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A POSTERIORI ERROR ESTIMATES FOR TWO-PHASE OBSTACLE PROBLEM

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For the two–phase obstacle problem we derive the basic error identity which yields natural measure of the distance to the exact solution. For this measure we derive a computable majorant valid for any function in the admissible (energy) class of functions. It is proved that the majorant vanishes if and only if the function coincides with the minimizer. It is shown that the respective estimate has no gap, so that accuracy of any approximation can be evaluated with any desired accuracy. Bibliography: 10 titles. Illustrations: 1 figure.

Dedicated to dear Nina Nikolaevna Uraltseva in occasion of her jubilee

1 Two–Phase Obstacle Type Problem

Let $\Omega \subset \mathbb{R}^d$ $(d \geqslant 1)$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. We consider the following two-phase obstacle problem, which was introduced and studied in [1]–[3] and other papers cited therein.

Problem 1.1 (primal problem). Find $u \in K$ such that where

$$J(v) := \int\limits_{\Omega} \Big(\frac{1}{2} A\nabla v \cdot \nabla v - fv + \alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}\{v\}_{\ominus}\Big)\,\mathrm{d}x, \tag{1.1}$$

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$\{v\}_\oplus$ and $\{v\}_\oplus$ denote the positive and negative parts of v, i.e., $\{v\}_\oplus := \max\{0, v\}$, $\{v\}_\ominus := \max\{-v, 0\}$, and $K := \{v \in H^1(\Omega) : v|_{\partial\Omega} = u_0\}$. The function $u_0 \in H^1(\Omega)$ defines the prescribed Dirichlet boundary condition, which may attain both positive and negative values.

It is assumed that the coefficients $\alpha_\oplus, \alpha_\ominus : \Omega \to \mathbb{R}$ are positive Lipschitz continuous functions, $f \in L^2(\Omega)$, and $A \in L^2(\Omega, \mathbb{R}^{d\times d})$ is a symmetric matrix with bounded coefficients satisfying the condition

$$A(x)\xi \cdot \xi \geqslant c_1 |\xi|^2, \quad c_1 > 0, \quad \forall \xi \in \mathbb{R}^d$$

almost everywhere in Ω. Under these assumptions, the functional $J(v)$ is strictly convex and continuous on V. Hence there exists a unique minimizer $u \in K$ of Problem 1.1. A physical interpretation of the problem is presented by an elastic membrane touching the planar phase boundary between two liquid/gaseous phases (cf., for example, [1]). Properties of minimizers were studied in [1]–[3] and some other papers. It was shown that the correspoding Euler-Lagrangian equation has the form

$$\operatorname{div}(A\nabla u) = -f + \alpha_\oplus \chi_{\{u>0\}} - \alpha_\ominus \chi_{\{u<0\}}, \quad u|_{\partial\Omega} = u_0, \tag{1.2}$$

where χ denotes a characteristic set function.

We analyze Problem 1.1 in the context of a posteriori error estimates, i.e., our goal is to deduce a fully computable and guaranteed bound of the difference between the exact solution $u \in K$ and an approximation $v \in K$ measured in terms of the natural (energy) norm. For this purpose we use general theory presented in [4, 5] and [7, Chapter 7], where estimates of the distance to the minimizer were derived for a class of convex variational problems generated by functionals $J : V \to \mathbb{R}$ of type

$$J(v) = G(\Lambda v) + F(v).$$

Here $\Lambda : Y^* \to \mathbb{R}$ is a bounded linear operator, $G : Y \to \mathbb{R}$ is a convex, coercive, and lower semicontinuous functional, $F : V \to \mathbb{R}$ is another convex lower semicontinuous functional, and Y and V are reflexive Banach spaces. The functional J is assumed to be proper (in the termiology of [6]). The respective conjugate (dual) spaces are denoted by Y^* and V^* respectively and the duality pairing are denoted by (y^*, y) and $< v^*, v >$. The starting point of our analysis is the identity

$$D_F(v, -\Lambda^* p^*) + D_G(\Lambda v, p^*) = J(v) - I^*(p^*), \tag{1.3}$$

which is a particular form of the identity (7.2.13) in [7]. Here p^* is the exact solution of the dual variational problem, which is to maximize the functional $I^*(y^*) := -G^*(y^*) - F^*(-\Lambda^* y^*)$ over the space Y^* topologically dual to Y, $G^* : Y^* \to \mathbb{R}$ and $F^* : V^* \to \mathbb{R}$ are the Young–Fenchel transforms of G and F, respectively, and

$$D_F(v, -\Lambda^* p^*) := F(v) + F^*(v^*) - \langle v^*, v\rangle,$$
$$D_G(\Lambda v, p^*) := G(\Lambda v) + G^*(p^*) - (p^*, \Lambda v).$$

In view of the duality relation $J(u) = I^*(p^*)$, the identity (1.3) is equivalent to

$$D_F(v, -\Lambda^* p^*) + D_G(\Lambda v, p^*) = J(v) - J(u). \tag{1.4}$$

We apply this identity to Problem 1.1 in which Λ is defined as the gradient operator acting from $H^1(\Omega)$ to $Y := L^2(\Omega, \mathbf{R}^d)$,

$$G(\Lambda v) := \frac{1}{2} \int_\Omega A\nabla v \cdot \nabla v \, dx,$$

$$F(v) := \int_\Omega \Big(- fv + \alpha_\oplus \{v\}_\oplus + \alpha_\ominus \{v\}_\ominus \Big) \, dx.$$

In this case, $V = V_0 := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}$ and V^* is $H^{-1}(\Omega)$. The identities (1.3) and (1.4) also hold if v is defined on an affine set $V_0 + u_0$ (cf. [7, Section 7.3]). The identity (1.4) plays an important role in our subsequent analysis. First of all, the left-hand side of (1.4) yields the measure of the distance to the exact solution, which is natural for the considered class of variational problems. This measure contains two parts. The first part is an integral (energy) norm of $v - u$ generated by the main part of the functional (associated with G). The second part is a certain measure of the error associated with free boundary. These questions are discussed in Section 2. In Section 3, we estimate the right-hand side of (1.4) from above by a fully computable quantity, which yields a majorant of the above discussed error measure. In the last section, we present an example, which shows that the estimates indeed provide correct estimates of the distance to the minimizer.

2 Measure of the Distance Between v and u

Let

$$H(v) := \int_\Omega h(z) \, \mathrm{d}x, \quad h(z) := \alpha_\oplus \{v\}_\oplus + \alpha_\ominus \{v\}_\ominus.$$

First, we find $H^*(v^*)$, where $v^* \in L^2(\Omega)$. In this case, finding $H^*(v^*)$ is reduced to finding the polar to h (cf., for example, [6]) and we need to find

$$\sup_{z \in \mathbb{R}} \{ z^* z - \alpha_\ominus \{z\}_\ominus - \alpha_\oplus \{z\}_\oplus \},$$

where $z^* \in \mathbb{R}$. If $z^* > -\alpha_\oplus$, then the expression tends to $+\infty$ as $z \to +\infty$. Analogously, if $z^* < \alpha_\ominus$, then the expression tends to $+\infty$ as $z \to -\infty$. If $0 \leqslant z^* \leqslant \alpha_\oplus$, then for $z > 0$ we have $z^* z - \alpha_\oplus \{z\}_\oplus \leqslant 0$ and for $z < 0$ we also have $z^* z - \alpha_\ominus \{z\}_\ominus \leqslant 0$. Therefore, in this case, the supremum is equal to zero. Analogous arguments show that for $-\alpha_\ominus \leqslant z^* \leqslant 0$, the supremum is also equal to zero. Hence

$$\sup_{z \in \mathbb{R}} \big\{ z^* z - \alpha_\ominus \{z\}_\ominus - \alpha_\oplus \{z\}_\oplus \big\} = h^*(z^*) := \begin{cases} +\infty, & z^* < -\alpha_\ominus, \\ 0, & -\alpha_\ominus \leqslant z^* \leqslant \alpha_\oplus, \\ +\infty, & z^* > \alpha_\oplus. \end{cases}$$

Since

$$F(v) = \int_\Omega (h(v) - fv) dx$$

and

$$\sup_{z\in\mathbb{R}}\{z^*z+fz-h(z)\}=h^*(z^*+f),$$

we conclude that

$$F^*(v^*)=\int\limits_{\Omega}h^*(v^*+f)\,dx.$$

Note that $h^*(v^*+f)(x)=0$ if $v^*(x)$ is in the interval $[-\alpha_\ominus-f,\alpha_\oplus-f]$. The corresponding compound functional is finite if this condition holds for almost all x and it has the form

$$D_F(v,v^*)=\int\limits_{\Omega}(\alpha_\oplus\{v\}_\oplus+\alpha_\ominus\{v\}_\ominus-(v^*+f)v)dx.$$

Now, we apply (1.3), where

$$p^*:=A\nabla u. \tag{2.1}$$

In our case, $\Lambda=\nabla$, $\Lambda^*=-\mathrm{div}$, and (note that $\mathrm{div}p^*+f\in L^2(\Omega)$)

$$D_F(v,-\Lambda^*p^*)=\int\limits_{\Omega}(\alpha_\oplus\{v\}_\oplus+\alpha_\ominus\{v\}_\ominus-(\mathrm{div}p^*+f)v)dx$$

provided that

$$-\alpha_\ominus\leqslant\mathrm{div}p^*+f\leqslant\alpha_\oplus.$$

In view of (1.2), this condition holds. We introduce two decompositions of Ω associated with the minimizer u and the function v:

$$\Omega_u^+:=\{x\in\Omega:u(x)>0\},\quad\Omega_u^-:=\{x\in\Omega:u(x)<0\},\quad\Omega_u^0:=\{x\in\Omega:u(x)=0\}$$

and

$$\Omega_v^+:=\{x\in\Omega:v(x)>0\},\quad\Omega_v^-:=\{x\in\Omega:v(x)<0\},\quad\Omega_v^0:=\{x\in\Omega:v(x)=0\}.$$

First, we see that

$$D_F(v,-\Lambda^*p^*)=\int\limits_{\Omega_{v>0}}(\alpha_\oplus-(\mathrm{div}p^*+f))v\,dx+\int\limits_{\Omega_{v<0}}(-\alpha_\ominus-(\mathrm{div}p^*+f))v\,dx\geqslant0. \tag{2.2}$$

Thus, the value of the compound functional D_F is always nonnegative. If $\mathrm{sgn}\,(u)=\mathrm{sgn}\,(v)$ (i.e., if Ω_u^+ and Ω_u^- coincide with Ω_v^+ and Ω_v^- respectively) then

$$\mathrm{div}p^*+f=a_\oplus\quad\text{on}\quad\Omega_v^+,$$
$$\mathrm{div}p^*+f=-a_\ominus\quad\text{on}\quad\Omega_v^-$$

due to (1.2), so that $D_F(v,-\Lambda^*p^*)=0$. In all other cases, $D_F(v,-\Lambda^*p^*)>0$. Below we consider all possible variants related to this situation.

If the set $\Omega_u^+\cap\Omega_v^-$ has a positive measure, then

$$\int\limits_{\Omega_u^+\cap\Omega_v^-}(\alpha_\oplus\{v\}_\oplus+\alpha_\ominus\{v\}_\ominus-(\mathrm{div}p^*+f)v)\,dx=-\int\limits_{\Omega_u^+\cap\Omega_v^-}(\alpha_\ominus+\alpha_\oplus)v\,dx>0. \tag{2.3}$$

Analogously, if the set $\Omega_u^- \cap \Omega_v^+$ has a positive measure, then

$$\int\limits_{\Omega_u^- \cap \Omega_v^+} (\alpha_\oplus \{v\}_\oplus + \alpha_\ominus \{v\}_\ominus - (\mathrm{div} p^* + f)v)dx = (\alpha_\oplus + \alpha_\ominus) \int\limits_{\Omega_u^- \cap \Omega_v^+} vdx > 0. \tag{2.4}$$

Hence for the set $\omega_\pm := \{\Omega_u^+ \cap \Omega_v^-\} \cup \{\Omega_u^- \cap \Omega_v^+\}$

$$\int\limits_{\omega_\pm} (\alpha_\oplus \{v\}_\oplus + \alpha_\ominus \{v\}_\ominus - (\mathrm{div} p^* + f)v)dx = (\alpha_\oplus + \alpha_\ominus) \int\limits_{\omega_\pm} |v|dx > 0. \tag{2.5}$$

If the set $\omega_-^0 := \Omega_u^0 \cap \Omega_v^-$, has a positive measure, then

$$\int\limits_{\omega_-^0} (\alpha_\oplus \{v\}_\oplus + \alpha_\ominus \{v\}_\ominus - (\mathrm{div} p^* + f))vdx = -\alpha_\ominus \int\limits_{\omega_-^0} v\,dx > 0. \tag{2.6}$$

Analogously, if the set $\omega_+^0 := \Omega_u^0 \cap \Omega_v^+$, has a positive measure, then

$$\int\limits_{\omega_+^0} (\alpha_\oplus \{v\}_\oplus + \alpha_\ominus \{v\}_\ominus - (\mathrm{div} p^* + f))vdx = \alpha_\oplus \int\limits_{\omega_+^0} v\,dx > 0. \tag{2.7}$$

Now, we can rewrite $D_F(v, -\Lambda^* p^*)$ in a more compact way as

$$D_F(v, -\Lambda^* p^*) = m_\omega(v) := \int\limits_\Omega \alpha(x)|v|dx. \tag{2.8}$$

Here,

$$\alpha(x) = \begin{cases} 0, & x \in \Omega \setminus \omega, \quad \omega := \omega_-^0 \cup \omega_+^0 \cup \omega_\pm, \\ \alpha_\ominus, & x \in \omega_-^0, \\ \alpha_\oplus, & x \in \omega_+^0, \\ \alpha_\ominus + \alpha_\oplus, & x \in \omega_\pm, \end{cases}$$

and $m_\omega(v)$ is a nonnegative functional, which is positive on the set $\omega := \omega_-^0 \cup \omega_+^0 \cup \omega_\pm$. If the sets Ω_v^- and Ω_v^+ are well defined (i.e., for v they are the same as for the exact solution u), then $\omega = \varnothing$ and $m_\omega(v) = 0$. Hence we can view $m_\omega(v)$ as a certain accuracy measure related to approximations of free boundaries.

The second term on the right-hand side of (1.4) generates another measure of the error. Indeed,

$$\begin{aligned} D_G(\Lambda v, p^*) &= \int\limits_\Omega \left(\frac{1}{2} A\nabla v \cdot \nabla v + \frac{1}{2} A^{-1} p^* \cdot p^* - \nabla v \cdot p^*\right) dx \\ &= \frac{1}{2} \int\limits_\Omega A\nabla(u - v) \cdot \nabla(u - v) dx = \frac{1}{2} \|\nabla(u - v)\|^2_{\Omega, A}, \end{aligned} \tag{2.9}$$

and this term is equal to the energy norm of $u - v$.

Now, the general error identity (1.4) implies the following error identity for approximate solutions to Problem 1.1:

$$\frac{1}{2}\|\nabla(u-v)\|^2_{\Omega,A} + m_\omega(v) \;=\; J(v) - J(u) \quad \forall\, v \in K, \tag{2.10}$$

The left-hand side of (2.10) presents the *full error measure*

$$\frac{1}{2}\|\nabla(u-v)\|^2_{\Omega,A} + m_\omega(v) \tag{2.11}$$

naturally associated with the considered nonlinear problem. This entity is related to the question which error measure correctly reflects accuracy of an approximation for nonlinear problems (cf. [5]).

The estimate (2.10) does not represent an a posteriori error estimate yet since it contains the unknown term $J(u)$ on the right-hand side. By techniques of Calculus of variation and concepts of perturbed and dual problems, we replace this difference of energies by a fully computable term independent of u.

3 Majorant of the Difference Between $J(v)$ and $J(u)$

Now, our goal is to estimate the right-hand side of (2.10). For this purpose we define a new (perturbed) functional

$$J_\mu(v) := J_{(\mu_\oplus,\mu_\ominus)}(v) = \int_\Omega \Big(\frac{1}{2}A\nabla v\cdot\nabla v - f_\mu v\Big)\,\mathrm{d}x, \tag{3.1}$$

where $f_\mu := f - \alpha_\oplus\mu_\oplus + \alpha_\ominus\mu_\ominus$. It is easy to see that

$$J(v) = \sup_{\mu\in\Lambda} J_\mu(v) \quad \forall\, v\in K, \tag{3.2}$$

where $\Lambda := \big\{(\mu_\oplus,\mu_\ominus)\in L^\infty(\Omega,\mathbb{R}^2) : \mu_\oplus(x),\mu_\ominus(x)\in[0,1] \text{ a.e. in}\Omega\big\}$. Since

$$J(u) = \inf_{v\in K}\sup_{\mu\in\Lambda} J_\mu(v) \geqslant \sup_{\mu\in\Lambda}\inf_{v\in K} J_\mu(v) \geqslant \inf_{v\in K} J_\mu(v) \quad \forall\,\mu\in\Lambda, \tag{3.3}$$

the value of $J(u)$ is estimated from below by the value of the problem on the right-hand side of (3.3).

Problem 3.1 (perturbed problem)**.** For a given $\mu\in\Lambda$ find $u_\mu\in K$ such that

$$J_\mu(u_\mu) = \inf_{v\in K} J_\mu(v). \tag{3.4}$$

For any given $\mu\in\Lambda$ the perturbed problem is uniquely solvable by the same reasons as those for Problem 1.1. In view of (3.3), the minimal perturbed energy $J_\mu(u_\mu)$ serves as the lower bound of $J(u)$. We find a computable lower bound of $J_\mu(u_\mu)$ by means of the dual counterpart of the perturbed problem. The dual problem is generated by the Lagrangian

$$L_\mu(v,\tau^*) = \int_\Omega (\nabla v\cdot\tau^* - \frac{1}{2}A^{-1}\tau^*\cdot\tau^* - f_\mu v)\,\mathrm{d}x,$$

where $\tau^* \in L^2(\Omega, \mathbb{R}^d)$. It is easy to see that

$$\sup_{\tau^* \in L^2(\Omega,\mathbb{R}^d)} L_\mu(v, \tau^*) = J_\mu(v).$$

Moreover,

$$\begin{aligned} J_\mu(u_\mu) &= \inf_{v \in K} J_\mu(v) = \inf_{v \in K} \sup_{\tau^* \in L^2(\Omega,\mathbb{R}^d)} L_\mu(v, \tau^*) \\ &= \sup_{\tau^* \in L^2(\Omega,\mathbb{R}^d)} \inf_{v \in K} L_\mu(v, \tau^*) = \sup_{\tau^* \in Q_{f_\mu}} J^*_\mu(\tau^*), \end{aligned} \tag{3.5}$$

where

$$J^*_\mu(\tau^*) := \inf_{v \in K} L_\mu(v, \tau^*) = \int_\Omega \Big(-\frac{1}{2} A^{-1}\tau^* \cdot \tau^* + \nabla u_0 \cdot \tau - f_\mu u_0 \Big) \, \mathrm{d}x, \tag{3.6}$$

$$Q_{f_\mu} = \Big\{ \tau^* \in L^2(\Omega, \mathbb{R}^d) : \int_\Omega (\tau^* \cdot \nabla w - f_\mu w) \, \mathrm{d}x = 0 \quad \forall \, w \in V_0 \Big\}. \tag{3.7}$$

The right-hand side of (3.5) gives rise to the dual problem.

Problem 3.2 (dual perturbed problem)**.** For a given $\mu \in \Lambda$, find $\sigma^* \in Q_{f_\mu}$ such that

$$J^*_\mu(\sigma^*) = \sup_{\tau^* \in Q_{f_\mu}} J^*_\mu(\tau^*). \tag{3.8}$$

Problem 3.2 is a quadratic maximization problem with a strictly concave and continuous functional. It has a unique maximizer in the affine subspace $\tau^* \in Q_{f_\mu}$. Due to (3.3) and (3.5), we obtain the estimate

$$J(v) - J(u) \leqslant J(v) - J^*_\mu(\tau^*), \tag{3.9}$$

which is valid for all $v \in K$, $\mu \in \Lambda$, $\tau^* \in Q_{f_\mu}$. Due to (1.1), (3.6), and (3.7),

$$J(v) - J^*_\mu(\tau^*) = \frac{1}{2} ||A\nabla v - \tau^*||^2_{\Omega, A^{-1}} + \int_\Omega \big(\alpha_\oplus (\{v\}_\oplus - \mu_\oplus v) + \alpha_\ominus (\{v\}_\ominus + \mu_\ominus v) \big) \, \mathrm{d}x. \tag{3.10}$$

The right-hand size of (3.10) is fully computable, but it requires that τ^* satisfies the constraint $\tau^* \in Q_{f_\mu}$. Our goal is to replace (3.10) by a more general estimate for a function $\eta^* \in H(\Omega, \mathrm{div})$, which may not satisfy the constraint $\eta^* \in Q_{f_\mu}$. We apply the triangle inequality

$$||A\nabla v - \tau^*||_{\Omega, A^{-1}} \leqslant ||A\nabla v - \eta^*||_{\Omega, A^{-1}} + ||\eta^* - \tau^*||_{\Omega, A^{-1}}$$

and note that the estimate

$$\begin{aligned} \inf_{\tau^* \in Q_{f_\mu}} ||A\nabla v - \tau^*||_{\Omega, A^{-1}} &\leqslant \{ ||A\nabla v - \eta^*||_{\Omega, A^{-1}} + \inf_{\tau^* \in Q_{f_\mu}} ||\eta^* - \tau^*||_{\Omega, A^{-1}} \} \\ &\leqslant ||A\nabla v - \eta^*||_{\Omega, A^{-1}} + C_\Omega ||\mathrm{div}\, \eta^* + f_\mu||_\Omega \end{aligned} \tag{3.11}$$

is valid for any $\eta^* \in H(\Omega, \mathrm{div})$. In this estimate, we used a projection-type inequality (cf., for example, [8, Chapter 3])

$$\inf_{\tau^* \in Q_{f_\mu}} ||\eta^* - \tau^*||_{\Omega, A^{-1}} \leqslant C_\Omega ||\mathrm{div}\, \eta^* + f_\mu||_\Omega \quad \forall \, \eta^* \in H(\Omega, \mathrm{div}).$$

Here, the constant $C_\Omega > 0$ originates from the generalized Friedrichs inequality

$$||v||_\Omega \leqslant C_\Omega ||\nabla v||_{\Omega,A}$$

valid for all $v \in V_0$. The Young estimate with a parameter $\beta > 0$ yields

$$(||A\nabla v - \eta^*||_{\Omega,A^{-1}} + C_\Omega ||\operatorname{div} \eta^* + f_\mu||_\Omega)^2$$
$$\leqslant (1+\beta)||A\nabla v - \eta^*||^2_{\Omega,A^{-1}} + \Big(1 + \frac{1}{\beta}\Big) C^2_\Omega ||\operatorname{div} \eta^* + f_\mu||^2_\Omega.$$

Hence the combination of (3.9), (3.10), and (3.11) yields the *majorant estimate*

$$J(v) - J(u) \leqslant M_+(v; \beta, \eta^*, \mu_\oplus, \mu_\ominus), \tag{3.12}$$

where

$$M_+(v; \beta, \eta^*, \mu_\oplus, \mu_\ominus) := \frac{1}{2}(1+\beta)||A\nabla v - \eta^*||^2_{\Omega,A^{-1}} + \frac{1}{2}(1 + \frac{1}{\beta})C^2_\Omega ||\operatorname{div} \eta^* + f_\mu||^2_\Omega$$
$$+ \int\limits_\Omega \big(\alpha_\oplus\big(\{v\}_\oplus - \mu_\oplus v\big) + \alpha_\ominus\big(\{v\}_\ominus + \mu_\ominus v\big)\big)\, \mathrm{d}x, \tag{3.13}$$

$M_+(v; \beta, \eta^*, \mu_\oplus \mu_\ominus)$ is a nonnegative functional (error majorant) which reflects natural properties of the original problem (Problem 1.1). We recall (2.10) and arrive at the main result.

Theorem 3.1. *For all $v \in K$, $\beta > 0$, $(\mu_\oplus, \mu_\ominus) \in \Lambda$, and $\eta^* \in H(\Omega, \operatorname{div})$*

$$\frac{1}{2}\|\nabla(u-v)\|^2_\Omega, A + m_\omega(v) \;\leqslant\; M_+(v; \beta, \eta^*, \mu_\oplus, \mu_\ominus). \tag{3.14}$$

In view of (3.14), we see that if $M_+(v; \beta, \eta^*, \mu_\oplus, \mu_\ominus) = 0$ then $v = u$ almost everywhere in Ω. Moreover, in this case,

$$\eta^* = A\nabla u, \tag{3.15}$$
$$\operatorname{div} \eta^* = -f_\mu, \tag{3.16}$$
$$\{v\}_\oplus = \mu_\oplus v, \tag{3.17}$$
$$\{v\}_\ominus = -\mu_\ominus v \tag{3.18}$$

almost everywhere in Ω. We can also justify the opposite statement. Indeed, if $v = u$ and other arguments of the majorant are generated by the minimizer, i.e.,

$$\eta^* = A\nabla u, \quad \mu_\oplus = \chi\{u > 0\}, \quad \mu_\ominus = \chi\{u < 0\} \quad \text{a.e. in } \Omega, \tag{3.19}$$

then $M_+(v; \beta, \eta^*, \mu_\oplus, \mu_\ominus) = 0$ for all $\beta > 0$. In other words, for any $\beta > 0$ and $(\mu_\oplus, \mu_\ominus) \in \Lambda$, the majorant $M_+(v; \beta, \eta^*, \mu_\oplus, \mu_\ominus)$ vanishes if and only if $v = u$ and $\eta^* = A\nabla u$.

It is worth noticing one other important property of the majorant. If the functions η^*, $\mu_\oplus$, and $\mu_\ominus$ are selected in accordance with the exact solution by (3.19) then for any $v \in K$ the majorant coincides with the error measure on the left-hand side of (3.14). Indeed, in this case, $\operatorname{div} \eta^* + f_\mu = 0$ and we can set $\beta = 0$. Then

$$\|A\nabla v - \eta^*\|^2_{\Omega,A^{-1}} = \|\nabla(u-v)\|^2_{\Omega,A}.$$

Consider the last term of the majorant

$$\int_{\Omega} \big(\alpha_{\oplus}(\{v\}_{\oplus} - \mu_{\oplus} v) + \alpha_{\ominus}(\{v\}_{\ominus} + \mu_{\ominus} v)\big)\,\mathrm{d}x$$

$$= \int_{\Omega_-^u} \big(\alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}(\{v\}_{\ominus} + v)\big)dx + \int_{\Omega_+^u} \big(\alpha_{\oplus}(\{v\}_{\oplus} - v) + \alpha_{\ominus}\{v\}_{\ominus}\big)dx$$

$$+ \int_{\Omega_0^u} \big(\alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}\{v\}_{\ominus}\big)dx.$$

Here,

$$\int_{\Omega_-^u} \big(\alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}(\{v\}_{\ominus} + v)\big)dx = \int_{\Omega_-^u \cap \Omega_+^v} \Big(\alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}(\{v\}_{\ominus} + v)\big)dx$$

$$+ \int_{\Omega_-^u \cap \Omega_-^v} \big(\alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}(\{v\}_{\ominus} + v)\big)dx + \int_{\Omega_-^u \cap \Omega_0^v} \big(\alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}(\{v\}_{\ominus} + v)\big)dx$$

$$= \int_{\Omega_-^u \cap \Omega_+^v} \Big(\alpha_{\oplus} + \alpha_{\ominus})v dx.$$

Analogously,

$$\int_{\Omega_+^u} \big(\alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}(\{v\}_{\ominus} + v)\big)dx = \int_{\Omega_+^u \cap \Omega_-^v} \Big(\alpha_{\oplus} + \alpha_{\ominus})v dx.$$

Now, we split the third term in a similar way

$$\int_{\Omega_0^u} \big(\alpha_{\oplus}\{v\}_{\oplus} + \alpha_{\ominus}\{v\}_{\ominus}\big)dx = -\int_{\Omega_0^u \cap \Omega_-^v} \alpha_{\ominus} v dx + \int_{\Omega_0^u \cap \Omega_+^v} \alpha_{\oplus} v dx$$

and find that the last term of the majorant coincides with $m_{\omega}(v)$. Therefore, the estimate (3.14) has no gap between the left- and right-hand sides and, in principle, we can always select the parameters of the majorant such that it is arbitrarily close to the error.

Remark 3.1. Under the special choice $\mu_{\oplus} = \chi\{v > 0\}$ and $\mu_{\ominus} = \chi\{v < 0\}$, the majorant can be presented in a simplified form

$$\widetilde{M_+}(v; \beta, \eta^*) = \frac{1}{2}(1+\beta)||A\nabla v - \eta^*||^2_{\Omega, A^{-1}}$$
$$+ \frac{1}{2}\Big(1 + \frac{1}{\beta}\Big) C_{\Omega}^2 ||\operatorname{div} \eta^* + f - \alpha_{\oplus}\chi\{v > 0\} + \alpha_{\ominus}\chi\{v < 0\}||^2_{\Omega}. \qquad (3.20)$$

It is easy to see that this majorant vanishes if and only if v coincides with the exact minimizer.

4 Example

We verify the error identity (2.10) and the majorant estimate (3.12) on a benchmark example taken from [9]. In this example, $\Omega = (-1, 1)$, $f = 0$, $\alpha_{\oplus} = \alpha_{\ominus} = 8$ and the equation is supplied with the Dirichlet boundary conditions $u(-1) = -1, u(1) = 1$. The exact solution is given by the relation

$$u(x) = \begin{cases} -4x^2 - 4x - 1, & x \in \langle -1, -0.5 \rangle, \\ 0, & x \in \langle < -0.5, 0.5 \rangle >, \\ 4x^2 - 4x + 1, & x \in \langle < 0.5, 1 \rangle > \end{cases}$$

and the respective exact minimum is $J(u) = 5\frac{1}{3}$.

The error identity (2.10) is valid for any approximation $v \in H^1(-1,1)$ satisfying boundary conditions $v(-1) = -1, v(1) = 1$. In order to illustrate the performance of the error majorant in different situations, we first consider a very bad quality approximation

$$v(x) = x, \quad x \in \langle -1, 1 \rangle .$$

In this case, $J(v) = 9$. By a direct computation, we find

$$J(v) - J(u) = 3\frac{2}{3}, \quad \frac{1}{2}\|\nabla(u-v)\|^2_{\Omega,A} = 1\frac{2}{3}, \quad m_\omega(v) = 2.$$

Hence the basic error identity

$$\frac{1}{2}\|\nabla(u-v)\|^2_{\Omega,A} + m_\omega(v) = J(v) - J(u)$$

holds. The quantity $m_\omega(v)$ significantly contributes to the error estimate and detailed information shows that $\omega^0_+(u,v) = (0, 0.5)$, $\omega^0_-(u,v) = (-0.5, 0)$, $\omega_\pm(u,v) = \varnothing$ and

$$\alpha_\oplus \int\limits_{\omega^0_+} v dx = 4\frac{1}{2}, \quad -\alpha_\ominus \int\limits_{\omega^0_-} v dx = 4\frac{1}{2}, \quad (\alpha_\oplus + \alpha_\ominus) \int\limits_{\omega_\pm} |v| dx = 0.$$

TABLE 1. The error identity and the majorant estimate computed on various uniform meshes with mesh size h.

h	$\frac{1}{2}\|\nabla(u-v)\|^2_{\Omega,A}$	$m_\omega(v)$	$J(v) - J(u)$	$\mathscr{M}(v,\dots)$	$\frac{\sqrt{2\mathscr{M}(v,\dots)}}{\|\nabla(v-u)\|_{\Omega,A}}$
1/4	6.67e-01	1.93e-06	6.67e-01	8.36e-01	1.12
1/8	1.67e-01	1.04e-06	1.67e-01	1.89e-01	1.07
1/16	4.17e-02	1.12e-06	4.17e-02	4.45e-02	1.03
1/32	1.04e-02	1.37e-06	1.04e-02	1.08e-02	1.02
1/64	2.60e-03	1.73e-07	2.60e-03	2.68e-03	1.01

For a better quality approximation v, the value of the gap $m_\omega(v)$ becomes less important. We construct v by the finite elements method on an equidistant partition of $\Omega = (-1,1)$ with mesh size $h \in \{1/4, 1/8, 1/16, 1/32, 1/64\}$. The approximation v is constructed as a nodal and piecewise linear function. The nodal values of v are obtained by the MATLAB automatic unconstrained minimization method *fminunc*. Table 1 reports on results for considered uniform meshes. The squared approximation error $\frac{1}{2}\|\nabla(u-v)\|^2_{\Omega,A}$ decreases by the factor 4 as the interval size h gets halved. This indicates the linear convergence of the approximation error

$\|\nabla(u-v)\|_{\Omega,A}$ expected for the nodal piecewise linear approximation v. The gap $m_\omega(v)$ is indeed very low for all interval partitions. In verification of the majorant estimate (3.12), the gradient (flux) field τ is constructed as a nodal and piecewise linear function. The Lagrange multipliers $\mu_\ominus$ and $\mu_\oplus$ are sought as a piecewise constant function. The minimization of the functional majorant providing optimal values of $\tau, \mu_\ominus, \mu_\oplus$ is obtained by the MATLAB automatic constrained minimization method $fmincon$. Table 1 confirms that the functional majorant $\mathscr{M}(v,\dots)$ serves as a sharp error estimate. Local distributions of the (squared) error and the functional majorant are compared in Figure 1 for the finest interval partition. The MATLAB software inspired by [10] is available at http://www.mathworks.com/matlabcentral/fileexchange/47966.

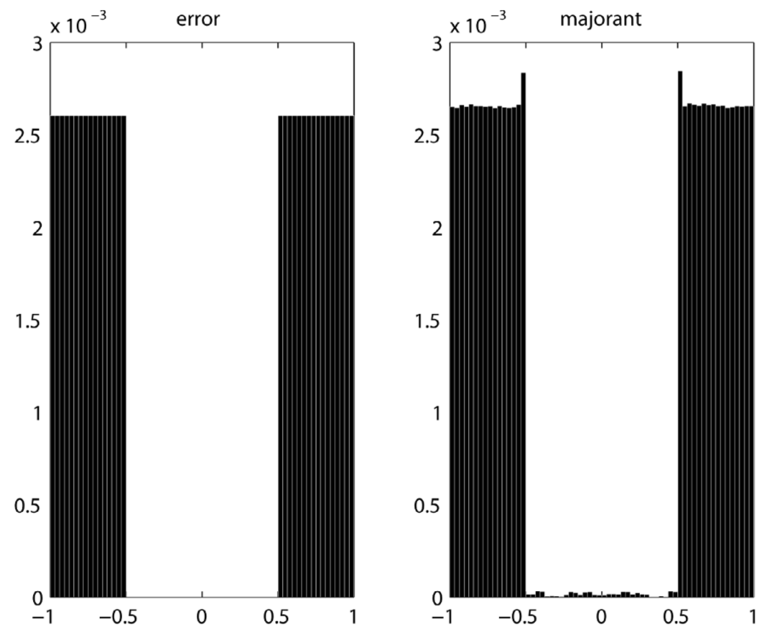

FIGURE 1. Distribution of the error $\frac{1}{2}\|\nabla(v-u)\|^2_{\Omega,A}$ (left) and the majorant (right) for $h = 1/64$.

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