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General solution of Chládek's functional equations

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In this contribution we find the solution $F : I \times P_2(I) \times M \times M \to M$ of the following functional equations $F(\tau, \gamma, \delta, F(\gamma, \alpha, \beta, a, b), F(\delta, \alpha, \beta, a, b)) = F(\tau, \alpha, \beta, a, b)$, $F(\alpha, \alpha, \beta, a, b) = a$, $F(\beta, \alpha, \beta, a, b) = b$. Since we do not insist on any conditions for the solution, the obtained result is greatly general.

Keywords: functional equations; general solution; ordinary differential equations; Dirichlet conditions

AMS Subject Classification: 39B52

1. Introduction

In 2005 Chládek [5] derived the following functional equations

$$F(\tau, \gamma, \delta, F(\gamma, \alpha, \beta, a, b), F(\delta, \alpha, \beta, a, b)) = F(\tau, \alpha, \beta, a, b), \tag{1}$$

$$F(\alpha, \alpha, \beta, a, b) = a, \quad F(\beta, \alpha, \beta, a, b) = b \tag{2}$$

describing the dependence $F(\tau, \alpha, \beta, a, b)$ of the solution x of the ordinary differential equation (ODE) of second order on the independent τ and the Dirichlet conditions $x(\alpha) = a, x(\beta) = b$ where $\alpha \neq \beta$. With this achievement Chládek has pursued the effort to find a functional formulation of ODEs (see Abdelkader [1], Castillo and Ruiz-Cobo [4], Eshukov [6]). Moreover, he explained the relation of his functional equations with the theory of geodesics [10], with the theory of conics [3], with Jensen's functional equation (page 43 in [2]) and with Neuman's results concerning ordinary linear differential equations [9].

Concerning the first-order differential equations it is already known that there exist functional equations

$$F(\tau, \gamma, F(\gamma, \alpha, a)) = F(\tau, \alpha, a), \quad F(\alpha, \alpha, a) = a \tag{3}$$

describing the dependence $F(\tau, \alpha, a)$ of the solution x of the ODE of first order on the independent τ and the Cauchy condition $x(\alpha) = a$ (e.g. [8], problem 17-15). Equations (3) do not contain derivatives and are therefore useful in the non-differentiable or even

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discontinuous case and as such they also cover much wider application field than differential equations. These Equations (3) represent a generalization of the Sincov's equation 8.1.3(8)[2] and are used, i.e. in the theory of stochastic processes. Clearly, we can use the relations (3) also for ODEs of second order. But since in this case the Cauchy condition $(x(\alpha), \dot{x}(\alpha)) = a$ contains the derivative $\dot{x}$, it does not give any information in non-differentiable or discontinuous cases. This is accessible only using the Dirichlet conditions by Equations (1) and (2).

Equations (1) and (2) represent the functional expression of existence and uniqueness conditions of the solution of the Dirichlet problem. In particular, Equation (2) guarantees the existence of the solution of any Dirichlet problem and Equation (1) its uniqueness. Hence, the equations hold if and only if the existence and uniqueness are ensured. In the situation described, i.e. in Theorem 4.1 in [7] on page 423 Equations (1) and (2) surely hold but can also hold if the assumptions of the Theorem 4.1 are not satisfied and moreover, it is not necessary to be limited to differential equations.

The aim of this contribution is to find the most general solution of Equations (1) and (2). For this purpose we consider these equations in the most abstract sense (see page 1 in [2]). Therefore, contrary to [5], we impose on the desired solutions neither continuity and differentiability requirements, nor topological or algebraical requirements on their domains.

2. General solution

By a solution of Equations (1) and (2) we understand a map

$$F : I \times P_2(I) \times M \times M \to M,$$

where I and M are sets, $P_2(I) \subset I \times I$ the set of all 2-permutations (without repetition) of elements of I, such that for all $a, b \in M$, $\tau, \alpha, \beta, \gamma, \delta \in I$, $\alpha \neq \beta$, $\gamma \neq \delta$ Equations (1) and (2) hold true.

THEOREM 2.1.

(a) *Any solution of Equations (1) and (2) is of the form*

$$F : (\tau, \alpha, \beta, a, b) \mapsto G_\tau \circ ((G_\alpha \times G_\beta) \circ \nu_W)^{-1}(a, b), \tag{4}$$

where W is a set, $G : I \times W \to M$ a map, $G_\tau : W \ni w \mapsto G(\tau, w) \in M$, $G_\alpha : W \ni w \mapsto G(\alpha, w) \in M$ and $G_\beta : W \ni w \mapsto G(\beta, w) \in M$ its partial maps, $G_\alpha \times G_\beta : W \times W \ni (w_1, w_2) \mapsto (G_\alpha(w_1), G_\beta(w_2)) \in M \times M$ the Cartesian product of the corresponding partial maps and $\nu_W : W \ni w \mapsto (w, w) \in W \times W$ the diagonal embedding.

(b) *If $G : I \times W \to M$ is a map such that for all $(\alpha, \beta) \in P_2(I) \subset I \times I$ the map $(G_\alpha \times G_\beta) \circ \nu_W$ is a bijection then the map* (4) *is a solution of Equations (1) and (2).*

(c) *Let F be a solution of Equations (1) and (2) and let $G^1 : I \times W^1 \to M$, $G^2 : I \times W^2 \to M$ be two maps both satisfying* (4)*. Then there exists such a bijection $S^{12} : W^2 \to W^1$ that $G^2 : (\tau, w) \mapsto G^1(\tau, S^{12}(w))$.*

Proof.

(a) Suppose that F is a solution of Equations (1) and (2), fix a pair $(\gamma, \delta) \in P_2(I)$ and put

$$W = M \times M, \quad G : I \times W \ni (\tau, (c, d)) \mapsto F(\tau, \gamma, \delta, c, d) \in M.$$

Thus it holds that

$$(G_\alpha \times G_\beta) \circ \nu_W : (c, d) \mapsto (F(\alpha, \gamma, \delta, c, d), F(\beta, \gamma, \delta, c, d)).$$

Consider now such pairs $(c_1, d_1) \in M \times M$, $(c_2, d_2) \in M \times M$ that $(G_\alpha \times G_\beta) \circ \nu_W(c_1, d_1) = (G_\alpha \times G_\beta) \circ \nu_W(c_2, d_2)$. Hence, by (1) and (2), we obtain

$$\begin{aligned}(c_1, d_1) &= (F(\gamma, \gamma, \delta, c_1, d_1), F(\delta, \gamma, \delta, c_1, d_1)) \\ &= (F(\gamma, \alpha, \beta, F(\alpha, \gamma, \delta, c_1, d_1), F(\beta, \gamma, \delta, c_1, d_1)), \\ &\quad F(\delta, \alpha, \beta, F(\alpha, \gamma, \delta, c_1, d_1), F(\beta, \gamma, \delta, c_1, d_1))) \\ &= (F(\gamma, \alpha, \beta, F(\alpha, \gamma, \delta, c_2, d_2), F(\beta, \gamma, \delta, c_2, d_2)), \\ &\quad F(\delta, \alpha, \beta, F(\alpha, \gamma, \delta, c_2, d_2), F(\beta, \gamma, \delta, c_2, d_2))) \\ &= (F(\gamma, \gamma, \delta, c_2, d_2), F(\delta, \gamma, \delta, c_2, d_2)) = (c_2, d_2).\end{aligned}$$

This proves the injectivity of the map $(G_\alpha \times G_\beta) \circ \nu_W$. Further by (1) and (2) it also follows that for any pair $(a, b) \in M \times M$ we have

$$\begin{aligned}(a, b) &= (F(\alpha, \alpha, \beta, a, b), F(\beta, \alpha, \beta, a, b)) \\ &= (F(\alpha, \gamma, \delta, F(\gamma, \alpha, \beta, a, b), F(\delta, \alpha, \beta, a, b)), \\ &\quad F(\beta, \gamma, \delta, F(\gamma, \alpha, \beta, a, b), F(\delta, \alpha, \beta, a, b))) \\ &= (G_\alpha \times G_\beta) \circ \nu_W(F(\gamma, \alpha, \beta, a, b), F(\delta, \alpha, \beta, a, b)).\end{aligned}$$

This proves the surjectivity of the map $(G_\alpha \times G_\beta) \circ \nu_W$. Therefore we have shown that the map $(G_\alpha \times G_\beta) \circ \nu_W$ is a bijection and that its inverse is

$$((G_\alpha \times G_\beta) \circ \nu_W)^{-1} : (a, b) \mapsto (F(\gamma, \alpha, \beta, a, b), F(\delta, \alpha, \beta, a, b)).$$

Since $G_\tau \circ ((G_\alpha \times G_\beta) \circ \nu_W)^{-1} : (a, b) \mapsto F(\tau, \alpha, \beta, a, b)$ it is proven that any solution of Equations (1) and (2) is of the form (4).

(b) Suppose that $G : I \times W \to M$ is a map such that for any $(\alpha, \beta) \in P_2(I)$ the map $(G_\alpha \times G_\beta) \circ \nu_W$ is a bijection. In this case there exists an inverse map $((G_\alpha \times G_\beta) \circ \nu_W)^{-1}$. Hence, we can use (4) to obtain

$$\begin{aligned}&(F(\alpha, \alpha, \beta, a, b), F(\beta, \alpha, \beta, a, b)) \\ &\quad = (G_\alpha \circ ((G_\alpha \times G_\beta) \circ \nu_W)^{-1}(a, b), G_\beta \circ ((G_\alpha \times G_\beta) \circ \nu_W)^{-1}(a, b)) \\ &\quad = ((G_\alpha \times G_\beta) \circ \nu_W) \circ ((G_\alpha \times G_\beta) \circ \nu_W)^{-1}(a, b) \\ &\quad = \mathrm{id}_{M \times M}(a, b) = (a, b).\end{aligned}$$

It shows that the map (4) is a solution of Equations (2) and

$$
\begin{aligned}
&F(\tau,\gamma,\delta,F(\gamma,\alpha,\beta,a,b),F(\delta,\alpha,\beta,a,b))\\
&\quad= G_\tau\circ((G_\gamma\times G_\delta)\circ\nu_W)^{-1}((G_\gamma\times G_\delta)\circ\nu_W\circ((G_\alpha\times G_\beta)\circ\nu_W)^{-1}(a,b))\\
&\quad= G_\tau\circ\mathrm{id}_W\circ((G_\alpha\times G_\beta)\circ\nu_W)^{-1}(a,b)\\
&\quad= G_\tau\circ((G_\alpha\times G_\beta)\circ\nu_W)^{-1}(a,b)=F(\tau,\alpha,\beta,a,b).
\end{aligned}
$$

Thus, the map (4) is also solution of (1).

(c) Finally, suppose that F is a solution of (1) and (2) and suppose that $G^1 : I\times W^1\to M$ and $G^2 : I\times W^2\to M$ are maps both satisfying (4). Then it holds $G^1_\tau\circ\left(\left(G^1_\alpha\times G^1_\beta\right)\circ\nu_{W^1}\right)^{-1}=G^2_\tau\circ\left(\left(G^2_\alpha\times G^2_\beta\right)\circ\nu_{W^2}\right)^{-1}$. Consequently,

$$
\begin{aligned}
G^2_\tau = G^2_\tau\circ\mathrm{id}_{W^2} &= G^2_\tau\circ\left(\left(G^2_\gamma\times G^2_\delta\right)\circ\nu_{W^2}\right)^{-1}\circ\left(\left(G^2_\gamma\times G^2_\delta\right)\circ\nu_{W^2}\right)\\
&= G^1_\tau\circ\left(\left(G^1_\gamma\times G^1_\delta\right)\circ\nu_{W^1}\right)^{-1}\circ\left(\left(G^2_\gamma\times G^2_\delta\right)\circ\nu_{W^2}\right)=G^1_\tau\circ S^{12}
\end{aligned}
$$

where

$$
S^{12}=\left(\left(G^1_\gamma\times G^1_\delta\right)\circ\nu_{W^1}\right)^{-1}\circ\left(\left(G^2_\gamma\times G^2_\delta\right)\circ\nu_{W^2}\right)
$$

for some fixed pair $(\gamma,\delta)\in P_2(I)$. Since by (4) for any pair $(\alpha,\beta)\in P_2(I)$ there exist inverse maps $\left(\left(G^1_\alpha\times G^1_\beta\right)\circ\nu_{W^1}\right)^{-1}$ and $\left(\left(G^2_\alpha\times G^2_\beta\right)\circ\nu_{W^2}\right)^{-1}$ then $\left(G^1_\alpha\times G^1_\beta\right)\circ\nu_{W^1}$ and $\left(G^2_\alpha\times G^2_\beta\right)\circ\nu_{W^2}$ are bijections. Therefore, the composition S^{12} is a bijection too. This concludes the proof.

□

3. Conclusion

Any map G for which the relation (4) makes sense generates by means of this relation exactly one solution F of functional Equations (1) and (2), and on the other side, for any solution F of Equations (1) and (2) there exists by (4) a generating map G that is unique up to bijection.

4. Notes on significance for the theory of ODE's

In the differentiable case $G(\tau,w)$ represents the dependence of the solution x of the ODE of second order on the independent variable τ and on some complete set of integrals of motion w. Part (b) of the Theorem 2.1 describes the uniqueness condition for solvability of all Dirichlet problems, part (a) describes the form of solutions of these Dirichlet problems and part (c) the transition to a different complete set of integrals of motion.

The Theorem 2.1 does not play any role for solving a given Dirichlet problem of a given ODE of second order. Actually, if we have a complete set of integrals of motion there is nothing to solve.

In the differentiable case the benefit of the Theorem 2.1 lies mainly in the fact that it provides the relation between uniqueness conditions for solvability of different types of problems of the same second order differential equation. For example, knowing the sufficient uniqueness conditions for solvability of Cauchy problems we can use the Theorem 2.1 to obtain sufficient conditions for unique solutions of Dirichlet problems. Also, by (13) in [5] page 267, we can generate all ODEs of second order guaranteeing the existence and the uniqueness of the solution of any Dirichlet problem.

In the non-differentiable case the benefit of the Theorem 2.1 is in the fact it concerns problems that are not considered in the theory of ODEs.

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