Chapter XVII: States of MV-algebras

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1 Introduction

This handbook series is devoted to many-valued logics and the associated mathematical structures. States of MV-algebras constitute the object of study of a rapidly growing discipline, which involves and takes inspiration from several areas of mathematics: Łukasiewicz logic, ordered algebraic structures, artificial intelligence, game theory and others.

States are functions from an MV-algebra into the real unit interval satisfying a normalization condition and a variant of the classical law of finite additivity of probability measures, making clear the intimate relation with finitely additive probability measures. In our exposition we underline several other interpretations of states. Namely the integral representation of states by regular Borel probability measures and the bookmaking theorem for infinite-valued events. We also discuss the relation of states to real homomorphisms of unital lattice ordered Abelian groups, finitely presented MV-algebras, piecewise linear geometry, and conditional probability in non-classical setting.

The chapter is organized as follows. In Section 2 we shall recall the basic notions and results used in the rest of this chapter. Namely we summarize the main mathematical tools involving MV-algebras, Łukasiewicz logic, probability measures and compact convex sets. In Section 3 we will introduce states of MV-algebras and discuss their elementary properties (Section 3.1). We will include basic examples of states (Section 3.2) and characterize completely the states of finitely presented MV-algebras (Section 3.3). In Section 3.4 the connection of states to lattice ordered groups is made clear.

One of the main results of this chapter, the integral representation theorem, is proved in Section 4. The set of all states (the state space) forms a Bauer simplex (Section 4.1). Section 4.2 is about the existence of invariant faithful states.

De Finetti's coherence criterion for many-valued events and states is discussed in Section 5: in Section 5.1 we concentrate on coherent books on free MV-algebras and in Section 5.2 we explore the computational complexity for the problem of deciding if a book on formulas of Łukasiewicz logic is coherent.

Section 6 introduces an algebraic setting for states of MV-algebras, the variety of SMV-algebras. In the ensuing subsections we study the algebraic properties of this variety and we present a way to characterize de Finetti's coherence criterion on many-valued events in a purely algebraic way. Conditional states and an extended coherence criterion for conditional many-valued events are the content of Section 7.

No survey chapter can deliver a fully accurate portrait of a given subject. Section 8 contains the brief discussion of omitted results and topics, accompanied with the detailed list of further reading about states and numerous references to the current literature.

2 Basic notions

We summarize basic notions and results which are used in this chapter. The scope of involved mathematical apparatus is somewhat extensive, ranging from algebraic semantics of Łukasiewicz logic (Section 2.1-2.2) over measure theory (Section 2.3) to infinite-dimensional convex sets (Section 2.4). Therefore we confine to discussing only the most essential concepts of those disciplines. The reader is invited to consult the cited references for details, if needed.

2.1 MV-algebras

Our notation and definitions are according to the Chapter VI of this Handbook. The standard reference about MV-algebras is [15].

DEFINITION 2.1.1. *An* MV-algebra *is a structure* $\boldsymbol{A} = \langle A, \oplus, \neg, 0\rangle$, *where* $\oplus$ *is a binary operation,* $\neg$ *is a unary operation and* 0 *is a constant, such that the following conditions are satisfied for every* $a, b \in A$*:*

(i) $\langle A, \oplus, 0\rangle$ *is an Abelian monoid*

(ii) $\neg(\neg a) = a$

(iii) $\neg 0 \oplus a = \neg 0$

(iv) $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a.$

The class of MV-algebras forms a variety that we shall denote by $\mathbb{MV}$. We introduce the new constant 1 and three additional operations $\odot$, $\ominus$ and $\rightarrow$ as follows:

$$1 = \neg 0 \qquad a \odot b = \neg(\neg a \oplus \neg b) \qquad a \ominus b = a \odot \neg b \qquad a \rightarrow b = \neg a \oplus b.$$

The *Chang distance* is the binary operation

$$d(a, b) = (a \ominus b) \oplus (b \ominus a). \tag{1}$$

In the rest of this chapter we shall always assume that any MV-algebra has at least two elements and thus $0 \neq 1$.

For every MV-algebra $\boldsymbol{A}$, the binary relation $\leq$ on A given by

$$a \leq b \quad \text{whenever} \quad a \rightarrow b = 1$$

is a partial order. As a matter of fact, $\leq$ is a lattice order induced by the join $\vee$ and the meet $\wedge$ defined by

$$a \vee b = \neg(\neg a \oplus b) \oplus b \quad \text{and} \quad a \wedge b = \neg(\neg a \vee \neg b),$$

respectively. The lattice reduct of $\boldsymbol{A}$ then becomes a distributive lattice with the top element 1 and the bottom element 0. If the order $\leq$ of $\boldsymbol{A}$ is total, then $\boldsymbol{A}$ is said to be an MV-*chain*.

EXAMPLE 2.1.2. The basic example of an MV-algebra is the *standard* MV-*algebra* $[0,1]_Ł$, which is just the real unit interval $[0,1]$ equipped with the operations

$$a \oplus b = \min(1, a+b) \qquad a \odot b = \max(0, a+b-1) \qquad \neg a = 1-a. \tag{2}$$

The partial order of the standard MV-algebra coincides with the usual order of real numbers from $[0,1]$. It is worth mentioning that the standard MV-algebra $[0,1]_Ł$ is generic for the variety $\mathbb{MV}$, that is, $\mathbb{MV} = \mathbf{V}([0,1]_Ł)$.

EXAMPLE 2.1.3. Every Boolean algebra is an MV-algebra with respect to the operations $\oplus = \vee$, $\odot = \wedge$, and the complement $\neg$.

EXAMPLE 2.1.4. For every natural number d, the set $\mathbf{Ł}_d = \{0, 1/d, \dots, (d-1)/d, 1\}$ endowed with the restriction of the operations of $[0,1]_Ł$ is a finite MV-chain.

MV-algebras generalize Boolean algebras in the following sense: an MV-algebra $\boldsymbol{A}$ is a Boolean algebra if and only if $a \oplus a = a$ for every $a \in A$. Hence MV-algebras are particular non-idempotent generalizations of Boolean algebras. For any MV-algebra $\boldsymbol{A}$, we denote

$$B(A) = \{a \in A \mid a \oplus a = a\}$$

and call $B(A)$ the *Boolean center* (or the *Boolean skeleton*) of A. It follows that the structure $\langle B(A), \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra.

Let $\boldsymbol{A} = \langle A, \oplus_A, \neg_A, 0_A\rangle$ and $\boldsymbol{B} = \langle B, \oplus_B, \neg_B, 0_B\rangle$ be MV-algebras. A *homomorphism* from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a mapping $h\colon A \to B$ such that, for every $a_1, a_2 \in A$,

(i) $h(a_1 \oplus_A a_2) = h(a_1) \oplus_B h(a_2)$

(ii) $h(\neg_A a_1) = \neg_B h(a_1)$

(iii) $h(0_A) = 0_B$.

Let us define

$$\mathcal{H}(\boldsymbol{A}, \boldsymbol{B}) = \{h \mid h \text{ is a homomorphism from } \boldsymbol{A} \text{ to } \boldsymbol{B}\}.$$

In case that $\boldsymbol{B} = [0,1]_Ł$, we write simply $\mathcal{H}(\boldsymbol{A})$ in place of $\mathcal{H}(\boldsymbol{A}, [0,1]_Ł)$. An *isomorphism* $h \in \mathcal{H}(\boldsymbol{A}, \boldsymbol{B})$ is a bijective homomorphism.

Let X be a nonempty set. The set $[0,1]^X$ of all functions $X \to [0,1]$ becomes an MV-algebra if the operations $\oplus$, $\neg$, and the element 0 as in (2) are defined pointwise. The corresponding lattice operations $\vee$ and $\wedge$ are then the pointwise maximum and the pointwise minimum of two functions $X \to [0,1]$, respectively.

DEFINITION 2.1.5. *Let X be a nonempty set. A* clan *over X is an* MV-*algebra* $\boldsymbol{A}_X = \langle A_X, \oplus, \neg, 0\rangle$*, where $A_X \subseteq [0,1]^X$ is a nonempty set of functions $X \to [0,1]$, endowed with the pointwise defined operations of the standard* MV-*algebra.*

We say that a clan $\boldsymbol{A}_X$ is *separating* whenever the following condition is satisfied: if $x, y \in X$ with $x \neq y$, then there exists $a \in A_X$ such that $a(x) \neq a(y)$.

It turns out that the clans of $[0,1]$-valued continuous functions over some compact Hausdorff space are the prototypes of an important class of MV-algebras. We will introduce the necessary algebraic machinery in order to formulate the corresponding representation theorem. An *ideal* in an MV-algebra $\boldsymbol{A}$ is a subset $I \subseteq A$ such that

(i) $0 \in I$.

(ii) If $a, b \in I$, then $a \oplus b \in I$.

(iii) If $b \in I$ and $b \geq a \in A$, then $a \in I$.

An ideal I is *proper* if $I \neq A$. We say that a proper ideal M is *maximal* if M is not strictly included in any proper ideal of $\boldsymbol{A}$. If $\boldsymbol{A}$ is an MV-algebra and S is a subset of A, then the ideal generated by S coincides with

$$(S] = \{a \in A \mid a \leq i_1 \oplus \cdots \oplus i_n \text{ for some } n \in \mathsf{N} \text{ and } i_1, \ldots, i_n \in S\}.$$

An ideal I is said to be *finitely generated* if there exists a finite subset S of A such that $I = (S]$. A *filter* in an MV-algebra $\boldsymbol{A}$ is a subset $F \subseteq A$ closed with respect to $\odot$ and such that $1 \in F$, and if $b \in F$ and $b \leq a \in A$, then $a \in F$. Ideals and filters are dual objects in the setting of MV-algebras. Indeed, there is a one-to-one correspondence between ideals and filters: if I is an ideal of an MV-algebra $\boldsymbol{A}$, the set

$$F_I = \{a \in A \mid \neg a \in I\}$$

is a filter in $\boldsymbol{A}$. Conversely, for every filter F,

$$I_F = \{b \in A \mid \neg b \in F\}$$

is an ideal in $\boldsymbol{A}$. The notions of proper and maximal filter, respectively, are defined as usual.

Let $\mathrm{Max}(\boldsymbol{A})$ be the nonempty set of all maximal ideals of $\boldsymbol{A}$, which we call the *maximal ideal space of* $\boldsymbol{A}$. Given any ideal I of $\boldsymbol{A}$, put

$$O_I = \{M \in \mathrm{Max}(\boldsymbol{A}) \mid I \not\subseteq M\}.$$

THEOREM 2.1.6 ([36, Theorem 3.6.10]). *For every* MV*-algebra* $\boldsymbol{A}$*, its maximal ideal space* $\mathrm{Max}(\boldsymbol{A})$ *is a compact Hausdorff space whose family of open sets coincides with* $\{O_I \mid I \text{ is an ideal of } \boldsymbol{A}\}$.

Let us consider the set

$$\mathrm{Rad}(\boldsymbol{A}) = \bigcap \{M \mid M \in \mathrm{Max}(\boldsymbol{A})\}$$

called the *radical of* $\boldsymbol{A}$. We say that $\boldsymbol{A}$ is *semisimple* if $\mathrm{Rad}(\boldsymbol{A}) = \{0\}$. It can be shown [36, Lemma 4.2.3] that semisimplicity is equivalent to non-existence of infinitesimal elements in $\boldsymbol{A}$, that is, for every $a \in A$ and every $n \in \mathsf{N}$, the condition $\bigoplus_{i=1}^{n} a \leq \neg a$ implies $a = 0$.

The representation of semisimple MV-algebras is one of the crucial tools employed in the study of states.

THEOREM 2.1.7 ([36, Theorem 5.4.7]). *Let $\boldsymbol{A}$ be a semisimple* MV*-algebra. Then $\boldsymbol{A}$ is isomorphic to a separating clan of* $[0,1]$*-valued continuous functions defined on the compact Hausdorff space* $\mathrm{Max}(\boldsymbol{A})$.

2.2 Łukasiewicz logic

In this section we provide a survey of Łukasiewicz infinite-valued propositional logic and its associated Lindenbaum algebra. For further details see the Chapter VI and XI of this Handbook, or consult the book [15].

We use the algebraic semantics based on MV-algebras, which enables us to use the same notation for the logical connectives and the corresponding MV-algebraic operations in the following paragraphs. Formulas $\phi, \psi, \dots$ are constructed from propositional variables $A_1, A_2, \dots$ by applying the standard rules known in Boolean logic. The connectives are negation $\neg$, disjunction $\oplus$, and conjunction $\odot$. The implication $\phi \to \psi$ can be defined as $\neg\phi \oplus \psi$. The set of all propositional formulas is denoted by Form.

The standard semantics for connectives of Łukasiewicz logic is defined by the corresponding operations of the standard MV-algebra $[0,1]_{Ł}$. A *valuation* is a mapping $V\colon \mathsf{Form} \to [0,1]$ such that, for each $\phi, \psi \in \mathsf{Form}$,

$$V(\neg\phi) = 1 - V(\phi)$$
$$V(\phi \oplus \psi) = V(\phi) \oplus V(\psi)$$
$$V(\phi \odot \psi) = V(\phi) \odot V(\psi).$$

For every two formulas $\phi, \psi \in \mathsf{Form}$, we define the relation $\equiv$ by the following stipulation: we say that $\phi \equiv \psi$ iff $V(\phi) = V(\psi)$ for every valuation V. It turns out that $\equiv$ is an equivalence relation on Form. For every $\phi \in \mathsf{Form}$ we denote $[\phi]$ the equivalence class of ϕ with respect to $\equiv$.

The Lindenbaum algebra of Łukasiewicz logic is the MV-algebra $\boldsymbol{F}$ of equivalence classes $\{[\phi] \mid \phi \in \mathsf{Form}\}$ endowed with the canonical operations $\neg$, $\oplus$, and $\odot$:

$$\neg[\phi] = [\neg\phi]$$
$$[\phi] \oplus [\psi] = [\phi \oplus \psi]$$
$$[\phi] \odot [\psi] = [\phi \odot \psi].$$

By Chang's completeness theorem [36, Theorem 5.3.7], we may identify $\boldsymbol{F}$ with the free MV-algebra over countably many generators, which is a sub-MV-algebra of $[0,1]^{[0,1]^{\mathsf{N}}}$.

Assume that Form_n is the set of all formulas $\phi \in \mathsf{Form}$ containing only the propositional variables from the list $A_1, \dots, A_n$. Let $\boldsymbol{F}_n$ be the corresponding Lindenbaum algebra, which coincides with the free n-generated MV-algebra. By McNaughton's theorem [1, Theorem 2.1.20], we know that $\boldsymbol{F}_n$ is isomorphic to the clan of functions $f\colon [0,1]^n \to [0,1]$ such that each f is

(i) continuous,

(ii) piecewise linear, and

(iii) each linear piece has integer coefficients only.

We call each function $[0,1]^n \to [0,1]$ satisfying (i)–(iii) a *McNaughton function*.

REMARK 2.2.1. *In this context we use the term "linear" as a synonym for "affine". There are more non-equivalent definitions of a piecewise linear function appearing in literature. For the purposes of this chapter, we adopt the weaker but the usual definition used in* MV*-algebraic context; see [1]. A function* $f\colon [0,1]^n \to [0,1]$ *is* piecewise linear *if there exist linear functions* $f_1, \dots, f_m$ *on* $[0,1]^n$ *such that for each* $\mathbf{x} \in [0,1]^n$ *there is some* $j \in \{1, \dots, m\}$ *with* $f_j(\mathbf{x}) = f(\mathbf{x})$*. Other authors may use a stronger definition that implies continuity; cf. [7, Definition 7.10].*

An MV-algebra $\boldsymbol{A}$ is said to be *finitely presented* if it is isomorphic to the quotient $\boldsymbol{F}_n/I$ for some natural number n and some finitely generated ideal I of $\boldsymbol{F}_n$.

Łukasiewicz logic is an algebraizable logic in the sense of Blok and Pigozzi. As a consequence, a finitely presented MV-algebra is the Lindenbaum algebra of a finitely axiomatizable theory[1] in Łukasiewicz logic and, moreover, it has a completely geometric characterization—see [48, Theorem 6.3]. Indeed, let $P \subseteq [0,1]^n$ be a nonempty rational polyhedron[2] and $\boldsymbol{M}(P)$ be the MV-algebra of all the restrictions to P of n-variable McNaughton functions in $\boldsymbol{F}_n$.

PROPOSITION 2.2.2. *Let* $P \subseteq [0,1]^n$*. Then the following are equivalent:*

(i) P *is a rational polyhedron.*

(ii) $P = f^{-1}(1)$ *for some McNaughton function* $f \in \boldsymbol{F}_n$*.*

The following characterizations of a finitely presented MV-algebra are used freely in this chapter.

THEOREM 2.2.3. *Let* $\boldsymbol{A}$ *be an* MV*-algebra. Then the following are equivalent:*

(i) $\boldsymbol{A}$ *is a finitely presented* MV*-algebra.*

(ii) $\boldsymbol{A}$ *is isomorphic to the Lindenbaum algebra of a theory with a single axiom.*

(iii) $\boldsymbol{A}$ *is isomorphic to* $\boldsymbol{M}(P)$*, where* P *is a nonempty rational polyhedron in* $[0,1]^n$*, for some* $n \in \mathsf{N}$*.*

2.3 Probability measures

We will need the basic notions of measure theory as appearing in [10] and [54], for example. The definitions given below apply to the case of Boolean algebras of sets, which is sufficient for most of our purposes.

Let X be a nonempty set. An *algebra of sets* $\mathfrak{A}$ is any Boolean algebra of subsets of X. A *finitely additive probability* is a function $\mu\colon \mathfrak{A} \to [0,1]$ such that:

(i) If $A, B \in \mathfrak{A}$, where $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

(ii) $\mu(\emptyset) = 0$ and $\mu(X) = 1$.

[1] In Łukasiewicz logic, we can replace any finite number of axioms with a single one. It has the effect that a finitely generated ideal I corresponds, through the algebraizability of Łukasiewicz logic, to a finitely axiomatizable theory T_I, which, in turn, corresponds to a single formula ϑ. Thus, algebraically, finitely generated and principal ideals in MV-algebras coincide.

[2] By a *rational polyhedron* we mean a finite point-set union of simplices with rational vertices in R^n; see Section 2.4.

REMARK 2.3.1. *The terminology oscillates: for example, Rao and Rao [54] call the above function μ a* probability charge. *Moreover, we will see that μ is at the same time a special example of state of an* MV*-algebra; cf. Definition 3.0.1.*

The condition (i), which is called *finite additivity*, must be strengthened in most applications. To this end, we require $\mathfrak{A}$ to be closed with respect to countable point-set unions. A *σ-algebra* is an algebra of sets $\mathfrak{A}$ such that the condition of *σ-completeness* holds true:

$$\text{if } A_1, A_2, \ldots \in \mathfrak{A}, \text{ then } \bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}.$$

Let $\mathfrak{A}$ be a σ-algebra. A finitely additive probability $\mu\colon \mathfrak{A} \to [0,1]$ is a *probability measure* if it is *σ-additive*, that is:

$$\text{if } A_1, A_2, \ldots \in \mathfrak{A} \text{ with } A_i \cap A_j = \emptyset \text{ for } i \neq j, \text{ then } \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

In case that the universe X is a topological space, the most natural choice of the σ-algebra over X is the *Borel σ-algebra* $\mathfrak{B}(X)$. Namely, the family $\mathfrak{B}(X)$ is the smallest σ-algebra over X containing every open set in X. Let μ be a probability measure defined on $\mathfrak{B}(X)$. Then there is no ambiguity in referring to μ as a *Borel probability measure* on X. Assume that X is a compact Hausdorff space. The *support* of Borel probability measure μ on X is the closed set $\bigcap\{A \subseteq X \mid \mu(A) = 1, A \text{ closed}\}$. Let $x \in X$. Then the Borel probability measure defined by

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is called the *Dirac measure* at x. The support of δ_x is the singleton $\{x\}$. We say that a Borel probability measure μ on X is *regular* whenever for every Borel set $A \subseteq X$,

$$\mu(A) = \sup\{\mu(B) \mid B \subseteq A, B \text{ compact}\}.$$

In general, not every Borel probability measure on a compact Hausdorff space X is regular. On the other hand, if every open set in the compact Hausdorff space X is an *F_σ set*, that is, a countable union of closed sets, then every Borel probability measure is necessarily regular. In particular, this applies to a compact Hausdorff space X that is metrizable.

2.4 Compact convex sets

The detailed exposition on compact convex sets in a space with weak topology is the subject of [5] or [27]. Our goal is to introduce the notion of infinite-dimensional simplex, the so-called Choquet simplex, that is a faithful generalization of an n-dimensional simplex.

Let E be a real linear space. A *convex set* in E is any subset K of E that is closed under *convex combinations:* if $x_1, \ldots, x_n \in K$ and $\alpha_i \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, then $\alpha_1 x_1 + \cdots + \alpha_n x_n \in K$. Given any set $X \subseteq E$, the *convex hull of* X is the set $\mathrm{co}(X)$ of all convex combinations of elements in X.

By an *affine combination* we mean a linear combination $\alpha_1 x_1 + \cdots + \alpha_n x_n$ with $\alpha_1 + \cdots + \alpha_n = 1$, where $\alpha_i \in \mathsf{R}$. A subset A of E is said to be *affinely independent* if there does not exist an element $a \in A$ that can be expressed as an affine combination of elements from $A \setminus \{a\}$. An *affine subspace* of E is any subset of E that is closed under affine combinations. An *n-dimensional simplex* is the convex hull of $n+1$ affinely independent points in E.

Let K_1 and K_2 be convex sets in linear spaces E_1 and E_2, respectively. A mapping $f : K_1 \to K_2$ is said to be *affine* if $f(\alpha_1 x_1 + \cdots + \alpha_n x_n) = \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$, for every convex combination $\alpha_1 x_1 + \cdots + \alpha_n x_n \in K_1$. If f is a bijection, then the inverse mapping f^{-1} is also affine and we call f an *affine isomorphism* of K_1 and K_2.

A *convex cone* in E is a subset C of E such that $0 \in C$ and $\alpha_1 x_1 + \alpha_2 x_2 \in C$ for any non-negative $\alpha_1, \alpha_2 \in \mathsf{R}$ and $x_1, x_2 \in C$. A *strict convex cone* is any convex cone C that satisfies this condition: if both $x \in C$ and $-x \in C$, then $x = 0$. A *base* for a convex cone C is any convex subset K of C such that every non-zero element $y \in C$ may be uniquely expressed as $y = \alpha x$ for some $\alpha \geq 0$ and some $x \in K$. The bases of cones can be easily visualized geometrically: the following characterization is true even for infinite-dimensional spaces.

PROPOSITION 2.4.1. *Let K be a non-empty convex subset of a linear space E and $C = \{\alpha x \mid \alpha \geq 0,\ x \in K\}$. Then C is a convex cone in E and the following conditions are equivalent:*

(i) K is a base for C.

(ii) K is contained in an affine subspace A of E such that $0 \notin A$.

Strict convex cones determine a partial order on linear spaces: if C is a strict convex cone in a linear space E, then the relation $\leq_C$ defined by $x \leq_C y$ iff $y - x \in C$ for any $x, y \in E$ makes E into a partially ordered linear space and, moreover, we have that $C = \{x \in E \mid 0 \leq_C x\}$. A *lattice cone* is any strict convex cone C in E such that the set C endowed with a partial order $\leq_C$ is a lattice. A *simplex* in a linear space E is any convex subset S of E that is affinely isomorphic to a base for a lattice cone in some linear space.

The deepest results about convex sets in infinite-dimensional spaces are attained when we assume that the linear space E is a locally convex Hausdorff space. We say that a subset S of E is a *Choquet simplex* if it is a simplex that is a compact set. Clearly, since $E = \mathsf{R}^k$ is locally convex, every n-dimensional simplex is a Choquet simplex.

An *extreme point* of a convex set K is a point $e \in K$ such that the set $K \setminus \{e\}$ remains convex. The set

$$\partial(K) = \{e \in K \mid e \text{ is an extreme point of } K\}$$

is called an *extreme boundary of* K. For any set $X \subseteq E$, let $\overline{\mathrm{co}}(X)$ be the topological closure in E of the convex hull $\mathrm{co}(X)$. The next theorem is the well-known characterization [29] of every compact convex set by its extreme boundary.

THEOREM 2.4.2 (Krein-Milman). *If K is a compact convex subset of a locally convex Hausdorff space E, then $K = \overline{\mathrm{co}}\, \partial(K)$.*

Let K_1 and K_2 be two compact convex subsets of E. An *affine homeomorphism* h of K_1 and K_2 is an affine isomorphism $h\colon K_1 \to K_2$ that is simultaneously a homeomorphism.

Choquet simplices whose extreme boundaries satisfy additional topological conditions are of particular interest.

DEFINITION 2.4.3. *Let E be a locally convex Hausdorff space. A Choquet simplex $S \subseteq E$ is called a* Bauer simplex *if its extreme boundary $\partial(S)$ is a closed subset of E.*

Every n-dimensional simplex S is a Bauer simplex since its extreme boundary ∂S has only finitely many extreme points. More generally, let X be a compact Hausdorff space and let $\mathcal{M}(X)$ denote the convex set of all regular Borel probability measures over X. We endow the set $\mathcal{M}(X)$ with the so-called *weak* topology*: a net (μ_γ) in $\mathcal{M}(X)$ weak* converges to $\mu \in \mathcal{M}(X)$ if and only if

$$\int_X f \, \mathrm{d}\mu_\gamma \to \int_X f \, \mathrm{d}\mu \quad \text{for every continuous function } f\colon X \to \mathsf{R}. \tag{3}$$

The properties of $\mathcal{M}(X)$ are collected in the next theorem for a future reference; see [5, Corollary II.4.2] for further details.

THEOREM 2.4.4. *Let X be a compact Hausdorff space.*

(i) The set of all Borel probability measures $\mathcal{M}(X)$ endowed with the weak topology is a Bauer simplex.*

(ii) The extreme boundary of $\mathcal{M}(X)$ is $\partial\mathcal{M}(X) = \{\delta_x \mid x \in X\}$.

(iii) The mapping $x \mapsto \delta_x$ is a homeomorphism of X and $\partial\mathcal{M}(X)$.

As we recalled in the end of Section 2.3, if X is a compact set in a finite-dimensional Euclidean space, then every Borel probability measure in $\mathcal{M}(X)$ is regular. Moreover, the compact space $\mathcal{M}(X)$ is metrizable in its weak* topology.

3 States

DEFINITION 3.0.1. *Let $\boldsymbol{A}$ be an* MV*-algebra. A mapping $s\colon A \to [0,1]$ is a* state *of $\boldsymbol{A}$ whenever $s(1) = 1$ and for every $a, b \in A$ the following condition is satisfied:*

$$\textit{if } a \odot b = 0, \textit{ then } s(a \oplus b) = s(a) + s(b). \tag{4}$$

The condition (4) means *additivity* with respect to Łukasiewicz sum $\oplus$. Indeed, the requirement $a \odot b = 0$ is analogous to disjointness of a pair of elements in a Boolean algebra. Thus states can be thought of as generalizations of finitely additive probabilities: every finitely additive probability on a Boolean algebra is a state as a special case of the above definition. In particular, every Borel probability measure is a state as well.

REMARK 3.0.2. *For historical reasons going back to the relation between states of ℓ-groups and* MV*-algebraic states, which in turns can be traced back to quantum mechanics, we always refer to states* of *an* MV*-algebra and not to states* on *it.*

3.1 Basic properties

We will summarize the basic properties of states with respect to the operations and the lattice order of an MV-algebra $\boldsymbol{A}$.

PROPOSITION 3.1.1. *Let s be a state of an* MV*-algebra $\boldsymbol{A}$. For every $a, b \in A$:*

(i) $s(0) = 0$.

(ii) $s(\neg a) = 1 - s(a)$.

(iii) If $a \leq b$, then $s(b \ominus a) = s(b) - s(a)$. *(subtractivity)*

(iv) $s(a \oplus b) + s(a \odot b) = s(a) + s(b)$. *(strong modularity)*

(v) $s(a \vee b) + s(a \wedge b) = s(a) + s(b)$. *(weak modularity)*

(vi) If $a \leq b$, then $s(a) \leq s(b)$. *(monotonicity)*

(vii) $s(a \vee b) \leq s(a \oplus b) \leq s(a) + s(b)$. *(subadditivity)*

(viii) If $a \in \mathrm{Rad}(\boldsymbol{A})$, then $s(a) = 0$.

Proof. (i) We have $s(1) = s(1 \oplus 0) = s(1) + s(0)$, which yields $s(0) = 0$. The identity (ii) follows from $s(a) + s(\neg a) = s(a \oplus \neg a) = s(1) = 1$ since in every MV-algebra, $a \odot \neg a = 0$ holds true. In order to prove (iii), observe that $a \leq b$ implies $(b \ominus a) \oplus a = b$. Since $(b \ominus a) \odot a = b \odot \neg a \odot a = 0$, we get $s(b) = s((b \ominus a) \oplus a) = s(b \ominus a) + s(a)$. Property (iv): as a consequence of [36, Proposition 2.1.2(h)], the identity $(a \oplus b) \odot \neg b = (\neg a \oplus \neg b) \odot a$ holds true. This, together with subtractivity, yields

$$\begin{aligned} s(a \oplus b) - s(b) &= s((a \oplus b) \ominus b) = s((a \oplus b) \odot \neg b) = s((\neg a \oplus \neg b) \odot a) \\ &= s(\neg(a \odot b) \odot a) = s(a \ominus (a \odot b)) = s(a) - s(a \odot b). \end{aligned}$$

Weak modularity (v) is a consequence of (iv) and [36, Proposition 2.1.2(d)–(e)], which says that $a \oplus b = (a \vee b) \oplus (a \wedge b)$ and $a \odot b = (a \vee b) \odot (a \wedge b)$. Thus

$$\begin{aligned} s(a) + s(b) &= s(a \oplus b) + s(a \odot b) \\ &= s((a \vee b) \oplus (a \wedge b)) + s((a \vee b) \odot (a \wedge b)) \\ &= s(a \vee b) + s(a \wedge b). \end{aligned}$$

Monotonicity (vi) is a direct consequence of subtractivity (iii). Subadditivity (vii) results from (vi) by considering that $a \vee b \leq a \oplus b$ together with modularity (iv) and nonnegativity of state. (viii) Let $a \in Rad(\boldsymbol{A})$. Reasoning by contradiction, assume that $s(a) > 0$. Then [36, Lemma 3.5.2(a)] gives $a \odot a = 0$. Hence $s(a \oplus a) = 2s(a)$. Proceeding by induction on the number of summands a, we can analogously derive the identity

$$s\left(\bigoplus_{i=1}^{n} a\right) = ns(a), \quad \text{for every } n \in \mathsf{N}.$$

But this means that there exists some $n_0 \in \mathsf{N}$ for which we obtain

$$s\left(\bigoplus_{i=1}^{n_0} a\right) = n_0 s(a) > 1,$$

a contradiction. □

COROLLARY 3.1.2. *Let $\boldsymbol{A}$ be an* MV*-algebra. The following are equivalent for a function $s\colon A \to [0,1]$ with $s(1) = 1$:*

(i) s is a state and

(ii) $s(a \oplus b) + s(a \odot b) = s(a) + s(b)$, for every $a, b \in A$.

Proof. An easy consequence of Proposition 3.1.1(iv). □

REMARK 3.1.3. *Every state satisfies the two modularity laws* (iv)–(v) *whose combination gives $s(a \oplus b) - s(a \vee b) = s(a \wedge b) - s(a \odot b)$. In particular, the weak modularity* (v) *expresses the fact that any state is a* monotone valuation *in the sense of lattice theory [11, Chapter V]. The converse clearly fails: not every monotone valuation on the lattice reduct of an* MV*-algebra $\boldsymbol{A}$ is a state of $\boldsymbol{A}$. Indeed, consider $\boldsymbol{A} = [0,1]_{Ł}$ and a lattice homomorphism $h\colon [0,1] \to [0,1]$ such that $h(a) = 1$ if $a = 1$, and $h(a) = 0$ otherwise. Then h is a monotone valuation but not a state of $[0,1]_{Ł}$.*

The next assertion is an immediate consequence of the properties of the Boolean center of any MV-algebra and the definition of state.

PROPOSITION 3.1.4. *For every state s of an* MV*-algebra $\boldsymbol{A}$, the restriction of s to the Boolean center $B(A)$ is a finitely additive probability.*

The restriction of s to $B(A)$ may carry little information about the state s. Indeed, when $\boldsymbol{A} = \boldsymbol{F}_n$ is the MV-algebra of n-variable McNaughton functions, its Boolean center contains only two elements: the functions 0 and 1.

There is no stateless MV-algebra.

PROPOSITION 3.1.5. *Every* MV*-algebra $\boldsymbol{A}$ carries at least one state s.*

Proof. By [36, Proposition 3.4.5], the collection $\mathrm{Max}(\boldsymbol{A})$ of all maximal ideals of $\boldsymbol{A}$ is nonempty. Let $M \in \mathrm{Max}(\boldsymbol{A})$. Then the quotient MV-algebra $\boldsymbol{A}/M$ is simple [36, Proposition 4.2.10] and thus isomorphic to a subalgebra of the standard MV-algebra $[0,1]_{Ł}$ by [36, Proposition 5.4.1]. Hence we can compose the natural epimorphism $e\colon \boldsymbol{A} \to \boldsymbol{A}/M$ with the embedding $\iota\colon \boldsymbol{A}/M \to [0,1]$ and put $s = \iota \circ e$. Since both e and ι are homomorphisms, their composition s is a homomorphism of $\boldsymbol{A}$ into $[0,1]_{Ł}$ and thus a state. □

REMARK 3.1.6. *Notice that the proof of Proposition 3.1.5 uses the fact that homomorphisms into the standard* MV*-algebra are states. This simple observation will play a key role in the characterization of the state space; cf. Example 3.2.2 and Section 4.1.*

The set of all states on an MV-algebra, which can be endowed with a topology, is called the *state space*. We will establish its basic geometric properties and show that semisimple MV-algebras have the largest possible state spaces among all MV-algebras.

PROPOSITION 3.1.7. *Let $\mathcal{S}(\boldsymbol{A})$ be the family of all states of an* MV*-algebra* $\boldsymbol{A}$*. Then:*

(i) $\mathcal{S}(\boldsymbol{A})$ *is a nonempty compact convex subset of* $[0,1]^A$.

(ii) The state spaces $\mathcal{S}(\boldsymbol{A})$ *and* $\mathcal{S}(\boldsymbol{A}/\operatorname{Rad}(\boldsymbol{A}))$ *are affinely homeomorphic.*

Proof. (i) That $\mathcal{S}(\boldsymbol{A}) \neq \emptyset$ is a consequence of Proposition 3.1.5. By Tychonoff's theorem (see [6, Theorem 2.61], for example), the product space $[0,1]^A$ is a compact subspace of the locally convex space R^A. It can routinely be verified that $\mathcal{S}(\boldsymbol{A}) \subseteq [0,1]^A$ is closed and thus compact. To check that a convex combination of two states is a state is straightforward as well.

(ii) The assertion is trivial if $\boldsymbol{A}$ is semisimple. Assume that $\boldsymbol{A}$ is not semisimple so that $\operatorname{Rad}(\boldsymbol{A}) \neq \{0\}$. Let $s \in \mathcal{S}(\boldsymbol{A})$. First, we will show that $s(a) = s(b)$, whenever $a, b \in A$ are such that $a/\operatorname{Rad}(\boldsymbol{A}) = b/\operatorname{Rad}(\boldsymbol{A})$. The last condition implies $d(a,b) \in \operatorname{Rad}(\boldsymbol{A})$, where d is the Chang distance given by (1), and Proposition 3.1.1(viii) yields $s(d(a,b)) = 0$. Strong modularity together with monotonicity give

$$\begin{aligned} s(a \ominus b) + s(b \ominus a) &= s((a \ominus b) \odot (b \ominus a)) + s((a \ominus b) \oplus (b \ominus a)) \\ &= s((a \ominus b) \odot (b \ominus a)) + s(d(a,b)) = 0. \end{aligned}$$

Hence necessarily $s(a \ominus b) = s(b \ominus a) = 0$. Using strong modularity again, we can now write

$$s(\neg a) + s(b) = s(\neg a \oplus b) = 1 = s(a \oplus \neg b) = s(a) + s(\neg b),$$

from which results $s(a) = s(b)$. It is thus correct to define $s' \colon \boldsymbol{A}/\operatorname{Rad}(\boldsymbol{A}) \to [0,1]$ by $s'(a/\operatorname{Rad}(\boldsymbol{A})) = s(a)$, for every $a \in A$. Then $s' \in \mathcal{S}(\boldsymbol{A}/\operatorname{Rad}(\boldsymbol{A}))$. It is easily seen that the mapping $s \mapsto s'$ is an affine isomorphism. In order to show that the mapping $s \mapsto s'$ is a homeomorphism, we need only check that it is continuous. Let (s_γ) be a net (generalized sequence) of elements in $\mathcal{S}(\boldsymbol{A})$ such that $s_\gamma \to s$ in $\mathcal{S}(\boldsymbol{A})$. This means that

$$\lim_\gamma s_\gamma(a) = s(a), \quad \text{for every } a \in A.$$

Then, for every $a \in A$, we have $s'_\gamma(a/\operatorname{Rad}(\boldsymbol{A})) = s_\gamma(a) \to s(a) = s'(a/\operatorname{Rad}(\boldsymbol{A}))$ in $[0,1]$. Hence $s \mapsto s'$ is continuous. □

We say that a state s of $\boldsymbol{A}$ is *faithful* if s is strictly positive, that is, we have $s(a) > 0$ whenever $a \in A$ is nonzero.

PROPOSITION 3.1.8. *Let* $\boldsymbol{A}$ *be an* MV*-algebra. Then:*

(i) A state s of $\boldsymbol{A}$ is faithful iff $s(a) < s(b)$, for every $a, b \in A$ with $a < b$.

(ii) If $\boldsymbol{A}$ carries a faithful state, then $\boldsymbol{A}$ is semisimple.

Proof. (i) If s is faithful, then the condition $a < b$ implies $b \ominus a > 0$ and subtractivity gives $s(b) - s(a) = s(b \ominus a) > 0$. Conversely, if $a \neq 0$, then $s(a) = s(a \ominus 0) = s(a) - s(0) > 0$.

(ii) Suppose that $\boldsymbol{A}$ is not semisimple. Then there exists a nonzero $a \in \operatorname{Rad}(\boldsymbol{A})$. However, Proposition 3.1.1(viii) says that $s(a) = 0$ for every state s of $\boldsymbol{A}$. □

3.2 Examples of states

We will discuss basic examples of states of various MV-algebras. The next sections then reveal the general pattern common to all of those examples. We have already proved (Proposition 3.1.5) that an arbitrary MV-algebra carries at least one state; in fact, the examples suggest that MV-algebras are abundant in states.

EXAMPLE 3.2.1 (Finitely-additive probability). We know that any Boolean algebra $\boldsymbol{B}$ is an MV-algebra in which the MV-operations $\oplus$ and $\odot$ coincide with the lattice operations $\vee$ and $\wedge$, respectively. Thus every state s of $\boldsymbol{B}$ is a finitely additive probability since the condition (4) reads as follows:

$$\text{if } a \wedge b = 0, \text{ then } s(a \vee b) = s(a) + s(b).$$

EXAMPLE 3.2.2 (Homomorphism). Every homomorphism h of an MV-algebra $\boldsymbol{A}$ into the standard MV-algebra $[0,1]_{Ł}$ is a state of $\boldsymbol{A}$. In particular, whenever $\boldsymbol{A}$ is a subalgebra of the MV-algebra $[0,1]^X$ of all functions $X \to [0,1]$. For any $x \in X$ the evaluation mapping $s_x \colon A \to [0,1]$ given by

$$s_x(f) = f(x), \quad f \in A, \tag{5}$$

is a state of $\boldsymbol{A}$.

Both examples above are special cases of the following construction. Let $\boldsymbol{A}$ be a semisimple MV-algebra with the maximal ideal space $\mathrm{Max}(\boldsymbol{A})$ and assume that $\boldsymbol{A}^*$ is the separating clan of continuous functions $\mathrm{Max}(\boldsymbol{A}) \to [0,1]$ that is isomorphic to $\boldsymbol{A}$ (Theorem 2.1.7). For every $a \in A$, let $a^* \in A^*$ be the function corresponding to a via the isomorphism. Consider any regular Borel probability measure μ on the compact Hausdorff space $\mathrm{Max}(\boldsymbol{A})$. Put

$$s_\mu(a) = \int_{\mathrm{Max}(\boldsymbol{A})} a^* \,\mathrm{d}\mu, \quad \text{for every } a \in A. \tag{6}$$

It can be routinely checked that s_μ is a state of $\boldsymbol{A}$. Observe that (5) is a special case of (6) upon putting $\mu = \delta_M$, where δ_M is the Dirac measure supported by a maximal ideal $M \in \mathrm{Max}(\boldsymbol{A})$.

EXAMPLE 3.2.3 (Lebesgue state). As an important special case of (6), consider $\boldsymbol{A}$ to be a finitely presented MV-algebra. Then there exists $n \in \mathsf{N}$ and a nonempty rational polyhedron $P \subseteq [0,1]^n$, $\dim(P) = n$, such that $\boldsymbol{A}$ is isomorphic to the MV-algebra $\boldsymbol{M}(P)$ of restrictions of McNaughton functions $f \in \boldsymbol{F}_n$ to P (Theorem 2.2.3). Let λ be the n-dimensional Lebesgue measure on $[0,1]^n$. Since $\lambda(P) > 0$ it makes sense to define a state of $\boldsymbol{M}(P)$ by putting

$$s_\lambda(f) = \frac{\int_P f \,\mathrm{d}\lambda}{\lambda(P)}, \quad f \in M(P). \tag{7}$$

The MV-algebra $\boldsymbol{M}(P)$ is isomorphic to the Lindenbaum algebra of theory $\{\vartheta\}$ for some satisfiable formula $\vartheta \in \mathsf{Form}_n$, so we arrive at the following interpretation of s_λ. The number (7) is the average truth value (with respect to all the models of $\{\vartheta\}$) of any formula φ such that $[\varphi]$ coincides with the restriction of f to P. Indeed, $s_\lambda(f)$ is the expected value of f with respect to the uniform distribution λ over the possible worlds in P.[3]

EXAMPLE 3.2.4 (States of Lindenbaum algebra). Another example interesting from the logical viewpoint is the Lindenbaum algebra $\boldsymbol{F}$ of Łukasiewicz logic over countably-many propositional variables. For every $f \in \boldsymbol{F}$ there exist $n \in \mathsf{N}$, $g \in \boldsymbol{F}_n$ and a nonempty set $I \subseteq \{1, \ldots, n\}$ such that $f(\langle x_i \rangle_{i \in \mathsf{N}}) = g(\langle x_i \rangle_{i \in I})$, for every sequence $\langle x_i \rangle_{i \in \mathsf{N}} \in [0,1]^{\mathsf{N}}$. Thus every element of $\boldsymbol{F}$ is a function of n variables only and coincides with some McNaughton function.

This enables us to simplify the computation of states of $\boldsymbol{F}$ defined by (6) since, for any Borel probability measure μ on the compact Hausdorff space (in the product topology) $[0,1]^{\mathsf{N}}$, we have

$$s_\mu(f) = \int_{[0,1]^{\mathsf{N}}} f \, \mathrm{d}\mu = \int_{[0,1]^n} g \, \mathrm{d}\mu',$$

where μ' is the Borel measure on $[0,1]^n$ given by $\mu'(A) = \mu\left(\pi^{-1}(A)\right)$, A is a Borel set in $[0,1]^n$ and $\pi \colon [0,1]^{\mathsf{N}} \to [0,1]^n$ is the projection function. In conclusion, when it comes to computing the average truth value (6), there is no loss of generality in replacing the Lindenbaum algebra $\boldsymbol{F}$ of Łukasiewicz logic with the MV-algebra of McNaughton functions $\boldsymbol{F}_n$, for some $n \in \mathsf{N}$.

EXAMPLE 3.2.5. Let ${}^*[0,1]_{Ł}$ be a non-trivial ultraproduct of the standard MV-algebra $[0,1]_{Ł}$ and choose a positive infinitesimal $c \in {}^*[0,1]$. The *Chang* MV-*algebra* $\boldsymbol{C}$ is (up to an isomorphism) the MV-subalgebra of ${}^*[0,1]_{Ł}$ generated by the set $\{0, c\}$ (see [36, Example 2.4.5] for a detailed analysis of Chang MV-algebra). It is known that $\boldsymbol{C}$ has the universe $\mathrm{Rad}(\boldsymbol{C}) \cup \mathrm{Rad}(\boldsymbol{C})^*$, where $\mathrm{Rad}(\boldsymbol{C}) = \{0, c, c \oplus c, c \oplus c \oplus c, \ldots\}$ is the radical of $\boldsymbol{C}$ and $\mathrm{Rad}(\boldsymbol{C})^* = \{1, 1-c, 1-(c \oplus c), 1-(c \oplus c \oplus c), \ldots\}$ is the co-radical of $\boldsymbol{C}$.

It is worth to notice that, by Proposition 3.1.1 (viii) and (ii), Chang algebra has only one trivial state. Namely $\boldsymbol{C}$ has a unique state s such that $s(x) = 0$ for every $x \in \mathrm{Rad}(\boldsymbol{C})$ and $s(x) = 1$ for every $x \in \mathrm{Rad}(\boldsymbol{C})^*$.

In general, having only one trivial state is a property that characterizes a wide class of MV-algebras, the so-called *perfect* MV-*algebras*, which are MV-algebras $\boldsymbol{A}$ having $\mathrm{Rad}(\boldsymbol{A}) \cup \mathrm{Rad}(\boldsymbol{A})^*$ as the universe (see [36, Section 4.3]).

3.3 States of finitely presented MV-algebras

In this section we completely characterize the states of any finitely presented MV-algebra. Although the result proved herein is a special case of integral representation theorem developed below (Section 4), we consider its separate treatment a worthwhile digression on the way to understanding the structure and the properties of states of any MV-algebra.

[3] With a small abuse of notation, we will identify the possible worlds of an MV-algebra $\boldsymbol{A}$ with the elements of $\mathcal{H}(\boldsymbol{A})$. This identification will be made explicit in Section 5.

First, we will recall the basic results concerning Schauder hats and bases; see [1] or [48]. Let $T \subseteq [0,1]^n$ be an n-dimensional simplex with rational vertices. Let $\mathbf{x} = \langle a_1/d, \ldots, a_n/d \rangle$ be a vertex of T, for uniquely determined relatively prime integers $a_1, \ldots, a_n, d$ with $d \geq 1$. Call $\langle a_1, \ldots, a_n, d \rangle$ the *homogeneous coordinates* of $\mathbf{x}$, and let $\mathrm{den}(\mathbf{x}) = d$ be the *denominator* of $\mathbf{x}$. We say that simplex T is *unimodular* if the determinant of the integer square matrix having the homogeneous coordinates of all the vertices of T in its rows is equal to ± 1. An r-dimensional simplex with $r \leq n$ is unimodular if it is a face of some unimodular n-dimensional simplex.

Let $p_1, \ldots, p_l$ be the linear pieces of an n-variable McNaughton function f. For every permutation π of $\{1, \ldots, l\}$, put:

$$P_\pi = \{\mathbf{x} \in [0,1]^n \mid p_{\pi(1)}(\mathbf{x}) \leq p_{\pi(2)}(\mathbf{x}) \leq \cdots \leq p_{\pi(l)}(\mathbf{x})\}, \tag{8}$$

$$C = \{P_\pi \mid P_\pi \text{ is } n\text{-dimensional}\}. \tag{9}$$

Clearly C is a finite set of n-dimensional polytopes with rational vertices, that is, every $P_\pi \in C$ is the convex hull of a finite set of rational points in $[0,1]^n$. It is well-known that C can be refined to a *unimodular triangulation* of $[0,1]^n$ that linearizes f, which is a finite set Σ of n-dimensional unimodular simplices over the rational vertices such that:

(i) the union of all simplices in Σ is equal to $[0,1]^n$,

(ii) any two simplices in Σ intersect in a common face and

(iii) for each simplex $T \in \Sigma$, there exists $j = 1, \ldots, l$ such that the restriction of f to T coincides with p_j (we also say that f is *linear* over Σ).

Let $\mathrm{V}_\Sigma = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ be the set of vertices of all simplices in Σ. The *Schauder hat* at $\mathbf{x}_i \in \mathrm{V}_\Sigma$ is the McNaughton function $h_{\mathbf{x}_i} = h_i$ linearized by Σ such that $h_i(\mathbf{x}_i) = 1/\mathrm{den}(\mathbf{x}_i)$ and $h_i(\mathbf{x}_j) = 0$ for every vertex $\mathbf{x}_j$ distinct from $\mathbf{x}_i$ in Σ. The set $H_\Sigma = \{h_1, \ldots, h_m\}$ is called a *Schauder basis for* $\boldsymbol{F}_n$. The *normalized* Schauder hat at $\mathbf{x}_i$ is the McNaughton function

$$\hat{h}_{\mathbf{x}_i} = \hat{h}_i = \mathrm{den}(\mathbf{x}_i) \cdot h_i.$$

We denote by $\hat{H}_\Sigma = \{\hat{h}_1, \ldots, \hat{h}_m\}$ the subset of $\boldsymbol{F}_n$ consisting of all the normalized Schauder hats. The set $\hat{H}_\Sigma$ is also called a *normalized Schauder basis*.

Note that every McNaughton function that is linear over Σ is a linear combination of the family of Schauder hats corresponding to Σ, where each hat has a uniquely determined integer coefficient between 0 and $\mathrm{den}(\mathbf{x}_i)$. Thus

$$f = \sum_{\mathbf{x}_i \in \mathrm{V}_\Sigma} a_{\mathbf{x}_i} \cdot h_i, \tag{10}$$

for uniquely determined integers $0 \leq a_{\mathbf{x}_i} \leq \mathrm{den}(\mathbf{x}_i)$.

This argument can be easily generalized to the case of finitely many McNaughton functions. In particular, if $f_1, \ldots, f_k$ are McNaughton functions on the n-cube $[0,1]^n$, we can find a unimodular triangulation Σ of $[0,1]^n$ with vertices $\mathrm{V}_\Sigma = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ such that each f_i is linear over each simplex of Σ. In what follows we will need the following result.

LEMMA 3.3.1. *Let $f_1, \ldots, f_k$, Σ, V_Σ and $\hat{H}_\Sigma$ be as above. Then:*

(i) For distinct $\hat{h}_i, \hat{h}_j \in \hat{H}_\Sigma$, $\hat{h}_i \odot \hat{h}_j = 0$.

(ii) $\displaystyle\bigoplus_{t=1}^{m} \hat{h}_t = 1.$

(iii) $f_i = \displaystyle\bigoplus_{t=1}^{m} f_i(\mathbf{x}_t) \cdot \hat{h}_t$*, for each $i = 1, \ldots, k$.*

For the proof of Theorem 3.3.4 we prepare two more lemmas.

LEMMA 3.3.2. *Let $P \subseteq [0,1]^n$ be a nonempty rational polyhedron. If μ and ν are Borel probability measures on P such that $\mu \neq \nu$, then $s_\mu \neq s_\nu$, where s_μ and s_ν are given by (6).*

Proof. By way of contradiction, suppose $s_\mu = s_\nu$. The Borel subsets of P are generated by the collection of all closed (in the subspace Euclidean topology of P) rational polyhedra. Indeed, every closed subset in P can be written as a countable intersection of such polyhedra. Since the set of all rational polyhedra is closed under finite intersections, [10, Theorem 3.3] yields the existence of a rational polyhedron $R \subseteq P$ with $\mu(R) \neq \nu(R)$.

There is $f \in M(P)$ such that $R = f^{-1}(1)$ by Proposition 2.2.2. Let χ_R denote the characteristic function of R. Then we obtain

$$\chi_R = \bigwedge_{m \in \mathsf{N}} \bigodot_{i=1}^{m} f. \tag{11}$$

For every $m \in \mathsf{N}$, the function $\bigodot_{i=1}^{m} f$ belongs to $M(P)$. The contradiction now follows from (11) together with the Lebesgue's dominated convergence theorem [6, Theorem 11.21]:

$$\begin{aligned}\mu(R) = \int_P \chi_R \,\mathrm{d}\mu = \bigwedge_{m \in \mathsf{N}} \int_P \bigodot_{i=1}^{m} f \,\mathrm{d}\mu = \bigwedge_{m \in \mathsf{N}} s_\mu \left(\bigodot_{i=1}^{m} f \right) \\ = \bigwedge_{m \in \mathsf{N}} s_\nu \left(\bigodot_{i=1}^{m} f \right) = \bigwedge_{m \in \mathsf{N}} \int_P \bigodot_{i=1}^{m} f \,\mathrm{d}\nu = \int_P \chi_R \,\mathrm{d}\nu = \nu(R).\end{aligned}$$ □

LEMMA 3.3.3. *Let $P \subseteq [0,1]^n$ be a rational polyhedron, Σ be any unimodular triangulation of P and $\hat{H}_\Sigma$ be a normalized Schauder basis. If $a\colon \hat{H}_\Sigma \to [0,1]$ is a function such that $\sum_{\mathbf{x} \in \mathrm{V}_\Sigma} a(\hat{h}_\mathbf{x}) = 1$, then there exists a Borel probability measure δ with finite support on P satisfying $a(\hat{h}_\mathbf{x}) = s_\delta(\hat{h}_\mathbf{x})$, for each $\mathbf{x} \in \mathrm{V}_\Sigma$.*

Proof. Let $\delta_\mathbf{x}$ be the Dirac measure concentrated at a vertex $\mathbf{x} \in \mathrm{V}_\Sigma$. Put

$$\delta = \sum_{\mathbf{x} \in \mathrm{V}_\Sigma} a(\hat{h}_\mathbf{x}) \delta_\mathbf{x}.$$

Then, for each vertex $\mathbf{x} \in \mathrm{V}_\Sigma$, we get

$$s_\delta(\hat{h}_\mathbf{x}) = \int_P \hat{h}_\mathbf{x}\, \mathrm{d}\delta \; = \sum_{\mathbf{x}' \in \mathrm{V}_\Sigma} \int_P a(\hat{h}_{\mathbf{x}'})\hat{h}_\mathbf{x}\, \mathrm{d}\delta_{\mathbf{x}'}$$

$$= \sum_{\mathbf{x}' \in \mathrm{V}_\Sigma} a(\hat{h}_{\mathbf{x}'})\hat{h}_\mathbf{x}(\mathbf{x}') = a(\hat{h}_\mathbf{x})\hat{h}_\mathbf{x}(\mathbf{x}) = a(\hat{h}_\mathbf{x}). \qquad \square$$

THEOREM 3.3.4. *Let s be a state of the finitely presented* MV*-algebra* $\boldsymbol{M}(P)$*, where P is a nonempty rational polyhedron. Then there exists a unique Borel probability measure μ on P such that $s = s_\mu$.*

Proof. Let s be a state of $\boldsymbol{M}(P)$. For any normalized Schauder basis $\hat{H}_\Sigma$, let us define the set of Borel probability measures

$$\mathcal{M}_\Sigma = \{\mu \mid s(\hat{h}_\mathbf{x}) = s_\mu(\hat{h}_\mathbf{x}),\ \text{for each}\ \mathbf{x} \in \mathrm{V}_\Sigma\}.$$

Note that $\mathcal{M}_\Sigma \neq \emptyset$ by Lemma 3.3.3. It follows directly from the definition of weak* topology (see Section 2.4) on the set of all Borel probability measures $\mathcal{M}(P)$ over P that $\mathcal{M}_\Sigma$ is weak* closed in $\mathcal{M}(P)$. Indeed, consider any net (μ_γ) in $\mathcal{M}_\Sigma$ weak* converging to some $\mu \in \mathcal{M}(P)$. For each $\hat{h}_\mathbf{x} \in \hat{H}_\Sigma$, this means by (3) that

$$s(\hat{h}_\mathbf{x}) = s_{\mu_\gamma}(\hat{h}_\mathbf{x}) \to s_\mu(\hat{h}_\mathbf{x}),$$

and $\mathcal{M}_\Sigma$ is thus weak* closed.

Let $\mathfrak{T}$ denote the family of all unimodular triangulations of polyhedron P. We are going to show that

$$\bigcap_{\Sigma \in \mathfrak{T}} \mathcal{M}_\Sigma \neq \emptyset. \tag{12}$$

Since $\mathcal{M}(P)$ is weak* compact by Theorem 2.4.4, $\mathcal{M}_\Sigma$ is weak* compact too, so it suffices to prove $\bigcap_{\Sigma' \in \mathfrak{T}'} \mathcal{M}_{\Sigma'} \neq \emptyset$ for every finite subset $\mathfrak{T}' \subseteq \mathfrak{T}$. It follows from [1, Lemma 2.1.7] that finitely-many unimodular triangulations—ergo Schauder bases—can always be jointly refined. Specifically, this means that there exists a Schauder basis H_Σ such that: for every $\Sigma' \in \mathfrak{T}'$ with $\mathfrak{T}'$ finite and for each normalized Schauder hat $\hat{h}_\mathbf{x} \in \hat{H}_{\Sigma'}$, there exists a uniquely determined nonnegative integer vector $(\beta_\mathbf{y})_{\mathbf{y} \in \mathrm{V}_\Sigma}$ such that $\hat{h}_\mathbf{x} = \sum_{\mathbf{y} \in \mathrm{V}_\Sigma} \beta_\mathbf{y} h_\mathbf{y}$. Let $\delta = \sum_{\mathbf{y} \in \mathrm{V}_\Sigma} s(\hat{h}_\mathbf{y})\delta_\mathbf{y}$. Clearly, $\delta \in \mathcal{M}(P)$. Linearity of Lebesgue integral gives

$$s_\delta(\hat{h}_\mathbf{x}) = \int_P \hat{h}_\mathbf{x}\, \mathrm{d}\delta = \sum_{\mathbf{y}' \in \mathrm{V}_\Sigma} s(\hat{h}_{\mathbf{y}'}) \int_P \sum_{\mathbf{y} \in \mathrm{V}_\Sigma} \frac{\beta_\mathbf{y}}{\mathrm{den}(\mathbf{y})} \hat{h}_\mathbf{y}\, \mathrm{d}\delta_{\mathbf{y}'}. \tag{13}$$

For every $\mathbf{y} \in \mathrm{V}_\Sigma$,

$$\frac{s(\hat{h}_\mathbf{y})}{\mathrm{den}(\mathbf{y})} = \frac{s(\mathrm{den}(\mathbf{y})h_\mathbf{y})}{\mathrm{den}(\mathbf{y})} = s(h_\mathbf{y}).$$

Then additivity of states enables us to express the right-hand side of (13) as

$$\sum_{\mathbf{y}'\in \mathrm{V}_\Sigma} s(\hat{h}_{\mathbf{y}'}) \sum_{\mathbf{y}\in \mathrm{V}_\Sigma} \frac{\beta_{\mathbf{y}}}{\mathrm{den}(\mathbf{y})}\hat{h}_{\mathbf{y}}(\mathbf{y}') = \sum_{\mathbf{y}'\in \mathrm{V}_\Sigma} s(\hat{h}_{\mathbf{y}'})\frac{\beta_{\mathbf{y}'}}{\mathrm{den}(\mathbf{y}')} = \sum_{\mathbf{y}'\in \mathrm{V}_\Sigma} s\left(\frac{\beta_{\mathbf{y}'}}{\mathrm{den}(\mathbf{y}')}\hat{h}_{\mathbf{y}'}\right)$$

$$= s\left(\sum_{\mathbf{y}\in \mathrm{V}_\Sigma} \beta_{\mathbf{y}} h_{\mathbf{y}}\right) = s(\hat{h}_{\mathbf{x}}).$$

Thus we have shown that $\delta \in \bigcap_{\Sigma'\in\mathfrak{T}'} \mathcal{M}_{\Sigma'}$, which implies (12), as $\mathfrak{T}'$ was an arbitrary finite set of unimodular triangulations of P.

We will prove that $s_\mu = s$ for every $\mu \in \bigcap_{\Sigma\in\mathfrak{T}} \mathcal{M}_\Sigma$. Indeed, given a function $f \in \boldsymbol{M}(P)$, find $\Pi \in \mathfrak{T}$ and a Schauder basis H_Π such that $f = \sum_{\mathbf{x}\in \mathrm{V}_\Pi} \alpha_{\mathbf{x}} h_{\mathbf{x}}$, for uniquely determined nonnegative integers $\alpha_{\mathbf{x}}$ [1, Lemma 2.1.19]. It results that

$$s(f) = s\left(\sum_{\mathbf{x}\in \mathrm{V}_\Pi} \alpha_{\mathbf{x}} h_{\mathbf{x}}\right) = \sum_{\mathbf{x}\in \mathrm{V}_\Pi} \alpha_{\mathbf{x}} s(h_{\mathbf{x}}) = \sum_{\mathbf{x}\in \mathrm{V}_\Pi} \alpha_{\mathbf{x}} s_\mu(h_{\mathbf{x}})$$

$$= s_\mu\left(\sum_{\mathbf{x}\in \mathrm{V}_\Pi} \alpha_{\mathbf{x}} h_{\mathbf{x}}\right) = \sum_{\mathbf{x}\in \mathrm{V}_\Pi} \alpha_{\mathbf{x}} s(h_{\mathbf{x}}) = s_\mu(f).$$

Finally, the set $\bigcap_{\Sigma\in\mathfrak{T}} \mathcal{M}_\Sigma$ contains a single element by Lemma 3.3.2. □

REMARK 3.3.5. *The unique Borel probability measure μ on the rational polyhedron P from Theorem 3.3.4 is regular since $P \subseteq [0,1]^n$.*

The previous proof highlights the central role of Schauder bases as the basic building blocks of any element in the finitely presented algebra. As a matter of fact, the resulting measure μ such that $s = s_\mu$ is the "finest" probability measure among all the probabilities agreeing with s over all Schauder bases or, equivalently, over the collection of all unimodular triangulations of the rational polyhedron P. Theorem 3.3.4 is substantially generalized into an integral representation for states of any MV-algebra in Section 4.

3.4 States of ℓ-groups

The states of MV-algebras are closely related to normalized positive real homomorphisms of lattice ordered Abelian groups (further abbreviated as ℓ-groups); see [27]. We will use the construction of unital ℓ-group associated with an MV-algebra—the details can be found in [36, Section 5]. Let $\langle G, 1\rangle$ be an Abelian ℓ-group with strong unit 1 and neutral element 0 (a unital ℓ-group, for short). Then the order interval $\Gamma(G,1) = \{a \in G \mid 0 \le a \le 1\}$ becomes an MV-algebra with the induced operations $a \oplus b = (a+b) \wedge 1$ and $\neg a = 1 - a$. The group operation $+$ of $\langle G, 1\rangle$ and the MV-algebraic operations of $\Gamma(G,1)$ are related as follows:

$$a + b = (a \oplus b) + (a \odot b), \quad \text{for every } a, b \in \Gamma(G,1). \tag{14}$$

Conversely, given some MV-algebra $\boldsymbol{A}$, Mundici [43] constructed the unital Abelian ℓ-group $\langle G_{\boldsymbol{A}}, 1\rangle$ such that $\boldsymbol{A}$ is isomorphic to $\Gamma(G_{\boldsymbol{A}}, 1)$ and showed that Γ provides the categorical equivalence between the category of MV-algebras and that of unital ℓ-groups.

By G^+ we denote the partially ordered monoid of all positive elements ($a \geq 0$) in a unital ℓ-group $\langle G, 1\rangle$. A *state of a unital ℓ-group* $\langle G, 1\rangle$ is a group homomorphism $s\colon G \to \mathsf{R}$ such that $s(a) \geq 0$, for every $a \in G^+$, and $s(1) = 1$. By $\mathcal{S}(G, 1)$ we denote the set of all states of $\langle G, 1\rangle$. It turns out that measuring the elements of an MV-algebra $\boldsymbol{A}$ with a state s is essentially the same as specifying a state of the corresponding unital ℓ-group $\langle G_{\boldsymbol{A}}, 1\rangle$.

PROPOSITION 3.4.1. *Let $\langle G, 1\rangle$ be a unital ℓ-group and $\boldsymbol{A} = \Gamma(G, 1)$ be the associated MV-algebra. Then:*

(i) For every state s of $\langle G, 1\rangle$, the restriction $\bar{s}$ of s to $\boldsymbol{A}$ is a state of $\boldsymbol{A}$.

(ii) The mapping $s \mapsto \bar{s}$ is an affine isomorphism of $\mathcal{S}(G, 1)$ onto $\mathcal{S}(\boldsymbol{A})$.

Proof. (i) Clearly, $\bar{s}(1) = s(1) = 1$. Let $a, b \in A$ be such that $a \odot b = 0$. By (14) this means that $a \oplus b = a + b$, where $+$ is the addition in $(G, 1)$. Therefore

$$\bar{s}(a \oplus b) = \bar{s}(a + b) = s(a + b) = s(a) + s(b) = \bar{s}(a) + \bar{s}(b).$$

(ii) We need to invert the mapping $s \mapsto \bar{s}$ sending a state s of $\langle G, 1\rangle$ to its restriction $\bar{s}$ on $\boldsymbol{A}$. To this end, assume that r is a state of $\boldsymbol{A}$. First, we extend state r to a monoid homomorphism $\tilde{r}\colon G^+ \to [0, \infty)$ in a unique way as follows. We can identify G^+ with the monoid of good sequences [36, Proposition 5.1.14]. Specifically, for every $a \in G^+$ there exists a unique tuple $\langle g_1, \ldots, g_n\rangle \in A^n$ (up to appending a finite sequence of 0s) such that

(i) $a = g_1 + \cdots + g_n$,

(ii) $g_i \oplus g_{i+1} = g_i$,

(iii) $g_i \odot g_{i+1} = g_{i+1}$.

Define $\tilde{r}(a) = \tilde{r}(\langle g_1, \ldots, g_n\rangle) = r(g_1) + \cdots + r(g_n)$. Clearly $\tilde{r}(0) = r(0) = 0$. We need only show that $\tilde{r}(a + b) = \tilde{r}(a) + \tilde{r}(b)$ for every $a \in G^+$ and every $b \in A$. Put

$$\begin{aligned}
g'_1 &= g_1 \oplus b,\\
g'_2 &= g_2 \oplus g_1 \odot b,\\
g'_3 &= g_3 \oplus g_2 \odot b,\\
&\vdots\\
g'_n &= g_n \oplus g_{n-1} \odot b,\\
g'_{n+1} &= g_n \odot b.
\end{aligned}$$

It can be shown that the good sequence $\langle g'_1, \ldots, g'_{n+1} \rangle$ represents $a + b \in G^+$. Then

$$\begin{aligned}\tilde{r}(a+b) &= \tilde{r}(\langle g'_1, \ldots, g'_{n+1} \rangle) = r(g'_1) + r(g'_2) + \cdots + r(g'_{n+1}) \\ &= r(g_1 \oplus b) + r(g_2 \oplus g_1 \odot b) + \cdots + r(g_n \odot b).\end{aligned}$$

The application of strong modularity (Proposition 3.1.1(iv)) to each summand makes it possible to write the last sum as

$$\begin{aligned}& r(g_1) + r(b) - r(g_1 \odot b) \\ & + r(g_2) + r(g_1 \odot b) - r(\underbrace{g_2 \odot g_1}_{g_2} \odot b) \\ & + r(g_3) + r(g_2 \odot b) - r(\underbrace{g_3 \odot g_2}_{g_3} \odot b) \\ & + \cdots \\ & + r(g_n) + r(g_{n-1} \odot b) - r(\underbrace{g_n \odot g_{n-1}}_{g_n} \odot b) \\ & + r(g_n \odot b) \\ & = r(g_1) + \cdots + r(g_n) + r(b) \\ & = \tilde{r}(a) + \tilde{r}(b).\end{aligned}$$

This shows that $\tilde{r}$ is a monoid homomorphism. Since $G = G^+ - G^+$, we can put $\hat{r}(a - b) = \tilde{r}(a) - \tilde{r}(b)$ for every $a, b \in G$ and routinely show that the definition is correct. The state $\hat{r}$ is the sought unique extension of r. Consequently, there exists a one-to-one correspondence between $\mathcal{S}(G, 1)$ and $\mathcal{S}(\boldsymbol{A})$ given by the restriction map $s \mapsto \bar{s}$. This map is easily seen to be affine. □

Thus every state of an MV-algebra $\boldsymbol{A}$ can be lifted to the unique state of the enveloping Abelian ℓ-group $G_{\boldsymbol{A}}$. This fact has, among others, the following interesting consequence derived from the known results about states of ℓ-groups.

COROLLARY 3.4.2. *Let $\boldsymbol{A}$ be an* MV*-algebra and $\boldsymbol{B}$ be a sub-*MV*-algebra. For every state s of $\boldsymbol{B}$ there exists a state s' of $\boldsymbol{A}$ such that $s'(b) = s(b)$, $b \in B$.*

Proof. The ℓ-group $G_{\boldsymbol{B}}$ is a subgroup of the ℓ-group $G_{\boldsymbol{A}}$. The state s of $\boldsymbol{B}$ lifts to a unique state (also denoted by s) of $G_{\boldsymbol{B}}$. It suffices to show that there exists a state s' of $G_{\boldsymbol{A}}$ such that $s'(b) = s(b)$, $b \in G_{\boldsymbol{B}}$. The last claim is, however, the content of [27, Corollary 4.3]. □

4 Integral representation

In this section we are going to prove one of the main results of this chapter: the one-to-one correspondence of states to regular Borel probability measures over the maximal ideal space of an MV-algebra. This correspondence is realized via Lebesgue integral and turns out to have very strong geometrical and topological properties. Specifically, we will show that the integral states of the form s_μ introduced in (6) are the most general examples of states.

From now on, we always assume that $\mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$, the Bauer simplex of all regular Borel probability measures over $\mathrm{Max}(\boldsymbol{A})$, is equipped with the weak* topology (see Section 2.4).

THEOREM 4.0.1. *Let $\boldsymbol{A}$ be an* MV*-algebra and $\mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$ be the set of all regular Borel probability measures on $\mathrm{Max}(\boldsymbol{A})$. Then there is an affine homeomorphism*

$$\Phi\colon \mathcal{S}(\boldsymbol{A}) \to \mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$$

such that, for every $a \in A$,

$$s(a) = \int\limits_{\mathrm{Max}(\boldsymbol{A})} a^*(M)\, \mathrm{d}\mu_s(M), \qquad \textit{where } \mu_s = \Phi(s). \tag{15}$$

Proof. In the light of Proposition 3.1.7(ii), we may assume that $\boldsymbol{A}$ is semisimple, without loss of generality. In particular, $\boldsymbol{A}$ is isomorphic to a separating MV-algebra $\boldsymbol{A}^*$ of continuous functions $\mathrm{Max}(\boldsymbol{A}) \to [0,1]$ (see Theorem 2.1.7). Let $s \in \mathcal{S}(\boldsymbol{A}^*)$. We are going to extend s uniquely to a bounded linear functional over the Banach space $C(\mathrm{Max}(\boldsymbol{A}))$ of all continuous functions $\mathrm{Max}(\boldsymbol{A}) \to \mathsf{R}$ endowed with the supremum norm $\|\cdot\|$.

By Proposition 3.4.1, the state s of $\boldsymbol{A}^*$ uniquely corresponds to a state s' of unital ℓ-group $G_{\boldsymbol{A}^*}$ such that $\Gamma(G_{\boldsymbol{A}^*}, 1) = \boldsymbol{A}^*$. The unital ℓ-group $G_{\boldsymbol{A}^*}$ embeds in its divisible hull, i.e., the rational sub-vector lattice $H_{\boldsymbol{A}^*} = \{qa \mid a \in G_{\boldsymbol{A}^*}, q \in \mathsf{Q}\}$ (see, e.g. [9, Sections 1.6.8–1.6.9]). Putting $s''(qa) = qs'(a)$ for every $q \in \mathsf{Q}$ and $a \in G_{\boldsymbol{A}^*}$, it is easy to check that $s''\colon H_{\boldsymbol{A}^*} \to \mathsf{R}$ is a positive linear functional uniquely extending state s'. The functional s'' is bounded: if $b \in H_{\boldsymbol{A}^*}$ is such that $\|b\| \leq 1$, then $|s''(b)| \leq s''(1) = 1$. Hence we obtain a continuous linear functional s'' on the vector lattice $H_{\boldsymbol{A}^*}$.

The lattice version of Stone–Weierstrass Theorem [6, Theorem 9.12] says that $H_{\boldsymbol{A}^*}$ is a norm-dense subspace of the Banach space $C(\mathrm{Max}(\boldsymbol{A}))$. For every $b \in C(\mathrm{Max}(\boldsymbol{A}))$, there exists a sequence $\langle b_n \rangle \in H_{\boldsymbol{A}^*}^{\mathsf{N}}$ such that $\|b - b_n\| \to 0$ whenever $n \to \infty$. Therefore, we can uniquely extend s'' onto $C(\mathrm{Max}(\boldsymbol{A}))$ by letting $\hat{s}(b) = \lim_{n\to\infty} s(b_n)$. Since for $b \geq 0$ we can find the converging sequence with elements $b_n \geq 0$, the unique extension $\hat{s}$ is a positive linear functional on $C(\mathrm{Max}(\boldsymbol{A}))$.

In order to complete the proof, it suffices to apply the Riesz representation theorem [6, Theorem 14.14] to $\hat{s}$. This yields a unique regular Borel probability measure μ_s such that

$$\hat{s}(b) = \int_{\mathrm{Max}(\boldsymbol{A})} b \,\mathrm{d}\mu_s, \qquad \text{for every } b \in C(\mathrm{Max}(\boldsymbol{A})),$$

so that

$$s(a^*) = \hat{s}(a^*) = \int_{\mathrm{Max}(\boldsymbol{A})} a^* \,\mathrm{d}\mu_s. \tag{16}$$

Consider the mapping $\Phi\colon \mathcal{S}(\boldsymbol{A}) \to \mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$ sending each s to a unique $\mu_s = \Phi(s)$ such that (16) holds. It is easy to see that Φ is an affine mapping onto $\mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$. Let $\mu, \nu \in \mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$ and $\mu = \nu$. Then

$$s_\mu(a^*) = \int_{\mathrm{Max}(\boldsymbol{A})} a^* \,\mathrm{d}\mu = \int_{\mathrm{Max}(\boldsymbol{A})} a^* \,\mathrm{d}\nu = s_\nu(a^*),$$

for every $a \in A$. Thus $s_\mu = s_\nu$ and the mapping Φ is one-to-one. To finish the proof, we only need to check that the affine isomorphism Φ is also a homeomorphism. Since Φ is a bijection between compact Hausdorff spaces, it suffices to check that Φ^{-1} is continuous. Let (μ_γ) be a net in $\mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$ weak* converging to some $\mu \in \mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$. This means that

$$\int_{\mathrm{Max}(\boldsymbol{A})} f \,\mathrm{d}\mu_\gamma \to \int_{\mathrm{Max}(\boldsymbol{A})} f \,\mathrm{d}\mu \quad \text{for every } f \in C(\mathrm{Max}(\boldsymbol{A})). \tag{17}$$

However, setting $f = a^*$ for $a \in A$, formula (17) reads as

$$s_{\mu_\gamma}(a) \to s_\mu(a) \quad \text{for every } a \in A.$$

In other words, the net $(\Phi^{-1}(\mu_\gamma))$ converges to $\Phi^{-1}(\mu)$ in the state space $\mathcal{S}(\boldsymbol{A})$. This concludes the proof. □

COROLLARY 4.0.2. *Let $\boldsymbol{A}_X$ be a separating clan of continuous functions over a compact Hausdorff space X. Then for every state s of $\boldsymbol{A}_X$ there exists a unique regular Borel probability measure μ_s on X such that $s(a) = \int_X a \,\mathrm{d}\mu_s$, for every $a \in A_X$.*

REMARK 4.0.3. *The special case of Theorem 4.0.1 appeared in [32], where the formula* (15) *was proved with the assumption that $\boldsymbol{A}$ is semisimple and without the explicit proof of the uniqueness of the representing Borel probability measure μ_s. The main idea of the proof of Theorem 4.0.1 presented above belongs to Panti [52]. Independently, the uniqueness of representing probability measure μ_s for any state s of the semisimple* MV*-algebra $\boldsymbol{A}$ was established in [31], where also the proof that Φ is an affine homeomorphism appears. The result presented in [31] in fact shows that Theorem 4.0.1 is equivalent to characterizing the state space of $\boldsymbol{A}$ as a Bauer simplex.*

REMARK 4.0.4. *If X is a metrizable space, then every Borel probability measure on X is regular and we may thus drop the word "regular" in the statement of the above theorems. A case in point is the maximal ideal space* $\mathrm{Max}(\boldsymbol{A})$ *of every countable* MV*-algebra $\boldsymbol{A}$ since the corresponding* $\mathrm{Max}(\boldsymbol{A})$ *is second countable and thus metrizable.*

A special case of Theorem 4.0.1 for a Boolean algebra $\boldsymbol{B}$ shows somewhat unexpected result: every finitely additive probability corresponds to a unique regular Borel probability measure on the Stone space of $\boldsymbol{B}$.

COROLLARY 4.0.5. *Let $\boldsymbol{B}$ be a Boolean algebra. For every finitely additive probability ν of $\boldsymbol{B}$ there exists a unique regular Borel probability measure μ_ν on the Stone space* $\mathrm{Max}(\boldsymbol{B})$ *of $\boldsymbol{B}$ such that*

$$\nu(a) = \mu_\nu(a^*), \quad \textit{for every } a \in \boldsymbol{B},$$

where a^* is the clopen subset of $\mathrm{Max}(\boldsymbol{B})$ *corresponding to a.*

A word of caution is in order here: the statement of Corollary 4.0.5 does not express the wrong claim "every finitely additive probability is a probability measure". As in the proof of Theorem 4.0.1, the real content of Corollary 4.0.5 is the extension of the dual representation of finitely additive probability to a continuous linear functional on the Banach space of continuous functions over the Stone space.

REMARK 4.0.6. *To the best of the authors' knowledge, there is only one reference where the result of Corollary 4.0.5 is explicitly formulated. Namely, Nešetřil and Ossona de Mendez [50] use the correspondence of finitely additive probabilities and regular Borel probability measures in their model-theoretic study of graph limits.*

4.1 Characterization of state space

Using integral representation, we will slightly refine the description of state space provided by Proposition 3.1.7. Recall that we always consider the state space $\mathcal{S}(\boldsymbol{A})$ with the relative topology of the product space $[0,1]^A$.

THEOREM 4.1.1. *Let $\boldsymbol{A}$ be an* MV*-algebra. Then:*

(i) The state space $\mathcal{S}(\boldsymbol{A})$ is a Bauer simplex.

(ii) The extreme boundary $\partial\mathcal{S}(\boldsymbol{A})$ is compact and

$$\partial\mathcal{S}(\boldsymbol{A}) = \{h \in \mathcal{S}(\boldsymbol{A}) \mid h \text{ is a homomorphism } A \to [0,1]\}.$$

(iii) $\mathcal{S}(\boldsymbol{A}) = \overline{\mathrm{co}}(\mathcal{H}(\boldsymbol{A}))$.

Proof. Without loss of generality, we may assume that the algebra $\boldsymbol{A}$ is semisimple (Proposition 3.1.7(ii)). By Proposition 3.1.7(i), $\mathcal{S}(\boldsymbol{A})$ is a compact convex set in $[0,1]^A$. Hence, it suffices to show that $\mathcal{S}(\boldsymbol{A})$ is a Choquet simplex with $\partial\mathcal{S}(\boldsymbol{A})$ compact. By Theorem 4.0.1, the mapping $\Phi\colon \mathcal{S}(\boldsymbol{A}) \to \mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$ is an affine homeomorphism. This implies that Φ^{-1} is an affine homeomorphism also. Theorem 2.4.4 yields

$$\partial\mathcal{M}(\mathrm{Max}(\boldsymbol{A})) = \{\delta_M \mid M \in \mathrm{Max}(\boldsymbol{A})\}$$

so that every element $s \in \Phi^{-1}(\partial\mathcal{M}(\mathrm{Max}(\boldsymbol{A})))$ is of the form

$$s_M\colon a \in A \mapsto a^*(M) \in [0,1].$$

Thus s_M is a homomorphism. Conversely, every homomorphism of $\boldsymbol{A}$ into $[0,1]$ arises in this way. It follows that the extreme boundary $\partial\mathcal{S}(\boldsymbol{A})$ must be compact since Φ is a homeomorphism. Since Φ is an affine isomorphism and $\mathcal{M}(\mathrm{Max}(\boldsymbol{A}))$ is a simplex, the image $\Phi^{-1}(\mathcal{M}(\mathrm{Max}(\boldsymbol{A}))) = \mathcal{S}(\boldsymbol{A})$ is also a simplex. This establishes (i) and (ii). The part (iii) directly follows from (i)–(ii) and Krein-Milman theorem (Theorem 2.4.2). □

Theorem 4.1.1 says that these are the only examples of a state $s \in \mathcal{S}(\boldsymbol{A})$:

(i) A homomorphism $h \in \partial\mathcal{S}(\boldsymbol{A})$

(ii) A convex combination $g = \sum_{i=1}^n \alpha_i h_i$ for some $h_1, \dots, h_n \in \partial\mathcal{S}(\boldsymbol{A})$ and non-negative reals $\alpha_1, \dots, \alpha_n$ satisfying $\sum_{i=1}^n \alpha_i = 1$

(iii) the limit in $[0,1]^A$ of a generalized sequence (g_γ), where every g_γ is as in (ii).

Recall that an MV-algebra $\boldsymbol{A}$ is said to be *simple* whenever the only maximal ideal of $\boldsymbol{A}$ is $\{0\}$.

COROLLARY 4.1.2. *Let* $\boldsymbol{A}$ *be an* MV*-algebra. Then:*

(i) There are n *maximal ideals in* $\boldsymbol{A}$ *if and only if* $\mathcal{S}(\boldsymbol{A})$ *is affinely homeomorphic to the* $(n-1)$*-dimensional standard simplex.*

(ii) In particular, if $\boldsymbol{A}$ *is simple, then the only state of* $\boldsymbol{A}$ *is the unique embedding of* $\boldsymbol{A}$ *into the real unit interval* $[0,1]$.

In conclusion, simpliciality and the topology of state space enables us to claim that states of an MV-algebra and regular Borel probability measures over the compact Hausdorff maximal ideal space can be identified up to an affine homeomorphism. The affine homeomorphism is determined by the integral formula (15), which is frequently used in applications.

4.2 Existence of invariant states and faithful states

The integral representation theorem for states enables us to study MV-algebraic dynamics in analogy with the probabilistic dynamics. A *measure-theoretic dynamical system* is a quadruple $\langle X, \mathfrak{A}, \mu, T\rangle$, where X is a nonempty set, $\mathfrak{A}$ is a σ-algebra of subsets of X, a mapping $T\colon X \to X$ is $\mathfrak{A}$-$\mathfrak{A}$ measurable ($T^{-1}(A) \in \mathfrak{A}$, for every $A \in \mathfrak{A}$), and $\mu\colon \mathfrak{A} \to [0,1]$ is a probability measure invariant with respect to T, i.e,

$$\mu(T^{-1}(A)) = \mu(A), \quad \text{for every } A \in \mathfrak{A}.$$

Analogously, let e be an endomorphism of an MV-algebra $\boldsymbol{A}$ and $s \in \mathcal{S}(\boldsymbol{A})$. Put

$$s^e(a) = s(e(a)), \quad a \in A. \tag{18}$$

Then s^e is a state of $\boldsymbol{A}$. We call s an e*-invariant state* whenever $s^e = s$. Every endomorphism e of $\boldsymbol{A}$ induces a continuous transformation $T_e\colon \mathrm{Max}(\boldsymbol{A}) \to \mathrm{Max}(\boldsymbol{A})$:

$$T_e(M) = \{a \in A \mid e(a) \in M\}, \quad \text{for every } M \in \mathrm{Max}(\boldsymbol{A}).$$

In particular, the function T_e is $\mathfrak{B}(\mathrm{Max}(\boldsymbol{A}))$-$\mathfrak{B}(\mathrm{Max}(\boldsymbol{A}))$ measurable.

PROPOSITION 4.2.1. *Let* e *be an endomorphism of an* MV*-algebra* $\boldsymbol{A}$ *and* $s \in \mathcal{S}(\boldsymbol{A})$. *Then* s *is an* e*-invariant state if and only if* $\langle \mathrm{Max}(\boldsymbol{A}), \mathfrak{B}(\mathrm{Max}(\boldsymbol{A})), \mu_s, T_e\rangle$ *is a measure-theoretic dynamical system.*

Proof. Clearly, e is an endomorphism of the isomorphic image $\boldsymbol{A}^*$ of $\boldsymbol{A}$ and we have $e(a^*) = a^* \circ T_e$. Then Theorem 4.0.1 and the change of variables in the Lebesgue integral (see [6, Theorem 13.46]) yield

$$s^e(a) = s(e(a)) = \int_{\mathrm{Max}(\boldsymbol{A})} e(a^*)\,\mathrm{d}\mu_s = \int_{\mathrm{Max}(\boldsymbol{A})} a^* \circ T_e\,\mathrm{d}\mu_s = \int_{\mathrm{Max}(\boldsymbol{A})} a^*\,\mathrm{d}\left(\mu_s \circ T_e^{-1}\right).$$

Hence the equality

$$\int_{\mathrm{Max}(\boldsymbol{A})} a^*\,\mathrm{d}\left(\mu_s \circ T_e^{-1}\right) = \int_{\mathrm{Max}(\boldsymbol{A})} a^*\,\mathrm{d}\mu_s = s(a)$$

holds if and only if μ_s is invariant with respect to T_e. □

PROPOSITION 4.2.2. *Let e be an endomorphism of an* MV*-algebra $\boldsymbol{A}$. Then there exists an e-invariant state s of $\boldsymbol{A}$.*

Proof. Consider the mapping $\bar{e}\colon \mathcal{S}(\boldsymbol{A}) \to \mathcal{S}(\boldsymbol{A})$ defined by $\bar{e}(s) = s^e$, where s^e is as in (18). Then $\bar{e}$ is continuous. Indeed, for every $a \in A$ and every net (s_γ) in $\mathcal{S}(\boldsymbol{A})$ such that $s_\gamma \to s \in \mathcal{S}(\boldsymbol{A})$, we have

$$\bar{e}(s_\gamma)(a) = s_\gamma(e(a)) \to s(e(a)) = \bar{e}(s)(a).$$

Because $\bar{e}$ is a continuous mapping of the compact convex set $\mathcal{S}(\boldsymbol{A})$ into itself, the Brouwer–Schauder–Tychonoff fixed point theorem [6, Theorem 17.56] says that there must exist a fixed point of the mapping $\bar{e}$, a state $s \in \mathcal{S}(\boldsymbol{A})$ such that $\bar{e}(s) = s$. □

The existence of a state s invariant with respect to a single endomorphism e is the consequence of purely geometrical-topological properties of the state space. The situation becomes more interesting if we require invariance of s with respect to all the automorphisms of the MV-algebra: we call a state s of $\boldsymbol{A}$ *invariant* if it is α-invariant for every automorphism α of $\boldsymbol{A}$. The existence of invariant states possibly satisfying additional properties can directly be proved for the free n-generated MV-algebra.

THEOREM 4.2.3. *The free n-generated* MV*-algebra $\boldsymbol{F}_n$ has an invariant faithful state with rational values.*

Proof. The natural candidate for a state from the statement is the Lebesgue state s_λ (Example 3.2.3) given by the Riemann integral

$$s_\lambda(f) = \int_{[0,1]^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x}, \quad f \in F_n.$$

Let $f \in F_n$. By [1, Lemma 2.1.4], there exists a polyhedral complex Σ and n-dimensional convex polytopes $P_1, \ldots, P_m \in \Sigma$ with rational vertices such that f is linear over each P_i and $\bigcup_{i=1}^m P_i = [0,1]^n$. For every $i = 1, \ldots, m$, put

$$v_i(f) = \int_{P_i} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

and observe that $s_\lambda(f) = \sum_{i=1}^m v_i(f)$. Since each polytope P_i has rational vertices and the linear function f over P_i has Z coefficients, the value $v_i(f)$ is rational and thus $s_\lambda(f) \in [0,1] \cap \mathsf{Q}$.

State s_λ is faithful. Indeed, if $f \neq 0$, then there exists a polytope $P_i \in \Sigma$ of dimension n such that f is nonzero in the interior of P_i. Thus $s_\lambda(f) \geq v_i(f) > 0$.

Let α be an automorphism of $\boldsymbol{F}_n$. Then α is determined by its action on the free generators of $\boldsymbol{F}_n$, the i-th coordinate projection functions $\pi_i\colon [0,1]^n \to [0,1]$. Denote $q_i = \alpha(\pi_i)$, for each $i = 1, \ldots, n$, and put $T_\alpha(\mathbf{x}) = (q_1(\mathbf{x}), \ldots, q_n(\mathbf{x}))$, $\mathbf{x} \in [0,1]^n$. The two mappings α and T_α are related by the formula

$$\alpha(f) = f \circ T_\alpha, \quad f \in F_n.$$

It can be shown that T_α is a Z-homeomorphism of $[0,1]^n$, that is, T_α is a homeomorphism of $[0,1]^n$ such that the scalar components of T_α and T_α^{-1} are in F_n. This implies existence of a polyhedral complex Θ such that $Q_1,\ldots,Q_r \in \Theta$ are convex polytopes with rational vertices, $\bigcup_{i=1}^r Q_i = [0,1]^n$, and both T_α and T_α^{-1} are linear over each Q_i. Since the components of T_α and T_α^{-1} restricted to Q_i are linear functions with Z coefficients, the corresponding Jacobian matrix satisfies $|J_{T_\alpha}(\mathbf{x})| = 1$ for every $\mathbf{x}$ in the interior of Q_i. Thus, for every $f \in F_n$,

$$s_\lambda(\alpha(f)) = \int_{[0,1]^n} \alpha(f)(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{[0,1]^n} f(T_\alpha(\mathbf{x}))\,\mathrm{d}\mathbf{x} = \sum_{i=1}^r \int_{Q_i} f(T_\alpha(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$
$$= \sum_{i=1}^r \int_{\mathrm{int}\ Q_i} f(T_\alpha(\mathbf{x}))\cdot|J_{T_\alpha}(\mathbf{x})|\,\mathrm{d}\mathbf{x}.$$

Using the change of variable [6, Theorem 13.49], the last expression is equal to

$$\sum_{i=1}^r \int_{\mathrm{int}\ Q_i} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{[0,1]^n} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = s_\lambda(f). \qquad \square$$

The theorem underlines the importance of Lebesgue state for Łukasiewicz logic: the average truth value of a formula is invariant under all substitutions $A_i \mapsto \alpha(A_1,\ldots,A_n)$, $i = 1,\ldots,n$ such that the equivalence classes $[\alpha(A_1,\ldots,A_n)]$ generate $\boldsymbol{F}_n$. Moreover, the statement of Theorem 4.2.3 can easily be extended onto the class of all finitely presented MV-algebras.

5 De Finetti's coherence criterion for many-valued events

De Finetti's foundation of subjective probability theory is based on the notion of *coherent betting odds*. Two players, a *bookmaker* and a *gambler*, wager money on the occurrence of some *events of interest* $e_1,\ldots,e_k$. At the very first stage of the game, the bookmaker publishes a book β assigning a betting odd $\beta_i \in [0,1]$ to each event e_i and the gambler, once the book has been published, places *stakes* $\sigma_1,\ldots,\sigma_k \in \mathsf{R}$, one for each event e_i, and pays to the bookmaker the amount of $\sum_{i=1}^k \sigma_i \cdot \beta_i$ euros. Notice that each stake σ_i is positive for the gambler whenever the bet is placed, while it is negative for a bet accepted. In other words, the effect of gambler's decision to pay a negative amount σ_i on e_i, is that she will receive, already in this first stage of the game, $\sigma_i \cdot \beta_i$ euros from the bookmaker. We call this assumption *reversibility*.

At this stage of the game, the bookmaker and the gambler are obviously uncertain about the truth value of the events involved in the game. However, once a future possible world w is reached, every e_i is known to be either true or false in this possible world. For every event e_i, the bookmaker pays back to the gambler σ_i euros if e_i turns out to be true in w, or nothing if e_i is false in w.

Formally, if we denote by $w(e_i)$ the truth-value of e_i in the world w, the total balance for the bookmaker at the end of the game is calculated by $\sum_{i=1}^k \sigma_i(\beta_i - w(e_i))$ and hence, in the world w, the bookmaker gains money if his balance is positive, or he looses money if it is negative. Notice that the balance is calculated with respect to a specific possible world w, but if the bookmaker arranges his book β in a such a way that he is going to loose money in every possible world, then we say he incurred a *sure loss* and the book β is called *Dutch*, or *incoherent*. Conversely, the book β is called *coherent* if it does not ensure bookmaker to incur a sure loss, i.e., for every choice of stakes $\sigma_1, \ldots, \sigma_k$, there exists a world w in which the bookmaker's total balance is not negative.

A suitable formalization of classical de Finetti's betting game consists in interpreting events, books and possible worlds by the following stipulations:

- *events* are elements of an arbitrary Boolean algebra $\boldsymbol{B}$,
- a *book* on a finite subset $\{e_1, \ldots, e_k\} \subseteq B$ is a map $\beta \colon e_i \mapsto \beta_i \in [0,1]$, and
- a *possible world* is a Boolean homomorphism of $\boldsymbol{B}$ into the two element Boolean chain $\mathbf{2}$, that is, any element of $\mathcal{H}(\boldsymbol{B}, \mathbf{2})$.

Within this framework, de Finetti's coherence criterion reads as follows:

Classical Coherence Criterion. *Let $\boldsymbol{B}$ be a Boolean algebra and let $\{e_1, \ldots, e_k\}$ be a finite subset of B. A book $\beta \colon e_i \mapsto \beta_i$ is said to be* coherent *iff for each choice of $\sigma_1, \ldots, \sigma_k \in \mathsf{R}$, there exists $w \in \mathcal{H}(\boldsymbol{B}, \mathbf{2})$ such that*

$$\sum_{i=1}^{k} \sigma_i(\beta_i - w(e_i)) \geq 0.$$

The celebrated de Finetti's theorem can be stated as follows.

THEOREM 5.0.1 (de Finetti). *For every Boolean algebra $\boldsymbol{B}$, for every finite subset $\{e_1, \ldots, e_k\}$ and for every book β, the following are equivalent:*

(i) β is coherent.

(ii) There exists a finitely additive probability measure $\mu \colon \boldsymbol{B} \to [0,1]$ such that $\mu(e_i) = \beta_i$, for every $i = 1, \ldots, k$.

It is not difficult to generalize the Classical Coherence Criterion to events being elements of an MV-algebra $\boldsymbol{A}$ and possible worlds being MV-homomorphisms of $\boldsymbol{A}$ into the standard MV-algebra $[0,1]_{\text{Ł}}$. Within the MV-algebraic setting, the game played by the bettor and the bookmaker must take into account that events are evaluated by every possible world w taking values in $[0,1]$. Therefore, in a possible world w, the amount of money that the gambler will receive back from the bookmaker is calculated by *weighting* each stake σ_i with the truth-value $w(e_i) \in [0,1]$ of e_i. This leads to the many-valued version of the Classical Coherence Criterion.

Many-valued Coherence Criterion. *Let $\boldsymbol{A}$ be an MV-algebra and $A' = \{e_1, \ldots, e_k\}$ be a finite subset of A. We say that a book $\beta \colon e_i \mapsto \beta_i$ is* coherent *iff for each choice of $\sigma_1, \ldots, \sigma_k \in \mathsf{R}$, there exists $w \in \mathcal{H}(\boldsymbol{A})$ such that*

$$\sum_{i=1}^{k} \sigma_i(\beta_i - w(e_i)) \geq 0. \tag{19}$$

The aim of the first part of this section is to generalize de Finetti's theorem to many-valued events. First, we need some preliminary results.

LEMMA 5.0.2. *Let $\boldsymbol{A}$ be an* MV*-algebra and $A' = \{e_1, \ldots, e_k\}$ be a finite subset of A. Then, for every $\beta \in \mathrm{co}(\mathcal{H}(\boldsymbol{A}))$, its restriction to A' is a coherent book.*

Proof. Let $A' = \{e_1, \ldots, e_k\}$ be a finite subset of A. Let $\lambda_1, \ldots, \lambda_r \in \mathsf{R}^+$ be such that $\sum_{j=1}^{r} \lambda_j = 1$ and let $w_1, \ldots, w_r \in \mathcal{H}(\boldsymbol{A})$ be such that, for all $i = 1, \ldots, k$,

$$\beta_i = \beta(e_i) = \sum_{j=1}^{r} \lambda_j w_j(e_i). \tag{20}$$

Assume, by way of contradiction, that for some $\sigma_1, \ldots, \sigma_k \in \mathsf{R}$, it holds that, for all $w \in \mathcal{H}(\boldsymbol{A})$, $\sum_{i=1}^{k} \sigma_i(\beta_i - w(e_i)) < 0$. Then, in particular, for all $j = 1, \ldots, r$, one has

$$\sum_{i=1}^{k} \sigma_i(\beta_i - w_j(e_i)) = \sum_{i=1}^{k} \sigma_i \beta_i - \sum_{i=1}^{k} \sigma_i w_j(e_i) < 0.$$

Therefore, since $\sum_{j=1}^{r} \lambda_j = 1$,

$$\sum_{i=1}^{k} \sigma_i \beta_i - \sum_{j=1}^{r} \lambda_j \left(\sum_{i=1}^{k} \sigma_i w_j(e_i) \right) < 0,$$

that is, from (20),

$$\sum_{i=1}^{k} \sigma_i \left(\sum_{j=1}^{r} \lambda_j w_j(e_i) \right) - \sum_{j=1}^{r} \lambda_j \left(\sum_{i=1}^{k} \sigma_i w_j(e_i) \right) < 0,$$

a contradiction. □

LEMMA 5.0.3. *Let $\boldsymbol{A}$ be an* MV*-algebra, let $A' = \{e_1, \ldots, e_k\}$ be a finite subset of A and let β be a book on A'. If β is coherent, then there exists $\gamma \in \mathrm{co}(\mathcal{H}(\boldsymbol{A}))$ such that $\beta(e_i) = \gamma(e_i)$ for all $e_i \in A'$.*

Proof. Let

$$\mathcal{H} \restriction_{A'} = \{\mathbf{x} \in [0,1]^k \mid \mathbf{x} = (h(e_1), \ldots, h(e_k)) \text{ for } h \in \mathcal{H}(\boldsymbol{A})\}. \tag{21}$$

Since $\mathcal{H}(\boldsymbol{A})$ is closed in $[0,1]^A$, then both $\mathcal{H} \restriction_{A'}$ and $\mathrm{co}(\mathcal{H} \restriction_{A'})$ are compact subsets of $[0,1]^k$. Let $\mathbf{y} = \langle \beta(e_1), \ldots, \beta(e_k) \rangle \in [0,1]^k$ and assume, by way of contradiction, that $\mathbf{y} \notin \mathrm{co}(\mathcal{H} \restriction_{A'})$. Then, since $\mathrm{co}(\mathcal{H} \restriction_{A'})$ is convex, by the Separating Hyperplane Theorem [22, Lemma 3.5] there exist $\mathbf{p} \in \mathsf{R}^k$ and $r \in \mathsf{R}$ such that the affine hyperplane $H = \{\mathbf{a} \mid \mathbf{p} \circ \mathbf{a} = r\}$ strongly separates $\mathbf{y}$ and $\mathrm{co}(\mathcal{H} \restriction_{A'})$, meaning that the scalar product $\mathbf{p} \circ \mathbf{y} < r$ and, for all $\mathbf{x} \in \mathcal{H} \restriction_{A'}$, we have $\mathbf{p} \circ \mathbf{x} > r$. In particular, for every $\mathbf{x} \in \mathcal{H} \restriction_{A'}$, it follows that $\mathbf{p} \circ (\mathbf{y} - \mathbf{x}) < 0$ and hence, letting $\mathbf{p} = \langle \sigma_1, \ldots, \sigma_k \rangle$,

$$\sum_{i=1}^{k} \sigma_i(\beta(e_i) - h(e_i)) < 0$$

for every $h \in \mathcal{H}(\boldsymbol{A})$, contradicting the coherence of β. □

THEOREM 5.0.4. *Let $\boldsymbol{A}$ be an* MV*-algebra, $A' = \{e_1, \ldots, e_k\}$ be a finite subset of A and let β be a book on A'. Then the following are equivalent:*

(i) β is coherent.

(ii) There exists a state $s \in \mathcal{S}(\boldsymbol{A})$ such that s coincides with β over A'.

(iii) β can be extended to a convex combination of at most $k+1$ elements of $\mathcal{H}(A)$.

Proof. The fact that the last two claims are equivalent is a consequence of Carathéodory theorem [22, Theorem 2.3]. To prove the equivalence of the first two we shall use the fact that, for every MV-algebra $\boldsymbol{A}$, $\mathcal{S}(\boldsymbol{A}) = \overline{\mathrm{co}}(\mathcal{H}(\boldsymbol{A}))$ (Theorem 4.1.1(iii)).

(ii) $\Rightarrow$ (i): Let $s \in \overline{\mathrm{co}}(\mathcal{H}(\boldsymbol{A}))$ and assume, by way of contradiction, that β is not A'-coherent for some finite subset $A' = \{e_1, \ldots, e_k\}$ of A. So there are $\sigma_1, \ldots, \sigma_k \in \mathsf{R}$ such that, for every $w \in \mathcal{H}(\boldsymbol{A})$, $\sum_{i=1}^{k} \sigma_i(\beta_i - w(e_i)) < 0$, which gives

$$\sum_{i=1}^{k} \sigma_i \beta_i < \sum_{i=1}^{k} \sigma_i w(e_i).$$

From Theorem 4.1.1(ii), $\mathcal{H}(\boldsymbol{A})$ is closed and therefore, by the continuity of addition and multiplication,

$$\begin{aligned}
\min_{w \in \mathcal{H}(\boldsymbol{A})} \sum_{i=1}^{k} \sigma_i w(e_i) &= \sum_{i=1}^{k} \min_{w \in \mathcal{H}(\boldsymbol{A})} \sigma_i w(e_i) \\
&= \sum_{i=1}^{k} \sigma_i \min_{w \in \mathcal{H}(\boldsymbol{A})} w(e_i) \\
&= \sum_{i=1}^{k} \sigma_i \overline{w}(e_i)
\end{aligned}$$

for some $\overline{w} \in \mathcal{H}(\boldsymbol{A})$. Further, let

$$z = \sum_{i=1}^{k} \sigma_i \overline{w}(e_i) - \sum_{i=1}^{k} \sigma_i \beta_i > 0.$$

Since by hypothesis s is a state and so it belongs to $\overline{\mathrm{co}}(\mathcal{H}(\boldsymbol{A}))$, it holds

$$\forall \varepsilon > 0\ \exists \beta' \in \mathrm{co}(\mathcal{H}(\boldsymbol{A})) \text{ such that, } \forall i = 1, \ldots, k,\ |\beta_i - \beta'(e_i)| < \varepsilon.$$

Therefore, in particular, for all sufficiently small ε, there is $\beta' \in \mathrm{co}(\mathcal{H}(\boldsymbol{A}))$, such that, for all $w \in \mathcal{H}(\boldsymbol{A})$,

$$\sum_{i=1}^{k} \sigma_i \beta'(e_i) < \frac{z}{2} + \sum_{i=1}^{k} \sigma_i \beta_i < \sum_{i=1}^{k} \sigma_i \overline{w}(e_i) \leq \sum_{i=1}^{k} \sigma_i w(e_i)$$

and thus $\sum_{i=1}^{k} \sigma_i(\beta'(e_i) - w(e_i)) < 0$. Therefore β' is not coherent, contradicting Lemma 5.0.2.

(i) ⇒ (ii): Let β be coherent. Then, for every finite subset $A' \subseteq A$, Lemma 5.0.3 ensures that the set

$$\mathcal{D}(A') = \{\gamma \in \overline{\text{co}}(\mathcal{H}(\boldsymbol{A})) \mid \beta = \gamma \text{ on } A'\}$$

is a nonempty closed subset of $\overline{\text{co}}(\mathcal{H}(\boldsymbol{A}))$. Notice that, for every choice of finite subsets $A'_1, \dots, A'_m \subseteq A$,

$$\mathcal{D}(A'_1) \cap \cdots \cap \mathcal{D}(A'_m) = \mathcal{D}(A'_1 \cup \cdots \cup A'_m)$$

and hence the family $\{\mathcal{D}(A') \mid A' \text{ is a finite subset of } A\}$ has the finite intersection property. Therefore there exists $\gamma \in \overline{\text{co}}(\mathcal{H}(\boldsymbol{A}))$ such that

$$\gamma \in \bigcap\{\mathcal{D}(A') \mid A' \text{ is a finite subset of } A\}.$$

Thus $\gamma = \beta$ on A and $\beta \in \overline{\text{co}}(\mathcal{H}(\boldsymbol{A}))$ and so the claim follows by setting $\gamma = s$. □

5.1 Betting on formulas of Łukasiewicz logic

It is common to think about an *event* as a formula in the language of a logical calculus. For this reason, a suitable setting for many-valued events is the Lindenbaum algebra of Łukasiewicz propositional logic. By Form (Form_n) we denote the class of all formulas in Łukasiewicz logic (the class of all formulas containing at most the first n propositional variables), respectively. In Section 2.2 we recalled that the Lindenbaum algebra of Łukasiewicz propositional calculus with n variables is the free n-generated MV-algebra $\boldsymbol{F}_n$ which, in turn, is isomorphic to the MV-algebra of McNaughton functions on $[0,1]^n$ with pointwise operations of $\oplus$ and $\neg$.

In this section we are going to provide an alternative proof of Theorem 5.0.4 considering coherent assignments on finite subsets of the free n-generated MV-algebra $\boldsymbol{F}_n$. Moreover, we will show that the problem of establishing the coherence for a rational-valued book on formulas of Łukasiewicz logic is $\mathbf{NP}$-complete.

We now prepare some terminology and notation in view of the main theorem of this section. We assume a reasonably compact binary encoding of $\phi \in \mathsf{Form}$, such that the number $\mathrm{size}(\phi)$ of bits in the encoding of ϕ is bounded above by a polynomial $e_1 \colon \mathsf{N} \to \mathsf{N}$ of the number $c(\phi)$ of symbols $\odot, \to$ occurring in ϕ, that is,

$$\mathrm{size}(\phi) \leq e_1(c(\phi_i)).$$

Analogously, we assume that the length in bits of the encoding of a finite set of formulas $\{\phi_1, \dots, \phi_k\} \subseteq \mathsf{Form}$, in symbols $\mathrm{size}(\{\phi_1, \dots, \phi_k\})$, satisfies

$$\mathrm{size}(\{\phi_1, \dots, \phi_k\}) \leq e_2(\mathrm{size}(\phi_1) + \cdots + \mathrm{size}(\phi_k))$$

for some polynomial $e_2 \colon \mathsf{N} \to \mathsf{N}$. Also, letting $\beta \colon \{\phi_1, \dots, \phi_k\} \to [0,1]$ be a rational book such that $\beta(\phi_i) = n_i/d_i$ with n_i and d_i relatively prime integers for all i in $\{1, \dots, k\}$, we assume a binary encoding of β such that the number of bits in the encoding of β, in symbols, $\mathrm{size}(\beta)$, satisfies

$$\mathrm{size}(\beta) \leq e_3\left(\mathrm{size}(\{\phi_1, \dots, \phi_k\}) + k \cdot \log_2 \max\{d_1, \dots, d_k\}\right)$$

for some polynomial $e_3 \colon \mathsf{N} \to \mathsf{N}$.

PROPOSITION 5.1.1. *Let s be a state of $\boldsymbol{F}_n$ and let H_Σ be a Schauder basis for $\boldsymbol{F}_n$. If $h_i \in H_\Sigma$ is the Schauder hat at $\mathbf{x}_i$, then $s(\mathrm{den}(\mathbf{x}_i) \cdot h_i) = \mathrm{den}(\mathbf{x}_i) \cdot s(h_i)$.*

Proof. This is a direct consequence of additivity of s and the definition of a (normalized) Schauder hat. □

PROPOSITION 5.1.2. *Let $\phi_1, \dots, \phi_k \in \mathsf{Form}_n$ for some $k \geq 1$. Then there exist a unary polynomial $q\colon \mathsf{N} \to \mathsf{N}$ and a unimodular triangulation Σ of $[0,1]^n$ linearizing $[\phi_1], \dots, [\phi_k]$, such that each rational vertex $\mathbf{x}$ of Σ satisfies*

$$\log_2 \mathrm{den}(\mathbf{x}) \leq q(\mathrm{size}(\{\phi_1, \dots, \phi_k\})).$$

Proof. For all $i \in \{1, \dots, k\}$, let f_i be the n-ary McNaughton function $[\phi_i]$. Let $p_1, \dots, p_l$ be the list of all the linear pieces of the functions $f_1, \dots, f_k$, together with the projection functions $x_1, \dots, x_n$ and the constants $0, 1$, and define P_π as in (8) and C as in (9) based on these pieces. Let Σ be a unimodular triangulation produced from C without adding new vertices, as explained in Section 3.3. We show that Σ satisfies the statement.

First, since C includes all the linear domains of all the functions $f_1, \dots, f_k$ and Σ is a subdivision of C, it follows that Σ linearizes all the functions $f_1, \dots, f_k$. Second, by the definition of McNaughton function each piece p_i has the form

$$p_i(x_1, \dots, x_n) = c_{i,1}x_1 + \dots + c_{i,n}x_n + d_i$$

with $c_{i,1}, \dots, c_{i,n}, d_n \in \mathsf{Z}$. Thus, by inspection of (8), each vertex $\mathbf{x} \in \mathrm{V}_\Sigma$ is the rational solution of a system of n linear equations in n unknowns, each equation having one of the following forms

$$\begin{aligned} p_h(x_1, \dots, x_n) &= p_i(x_1, \dots, x_n) \\ p_h(x_1, \dots, x_n) &= 0 \\ p_h(x_1, \dots, x_n) &= 1 \end{aligned}$$

for $h, i \in \{1, \dots, l\}$, or $x_i = 0$, $x_i = 1$ for $i \in \{1, \dots, n\}$.

Suppose that p_i is a linear piece of f_j. A routine induction on ϕ_j shows that

$$|c_{i,1}|, \dots, |c_{i,n}| \leq \mathrm{size}(\phi_j).$$

Hence, the largest coefficient (in absolute value) of any linear piece amongst $p_1, \dots, p_l$ is bounded above by

$$\max\{\mathrm{size}(\phi_j) \mid j \in \{1, \dots, k\}\} \leq \mathrm{size}(\{\phi_1, \dots, \phi_k\}),$$

so that the m-th equation in the linear system having $\mathbf{x}$ as solution has the form

$$a_{m,1}x_1 + \dots + a_{m,n}x_n = b_m$$

with

$$|a_{m,1}|, \dots, |a_{m,n}| \leq 2 \cdot \mathrm{size}(\{\phi_1, \dots, \phi_k\}). \tag{22}$$

Since

$$\mathbf{x} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = A^{-1}\boldsymbol{B},$$

it follows that $\mathrm{den}(\mathbf{x}) \leq |\det(A)|$ by elementary linear algebra [56]. In light of (22), the application of Hadamard's inequality now yields the desired bound:

$$\begin{aligned} |\det(A)| &\leq \prod_{i\in\{1,\dots,m\}} (a_{i,1}^2 + \dots + a_{i,n}^2)^{1/2} \\ &\leq \prod_{i\in\{1,\dots,n\}} |a_{i,1}| + \dots + |a_{i,n}| \\ &\leq \prod_{i\in\{1,\dots,n\}} 2n \cdot \mathrm{size}(\{\phi_1,\dots,\phi_k\}) \\ &\leq 2^{2n\log_2 n \cdot \mathrm{size}(\{\phi_1,\dots,\phi_k\})} \\ &\leq 2^{q(\mathrm{size}(\{\phi_1,\dots,\phi_k\}))}. \end{aligned}$$

It is enough to put

$$q(m) = m^2$$

and notice that $n \leq \mathrm{size}(\{\phi_1,\dots,\phi_k\})$, since the size of a set of formulas over n distinct variables is greater than or equal to n. □

THEOREM 5.1.3. *Let* $\phi_1,\dots,\phi_k \in \mathsf{Form}_n$ *and* $\beta\colon [\phi_i] \mapsto \beta_i \in [0,1]\cap \mathsf{Q}$ *be a book. The following are equivalent:*

(i) β *is coherent.*

(ii) There exist a unary polynomial $p\colon \mathsf{N} \to \mathsf{N}$ *and* $l \leq k+1$ *homomorphisms* $q_1,\dots,q_l$ *in* $\mathcal{H}(\boldsymbol{F}_n)$ *satisfying the following. For all* $i \in \{1,\dots,l\}$, q_i *ranges in the finite* MV*-chain* Ł_{d_i}*,where*

$$\log_2 d_i \leq p(\mathrm{size}(\beta)),$$

and $\langle\beta(\phi_i)\rangle_{i\in\{1,\dots,k\}}$ *is a convex combination of*

$$\langle q_1(\phi_i)\rangle_{i\in\{1,\dots,k\}},\dots,\langle q_l(\phi_i)\rangle_{i\in\{1,\dots,k\}}.$$

Proof. (i) ⇒ (ii) Let s be a state of $\boldsymbol{F}_n$ satisfying Theorem 5.0.4 and let Σ be a unimodular triangulation satisfying Proposition 5.1.2. Let $\mathbf{x}_1,\dots,\mathbf{x}_m$ be the rational vertices of Σ and put $d_1 = \mathrm{den}(\mathbf{x}_1),\dots,d_n = \mathrm{den}(\mathbf{x}_m)$. Let $\hat{h}_i$ be the normalized Schauder hat at vertex $\mathbf{x}_i$, $i \in \{1,\dots,m\}$, and define $\lambda_1,\dots,\lambda_m \in \mathsf{R}^+$ by putting, for every $i \in \{1,\dots,m\}$,

$$\lambda_i = s(\hat{h}_i).$$

Then

$$\begin{aligned}
\lambda_1 + \cdots + \lambda_m &= s(\hat{h}_1) + \cdots + s(\hat{h}_m) \\
&= s(\hat{h}_1 \oplus \cdots \oplus \hat{h}_m) && \text{by Lemma 3.3.1(i) and additivity of } s \\
&= s(1) && \text{by Lemma 3.3.1(ii)} \\
&= 1.
\end{aligned}$$

Let $h = \hat{h}_i/d_i$ be the Schauder hat at vertex $\mathbf{x}_i$, $i \in \{1, \ldots, m\}$. For all $i \in \{1, \ldots, k\}$, f_i is a McNaughton function linearized by Σ, and for $j \in \{1, \ldots, m\}$, h_j is the Schauder hat at vertex $\mathbf{x}_j$ of Σ. Thus by (10) there is a unique choice of integers $0 \leq a_{i,j} \leq \mathrm{den}(\mathbf{x}_j) \leq 1$ such that

$$f_i = \sum_{j=1}^{n} a_{i,j} \cdot h_j.$$

For all $i \in \{1, \ldots, m\}$, let t_i be the homomorphism from $\boldsymbol{F}_n$ to $[0,1]_{Ł}$ defined by putting, for every $f \in \boldsymbol{F}_n$,

$$t_i(f) = f(\mathbf{x}_i).$$

Note that t_i ranges in $\{0, 1/d_i, \ldots, (d_i - 1)/d_i, 1\}$, and by Proposition 5.1.2,

$$\log_2 d_i = \log_2 \mathrm{den}(\mathbf{x}_i) \leq q(\mathrm{size}(\{\phi_1, \ldots, \phi_k\})) \leq p(\mathrm{size}(\beta)),$$

letting

$$p(n) = q(n) = n^2,$$

as $\mathrm{size}(\{\phi_1, \ldots, \phi_k\}) \leq \mathrm{size}(\beta)$. For every $j \in \{1, \ldots, k\}$:

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i \cdot t_i(\phi_j) &= \sum_{i=1}^{m} \lambda_i \cdot f_j(\mathbf{x}_i) \\
&= \sum_{i=1}^{m} s(\hat{h}_i) \cdot a_{j,i}/d_i \\
&= \sum_{i=1}^{m} s(d_i \cdot h_i) \cdot a_{j,i}/d_i \\
&= \sum_{i=1}^{m} s(a_{j,i} \cdot h_i) && \text{by Proposition 5.1.1} \\
&= \sum_{i=1}^{m} s(a_{j,i}/d_i \cdot \hat{h}_i) \\
&= s\left(\bigoplus_{i=1}^{m} a_{j,i}/d_i \cdot \hat{h}_i\right) && \text{by Lemma 3.3.1(1) and additivity of } s \\
&= s(f_j).
\end{aligned}$$

As $\lambda_1 + \cdots + \lambda_n = 1$, the point $\langle \beta(\phi_i) \rangle_{i \in \{1,\ldots,k\}}$ is a convex combination of points $\langle t_1(\phi_i) \rangle_{i \in \{1,\ldots,k\}}, \langle t_2(\phi_i) \rangle_{i \in \{1,\ldots,k\}}, \ldots, \langle t_m(\phi_i) \rangle_{i \in \{1,\ldots,k\}}$. By Carathéodory theorem (see, e.g. [22, Theorem 2.3]), there exists a choice of $l \leq k+1$ homomorphisms $q_1, \ldots, q_l$ amongst $t_1, \ldots, t_m$ such that $(\beta(\phi_i))_{i \in \{1,\ldots,k\}}$ is a convex combination of

$$\langle q_1(\phi_i) \rangle_{i \in \{1,\ldots,k\}}, \ldots, \langle q_l(\phi_i) \rangle_{i \in \{1,\ldots,k\}},$$

and we are done.

(ii) $\Rightarrow$ (i) By Theorem 4.1.1, for every $\lambda_1, \ldots, \lambda_m \in \mathsf{R}^+$ satisfying $\sum_{i=1}^m \lambda_i = 1$, the map s from $\boldsymbol{F}_n$ to $[0,1]$ defined by putting

$$s([\varphi]) = \sum_{i=1}^{m} \lambda_i \cdot q_i([\varphi])$$

for every $\varphi \in \mathsf{Form}_n$, is a state. This means that β is coherent by Theorem 5.0.4. $\square$

5.2 Complexity

Let $\langle \beta \rangle$ denote the binary encoding of some rational Łukasiewicz assessment β. The problem of deciding coherence of rational Łukasiewicz assessments is defined as:

$$\mathsf{LUK\text{-}COH} = \{\langle \beta \rangle \mid \beta \text{ is a coherent book on Łukasiewicz formulas}\}.$$

THEOREM 5.2.1. LUK-COH *is* $\mathbf{NP}$*-complete.*

In the next two paragraphs we prove that LUK-COH is in $\mathbf{NP}$ (Lemma 5.2.2) and is $\mathbf{NP}$-hard (Lemma 5.2.3), thus proving Theorem 5.2.1.

Upper Bound. It is known that the feasibility problem of linear systems is decidable in polynomial time in the size of the binary encoding of the linear system [56]. Therefore, Theorem 5.1.3 directly furnishes a nondeterministic polynomial time algorithm for the coherence problem as follows.

LEMMA 5.2.2. LUK-COH *is in* $\mathbf{NP}$.

Proof. Let $\beta\colon \{\phi_1, \ldots, \phi_k\} \to [0,1] \cap \mathsf{Q}$ be a rational-valued book on Łukasiewicz formulas $\phi_1, \ldots, \phi_k$ over variables $A_1, \ldots, A_m$. Following Lemma 5.1.3, the algorithm guesses a natural number $l \leq k+1$ and, for all $i \in \{1, \ldots, l\}$, the algorithm guesses the denominator d_i, the restriction of homomorphism q_i to variables $A_1, \ldots, A_m$, and eventually checks the feasibility of the following linear system:

$$\begin{aligned} x_1 + \cdots + x_{l-1} + x_l &= 1 \\ q_1(\phi_1)x_1 + \cdots + q_{l-1}(\phi_1)x_{l-1} + q_l(\phi_1)x_l &= \beta(\phi_1) \\ &\vdots \\ q_1(\phi_k)x_1 + \cdots + q_{l-1}(\phi_k)x_{l-1} + q_l(\phi_k)x_l &= \beta(\phi_k). \end{aligned}$$

By Lemma 5.1.3, for all $i \in \{1,\ldots,l\}$, the denominator d_i has a polynomial-space encoding. It follows that the restriction of q_i to $A_1,\ldots,A_m$, as well as the coefficients $q_1(\phi_1),\ldots,q_l(\phi_k)$, are in $\{0,1/d_i,\ldots,(d_i-1)/d_i,1\}$. So the size of the system is polynomial in $\mathrm{size}(\beta)$, and the algorithm terminates in time polynomial in $\mathrm{size}(\beta)$. Notice that the linear system is feasible if and only if β is a convex combination of $q_1,\ldots,q_l$ if and only if β is coherent. □

Lower Bound. Let $\langle\phi\rangle$ denote the binary encoding of the formula $\phi \in \mathsf{Form}$. In [44] it is proved that the problem

$$\mathsf{LUK\text{-}SAT} = \{\langle\phi\rangle \mid \phi \text{ is satisfiable in Łukasiewicz logic}\}$$

is $\mathbf{NP}$-complete.

LEMMA 5.2.3. LUK-COH *is* $\mathbf{NP}$*-hard.*

Proof. We provide a logarithmic-space reduction from the $\mathbf{NP}$-hard problem LUK-SAT to LUK-COH.

Let $\phi \in \mathsf{Form}_n$. Let β be the book sending formulas $A_1 \oplus \neg A_1,\ \ldots,\ A_m \oplus \neg A_m$, and ϕ to 1, that is,

$$\beta(A_1 \oplus \neg A_1) = \ldots = \beta(A_m \oplus \neg A_m) = \beta(\phi) = 1.$$

The construction of the assessment β is feasible in space logarithmic in $\mathrm{size}(\phi)$. We show that β is coherent if and only if ϕ is satisfiable in Łukasiewicz logic.

($\Rightarrow$) Suppose that β is coherent. Let $b_i = -1$ for all $i \in [m+1]$, and let q be a homomorphism from $\boldsymbol{F}_n$ to $[0,1]_{Ł}$ such that (19) holds, that is,

$$\beta(\phi) - q(\phi) \leq \sum_{i=1}^{m}(q(A_i \oplus \neg A_i) - \beta(A_i \oplus \neg A_i)).$$

As $q(A_i \oplus \neg A_i) = 1 = \beta(A_i \oplus \neg A_i)$ for every $i \in \{1,\ldots,m\}$, the right-hand side vanishes so that

$$1 = \beta(\phi) \leq q(\phi) \leq 1.$$

($\Leftarrow$) Let $q \in \mathcal{H}(\boldsymbol{F}_n)$ be such that $q(\phi) = 1$. Then q is a state of $\boldsymbol{F}_n$ satisfying

$$q([\phi]) = 1 = \beta(\phi)$$

and for every $i \in \{1,\ldots,m\}$,

$$q([A_i \oplus \neg A_i]) = 1 = \beta(A_i \oplus \neg A_i).$$

Hence β is coherent by Example 3.2.2 and Theorem 5.0.4. □

6 MV-algebras with internal states

The content presented in the preceding sections shows that states of MV-algebras are tightly connected to Borel probability measures on the maximal spectral spaces. Moreover, de Finetti's theorem about coherent betting on Boolean events can be generalized to states and many-valued events. In this section we are going to enrich the existing perspectives with a purely algebraic approach to states. Namely we are going to introduce a class of algebras, called SMV-*algebras*, which provide a universal-algebraic framework for states. SMV-algebras are defined by expanding the signature of MV-algebras with a fresh unary symbol σ, which is equationally described in order to preserve the basic properties of a state. The mapping σ is called an *internal state* of an MV-algebra $\boldsymbol{A}$ and, as we will see, it can be successfully applied to cope with de Finetti's coherence criterion in a purely algebraic setting, among other applications.

DEFINITION 6.0.1. *An* MV-algebra with internal state *(*SMV-algebra*, for short) is an algebra* $\langle \boldsymbol{A}, \sigma\rangle = \langle A, \oplus, \neg, \sigma, 0\rangle$*, where* $\langle A, \oplus, \neg, 0\rangle$ *is an* MV*-algebra and* σ *is a unary operator on* $\boldsymbol{A}$ *satisfying the following conditions for every* $x, y \in A$*:*

($\sigma 1$) $\sigma(0) = 0$

($\sigma 2$) $\sigma(\neg x) = \neg(\sigma(x))$

($\sigma 3$) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$

($\sigma 4$) $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.

An SMV*-algebra* $\langle \boldsymbol{A}, \sigma\rangle$ *is said to be* faithful *if it satisfies the following quasi-equation:* $\sigma(x) = 0$ *implies* $x = 0$.

Clearly the class of SMV-algebras constitutes a variety, which is denoted by $\mathbb{SMV}$.

EXAMPLE 6.0.2. (a) We start with a trivial example. Let $\boldsymbol{A}$ be any MV-algebra and σ be the identity on $\boldsymbol{A}$. Then $\langle \boldsymbol{A}, \sigma\rangle$ is an SMV-algebra.

(b) Let σ be an idempotent endomorphism of an MV-algebra $\boldsymbol{A}$. For example, we may take $\boldsymbol{A}$ to be a non-trivial ultrapower of the standard MV-algebra $[0,1]_{\text{Ł}}$ and σ to be the standard part function. Then $\langle \boldsymbol{A}, \sigma\rangle$ is an SMV-algebra.

(c) This is a sufficiently general example for our purposes. Let $\boldsymbol{A}$ be the MV-algebra of all continuous and piecewise linear functions with real coefficients from $[0,1]^n$ into $[0,1]$. Then $\boldsymbol{A}$ is an MV-algebra endowed with the pointwise application of the operations $\oplus$ and $\neg$. For every $f \in A$ let $\sigma(f)$ be the function from $[0,1]^n$ to $[0,1]$ which is constantly equal to

$$\int_{[0,1]^n} f(x)\, \mathrm{d}x.$$

It follows that $\langle \boldsymbol{A}, \sigma\rangle$ is an SMV-algebra. It will become clear from the results of the next section that $\langle \boldsymbol{A}, \sigma\rangle$ is simple and thus subdirectly irreducible, but it is not totally ordered. This algebra is faithful, i.e., it satisfies the quasi-equation $\sigma(x) = 0$ implies $x = 0$.

LEMMA 6.0.3. *In any* SMV*-algebra* $\langle \boldsymbol{A}, \sigma \rangle$ *the following properties hold:*

(i) $\sigma(1) = 1$.

(ii) If $x \le y$*, then* $\sigma(x) \le \sigma(y)$.

(iii) $\sigma(x \oplus y) \le \sigma(x) \oplus \sigma(y)$*; and if* $x \odot y = 0$*, then* $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$.

(iv) $\sigma(x \ominus y) \ge \sigma(x) \ominus \sigma(y)$*; and if* $y \le x$*, then* $\sigma(x \ominus y) = \sigma(x) \ominus \sigma(y)$.

(v) $d(\sigma(x), \sigma(y)) \le \sigma(d(x, y))$*, where* d *is the Chang distance.*

(vi) $\sigma(x) \odot \sigma(y) \le \sigma(x \odot y)$*. Thus if* $x \odot y = 0$*, then* $\sigma(x) \odot \sigma(y) = 0$.

(vii) $\sigma(\sigma(x)) = \sigma(x)$.

(viii) The image $\sigma(A)$ *of* A *under* σ *is the domain of an* MV*-subalgebra* $\sigma(\boldsymbol{A})$ *of* $\boldsymbol{A}$.

Proof. (i) A direct consequence of (σ1) and (σ2).

(ii) If $x \le y$, then $y = x \oplus (y \ominus x)$, and hence $\sigma(y) = \sigma(x \oplus (y \ominus x))$. Since $x \odot (y \ominus x) = 0$, by ($\sigma$3) we get $\sigma(y) = \sigma(x \oplus (y \ominus x)) = \sigma(x) \oplus \sigma(y \ominus x) \ge \sigma(x)$.

(iii) By (ii), $\sigma(y) \ge \sigma(y \ominus (x \odot y))$, so

$$\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y)) \le \sigma(x) \oplus \sigma(y).$$

If $(x \odot y) = 0$, then $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y)) = \sigma(x) \oplus \sigma(y)$.

(iv) Using (σ2), (iii) and the order-reversing property of $\neg$, we obtain:

$$\begin{aligned}\sigma(x \ominus y) = \sigma(\neg(\neg x \oplus y)) = \neg(\sigma(\neg x \oplus y)) &\ge \neg(\neg\sigma(x) \oplus \sigma(y)) \\ = \neg(\neg(\sigma(x)) \oplus \sigma(y)) &= \sigma(x) \ominus \sigma(y).\end{aligned}$$

Moreover, if $y \le x$, then $\neg x \odot y = 0$. Hence again by (c),

$$\sigma(x \ominus y) = \neg(\sigma(\neg x \oplus y)) = \neg(\sigma(\neg x) \oplus \sigma(y)) = \sigma(x) \ominus \sigma(y).$$

(v) Since $(x \ominus y) \odot (y \ominus x) = 0$, by (iv) and (iii) we get $\sigma(d(x, y)) = \sigma(x \ominus y) \oplus \sigma(y \ominus x) \ge (\sigma(x) \ominus \sigma(y)) \oplus (\sigma(y) \ominus \sigma(x)) = d(\sigma(x), \sigma(y))$.

(vi) We have $x \odot y = x \ominus \neg y$ and thus (iv) and (σ2) yield

$$\sigma(x \odot y) = \sigma(x \ominus \neg y) \ge \sigma(x) \ominus \sigma(\neg y) = \sigma(x) \ominus \neg(\sigma(y)) = \sigma(x) \odot \sigma(y).$$

If $x \odot y = 0$, then $0 = \sigma(x \odot y) \ge \sigma(x) \odot \sigma(y)$. Therefore, $\sigma(x \odot y) = 0$.

(vii) By (i), $\sigma(0) = 0$, and using (σ4) we get

$$\sigma(\sigma(x)) = \sigma(\sigma(x) \oplus \sigma(0)) = \sigma(x) \oplus \sigma(0) = \sigma(x).$$

(viii) By (vii), the range of σ consists of all the fixed points of σ. Therefore, it is sufficient to prove that the set of the fixed points is closed under $\oplus$ and under $\neg$. Closure under $\oplus$ follows from (σ4). Concerning closure under $\neg$, using (σ2) and (vii), we get $\sigma(\neg(\sigma(x))) = \neg(\sigma(\sigma(x)) = \neg(\sigma(x))$. □

6.1 Subdirectly irreducible SMV-algebras

A σ*-filter* of an SMV-algebra $\langle \boldsymbol{A}, \sigma\rangle$ is a filter in the MV-algebra $\boldsymbol{A}$ closed under σ. Given a congruence θ of an SMV-algebra $\langle \boldsymbol{A}, \sigma\rangle$, we define

$$F_\theta = \{x \in A \mid \langle x, 1\rangle \in \theta\}.$$

Conversely, given a σ-filter F of $\langle \boldsymbol{A}, \sigma\rangle$, we define

$$\theta_F = \{\langle x, y\rangle \mid \neg(d(x, y)) \in F\},$$

where d is the Chang distance.

THEOREM 6.1.1. *The maps $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse isomorphisms between the lattice of congruences of an* SMV*-algebra $\langle \boldsymbol{A}, \sigma\rangle$, and the lattice of σ-filters of $\langle \boldsymbol{A}, \sigma\rangle$.*

Proof. The above defined maps are mutually inverse isomorphisms between the lattice of MV-congruences and the lattice of MV-filters. Therefore it suffices to prove that F is a σ-filter iff θ_F is an SMV-congruence. Since $\sigma(1) = 1$, the congruences classes of 1 are σ-filters. Conversely, let F be a σ-filter of an SMV-algebra $\langle \boldsymbol{A}, \sigma\rangle$. If $\langle x, y\rangle \in \theta_F$, then $\neg(d(x, y)) \in F$, so that $\sigma(\neg d(x, y)) \in F$, where F is a σ-filter. From Lemma 6.0.3 (v) we obtain $d(\sigma(x), \sigma(y)) \leq \sigma(d(x, y))$, and thus by the order-reversing property of $\neg$ it follows that $\neg(d(\sigma(x), \sigma(y))) \geq \neg(\sigma(d(x, y))) \in F$. Hence $\neg(d(\sigma(x), \sigma(y))) \in F$, and $\langle \sigma(x), \sigma(y)\rangle \in \theta_F$. In conclusion, θ_F is a congruence of $\langle \boldsymbol{A}, \sigma\rangle$. □

For every positive integer n and $a \in A$ we define

$$a^n = \underbrace{a \odot \cdots \odot a}_{n}.$$

LEMMA 6.1.2. *Let $\langle \boldsymbol{A}, \sigma\rangle$ be an* SMV*-algebra. Then the σ-filter $F_{\sigma(x)}$ generated by a single element $\sigma(x) \in \sigma(A)$ is $F_{\sigma(x)} = \{y \in A \mid \exists n \in \mathsf{N}(y \geq \sigma(x)^n)\}$.*

Proof. Let $H = \{y \in A \mid \exists n \in \mathsf{N}(y \geq \sigma(x)^n)\}$. By the definition of σ-filter, every element of H also belongs to $F_{\sigma(x)}$. Thus $H \subseteq F_{\sigma(x)}$. For the converse inclusion, it is sufficient to prove that H is a σ-filter and $\sigma(x) \in H$. Let us show that H is closed under σ. If $y \in H$, then there is $n \in \mathsf{N}$ such that $y \geq \sigma(x)^n$. By Lemma 6.0.3 (ii), (vi) and (vii), we get

$$\sigma(y) \geq \sigma(\sigma(x)^n) \geq (\sigma(\sigma(x)))^n \geq \sigma(x)^n.$$

Thus $\sigma(y) \in H$. That $\sigma(x) \in H$ is trivial. □

Now we are ready to prove the main result of this section.

THEOREM 6.1.3. *(i) If $\langle \boldsymbol{A}, \sigma\rangle$ is a subdirectly irreducible* SMV*-algebra, then $\sigma(\boldsymbol{A})$ is linearly ordered.*

(ii) If $\langle \boldsymbol{A}, \sigma\rangle$ is faithful, then $\langle \boldsymbol{A}, \sigma\rangle$ is a subdirectly irreducible SMV*-algebra iff $\sigma(\boldsymbol{A})$ is a subdirectly irreducible* MV*-algebra.*

Proof. (i) Let H be the smallest non-trivial σ-filter of $\langle \boldsymbol{A}, \sigma\rangle$ and let $x \in H \setminus \{1\}$. Suppose by contradiction that $\sigma(\boldsymbol{A})$ is not linearly ordered, and let $\sigma(a), \sigma(b) \in \sigma(A)$ be such that $\sigma(a) \not\leq \sigma(b)$ and $\sigma(b) \not\leq \sigma(a)$. Then the filters $F_{\sigma(a)\to\sigma(b)}$ and $F_{\sigma(b)\to\sigma(a)}$ generated by $\sigma(a) \to \sigma(b)$ and $\sigma(b) \to \sigma(a)$, respectively, are non-trivial. Hence they both contain H. In particular, $x \in F_{\sigma(a)\to\sigma(b)}$ and $x \in F_{\sigma(b)\to\sigma(a)}$. Since we know that $\sigma(a) \to \sigma(b) \in \sigma(A)$ and $\sigma(b) \to \sigma(a) \in \sigma(A)$, by Lemma 6.1.2 there is $n \in \mathsf{N}$ such that $x \geq (\sigma(a) \to \sigma(b))^n$ and $x \geq (\sigma(b) \to \sigma(a))^n$. Therefore,

$$x \geq (\sigma(a) \to \sigma(b))^n \vee (\sigma(b) \to \sigma(a))^n = 1.$$

Hence $x = 1$, which is a contradiction.

(ii) If $\langle \boldsymbol{A}, \sigma\rangle$ is faithful, then by definition $\sigma(x) = 0$ implies $x = 0$ and $\sigma(x) = 1$ implies $x = 1$. It follows that the intersection of a non-trivial σ-filter H of $\langle \boldsymbol{A}, \sigma\rangle$ with $\sigma(A)$ is a non-trivial σ-filter of $\sigma(\boldsymbol{A})$. Moreover, every filter of $\sigma(\boldsymbol{A})$ is closed under σ. Then every MV-filter of $\sigma(\boldsymbol{A})$ is indeed a σ-filter. Hence, if H is a minimal σ-filter of $\langle \boldsymbol{A}, \sigma\rangle$, then $H \cap \sigma(A)$ is a minimal non-trivial σ-filter of $\sigma(\boldsymbol{A})$. In fact, if F is another non-trivial filter of $\sigma(\boldsymbol{A})$, then the σ-filter F' of $\langle \boldsymbol{A}, \sigma\rangle$ generated by F contains H, and

$$F = F' \cap \sigma(A) \supseteq H \cap \sigma(A).$$

Hence $H \cap \sigma(A)$ is minimal. Therefore, if $\langle \boldsymbol{A}, \sigma\rangle$ is subdirectly irreducible, so is $\sigma(\boldsymbol{A})$.

Conversely, if H is the minimal non-trivial filter of $\sigma(\boldsymbol{A})$, then the σ-filter F of $\langle \boldsymbol{A}, \sigma\rangle$ generated by H is the minimal non-trivial σ-filter of $\langle \boldsymbol{A}, \sigma\rangle$. Indeed, if G is another non-trivial σ-filter of $\langle \boldsymbol{A}, \sigma\rangle$, then $G \cap \sigma(A) \supseteq F \cap \sigma(A) = H$. Then G contains the σ-filter generated by H, that is, $F \subseteq G$ and F is minimal. Thus $\langle \boldsymbol{A}, \sigma\rangle$ is subdirectly irreducible. □

REMARK 6.1.4. *In addition to the results contained in the previous theorem, the subdirectly irreducible* SMV*-algebras have been completely characterized by Dvurečenskij, Kowalski and Montagna. Their proof requires techniques from universal algebra which are out of the scope of this chapter. We invite the interested reader to consult the last section of this document, where we include an extensive list of references.*

REMARK 6.1.5. *The variety* $\mathbb{SMV}$ *is not generated by its linearly ordered algebras. Indeed, the equation* $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$ *is valid in any linearly ordered* SMV*-algebra, but it does not hold in general. It is enough to consider Example 6.0.2 (c) with* $f(x) = x$ *and* $g(x) = 1 - x$*. Then* $\sigma(f \vee g) = \frac{3}{4} > \sigma(f) \vee \sigma(g) = \frac{1}{2}$.

6.2 States of MV-algebras and internal states

In this section we relate the notion of an SMV-algebra and that of state of an MV-algebra. We will show that, starting from an SMV-algebra $\langle \boldsymbol{A}, \sigma\rangle$, one can define a state s of the MV-algebra $\boldsymbol{A}$. Conversely, starting from a state s of an MV-algebra $\boldsymbol{A}$, we shall recover an MV-algebra $\boldsymbol{T}$ containing $\boldsymbol{A}$ as an MV-subalgebra together with an internal state σ of $\boldsymbol{T}$.

Let us start with an SMV-algebra $\langle \boldsymbol{A}, \sigma\rangle$. By Lemma 6.0.3 (viii), $\langle \sigma(\boldsymbol{A}), \oplus, \neg, 0\rangle$ is an MV-subalgebra of $\boldsymbol{A}$, where $\oplus$ and $\neg$ denote respectively the restrictions of MV-algebraic operations of $\boldsymbol{A}$ to $\sigma(\boldsymbol{A})$. If M is a maximal filter of $\sigma(\boldsymbol{A})$, then the quotient MV-algebra $\sigma(\boldsymbol{A})/M$ is simple and thus there has to be a unique embedding of

$\sigma(\boldsymbol{A})/M$ into the standard MV-algebra $[0,1]_Ł$; see the proof of Proposition 3.1.5. Let $i\colon \sigma(\boldsymbol{A})/M \hookrightarrow [0,1]_Ł$ be such an embedding and let $\eta_M\colon \sigma(\boldsymbol{A}) \to \sigma(\boldsymbol{A})/M$ be the canonical MV-homomorphism induced by the maximal filter M. Finally, let us call s the map obtained by the composition

$$i \circ \eta_M \circ \sigma\colon \boldsymbol{A} \to [0,1]_Ł. \tag{23}$$

Then s is a state of $\boldsymbol{A}$ as the following theorem shows.

THEOREM 6.2.1. *Let $\langle \boldsymbol{A}, \sigma\rangle$ be any* SMV*-algebra and let $s\colon \boldsymbol{A} \to [0,1]_Ł$ be defined by (23). Then s is a state of $\boldsymbol{A}$.*

Proof. Since σ, i and η_M preserve 1, it is clear that $s(1) = 1$. To show that s is additive, let $x, y \in A$ be such that $x \odot y = 0$. By Lemma 6.0.3 (iii) one has $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$. Moreover, by Lemma 6.0.3 (f), $\sigma(x) \odot \sigma(y) = 0$, thus $s(x) \odot s(y) = 0$. Hence $s(x \oplus y) = s(x) \oplus s(y) = s(x) + s(y) - (s(x) \odot s(y)) = s(x) + s(y)$. □

Conversely, we shall obtain an SMV-algebra from an MV-algebra equipped with a state. To this purpose, we need to introduce an MV-*algebraic tensor product* (or simply a *tensor product*) between MV-algebras. Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be MV-algebras. A *bimorphism* from the direct product $\boldsymbol{A} \times \boldsymbol{B}$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ into $\boldsymbol{C}$, is a map Υ satisfying the following list of properties:

(i) $\Upsilon(1,1) = 1$ and for all $a \in A$ and $b \in B$, $\Upsilon(a,0) = \Upsilon(0,b) = 0$.

(ii) For all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$, $\Upsilon(a_1 \wedge a_2, b) = \Upsilon(a_1, b) \wedge \Upsilon(a_2, b)$, $\Upsilon(a_1 \vee a_2, b) = \Upsilon(a_1, b) \vee \Upsilon(a_2, b)$ and $\Upsilon(a, b_1 \wedge b_2) = \Upsilon(a, b_1) \wedge \Upsilon(a, b_2)$, $\Upsilon(a, b_1 \vee b_2) = \Upsilon(a, b_1) \vee \Upsilon(a, b_2)$.

(iii) For all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$, if $a_1 \odot a_2 = 0$, then $\Upsilon(a_1 \odot a_2, b) = \Upsilon(a_1, b) \odot \Upsilon(a_2, b)$ and $\Upsilon(a_1 \oplus a_2, b) = \Upsilon(a_1, b) \oplus \Upsilon(a_2, b)$. If $b_1 \odot b_2 = 0$, then $\Upsilon(a, b_1 \odot b_2) = \Upsilon(a, b_1) \odot \Upsilon(a, b_2)$ and $\Upsilon(a, b_1 \oplus b_2) = \Upsilon(a, b_1) \oplus \Upsilon(a, b_2)$.

Then the *tensor product* $\boldsymbol{A} \otimes \boldsymbol{B}$ of two MV-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ is an MV-algebra (unique up to an isomorphism) such that there is a universal bimorphism Υ from $A \times B$ into $\boldsymbol{A} \otimes \boldsymbol{B}$. Universality means that for any bimorphism $\Upsilon'\colon \boldsymbol{A} \times \boldsymbol{B} \to \boldsymbol{C}$, where $\boldsymbol{C}$ is an MV-algebra, there exists a unique homomorphism $\lambda\colon \boldsymbol{A} \otimes \boldsymbol{B} \to \boldsymbol{C}$ such that $\Upsilon' = \lambda \circ \Upsilon$.

In algebraic terms, $\boldsymbol{A} \otimes \boldsymbol{B}$ is constructed in the following way: let $\boldsymbol{F}(A \times B)$ be the free MV-algebra over the free generating set $A \times B$. Let I_T be the ideal of $\boldsymbol{F}(A \times B)$ generated by the following elements for every $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$:

(i) $d(\langle 1,1\rangle, 1)$

(ii) $d(\langle a,0\rangle, 0)$

(iii) $d(\langle 0,b\rangle, 0)$

(iv) $d(\langle a_1 \wedge a_2, b\rangle, \langle a_1, b\rangle \wedge \langle a_2, b\rangle)$

(v) $d(\langle a_1 \vee a_2, b\rangle, \langle a_1, b\rangle \vee \langle a_2, b\rangle)$

(vi) $d(\langle a, b_1 \wedge b_2\rangle, \langle a, b_1\rangle \wedge \langle a, b_2\rangle))$

(vii) $d(\langle a, b_1 \vee b_2\rangle, \langle a, b_1\rangle \vee \langle a, b_2\rangle)$,

(viii) $d(\langle a_1 \odot a_2, b\rangle, 0)$ whenever $a_1 \odot a_2 = 0$

(ix) $d(\langle a_1 \oplus a_2, b\rangle, \langle a_1, b\rangle \oplus \langle a_2, b\rangle)$ whenever $a_1 \odot a_2 = 0$

(x) $d(\langle a, b_1 \odot b_2\rangle, 0)$ whenever $b_1 \odot b_2 = 0$

(xi) $d(\langle a, b_1 \oplus b_2\rangle, \langle a, b_1\rangle \oplus \langle a, b_2\rangle)$ whenever $b_1 \odot b_2 = 0$.

Then we define $\boldsymbol{A} \otimes \boldsymbol{B}$ to be the MV-algebra $\boldsymbol{F}(A \times B)/I_T$.

In the following we consider the tensor products of the form $\boldsymbol{T} = [0,1]_Ł \otimes \boldsymbol{A}$, where $\boldsymbol{A}$ is an arbitrary MV-algebra. For every $\alpha \in [0,1]$ and $a \in A$, we shall denote $\Upsilon(\alpha, a)$ by $\alpha \otimes a$.

PROPOSITION 6.2.2. *Let $\boldsymbol{T} = [0,1]_Ł \otimes \boldsymbol{A}$. Then the following conditions hold for any $\alpha, \alpha_1, \alpha_2 \in [0,1]$ and any $a, a_1, a_2 \in A$:*

(i) $(\alpha_1 \oplus \alpha_2) \otimes 1 = (\alpha_1 \otimes 1) \oplus (\alpha_2 \otimes 1)$, *and* $1 \otimes (a_1 \oplus a_2) = (1 \otimes a_1) \oplus (1 \otimes a_2)$.

(ii) $\neg(\alpha \otimes 1) = (1 - \alpha) \otimes 1$, *and* $\neg(1 \otimes a) = 1 \otimes \neg a$.

(iii) The maps $\Phi\colon \alpha \mapsto (\alpha \otimes 1)$, and $\Psi\colon a \mapsto (1 \otimes a)$ are respectively embeddings of $[0,1]_Ł$ and $\boldsymbol{A}$ into $\boldsymbol{T}$.

(iv) If $\alpha_1 \odot \alpha_2 = 0$, then $(\alpha_1 + \alpha_2) \otimes a = (\alpha_1 \otimes a) \oplus (\alpha_2 \otimes a)$, and if $a_1 \odot a_2 = 0$, then $\alpha \otimes (a_1 \oplus a_2) = (\alpha \otimes a_1) \oplus (\alpha \otimes a_2)$.

(v) $\alpha \otimes (a_1 \ominus a_2) = (\alpha \otimes a_1) \ominus (\alpha \otimes a_2)$, *and* $(\alpha_1 \ominus \alpha_2) \otimes a = (\alpha_1 \otimes a) \ominus (\alpha_2 \otimes a)$.

(vi) $1 \otimes 1$ is the top element of $\boldsymbol{T}$, while for every $a \in A$ and every $\alpha \in [0,1]$, $0 \otimes a$ and $\alpha \otimes 0$ coincide with the bottom element of $\boldsymbol{T}$.

Proof. All the properties but (iii) are straightforward consequences of the tensor product construction explained above. For the proof of (iii) see Proposition 2.1 in [35]. □

Due to Proposition 6.2.2 (iii), for any $\alpha \in [0,1]$, we may denote $\alpha \otimes 1$ by α.

THEOREM 6.2.3. *Let $\boldsymbol{A}$ be an MV-algebra and s be a state of $\boldsymbol{A}$. Then there exists a state $\hat{s}\colon [0,1]_Ł \otimes A \to [0,1]$ making $\langle \boldsymbol{T}, \hat{s}\rangle$ an SMV-algebra. Moreover, if Φ and Ψ are the embeddings of $[0,1]_Ł$ and $\boldsymbol{A}$ into $\boldsymbol{T}$, respectively, then $\Phi(s(a)) = \hat{s}(\Psi(a))$ for each $a \in A$.*

Proof. Let us define the state s_1 of $\Psi(\boldsymbol{A})$ by the stipulation $s_1(\Psi(a)) = s(a)$ for every $a \in A$. Then, since $\Psi(\boldsymbol{A})$ is an MV-subalgebra of $\boldsymbol{T}$, by Corollary 3.4.2 the state s_1 can be extended to a state $s_2\colon [0,1]_Ł \otimes A \to [0,1]$. Finally, the map $\hat{s}\colon [0,1]_Ł \otimes A \to [0,1]_Ł \otimes A$ defined by

$$\hat{s}(t) = s_2(t) \otimes 1$$

makes $(\boldsymbol{T}, \hat{s})$ into an SMV-algebra. Moreover, for every $a \in A$,

$$\hat{s}(\Psi(a)) = s_2(\Psi(a)) \otimes 1 = \Phi(s(a))$$

and the claim is settled. □

6.3 Dealing with coherent books in SMV-algebraic theory

Let $\phi_1, \ldots, \phi_k$ be formulas of Łukasiewicz logic and let $\beta : [\phi_i] \mapsto \beta_i$ be a rational-valued book, that is, let us assume that the β_i's are rational numbers, say $\beta_i = \frac{n_i}{m_i}$. Moreover, let $x_1, \ldots, x_k$ be fresh variables, and consider for each $i = 1, \ldots, k$, the equations

$$\varepsilon_i : (m_i - 1)x_i = \neg x_i \qquad\qquad \delta_i : \sigma(\phi) = n_i x_i.$$

Then we can prove the following:

THEOREM 6.3.1. *Let $\beta : [\phi_i] \mapsto \frac{n_i}{m_i}$ be a rational book on the Łukasiewicz formulas $\phi_1, \ldots, \phi_k$. Then the following are equivalent:*

(i) β is coherent.

(ii) The equations ε_i and δ_i (for $i = 1, \ldots, k$) are satisfied in some non-trivial SMV-*algebra.*

Proof. As in Example 3.2.4, let $\boldsymbol{F}$ be the Lindenbaum algebra of Łukasiewicz logic over countably-many variables. By Theorem 5.0.4 it is sufficient to prove that (ii) is equivalent to the existence of a state s on $\boldsymbol{F}$ such that, for all $i = 1, \ldots, k$, $s([\phi_i]) = \frac{n_i}{m_i}$.

(i) ⇒ (ii). Let s be a state on $\boldsymbol{F}$ extending β. Recalling the tensor product construction (see Theorem 6.2.3), let $\boldsymbol{T} = [0,1]_Ł \otimes \boldsymbol{F}$ and let $\sigma : \boldsymbol{T} \to \boldsymbol{T}$ be defined as in the proof of Theorem 6.2.3. Since $1 \otimes [\phi_i] \in [0,1]_Ł \otimes \boldsymbol{F}$ and $\sigma(1 \otimes [\phi_i]) = s([\phi_i])$, for each $i = 1, \ldots, n$, it is clear that σ extends s (up to an isomorphism).

Let V be a valuation on $\langle \boldsymbol{T}, \sigma \rangle$ such that $V(x_i) = \frac{1}{m_i}$ for each $i = 1, \ldots, n$ (notice that $\frac{1}{m_i} = \frac{1}{m_i} \otimes [1] \in [0,1]_Ł \otimes \boldsymbol{F}$, whence V is a valuation on $\langle \boldsymbol{T}, \sigma \rangle$). Then V satisfies the equations ε_i because

$$(m_i - 1)V(x_i) = \frac{m_i - 1}{m_i} = 1 - \frac{1}{m_i} = V(\neg x_i).$$

Moreover, V satisfies the equations δ_i:

$$\sigma(1 \otimes [\phi_i]) = 1 \cdot s([\phi_i]) = s([\phi_i]) = \frac{n_i}{m_i} = n_i V(x_i).$$

Thus the equations ε_i and δ_i are satisfied in a non-trivial SMV-algebra as required.

(ii) ⇒ (i). Let $\langle \boldsymbol{A}, \sigma \rangle$ be an SMV-algebra and V be a valuation on $\langle \boldsymbol{A}, \sigma \rangle$ satisfying the equations ε_i and δ_i for each $i = 1, \ldots, k$. Without loss of generality, we may assume that $\boldsymbol{A}$ is finitely (or even countably) generated, so that there is an epimorphism $h_V : \boldsymbol{F} \to \boldsymbol{A}$ such that $h_V([x]) = V([x])$ for every propositional variable x. Then

$$(m_i - 1)h_V([x]) = \neg(h_V([x_i])) \quad \text{and} \quad \sigma(h_V([\phi_i])) = n_i h_V([x_i]).$$

As in the proof of Theorem 6.2.1, let M be a maximal MV-filter of $\sigma(\boldsymbol{A})$ and define, for each $[\psi] \in \boldsymbol{F}$,

$$s([\psi]) = \sigma(h_V([\psi]))/M.$$

Since quotients preserve identities, one has

$$s([\phi_i]) = (n_i h_V([x_i]))/M \quad \text{and} \quad (m_i - 1)(h_V([x_i])/M) = \neg(h_V([x_i])/M).$$

Hence the MV-homomorphism $\eta_M : \sigma(\boldsymbol{A})/M \to [0,1]_Ł$ maps $h_V([x_i])/M$ to $\frac{1}{m_i}$ and $s([\phi_i])$ to $\frac{n_i}{m_i}$, respectively.

It remains to be proved that s is a state of $\boldsymbol{F}$. First of all it is clear that $s([1]) = 1$. As for additivity, let $[\psi_1], [\psi_2] \in \boldsymbol{F}$ such that $[\psi_1] \odot [\psi_2] = 0$. Then

$$\begin{aligned} s([\psi_1] \oplus [\psi_2]) &= (\sigma(h_V([\psi_1]) \oplus h_V([\psi_2])))/M \\ &= (\sigma(h_V([\psi_1])))/M \oplus (\sigma(h_V([\psi_2])))/M \\ &= s([\psi_1]) \oplus s([\psi_2]) \\ &= s([\psi_1]) + s([\psi_2]), \end{aligned}$$

where the last equality follows from the following fact: if $[\psi_1] \odot [\psi_2] = 0$, then $h_V([\psi_1]) \odot h_V([\psi_2]) = 0$ in A, and so

$$s([\psi_1]) \odot s([\psi_2]) = (\sigma(h_V([\psi_1]))) \odot \sigma(h_V([\psi_2]))/M = 0/M = 0$$

and $s([\psi_1]) \oplus s([\psi_2]) = s([\psi_1]) + s([\psi_2])$. This implies that s is a state of $\boldsymbol{F}$ extending the assessment β. Therefore β is coherent. □

7 Conditional probability and Dutch Book argument

One of the main motivations for dealing with conditional probability in the classical Boolean setting is to quantify the uncertainty degree of an "event given an event". In this scenario, given two elements a, b of a Boolean algebra $\boldsymbol{B}$, the conditional probabilistic value of the conditional event "a given b"—denoted $a|b$—is computed with the help of a given unconditional finitely additive probability measure $\mu\colon B \to [0,1]$ as follows:

$$\mu(a|b) = \frac{\mu(a \wedge b)}{\mu(b)}. \tag{24}$$

Clearly, the value $\mu(a|b)$ is defined only when $\mu(b) \neq 0$. This means that we can define a conditional probability $\mu(\cdot\,|\,\cdot)$ as a partial map on the product $B \times B$ via (24). Every conditional probability satisfies the following fundamental property, which also motivates the definition of conditional states: whenever $b \in B$ is such that $\mu(b) \neq 0$, then the function $\mu(\cdot\,|b)\colon B \to [0,1]$ is a finitely additive probability measure.

In order to define conditional states, it is worth noticing that the direct generalization of (24) by substituting the operation $\wedge$ with the MV-operation $\odot$ fails to satisfy the additivity property of states. Indeed, if a_1, a_2, b are elements of an MV-algebra $\boldsymbol{A}$ such that $a_1 \odot a_2 = 0$ and s is a state of $\boldsymbol{A}$ with $s(b) > 0$, then

$$\frac{s((a_1 \oplus a_2) \odot b)}{s(b)} \neq \frac{s(a_1 \odot b) + s(a_2 \odot b)}{s(b)}$$

since $\odot$ does not distribute over $\oplus$.

A possible solution is to introduce a new MV-algebraic operation whose standard behavior has the features of the usual product between real numbers. The resulting algebraic structures are called PMV^+-algebras.

DEFINITION 7.0.1. *A* PMV^+*-algebra is a pair* $\langle \boldsymbol{A}, \cdot \rangle$ *such that* $\boldsymbol{A}$ *is an* MV*-algebra and* $\cdot$ *is a binary operation on* A *satisfying the following properties for all* $x, y, z \in A$*:*

(i) $\langle A, \cdot, 1\rangle$ *is a commutative monoid.*

(ii) $x \cdot (y \ominus z) = (x \cdot y) \ominus (x \cdot z)$.

(iii) If $x \cdot x = 0$*, then* $x = 0$.

The class of PMV^+-algebras forms a quasivariety, which can be generated by the standard algebra $[0,1]_{\mathrm{PMV}^+} = \langle [0,1]_{Ł}, \cdot \rangle$, where $\cdot$ is the ordinary product of real numbers in $[0,1]$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be PMV^+-algebras. A map $h\colon A \to B$ is a PMV^+-*homomorphism* if h is an MV-homomorphism and $h(a \cdot b) = h(a) \cdot h(b)$. For every PMV^+-algebra $\boldsymbol{A}$, we denote by $\mathcal{H}^+(\boldsymbol{A})$ the set of homomorphisms of $\boldsymbol{A}$ in $[0,1]_{\mathrm{PMV}^+}$. A *filter* of a PMV^+-algebra $\boldsymbol{A}$ is a subset F of A such that F is a filter of the MV-reduct $\boldsymbol{A}^-$ of $\boldsymbol{A}$ and for every $a, b \in F$, the condition $a \cdot b \in F$ holds. By a *state* of a PMV^+-algebra $\boldsymbol{A}$ we mean a state of its MV-reduct.

REMARK 7.0.2. *If* $\boldsymbol{A}$ *is a* PMV^+*-algebra, it is known that* $\boldsymbol{A}$ *and its* MV*-reduct* $\boldsymbol{A}^-$ *have the same congruences. In particular,* $\mathrm{Max}(\boldsymbol{A}) = \mathrm{Max}(\boldsymbol{A}^-)$*. Let* s *be a state of a* PMV^+*-algebra* $\boldsymbol{A}$*. Then, since* s *is a state of* $\boldsymbol{A}^-$*, by Theorem 4.0.1 there exists a unique probability measure* $\mu \in \mathcal{M}(\mathrm{Max}(\boldsymbol{A}^-))$ *such that, for every* $a \in A$*,*

$$s(a) = \int_{\mathrm{Max}(\boldsymbol{A}^-)} a^*(M)\,\mathrm{d}\mu(M) = \int_{\mathrm{Max}(\boldsymbol{A})} a^*(M)\,\mathrm{d}\mu(M).$$

Therefore, states of PMV^+*-algebras corresponds to integrals with respect to regular Borel probability measures.*

Analogously to the Many-valued Coherence Criterion for MV-algebras, we introduce the following notions. For every PMV^+-algebra $\boldsymbol{A}$ and a finite subset $A' = \{e_1, \ldots, e_k\}$ of A, a *book* on A' is defined to be a map from A' into $[0,1]$. An assessment β is called *coherent* if the bookmaker does not lose money in every possible world $w \in \mathcal{H}^+(\boldsymbol{A})$.

LEMMA 7.0.3. *Let* $e_1, \ldots, e_k$ *be elements of a* PMV^+*-algebra* $\boldsymbol{A}$ *and let* $\beta\colon e_i \mapsto \beta_i$ *be a book. Then the following are equivalent:*

(i) β *is coherent.*

(ii) β *extends to a state of* $\boldsymbol{A}$*.*

Proof. Adopting the same notation as in Remark 7.0.2, let $\boldsymbol{A}^-$ be the MV-reduct of $\boldsymbol{A}$. Then the book β (regarded as a partial map on A^-) is coherent if and only if there exists a state of $\boldsymbol{A}^-$ that extends it (cf. Theorem 5.0.4) if and only if, by the above Remark 7.0.2, there is a state of $\boldsymbol{A}$ that extends it. □

Similarly to the Boolean setting, any state s of a PMV^+-algebra $\boldsymbol{A}$ defines a conditional state, which can also be regarded as a partial map on $A \times A$ by the stipulation: for every $a, b \in A$,

$$s(a|b) = \frac{s(a \cdot b)}{s(b)}, \quad \text{whenever } s(b) > 0. \tag{25}$$

We may leave $s(a|b)$ undefined otherwise.

PROPOSITION 7.0.4. *Let $\boldsymbol{A}$ be any PMV^+-algebra, s be a state of $\boldsymbol{A}$ and let $b \in A$ be such that $s(b) \neq 0$. Then the map $s(\cdot\,|b) : A \to [0,1]$ is a state of $\boldsymbol{A}$.*

Proof. We have

$$s(1|b) = \frac{s(1 \cdot b)}{s(b)} = 1.$$

Let $a_1, a_2 \in A$ be such that $a_1 \odot a_2 = 0$. Then $(a_1 \cdot b) \odot (a_2 \cdot b) = 0$ and $(a_1 \oplus a_2) \cdot b = (a_1 \cdot b) \oplus (a_2 \cdot b)$. Therefore

$$\begin{aligned} s(a_1 \oplus a_2|b) = \frac{s((a_1 \oplus a_2) \cdot b)}{s(b)} &= \frac{s((a_1 \cdot b) \oplus (a_2 \cdot b))}{s(b)} = \frac{s(a_1 \cdot b) + s(a_2 \cdot b)}{s(b)} \\ &= s(a_1|b) + s(a_2|b). \end{aligned}$$

This means that $s(\cdot\,|b)$ is a state of the MV-reduct of $\boldsymbol{A}$ and our claim is settled. □

7.1 Bookmaking on many-valued conditional events

The classical coherence criterion discussed in Section 5 was extended by de Finetti [17] to a class $\{e_1|h_1, \ldots, e_k|h_k\}$ of conditional events by introducing an additional rule on which a bookmaker and a gambler must agree: any bet on a conditional event $e_i|h_i$ is ruled out in a possible world w, when w falsifies h_i, that is, $w(h_i) = 0$.

When moving from classical to many-valued events, it is reasonable to assume that the truth value $w(h_i)$ is neither 0 nor 1. Consider for instance the following example introduced by Franco Montagna: suppose that we are betting on the conditional event "The Barcelona soccer team will win the next match, provided that Messi plays". For convenience, let us denote by ϕ the event "the Barcelona soccer team will win" and by ψ the antecedent of the previous statement: "Messi will play", so that the above conditional event can be written as $\phi|\psi$. Assume that, during the soccer match (and hence in the possible world w), Messi plays the whole match except for the last 30 seconds. It would not make sense to completely invalidate the bet; instead it would be meaningful to think that the bet on that many-valued conditional event is true to the degree $w(\psi)$. Thus, if $w(\psi) = 1$, then the bet is completely valid. If $w(\psi) = 0$, then the bet is called off. In all the intermediate cases $0 < w(\psi) < 1$ the bet is partially valid with degree $w(\psi)$. Obviously, in order to cope with the partial validity of bets, we shall require our book to be *complete*, meaning that if the bookmaker chooses many-valued conditional events $e_1|h_1, \ldots, e_k|h_k$ to assign a betting odd, he will also assign a betting odd to the antecedent $h_1, \ldots, h_k$ of each conditional event.

REMARK 7.1.1. *In this section many-valued events will be identified with elements of any* PMV^+*-algebra. Therefore, when we will speak about a many-valued* conditional *event, we always refer to an ordered pair* $\langle e_i, h_i \rangle$ *(denoted by* $e_i|h_i$*) of elements of a* PMV^+*-algebra.*

Formally, let $C = \{e_1|h_1, \dots, e_k|h_k\}$ be a set of many-valued conditional events and $U = \{u_1, \dots, u_l\}$ (with $l \geq k$) be a set of many-valued unconditional events such that for all $i = 1, \dots, k$, there is $j = 1, \dots, l$ such that $h_i = u_j$. Further, let there be a complete book β such that $\beta(e_i|h_i) = \beta_i$ and $\beta(u_j) = \gamma_j$; if the gambler bets σ_i on $e_i|h_i$ and λ_j on u_j, respectively, then the bookmaker's balance with respect to the possible world w is computed as

$$\sum_{i=1}^{k} \sigma_i w(h_i)(\beta_i - w(e_i)) + \sum_{i=j}^{l} \lambda_j(\gamma_j - w(u_j)).$$

A many-valued coherence criterion can be formulated in the following way.

Many-valued Conditional Coherence Criterion. Let $\boldsymbol{A}$ be a PMV^+-algebra and let C and U be defined as above. A complete book such that $\beta(e_i|h_i) = \beta_i$ and $\beta(u_j) = \gamma_j$ is said to be *conditionally coherent* if and only if for every choice of $\sigma_1, \dots, \sigma_k, \lambda_1, \dots, \lambda_l \in \mathsf{R}$, there exists $w \in \mathcal{H}^+(\boldsymbol{A})$ that does not cause a sure loss, that is,

$$\sum_{i=1}^{k} \sigma_i w(h_i)(\beta_i - w(e_i)) + \sum_{j=1}^{l} \lambda_j(\gamma_j - w(u_j)) \geq 0. \tag{26}$$

In the rest of this section, we will always assume that β_i and γ_j are rational numbers and, moreover, we will use the following notation without danger of confusion:

(i) For all $i = 1, \dots, k$, β_i denotes the value that a complete book β assigns to the conditional events $e_i|h_i$ (for $i = 1, \dots, k$), while, for every $j = 1, \dots, l$, γ_j denotes the value $\beta(u_j)$.

(ii) When we will refer to a class C of many-valued conditional events and a class of many-valued unconditional events U, we will always understand that for each $e_i|h_i \in C$ there is $u_j \in U$ such that $h_i = u_j$. Therefore, we shall speak about a conditional book β on $C \cup U$ without loss of generality and in particular, unless otherwise specified, we will always assume that $C = \{e_1|h_1, \dots, e_k|h_k\}$ and $U = \{u_1, \dots, u_l\}$.

We are going to characterize complete coherent books in terms of conditional states. Clearly, since a conditional state is not defined for any conditional event $e|h$, where $s(h) = 0$, we have to ensure that all the antecedents h_i's were assigned positive betting odds in a complete book β. In this case, i.e., when $\gamma_i > 0$ for all h_i's, we will say that the complete book β is *positive*. In what follows we will show that this is not so restrictive. Nevertheless, for the sake of clarity, let us start by considering the case of positive complete books.

LEMMA 7.1.2. *Let* $\boldsymbol{A}$ *be a* PMV^+ *algebra and let* β *be a positive complete book on* $C \cup U$*. Then* β *avoids sure loss iff* $\beta' \colon h_i \mapsto \gamma_i, e_i \cdot h_i \mapsto \beta_i\gamma_i$ *(*$i \leq k$*) avoids sure loss.*

Proof. (⇐) We argue contrapositively. Suppose that betting $\lambda_1, ..., \lambda_k$, $\sigma_1, \ldots, \sigma_k$ on $h_1, \ldots, h_k$, $e_1|h_1, \ldots, e_k|h_k$ causes a sure loss. Then

$$\sum_{i=1}^{k} \lambda_i(\beta_i - w(h_i)) + \sum_{i=1}^{k} \sigma_i w(h_i)(\gamma_i - w(e_i)) < 0$$

for every valuation w. Adding and subtracting $\sigma_i\gamma_i\beta_i$ yield

$$\sum_{i=1}^{k} \lambda_i(\beta_i - w(h_i)) + \sum_{i=1}^{k} \sigma_i(\gamma_i\beta_i - w(e_i \cdot h_i)) + \sum_{i=1}^{k} \sigma_i\gamma_i(w(h_i - \beta_i) < 0,$$

so that

$$\sum_{i=1}^{k}(\lambda_i - \sigma_i\gamma_i)(\beta_i - w(h_i)) + \sum_{i=1}^{k} \sigma_i(\gamma_i\beta_i - w(e_i \cdot h_i)) < 0.$$

Therefore by betting $\lambda_i - \sigma_i\gamma_i$ on h_i and σ_i on $e_i h_i$ we cause a sure loss and β' is not coherent.

(⇒) Conversely, if β' is not coherent, then there are δ_i, μ_i ($i = 1, \ldots, k$) such that

$$\sum_{i=1}^{k} \delta_i \cdot (\beta_i - w(h_i)) + \sum_{i=1}^{k} \mu_i(\gamma_i\beta_i - w(e_i \cdot h_i)) < 0$$

for every valuation w. Adding and subtracting $\mu_i\gamma_i(\beta_i - w(h_i))$ give

$$\sum_{i=1}^{k}(\delta_i + \mu_i\gamma_i)(\beta_i - w(h_i)) + \sum_{i=1}^{k} \mu_i(\gamma_i\beta_i - w(e_i) \cdot w(h_i)) - \sum_{i=1}^{k} \mu_i\gamma_i(\beta_i - w(h_i)) < 0,$$

which implies

$$\sum_{i=1}^{k}(\delta_i + \mu_i\gamma_i)(\beta_i - w(h_i)) + \sum_{i=1}^{k} \mu_i \cdot w(h_i)(\gamma_i - w(e_i)) < 0.$$

It follows that betting $\delta_i + \mu_i\gamma_i$ on h_i and μ_i on $e_i|h_i$ causes a sure loss and hence β is not conditionally coherent. □

THEOREM 7.1.3. *Let $\boldsymbol{A}$ be a PMV^+-algebra and let β a positive complete book on $C \cup U$. Then the following are equivalent:*

(i) β is conditionally coherent.

(ii) There is a state s of $\boldsymbol{A}$ such that, for all $i = 1, \ldots, k$, $\beta_i s(h_i) = s(e_i \cdot h_i)$ and $\gamma_i = s(h_i)$, i.e., for all $i = 1, \ldots, k$, $\beta_i = s(e_i|h_i)$ and $\gamma_i = s(h_i)$.

Proof. It follows from Lemma 7.1.2 that β is conditionally coherent if and only if the book β' on $\{e_i \cdot h_i, h_i | i = 1, \ldots, k\} \subset A$, which assigns $\beta'(e_i \cdot h_i) = \beta_i\gamma_i$ and $\beta'(h_i) = \gamma_i$, is coherent as well. By Lemma 7.0.3, there is a state s of $\boldsymbol{A}$ such that, for all $i = 1, \ldots, k$, $s(e_i \cdot h_i) = \beta_i\gamma_i$ and $s(h_i) = \gamma_i$. Thus $s(h_i)\beta_i = s(e_i \cdot h_i)$ and the claim is settled. □

Let us now analyze the case of a not necessarily positive complete book β on $C \cup U$. We will assume that for some h_i's, $\beta(h_i) = 0$. Without loss of generality, let $h_1, \ldots, h_t$ (for $t < k$) be such that β assigns a strictly positive value to them, while $\beta(h_{t+1}) = \cdots = \beta(h_k) = 0$. In what follows, we shall denote by β^- the complete book obtained by removing from β all the occurrences of $e_i|h_i$ for which $\beta(h_i) = 0$. In other words, β^- will denote the complete book on $C' \cup U = \{e_1|h_1, \ldots, e_t|h_t, u_1, \ldots, u_l\}$ obtained from β by restriction.

LEMMA 7.1.4. *Let $\boldsymbol{A}$ be a PMV^+-algebra and β be a complete book on $C \cup U$. Then:*

(i) β is conditionally coherent if and only if so is β^- on $C' \cup U$.

(ii) There is a s state of $\boldsymbol{A}$ such that for all $i = 1, \ldots, k$, $s(e_i \cdot h_i) = \beta(e_i|h_i)s(h_i)$ if and only if there is a state s' of $\boldsymbol{A}$ such that for all $i = 1, \ldots, t$, $s'(e_i \cdot h_i) = \beta^-(e_i|h_i)s'(h_i)$.

Proof. (i) It is easy to see that if β is conditionally coherent, then so is β^-. Conversely, let us assume β^- to be conditionally coherent. Then, since β^- is positive, by Lemma 7.1.2 and Lemma 7.0.3 there are $k + l + 1 \geq t + l + 1$ homomorphisms w_s of $\boldsymbol{A}$ into $[0, 1]_{\mathrm{PMV}^+}$ and positive real numbers $\alpha_1, \ldots, \alpha_{k+l+1}$ such that $\sum_{s=1}^{k+l+1} \alpha_s = 1$ and the following holds:

(c1) For all u_j such that for some $i = 1, \ldots, k$, $u_j = h_i$ and $i \leq t$, we have

$$\beta^-(h_i) = \sum_{s=1}^{k+l+1} \alpha_s w_s(h_i).$$

(c2) If $i > t$, then $\beta^-(h_i) = \displaystyle\sum_{s=1}^{k+l+1} \alpha_s w_s(h_i) = 0$.

(c3) For all $i = 1, \ldots, t$, $\beta^-(e_i|h_i) = \dfrac{\sum\limits_{s=1}^{k+l+1} \alpha_s w_s(e_i) w_s(h_i)}{\sum\limits_{s=1}^{k+l+1} \alpha_s w_s(h_i)}$.

Let $\sigma_1, \ldots, \sigma_k, \lambda_1, \ldots, \lambda_l$ be any system of bets on β. Then there exists y with $0 \leq y \leq k + l + 1$ and such that the homomorphism w_y of $\boldsymbol{A}$ into $[0, 1]_{\mathrm{PMV}^+}$ satisfies

$$\sum_{i=1}^{t} \sigma_i w_y(h_i)(\beta(e_i|h_i) - w_y(e_i)) + \sum_{j=1}^{l} \lambda_j(\beta(u_j) - w_y(u_j)) \geq 0.$$

By way of contradiction, assume that for all $s = 1, \ldots, k + l + 1$,

$$\sum_{i=1}^{t} \sigma_i w_s(h_i)(\beta(e_i|h_i) - w_s(e_i)) + \sum_{j=1}^{l} \lambda_j(\beta(u_j) - w_s(u_j)) < 0.$$

Then, letting

$$w'(a) = \sum_{s=1}^{k+l+1} \alpha_s w_s(a)$$

we obtain

$$\sum_{i=1}^{t} \sigma_i w'(h_i)\beta(e_i|h_i) - \sum_{i=1}^{t} \sigma_i w'(h_i)w'(e_i) + \sum_{j=1}^{l} \lambda_j(\beta(u_j) - w'(u_j)) < 0.$$

On the other hand, from the above (c1), (c3) and the definition of w', we have

$$\sum_{i=1}^{t} \sigma_i w'(h_i)\beta(e_i|h_i) = \sum_{i=1}^{t} \sigma_i w'(h_i)w'(e_i) \text{ and } \sum_{j=1}^{l} \lambda_j(\beta(u_j) - w'(u_j)) = 0,$$

a contradiction.

Moreover, from (c2) is follows that for all $i \geq t$, $w_y(h_i) = \sum_{s=1}^{k+l+1} \alpha_s w_s(h_i) = 0$ and hence

$$\sum_{i=1}^{k} \sigma_i w_y(h_i)(\beta(e_i|h_i) - w_y(e_i)) + \sum_{j=1}^{l} \lambda_j(\beta(u_j) - w_y(u_j)) \geq 0$$

and β is conditionally coherent.

(ii) Since β extends β^-, each state extending β extends β^- as well. Conversely, assume that s is a state extending β^-. Then, for every $i > t$, we have $s(h_i) = 0$ and thus $s(e_i \cdot h_i) = 0$. Therefore s satisfies $s(e_i \cdot h_i) = \beta(e_i|h_i)s(h_i)$ for all $i = 1, \ldots, k$ and the claim is proved. □

The expected characterization theorem follows from Lemma 7.0.3 and 7.1.4.

THEOREM 7.1.5. *Let $\boldsymbol{A}$ be a PMV^+-algebra and β be a complete book on $C \cup U$. Then the following are equivalent:*

(i) β is conditionally coherent.

(ii) There is a state of $\boldsymbol{A}$ such that for all $i = 1, \ldots, k$,

$$s(e_i \cdot h_i) = \beta(e_i|h_i)s(h_i).$$

(iii) There are homomorphisms $w_1, \ldots, w_{k+l+1}$ and positive reals $\alpha_1, \ldots, \alpha_{k+l+1}$ such that $\sum_{s=1}^{k+l+1} \alpha_s = 1$, and

(a) for all $i \leq t$, $\gamma_i = \sum_{s=1}^{k+l+1} \alpha_s w_s(h_i)$,

(b) for all $i = 1, \ldots, k$, $\beta(e_i|h_i) \sum_{s=1}^{k+l+1} \alpha_s w_s(h_i) = \sum_{s=1}^{k+l+1} \alpha_s w_s(e_i)w_s(h_i)$.

8 Historical remarks and further reading

The states of MV-algebras were defined and studied by Mundici in [45] as averaging processes for truth values in Łukasiewicz logic. Since then the topic attracted a number of researchers in many-valued logics. In this chapter we made an effort to include state-of-the-art results and, in the same time, to present self-contained proofs together with some useful techniques for dealing with states. This approach unavoidably led to omitting some important developments; otherwise the scope of mathematical prerequisites would become too broad, ranging from piecewise linear topology to geometric measure theory. Thus many highly interesting parts of the theory are not discussed in this chapter, such as the rational measure of rational polyhedra, Rényi invariant conditional in Łukasiewicz logic, and the properties of Lebesgue state. The interested reader is referred to Mundici's recent book [48] for an in-depth treatment of those topics.

Three chapters of *Handbook of measure theory* published in 2002 are related to states of MV-algebras. Barbieri and Weber [8] studied MV-algebraic measures, which are bounded additive real functions on MV-algebras. The set of all such functions forms a Dedekind complete vector lattice such that the state space is the base of a lattice cone made of bounded additive and positive functions. *Probability on* MV*-algebras* is the chapter [55] by Riečan and Mundici in which an MV-algebraic counterpart of Boolean probability is thoroughly explored. The notion of a probability MV-algebra, which is a σ-complete MV-algebra equipped with a σ-order continuous state, is the framework for developing point-free versions of the central limit theorem, individual ergodic theorem, and Kolmogorov's construction of an infinite-dimensional sequence space. Butnariu and Klement [14] provide a survey on σ-continuous measures over the families of functions called T-tribes, where T is a t-norm. Since a T-tribe with Łukasiewicz t-norm T is a σ-complete MV-algebra, Butnariu and Klement deal with σ-states in particular. The main focus of their work is on integral representations of T-norm-based measures, which is of chief importance in theory of cooperative games with fuzzy coalitions [13]. A relatively recent treatment of tribes and their measures can be found in [49] by Navara.

The states of finitely presented algebras have attracted special attention. The results concerning invariant and faithful states presented in Section 4.2 are only scratching the surface of a rapidly developing subject—the dynamics of Z-homeomorphisms of the unit hypercube. The main results in this area include, but are not limited to: Panti's purely algebraic characterization of Lebesgue state and his study of Bernoulli automorphisms of the free finitely generated MV-algebra [51, 52]; the Haar theorem for lattice-ordered Abelian groups with order-unit, which can be directly applied to MV-algebras [47]; Marra's characterization of Lebesgue state [37]. The interpretation of state as a probability operator on formulas (or many-valued events) is discussed also by Marra [38].

Conditional probability over MV-algebras has been studied in several directions. The definition of conditioning involving the notion of algebraic product (25) was first used in [30]. PMV^{+}-algebras discussed in Section 7 were introduced in [39] and further studied in [40]. The approach to de Finetti theorem based on conditional events developed in Section 7 is based on Montagna's paper [41]. Further concepts of conditional probability were developed by Mundici [48, Chapter 15] and Montagna et al. [42].

The proof of Theorem 3.3.4—the integral representation for finitely presented MV-algebras—was published in [33]. The core of its proof is the refinement technique, which is used to recover a unique representing probability measure over all the Schauder bases. This idea goes back to a construction appearing in Pták's paper about extension of states on quantum logics [53].

The algebraic structures we discussed in Section 6, namely SMV-algebras, were introduced in [26] and they have been intensively studied since then. A particular attention has also been devoted to the case in which the internal state of an MV-algebra $\boldsymbol{A}$ is an MV-endomorphism of $\boldsymbol{A}$. The latter structures, called *state-morphism* MV*-algebras*, were introduced by Di Nola and Dvurečenskij in [18]. As we have already mentioned in Remark 6.1.4, subdirectly irreducible SMV-algebras and subdirectly irreducible state-morphisms MV-algebras were fully characterized in [21, Theorem 3.4] by Dvurečenskij, Kowalski, and Montagna.

The chapter was about states of MV-algebras, which are the algebras associated with Łukasiewicz infinite-valued logic. The systematic development of other many-valued logics was made possible by the pioneering work of Hájek [28]. The efforts to study states in other logics than Łukasiewicz are complicated by non-existence of the natural notion of addition and the discontinuity of logical operations, among other things. Convincing results were achieved mainly for Gödel and nilpotent minimum logics by Aguzzoli, Gerla, and Marra; see [2, 3]. The integral representation of states in Gödel logic was proved by the same authors in [4]. De Finetti style-theorem for the integral states was exhibited for the whole class of many-valued logics with continuous connectives in [34] by Kühr and Mundici. States on pseudo MV-algebras, non-commutative generalizations of MV-algebras, were introduced by Dvurečenskij [19]. The same author studied integral representation for a large class of algebras (including BL-algebras and effect algebras) [20]. Ciungu devotes several chapters of her book [16] to states of non-commutative structures, providing extensive bibliography.

The decision problem of coherence for rational books in infinite-valued Łukasiewicz logic was shown to be decidable by Mundici [46]. The NP-completeness result was achieved by Bova and Flaminio in [12].

Probability theory and states belong to the colorful mosaic consisting of calculi for uncertainty modeling and reasoning such as Dempster-Shafer theory or possibility theory, which are based on non-additive functions on Boolean algebras. Some classes of uncertainty measures have been generalized to MV-algebras. Since they are not the topic of this exposition, we confine ourselves to pointing the interested reader to the references [23–25] for the survey of current results.

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BIBLIOGRAPHY

[1] Stefano Aguzzoli, Simone Bova, and Brunella Gerla. Free algebras and functional representation for fuzzy logics. In Petr Cintula, Petr Hájek, and Carles Noguera, editors, *Handbook of Mathematical Fuzzy Logic - Volume 2*, volume 38 of *Studies in Logic, Mathematical Logic and Foundations*, pages 713–791. College Publications, London, 2011.

[2] Stefano Aguzzoli and Brunella Gerla. Probability measures in the logic of Nilpotent Minimum. *Studia Logica*, 94(2):151–176, 2010.

[3] Stefano Aguzzoli, Brunella Gerla, and Vincenzo Marra. De Finetti's no-Dutch-book criterion for Gödel logic. *Studia Logica*, 90(1):25–41, 2008.

[4] Stefano Aguzzoli, Brunella Gerla, and Vincenzo Marra. Defuzzifying formulas in Gödel logic through finitely additive measures. In *Fuzzy Systems, 2008. FUZZ-IEEE 2008. IEEE International Conference on Computational Intelligence*, pages 1886–1893, June 2008.

[5] Erik M. Alfsen. *Compact Convex Sets and Boundary Integrals*, volume 57 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, New York, 1971.

[6] Charalambos D. Aliprantis and Owen Burkinshaw. *Infinite Dimensional Analysis: a Hitchhiker's Guide*. Springer, Berlin, third edition, 2006.

[7] Charalambos D. Aliprantis and Rabee Tourky. *Cones and Duality*, volume 84 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007.

[8] Giuseppina Barbieri and Hans Weber. Measures on clans and on MV-algebras. In *Handbook of Measure Theory, Vol. I, II*, pages 911–945. North-Holland, Amsterdam, 2002.

[9] Alain Bigard, Klaus Keimel, and Samuel Wolfenstein. *Groupes at Anneaux Reticulés*, volume 608 of *Lectures Notes in Mathematics*. Springer, Columbia, 1977.

[10] Patrick Billingsley. *Probability and Measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, New York, third edition, 1995.

[11] Garrett Birkhoff. *Lattice Theory*. American Mathematical Society, Providence, RI, 1948.

[12] Simone Bova and Tommaso Flaminio. The coherence of Łukasiewicz assessments is NP-complete. *International Journal of Approximate Reasoning*, 51(3):294–304, 2010.

[13] Dan Butnariu and Erich Peter Klement. *Triangular Norm Based Measures and Games with Fuzzy Coalitions*. Kluwer, Dordrecht, 1993.

[14] Dan Butnariu and Erich Peter Klement. Triangular norm-based measures. In *Handbook of Measure Theory, Vol. I, II*, pages 947–1010. North-Holland, Amsterdam, 2002.

[15] Roberto Cignoli, Itala M.L. D'Ottaviano, and Daniele Mundici. *Algebraic Foundations of Many-Valued Reasoning*, volume 7 of *Trends in Logic*. Kluwer, Dordrecht, 1999.

[16] Lavinia Corina Ciungu. *Non-Commutative Multiple-Valued Logic Algebras*. Springer Monographs in Mathematics. Springer, 2014.

[17] Bruno de Finetti. *Wahrscheinlichkeitstheorie: Einführende Synthese mit Kritischem Anhang*. Oldenbourg, 1981.

[18] Antonio Di Nola and Anatolij Dvurečenskij. State-morphism MV-algebras. *Annals of Pure and Applied Logic*, 161:161–173, 2009.

[19] Anatolij Dvurečenskij. States on pseudo MV-algebras. *Studia Logica*, 68:301–327, 2001.

[20] Anatolij Dvurečenskij. States on quantum structures versus integrals. *International Journal of Theoretical Physics*, 50(12):3761–3777, 2011.

[21] Anatolij Dvurečenskij, Tomasz Kowalski, and Franco Montagna. State morphism MV-algebras. *International Journal of Approximate Reasoning*, 52(8):1215–1228, 2011.

[22] Günter Ewald. *Combinatorial Convexity and Algebraic Geometry*, volume 168 of *Graduated Texts in Mathematics*. Springer, New York, 1996.

[23] Tommaso Flaminio, Lluís Godo, and Tomáš Kroupa. Belief functions on MV-algebras of fuzzy sets: An overview. In V. Torra, Y. Narukawa, and M. Sugeno, editors, *Non-Additive Measures: Theory and Applications*, volume 310 of *Studies in Fuzziness and Soft Computing*, pages 173–200. Springer, 2014.

[24] Tommaso Flaminio, Lluís Godo, and Enrico Marchioni. On the logical formalization of possibilistic counterparts of states over n-valued Łukasiewicz events. *Journal of Logic and Computation*, 21(3): 429–446, 2011.

[25] Tommaso Flaminio, Lluís Godo, and Enrico Marchioni. Reasoning about uncertainty of fuzzy events: An overview. In Petr Cintula, Christian G. Fermüller, Lluís Godo, and Petr Hájek, editors, *Understanding Vagueness: Logical, Philosophical and Linguistic Perspectives*, volume 36 of *Studies in Logic*, pages 367–400. College Publications, London, 2011.

[26] Tommaso Flaminio and Franco Montagna. MV-algebras with internal states and probabilistic fuzzy logics. *International Journal of Approximate Reasoning*, 50(1):138–152, 2009.

[27] Ken R. Goodearl. *Partially Ordered Abelian Groups With Interpolation*, volume 20 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1986.

[28] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.

[29] Mark Krein and David Milman. On extreme points of regular convex sets. *Studia Mathematica*, 9: 133–138, 1940.

[30] Tomáš Kroupa. Conditional probability on MV-algebras. *Fuzzy Sets and Systems*, 149(2):369–381, 2005.

[31] Tomáš Kroupa. Every state on semisimple MV-algebras is integral. *Fuzzy Sets and Systems*, 157(20): 2771–2782, 2006.

[32] Tomáš Kroupa. Representation and extension of states on MV-algebras. *Archive for Mathematical Logic*, 45:381–392, 2006.

[33] Tomáš Kroupa. Note on construction of probabilities on many-valued events via Schauder bases and inverse limits. In *Multiple-Valued Logic (ISMVL), 2010 40th IEEE International Symposium on Multiple-Valued Logic*, pages 185–188, 2010.

[34] Jan Kühr and Daniele Mundici. De Finetti Theorem and Borel states in $[0,1]$-valued logic. *International Journal of Approximate Reasoning*, 46(3):605–616, 2007.

[35] Serafina Lapenta and Ioana Leuştean. Scalar extensions for algebraic structures of Łukasiewicz logic. Journal of Pure and Applied Algebra, 220(4):1538–1553, 2016.

[36] Ioana Leuştean and Antonio Di Nola. Łukasiewicz logic and MV-algebras. In Petr Cintula, Petr Hájek, and Carles Noguera, editors, *Handbook of Mathematical Fuzzy Logic - Volume 2*, volume 38 of *Studies in Logic, Mathematical Logic and Foundations*, pages 469–583. College Publications, London, 2011.

[37] Vincenzo Marra. The Lebesgue state of a unital Abelian lattice-ordered group, II. *Journal of Group Theory*, 12(6):911–922, 2009.

[38] Vincenzo Marra. Is there a probability theory of many-valued events? In *Probability, Uncertainty and Rationality*, volume 10 of *CRM Series*, pages 141–166. Ed. Norm., Pisa, 2010.

[39] Franco Montagna. An algebraic approach to propositional fuzzy logic. *Journal of Logic, Language and Information*, 9(1):91–124, 2000.

[40] Franco Montagna. Subreducts of MV-algebras with product and product residuation. *Algebra Universalis*, 53(1):109–137, 2005.

[41] Franco Montagna. A notion of coherence for books on conditional events in many-valued logic. *Journal of Logic and Computation*, 21(5):829–850, 2011.

[42] Franco Montagna, Martina Fedel, and Giuseppe Scianna. Non-standard probability, coherence and conditional probability on many-valued events. *International Journal of Approximate Reasoning*, 54(5):573–589, 2013.

[43] Daniele Mundici. Interpretations of AF $C^\star$-algebras in Łukasiewicz sentential calculus. *Journal of Functional Analysis*, 65(1):15–63, 1986.

[44] Daniele Mundici. Satisfiability in many-valued sentential logic is NP-complete. *Theoretical Computer Science*, 52(1–2):145–153, 1987.

[45] Daniele Mundici. Averaging the truth-value in Łukasiewicz logic. *Studia Logica*, 55(1):113–127, 1995.

[46] Daniele Mundici. Bookmaking over infinite-valued semantics. *International Journal of Approximate Reasoning*, 43:223–240, 2006.

[47] Daniele Mundici. The Haar theorem for lattice-ordered Abelian groups with order unit. *Discrete and Continuous Dynamical Systems*, 21(2):537–549, 2008.

[48] Daniele Mundici. *Advanced Łukasiewicz Calculus and MV-Algebras*, volume 35 of *Trends in Logic*. Springer, New York, 2011.

[49] Mirko Navara. Triangular norms and measures of fuzzy sets. In Erich Peter Klement and Radko Mesiar, editors, *Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms*, pages 345–390. Elsevier, 2005.

[50] Jaroslav Nešetřil and Patrice Ossona De Mendez. A model theory approach to structural limits. *Commentationes Mathematicae Universitatis Carolinae*, 53(4):581–603, 2012.

[51] Giovanni Panti. Bernoulli automorphisms of finitely generated free MV-algebras. *Journal of Pure and Applied Algebra*, 208(3):941–950, 2007.

[52] Giovanni Panti. Invariant measures in free MV-algebras. *Communications in Algebra*, 36(8):2849–2861, 2009.

[53] Pavel Pták. Extension of states on logics. *Polish Academy of Sciences. Bulletin. Mathematics*, 33(9-10): 493–497, 1985.

[54] Bhaskara K. P. S. Rao and Bhaskara M. Rao. *Theory of Charges*, volume 109 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, London, 1983.

[55] Beloslav Riečan and Daniele Mundici. Probability on MV-algebras. In Endre Pap, editor, *Handbook of Measure Theory*, volume II, pages 869–909. North-Holland, Amsterdam, 2002.

[56] Alexander Schrijver. *Theory of Linear and Integral Programming*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Chichester, 1998.

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