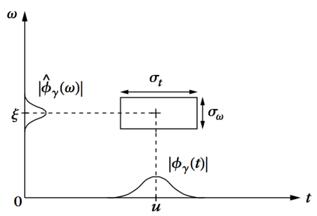

Figure 1 Heisenberg box of Gabor atom [1]

1.2 Windoved Fourier Transformation

Windoved Fourier Transformation uses time window $g(t)$ which is translated in time and frequency and has unit norm $||g|| = 1$. Each dictionary atom $g_{u,\xi}$ is projected by Windowed Fourier Transformation see Figure 2 (left side).

$$Sf(u, \xi) = \langle f, g_{u,\xi} \rangle = \int_{-\infty}^{+\infty} f(t).g(t - u).e^{-i\xi t} \tag{5}$$

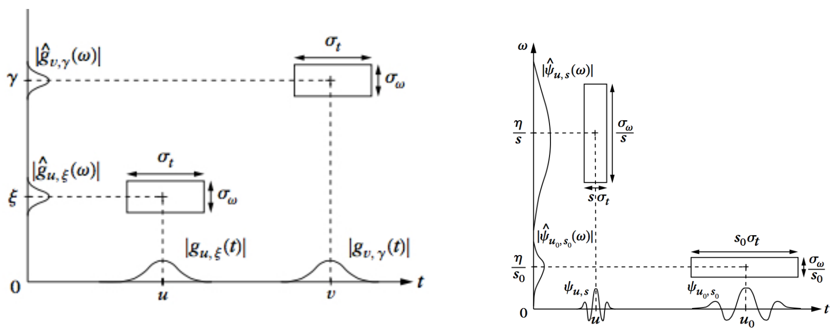

Figure 2 Energy spread of two windowed Fourier Transformation (left) and two Wavelets (right) [1]

1.3 Continuous Wavelet Transformation

Wavelet dictionary is based on the mother Wavelet ψ which is scaled by parameter s and translated by parameter u

$$\psi_{u,s} = \frac{1}{\sqrt{s}}.\psi\Big(\frac{t-u}{s}\Big)dt \tag{6}$$

and satisfying condition of zero mean

$$\int_{-\infty}^{+\infty} \psi(t)dt = 0 \tag{7}$$

The term $\frac{1}{\sqrt{s}}$ ensures energy conservation during scale shift. Continuous Wavelet Transformation projects function f at any scale and positon of mother Wavelet

$$Wf(u,s) = \langle f, \psi_{u,s}\rangle = \int_{-\infty}^{+\infty} f(t).\frac{1}{\sqrt{s}}.\psi^*\Big(\frac{t-u}{s}\Big)dt \tag{8}$$

Wavelet atom has Heisenberg box with fixed area where parameter s varies as is depicted in Figure 2 (right side). Wavelets have time and frequency localisation in comparation with Fourier atoms. When parameter s varies , the time and frequency changes, but area of box is still constant. For width window we have good resolution in frequency an low in time and vice versa for narrow window good resoltion int time and low in frequency.

2 Signals and their properties

Let $f(x)$ be a function defined on the interval $(-\pi, \pi)$ such that $f^2(x)$ has a finite integral on that interval. If a_n, b_n are the Fourier coefficients of the function $f(x)$ than Bessel's Inequality states

$$\pi\Big(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\Big) \leq \int_{-\pi}^{\pi} f^2(x)dx \tag{9}$$

The energy E of 2π periodic function $f(x)$ is defined

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx \tag{10}$$

and Bessel's inequality can be than written in following form:

$$\Big(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\Big) \leq E \tag{11}$$

where term $(a_n^2 + b_n^2)$ represents energy. We define discrete signal as a sequence of numbers $\{x(n)\}, n \in \mathbb{Z}$ satisfying following equation

$$\sum_{n\in\mathbb{Z}} |x(n)| < \infty \tag{12}$$

which states that signal has to be bounded. This condition is necessary prerequisite for Discrete Waveleet Transform. We also define for discrete signal $x(n)$ his energy $E = x_1^2 + x_2^2 + ... + x_n^2, n \in \mathbb{Z}$. If equation (12) is satisfied we speak about signal with finite energy. This definiton is based on Bessel's Inequality and energy definition for periodic function.

2.1 Signal comparison

Root Means Square Error (RMS Error) between two signals $x(n)$ and $y(n)$ is defined as

$$RMSError = \sqrt{\frac{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}{n}} \tag{13}$$

3 Discrete Wavelet Transformation - DWT

Wavelet function has two important parameters: scaling s and translation u and must satisfy admissibility and regularity condition. Admissibility means that Wavelet has zero average in time domain (must be oscillatory). Regularity requires smoothness and concentration in both time and frequency domains. Discrete Wavelet Transformation uses scaling parameter as a power of two ([5])

$$s = 2^{-j} \tag{14}$$

and the time shift becames

$$u = k.2^{-j} = k.s \tag{15}$$

Substituting equations (14) and (15) into base function in (6) we get

$$\psi_{u,s} = \psi\Big(\frac{t-u}{s}\Big) = \psi\Big(\frac{t-ku}{s}\Big) = \psi(s^{-1}t - k) = \psi(2^j.t - k) \tag{16}$$

Mother Wavelet at scale j and translation k is than defined as

$$\psi_{j,k} = 2^{j/2}\psi(2^j.t - k) \tag{17}$$

and similarly scaling function at scale j and translation k is defined

$$\phi_{j,k}(t) = 2^{j/2}\phi(2^j.t - k) \tag{18}$$

Scaling and mother function can be constructed as a liner combination of translations with the doubled frequency of a base scaling function $\phi(2t)$ and base mother function $\psi(2t)$ with equations

$$\phi(t) = \sum_{k=-\infty}^{+\infty} \sqrt{2}h_o(k)\phi(2t-k) \qquad \psi(t) = \sum_{k=-\infty}^{+\infty} \sqrt{2}h_1(k)\phi(2t-k) \tag{19}$$

where $h_0(k)$ and $h_1(k)$ are Wavelet filter coefficients. For scaling Haar Wavelet we have following coefficients $h_0(0) = h_0(1) = 1/\sqrt{2}$ and for Haar mother Wavelet we have $h_1(0) = 1/\sqrt{2}$ and $h_1(1) = -1/\sqrt{2}$. Daubechies 4 filter coefficients are $h_0(0) = 0.4830, h_0(1) = 0.8365, h_0(2) = 0.2241, h_0(3) = -0.1294$ and Daubechies 4 Wavelet coefficients are $h_1(0) = -0.1294, h_1(1) = -0.2241, h_1(2) = 0.8365, h_1(3) = -0.4830$.

We also notice that signal has to be bouded to perform DWT and that DWT ensures energy conservation on the first level of decompostition. First level of DWT is performed by computing first trend and first fluctuation coefficients from original signal. Second level DWT is performed by computing second trend and second fluctuation for the first trend only and so on for other decomposition levels.

4 Numerical results

Numerical part demonstrate computation corollaries of Heisenberg's Uncertainty Principle. As a data source we use daily price for Apple Inc. (AAPL) for the last 15 years. This sample has 3813 data points where each value represent value in USD and following graph shows stock price progress during evaluated period. From picture it is clear that this signal has finite energy thus we can use DWT where coefficients length depends on the input signal length and selected Wavelet filter.

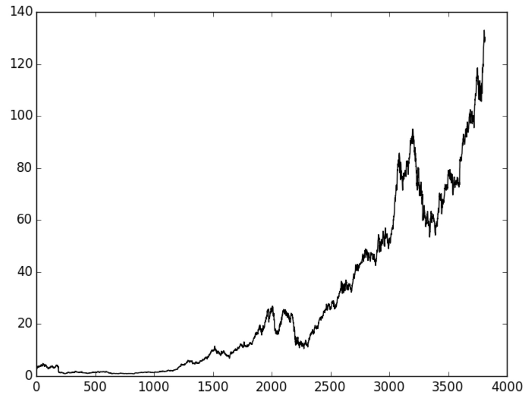

Figure 3 AAPL price in USD from 3.1.2000 to 27.2.2015.

We use for the Wavelet analysis a single prototype function (wavelet), which can be thought of as a bandpass filter. In the concrete filter coefficients for Haar are well known from theory and Daubechies 4 orthogonal wavelet with two vanishing moments with coefficients "in [1], which include values for different Wavelet filters". We use own code programmed in Python, because implemented Wavelet Transformations in various software packages use built in mechanism for energy preservation to get around consequences of Heisenberg's Uncertainty Principle. Verification can be done very easily. We select testing signal with length sufficient at minimum to level 4 of decomposition and compute trend coefficients with selected Wavelet package and with own program using filter described in section 3. Than we compare energy of coefficients on particular levels of decompostion. If energy is not approximately same than Wavelet package uses energy preservation algorithm.

Figure 4. depicts development of original AAPL price (top), Haar 1 and Haar 5 (bottom) trend coefficients where we can see how trend looks like with increasing level of Wavelet level decomposition thus coefficients reduction from 3813 to 119 (number of coefficients is on vertical axis).

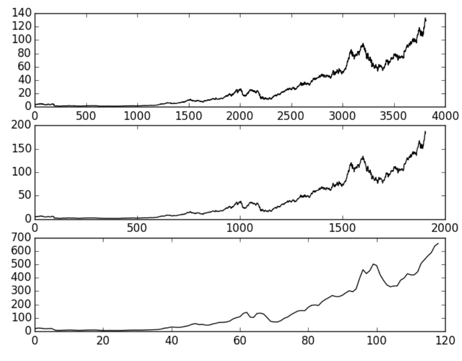

Figure 4 AAPL price in USD, Haar 1 and Haar 5 trend coefficients

Following table sumarises energy preservation in dependence of the number of decomposition. We note that maximum useful level of decomposition for the given input signal length and wavelet filter length is 9. Theoretically there is maximum level of decomposisiton equal to 11 with Haar Wavelet, but in this case energy of Wavelet trend coefficient drop to 20% of the original signal.

Level of decomposition	Haar	Daubechies 4
1	99.74	98.90
2	99.73	98.81
3	98.66	98.59
4	98.61	92.20
5	98.52	82.48
6	91.68	70.85
7	79.80	69.65
8	63.08	46.53
9	62.27	43.08

Table 1 Energy distribution on DWT decomposition level (trend coefficients)

Different values in table 1 have relation to various Wavelets families used during DWT and lenght of Wavelet compat support. For demostrative purposes we include in 1 row with RMS error for inverse DWT for Haar Wavelet.

5 Conclusion

In this paper we presented overview of Heisenbergs Uncertainty Principle and its relation to decomposition level of Discrete Wavelet Transformation. Heisenbergs Uncertainty Principle can be understood as well as a limitation of fixed amount energy localisation in small time interval. We cannot compact energy into decrescent time intervals. Numerical part analysed real stock data for Apple Inc. showing how energy percentage drop down for trend coefficients with increasing level of decomposition and different Wavelets used for analysis. We also note the fact that available Wavelet packages include built in mechanism for energy preservation and we expose simple algorithm how to check this fact.

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