Akademie věd České republiky
Ústav teorie informace a automatizace, v.v.i.

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RESEARCH REPORT

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Linear ARX and state-space model with uniform noise: computation of first and second moments

No.2357 September 2016

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Abstract

This report collects technical procedures used for computations of various estimates and keeps them in one place for internal purposes. The context concerns application of estimation of unknown parameters and states of linear model with uniformly distributed noise.

Keywords: uncertainty; bounded variable; uniform noise; linear model; model identification; state estimation;

Contents

1 Introduction

Linear models are frequently used in many applications. The noise term determines the method of estimation of unknown quantities and prediction of the model output. We consider a bounded noise, because some real-life applications require hard limits on either data or unknown parameters and/or states. This request initiated a research in theory and algorithms for models with noise that is bounded and, as the conceptually simplest case, uniformly distributed.

This report contains detailed procedures and computations of first and second moments of several pdfs as a formal solution of estimation and prediction tasks with the uniformly distributed noise.

The text is rather heterogenous, because it is composed of various notes that have been collected to record and archive hand-written remarks and inferences, as they appeared in time. The theory is outlined only roughly, the stress is put on the integrals. Although some steps may seem trivial and not worth so much writing, or, on the other hand, might be briefer and more elegant, we think that the current form and the content are useful.

2 Linear state space model with correlated noise

Estimation of states of a linear state space model with correlated noise is one of the promising tasks applicable in practice. Actually, the question of state distribution requires a finer treatment. The key issue is that a sum of two random quantities implies convolution of their probability density functions (pdf). Convolution of two uniform distributions is not a uniform distribution, unlike the normal one.

As one of possible approaches, the actual pdf can in each time step be approximated by a uniform pdf to keep the class of pdfs. Alternative approach may be to choose another wider class of pdfs with bounded domain, but these considerations are beyond the scope of this report.

2.1 Task description

Let us have a model

$$Ex_t = \tilde{x}_t + \nu_t, \tag{1}$$

where E is a known $n \times n$ upper triangular matrix with unit diagonal, $\tilde{x}_t$ is a known column vector, x_t is a column vector describing an unknown state in a discrete time instant time t, and ν_t is a uniform noise $\nu \sim \mathcal{U}(-\rho, \rho)$, where the vector elements $\rho_i > 0$, $i = 1, \ldots, n$. The idea of noise correlation is clearer from the alternative expression (18) or if the equation(1) is left-multiplied by E^{-1}.

For simplicity of the notation, the subscript t will be omitted.

Probability density function (pdf) of Ex is then

$$f(Ex|\tilde{x}, \rho) = \frac{1}{\tilde{J}}\, \chi\left(\tilde{x} - \rho \leq Ex \leq \tilde{x} + \rho\right), \tag{2}$$

where χ is a characteristic function (one for x fullfilling the condition, zero otherwise). $\tilde{J}$ is a normalizing integral. The pdf (2) is transformed from the variable $z = Ex$ to x, considering $\mathrm{d}z = |\det E|\, \mathrm{d}x$ and $J = \tilde{J}/|\det E|$,

$$f(x|\tilde{x}, \rho) = |\det E|\ f(Ex|\tilde{x}, \rho) = \frac{1}{J}\, \chi\left(\tilde{x} - \rho \leq Ex \leq \tilde{x} + \rho\right). \tag{3}$$

The task is to find

- the normalizing constant (integral) J,
- mean value $\mathcal{E}[x|\tilde{x}, \rho]$,
- covariance $\mathsf{cov}[x|\tilde{x}, \rho]$,
- a simple algorithm for inversion E^{-1} which occurs in the results.

2.2 Normalizing integral of $f(x|\tilde{x}, \rho)$

$$\begin{aligned} J &= \frac{\tilde{J}}{|\det E|} = \frac{1}{|\det E|} \int \chi\left(\tilde{x} - \rho \leq Ex \leq \tilde{x} + \rho\right) \mathrm{d}(Ex) = \\ &= \frac{1}{|\det E|} \int \chi\left(\tilde{x} - \rho \leq z \leq \tilde{x} + \rho\right) \mathrm{d}z = \frac{1}{|\det E|} \int\limits_{\tilde{x}-\rho}^{\tilde{x}+\rho} \mathrm{d}z = \frac{2^n}{|\det E|} \prod_{k=1}^{n} \rho_k. \end{aligned} \tag{4}$$

Note that, because of unit diagonal, $|\det E| = 1$.

2.3 Mean value

Using the substitution $z = Ex$ and a similar procedure as in (4),

$$\begin{aligned} \mathcal{E}[x|\tilde{x}, \rho] &= \frac{1}{J} \int x\chi\left(\tilde{x} - \rho \leq Ex \leq \tilde{x} + \rho\right) \mathrm{d}x = \left| \begin{array}{ll} z = Ex & x = E^{-1}z \\ \mathrm{d}z = |\det E|\, \mathrm{d}x & \mathrm{d}x = \frac{1}{|\det E|}\, \mathrm{d}z \end{array} \right| = \\ &= \frac{E^{-1}}{2^n \prod\limits_{k=1}^{n} \rho_k} \left[\begin{array}{c} \frac{(\tilde{x}_1+\rho_1)^2-(\tilde{x}_1-\rho_1)^2}{2} 2^{n-1} \prod\limits_{k=2}^{n} \rho_k \\ \vdots \\ \frac{(\tilde{x}_n+\rho_n)^2-(\tilde{x}_n-\rho_n)^2}{2} 2^{n-1} \prod\limits_{k=1}^{n-1} \rho_k \end{array} \right] = \frac{E^{-1}}{2^n \prod\limits_{k=1}^{n} \rho_k} 2^n \prod_{k=1}^{n} \rho_k \tilde{x} = \\ &= E^{-1}\tilde{x}. \end{aligned} \tag{5}$$

2.4 Covariance

Using the same substitution as in (5), we first calculate the second (non-central) moment (symbol $'$ means transposition).

$$\mathcal{E}[xx'|\tilde{x},\rho] \quad = \quad \frac{1}{J}\int xx'\chi\left(\tilde{x}-\rho \leq Ex \leq \tilde{x}+\rho\right)\mathrm{d}x = \frac{E^{-1}}{J|\det E|}\underbrace{\left(\int\limits_{\tilde{x}-\rho}^{\tilde{x}+\rho} zz'\,\mathrm{d}z\right)}_{\Delta}E^{-1\prime}. \tag{6}$$

Now let us focus on Δ.

$$\begin{aligned}\Delta \quad &= \int\limits_{\tilde{x}-\rho}^{\tilde{x}+\rho} zz'\,\mathrm{d}z = \int\limits_{\tilde{x}_n-\rho_n}^{\tilde{x}_n+\rho_n}\cdots\int\limits_{\tilde{x}_1-\rho_1}^{\tilde{x}_1+\rho_1}\begin{bmatrix} z_1z_1 & z_1z_2 & \dots & z_1z_n \\ z_2z_1 & z_2z_2 & \dots & z_2z_n \\ \vdots & \vdots & & \vdots \\ z_nz_1 & z_nz_2 & \dots & z_nz_n\end{bmatrix}\mathrm{d}z_1\dots\mathrm{d}z_n = \\ &= \int\limits_{\tilde{x}_n-\rho_n}^{\tilde{x}_n+\rho_n}\cdots\int\limits_{\tilde{x}_2-\rho_2}^{\tilde{x}_2+\rho_2}\begin{bmatrix} \frac{6\tilde{x}_1^2\rho_1+2\rho_1^3}{3} & 2\tilde{x}_1\rho_1z_2 & \dots & 2\tilde{x}_1\rho_1z_n \\ 2\tilde{x}_1\rho_1z_2 & 2\rho_1z_2^2 & \dots & 2\rho_1z_2z_n \\ \vdots & \vdots & & \vdots \\ 2\tilde{x}_1\rho_1z_n & 2\rho_1z_nz_2 & \dots & 2\rho_1z_n^2\end{bmatrix}\mathrm{d}z_2\dots\mathrm{d}z_n.\end{aligned} \tag{7}$$

Integration over z_k changes linear terms in k-th row and k-th column to $2\tilde{x}_k\rho_k$, the quadratic term in the Δ_{kk}-position to $\frac{6\tilde{x}_k^2\rho_k+2\rho_k^3}{3}$ and the others to $2\rho_k$.

After finishing the integration, as outlined above, we can express diagonal elements of Δ in this way.

$$\Delta_{ii} \quad = \quad \frac{6\tilde{x}_i^2\rho_i+2\rho_i^3}{3}\,\frac{2^{n-1}}{\rho_i}\prod_{k=1}^{n}\rho_k \quad = \quad \left(\tilde{x}_i^2+\frac{\rho_i^2}{3}\right)\,2^n\prod_{k=1}^{n}\rho_k. \tag{8}$$

The extradiagonal elements are

$$\Delta_{ij} \quad = \quad 2^n\prod_{k=1}^{n}\rho_k\tilde{x}_i\tilde{x}_j, \qquad i \neq j. \tag{9}$$

Generally, we can write

$$\Delta_{ij} \quad = \quad 2^n\prod_{k=1}^{n}\rho_k\,\left(\tilde{x}_i\tilde{x}_j+\delta_{ij}\frac{\rho_i^2}{3}\right), \tag{10}$$

where δ_{ij} is Kronecker symbol equal one if $i=j$, zero otherwise.

In the matrix form,

$$\Delta \quad = \quad 2^n\prod_{k=1}^{n}\rho_k\,\left(\tilde{x}\tilde{x}'+\frac{1}{3}GG'\right), \tag{11}$$

where G is a diagonal square matrix, $G_{ii}=\rho_i$ and $G_{ij}=0, i\neq j$.

To substitute (11) into (6),

$$\mathcal{E}[xx'|\tilde{x},\rho] = E^{-1}\left(\tilde{x}\tilde{x}'+\frac{1}{3}GG'\right)E^{-1\prime}. \tag{12}$$

Finally, using (5) and (12), covariance is by its definition obtained as

$$\mathsf{cov}[x|\tilde{x},\rho] = \mathcal{E}[xx'|\tilde{x},\rho]-\mathcal{E}[x|\tilde{x},\rho]\left(\mathcal{E}[x|\tilde{x},\rho]\right)' \;=\; \frac{1}{3}E^{-1}GG'E^{-1\prime}. \tag{13}$$

What remains is to express G using ρ and simple matrix operations.

2.5 Inversion of E

The matrix E is upper triangular with unit diagonal, therefore it is regular and its determinant equals 1. Let us denote $F = E^{-1}$. As $FE = EF = \mathbb{I}$, where $\mathbb{I}$ is a unit matrix, F is upper triangular with unit diagonal, too.

To construct the inversion, we use the definition $\sum_{k=1}^{n} F_{ik}E_{kj} = \delta_{ij}$, then $E_{ij} = F_{ij} = 0, i > j$, and start from the nn-element.

The result, for a general diagonal, is as follows.

Diagonal elements:

$$F_{jj} = \frac{1}{E_{jj}}. \tag{14}$$

As $E_{jj} = 1$, then also $F_{jj} = 1$.

Extradiagonal elements:

$$F_{n-j-l,n-j} = -\frac{1}{E_{n-j,n-j}} \sum_{k=n-j-l}^{n-j-1} F_{n-j-l,k}\, E_{k,n-j}. \tag{15}$$

The order of evaluation is given by the following procedure:

1) $l = 1,\ j = 0, 1, \ldots, n-2$,
2) $l = 2,\ j = 0, 1, \ldots, n-3$,
⋮
$n-1$) $l = n-1,\ j = 0$.

Generally:

$$F_{n-j-l,n-j} = \frac{1}{E_{n-j,n-j}} \left(\delta_{l0} - \sum_{k=n-j-l}^{n-j-1} F_{n-j-l,k}\, E_{k,n-j} \right), \tag{16}$$

The order of evaluation is given by the following procedure:

1) $l = 0$ (diagonal), $j = 0, 1, \ldots, n-1$ (the sum in (16) is zero, $\sum_{k=n}^{n-1} \ldots$),
2) $l = 1,\ j = 0, 1, \ldots, n-2$,
3) $l = 2,\ j = 0, 1, \ldots, n-3$,
⋮
n) $l = n-1,\ j = 0$.

Note that the sum must be initialized to zero. In the case of the diagonal element, the summing cycle will never execute (for $l = 0$) and only the first term of (16) remains.

This algorithm was verified. For matrices with unit diagonal, it performs comparably with the Matlab functions. However, numerical stability and precision with badly conditioned matrices has not been considered.

2.6 Alternative expression of the model

The matrix E in (1) can be expressed as

$$E = \mathbb{I} + \Lambda, \tag{17}$$

where $\mathbb{I}$ is a unit matrix and Λ is an upper triangular matrix with zero diagonal. The model (1) can be then written as

$$x_t = -\Lambda x_t + \tilde{x}_t + \nu_t. \tag{18}$$

The pdf (3) is then expressed (without time subscripts) as

$$f(x|\tilde{x},\rho) = \frac{1}{J}\,\chi\left(-\Lambda x + \tilde{x} - \rho \leq x \leq -\Lambda x + \tilde{x} + \rho\right). \tag{19}$$

This form is advantageous for other operations with the model and also that no inversion of E is needed because the matrix is decomposed additively. The disadvantage for the estimation may be that the random variable x is incorporated into integration bounds through Λ, which may complicate significantly evaluation of the moments.

Therefore, we will be using both (18) and (1).

3 Linear ARX model

We model a system with a scalar output y_t at a discrete time t. We define the linear ARX model with a uniform noise

$$y_t = \psi_t'\vartheta + \varepsilon_t, \tag{20}$$

$$f(\varepsilon_t|r) = \mathcal{U}_\varepsilon(0,r), \tag{21}$$

where

ψ_t is a finite-dimensional regression vector composed of past observed data and a current input in a known (recursively implementable) way,

ϑ is a vector of unknown regression coefficients,

ε_t is a uniformly distributed white noise at the time t (zero mean and uncorrelated with older observations),

$r > 0$ is an unknown positive scalar half-width of a noise range.

The pair of formulae (20) and (21) can be equivalently written

$$\begin{aligned} f(y_t|u_t,d(t-1),\vartheta,r) &\equiv f(y_t|\psi_t,\vartheta,r) = \mathcal{U}_{y_t}(\psi_t'\vartheta,r) \\ &= \frac{1}{2r}\,\chi_{y_t}(-r \leq y_t - \psi_t'\vartheta \leq r) \\ &= \frac{1}{2}\,\Theta_n\chi_{y_t}(-1 \leq \Psi_t'\Theta \leq 1) \end{aligned} \tag{22}$$

where

$$\Psi_t = \left[\psi_t', y_t\right]', \tag{23}$$

$$\Theta = \left[-\frac{\vartheta'}{r}, \frac{1}{r}\right]'. \tag{24}$$

The vectors Ψ_t and Θ are both n-dimensional.

The approximate posterior distribution of Θ is

$$f(\Theta|d(t)) \equiv f(\Theta|V_t,L_t,U_t,\nu_t) = \frac{\Theta_n^{\nu_t}\chi(L_t \leq V_t\Theta \leq U_t)}{\mathrm{J}(V_t,L_t,U_t,\nu_t)}, \tag{25}$$

where V_t is an $n \times n$-matrix, L_t and U_t are vectors of size n, ν_t is a positive integer scalar and $\mathrm{J}(V_t,L_t,U_t,\nu_t)$ is a normalizing factor (integral). For more details concerning motivation and theory, see e.g. [1].

The aim is to compute first and second central moments of Θ, ϑ and r, see (24).

3.1 Normalizing integral $\mathrm{J}(V_t, L_t, U_t, \nu_t)$

Let us denote $J_\nu = \mathrm{J}(V_t, L_t, U_t, \nu_t)$. Then

$$\begin{aligned} J_\nu &= \int \Theta_n^\nu \, \chi \left(L \le V\Theta \le U \right) \, \mathrm{d}\Theta = \left| \begin{array}{ll} x = V\Theta, & \mathrm{d}x = |\det V| \, \mathrm{d}\Theta \\ \Theta = V^{-1}x, & \mathrm{d}\Theta = \frac{1}{|\det V|} \, \mathrm{d}x \\ x_n = V_{nn}\Theta_n & \end{array} \right| = \frac{1}{|\det V| \, V_{nn}^\nu} \int\limits_L^U x_n^\nu \, \mathrm{d}x = \\ &= \frac{1}{|\det V| \, V_{nn}^\nu} \, \frac{U_n^{\nu+1} - L_n^{\nu+1}}{\nu+1} \, \prod_{i=1}^{n-1} (U_i - L_i). \end{aligned} \tag{26}$$

The product of the bounds L_i and U_i for $i = 1, \dots, n-1$ is given by implicit units in the corresponding items of the vector x in the integrand. Technical details are explained in the next sections.

If we introduce $\gamma = \frac{L_n}{U_n}$, then the alternative expression is

$$J_\nu = \frac{U_n^{\nu+1}}{|\det V| \, V_{nn}^\nu} \, \frac{1 - \gamma^{\nu+1}}{\nu+1} \, \prod_{i=1}^{n-1} (U_i - L_i). \tag{27}$$

3.2 Mean value of Θ

Let us introduce the symbol $\mathbb{I}_{n-1,n}$, which is an $(n-1)\times(n-1)$ identity matrix with added trailing zero column.

$$\begin{aligned}
\mathcal{E}[\Theta] &= \frac{1}{J_\nu}\int \Theta\Theta_n^\nu\,\chi\,(L\le V\Theta\le U)\;\mathrm{d}\Theta = \left|\begin{array}{ll} x=V\Theta, & \mathrm{d}x=|\det V|\,\mathrm{d}\Theta \\ \Theta=V^{-1}x, & \mathrm{d}\Theta=\frac{1}{|\det V|}\,\mathrm{d}x \\ x_n=V_{nn}\Theta_n & \end{array}\right| = \\
&= \frac{V^{-1}}{J_\nu\,|\det V|\,V_{nn}^\nu}\int_L^U x\,x_n^\nu\,\mathrm{d}x = \\
&= \frac{V^{-1}}{J_\nu\,|\det V|\,V_{nn}^\nu}\int_{L_n}^{U_n} x_n^\nu \int_{L_{n-1}}^{U_{n-1}} \dots \int_{L_1}^{U_1} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \mathrm{d}x_1\dots\mathrm{d}x_{n-1}\,\mathrm{d}x_n = \\
&= \frac{V^{-1}}{J_\nu\,|\det V|\,V_{nn}^\nu}\int_{L_n}^{U_n} x_n^\nu \int_{L_{n-1}}^{U_{n-1}} \dots \int_{L_2}^{U_2} \begin{bmatrix} \frac{U_1^2-L_1^2}{2} \\ \vdots \\ x_{n-1}(U_1-L_1) \\ x_n(U_1-L_1) \end{bmatrix} \mathrm{d}x_2\dots\mathrm{d}x_{n-1}\,\mathrm{d}x_n = \\
&= \frac{V^{-1}}{J_\nu\,|\det V|\,V_{nn}^\nu}\int_{L_n}^{U_n} x_n^\nu \int_{L_{n-1}}^{U_{n-1}} \dots \int_{L_2}^{U_2} \begin{bmatrix} \frac{U_1+L_1}{2}(U_1-L_1) \\ \vdots \\ x_{n-1}(U_1-L_1) \\ x_n(U_1-L_1) \end{bmatrix} \mathrm{d}x_2\dots\mathrm{d}x_{n-1}\,\mathrm{d}x_n = \\
&= \frac{V^{-1}}{J_\nu\,|\det V|\,V_{nn}^\nu}\int_{L_n}^{U_n} x_n^\nu \begin{bmatrix} \frac{U_1+L_1}{2}\prod_{i=1}^{n-1}(U_i-L_i) \\ \vdots \\ \frac{U_{n-1}+L_{n-1}}{2}\prod_{i=1}^{n-1}(U_i-L_i) \\ x_n\prod_{i=1}^{n-1}(U_i-L_i) \end{bmatrix} \mathrm{d}x_n = \\
&= \frac{V^{-1}\prod_{i=1}^{n-1}(U_i-L_i)}{J_\nu\,|\det V|\,V_{nn}^\nu}\int_{L_n}^{U_n} x_n^\nu \begin{bmatrix} \frac{U_1+L_1}{2} \\ \vdots \\ \frac{U_{n-1}+L_{n-1}}{2} \\ x_n \end{bmatrix} \mathrm{d}x_n = \frac{V^{-1}\prod_{i=1}^{n-1}(U_i-L_i)}{J_\nu\,|\det V|\,V_{nn}^\nu} \begin{bmatrix} \frac{U_n^{\nu+1}-L_n^{\nu+1}}{\nu+1}\mathbb{I}_{n-1,n}\frac{U+L}{2} \\ \\ \frac{U_n^{\nu+2}-L_n^{\nu+2}}{\nu+2} \end{bmatrix} = \\
&= V^{-1}\underbrace{\frac{\frac{U_n^{\nu+1}-L_n^{\nu+1}}{\nu+1}\prod_{i=1}^{n-1}(U_i-L_i)}{J_\nu\,|\det V|\,V_{nn}^\nu}}_{\text{reduces to 1}} \begin{bmatrix} \mathbb{I}_{n-1,n}\,\frac{U+L}{2} \\ \\ \frac{\nu+1}{U_n^{\nu+1}-L_n^{\nu+1}}\,\frac{U_n^{\nu+2}-L_n^{\nu+2}}{\nu+2} \end{bmatrix} = V^{-1} \begin{bmatrix} \mathbb{I}_{n-1,n}\,\frac{U+L}{2} \\ \\ U_n\,\frac{1-\gamma^{\nu+2}}{1-\gamma^{\nu+1}}\,\frac{\nu+1}{\nu+2} \end{bmatrix}.
\end{aligned}$$

3.3 Covariance of Θ

First, we compute the second noncentral moment

$$\begin{aligned}
\mathcal{E}[\Theta\Theta'] &= \frac{1}{J_\nu}\int \Theta\Theta'\Theta_n^\nu \chi\left(L \le V\Theta \le U\right) \mathrm{d}\Theta = \left| \begin{array}{ll} x = V\Theta, & \mathrm{d}x = |\det V|\,\mathrm{d}\Theta \\ \Theta = V^{-1}x, & \mathrm{d}\Theta = \frac{1}{|\det V|}\,\mathrm{d}x \\ x_n = V_{nn}\Theta_n & \end{array} \right| = \\
&= \frac{1}{J_\nu\,|\det V|\,V_{nn}^\nu}\int V^{-1}xx'V^{-1\prime}x_n^\nu\ \chi\left(L \le x \le U\right)\,\mathrm{d}x = \\
&= \frac{V^{-1}}{J_\nu\,|\det V|\,V_{nn}^\nu}\underbrace{\left(\int_L^U xx'x_n^\nu\,\mathrm{d}x\right)}_{A}\ V^{-1\prime}. \qquad (28)
\end{aligned}$$

The integral A equals

$$\begin{aligned}
A &= \int_L^U xx'x_n^\nu\,\mathrm{d}x = \int_{L_n}^{U_n} x_n^\nu \dots \int_{L_1}^{U_1} \begin{bmatrix} x_1x_1 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & x_2x_2 & \dots & x_2x_n \\ \vdots & \vdots & & \vdots \\ x_nx_1 & x_nx_2 & \dots & x_nx_n \end{bmatrix} \mathrm{d}x_1\dots\mathrm{d}x_n = \\
&= \int_{L_n}^{U_n} x_n^\nu \dots \int_{L_2}^{U_2} \begin{bmatrix} \frac{U_1^3-L_1^3}{3} & \frac{U_1^2-L_1^2}{2}x_2 & \dots & \frac{U_1^2-L_1^2}{2}x_n \\ \frac{U_1^2-L_1^2}{2}x_2 & (U_1-L_1)x_2^2 & \dots & (U_1-L_1)x_2x_n \\ \vdots & \vdots & & \vdots \\ \frac{U_1^2-L_1^2}{2}x_n & (U_1-L_1)x_2x_n & \dots & (U_1-L_1)x_n^2 \end{bmatrix} \mathrm{d}x_2\dots\mathrm{d}x_n = \\
&= \prod_{k=1}^{n-1}(U_k-L_k)\int_{L_n}^{U_n} x_n^\nu \begin{bmatrix} \frac{U_1^2+U_1L_1+L_1^2}{3} & \frac{U_1+L_1}{2}\frac{U_2+L_2}{2} & \dots & \frac{U_1+L_1}{2}x_n \\ \frac{U_1+L_1}{2}\frac{U_2+L_2}{2} & \frac{U_2^2+U_2L_2+L_2^2}{3} & \dots & \frac{U_2+L_2}{2}x_n \\ \vdots & \vdots & & \vdots \\ \frac{U_1+L_1}{2}x_n & \frac{U_2+L_2}{2}x_n & \dots & x_n^2 \end{bmatrix} \mathrm{d}x_n. \qquad (29)
\end{aligned}$$

The diagonal elements can be expressed as

$$\frac{U_i^2+U_iL_i+L_i^2}{3} = \left(\frac{U_i+L_i}{2}\right)^2 + \frac{1}{3}\left(\frac{U_i-L_i}{2}\right)^2, \qquad i = 1,\dots,n-1. \qquad (30)$$

Then,

$$\begin{aligned}
A &= \prod_{k=1}^{n-1}(U_k-L_k)\int_{L_n}^{U_n} x_n^\nu \begin{bmatrix} \left(\frac{U_1+L_1}{2}\right)^2 + \frac{1}{3}\left(\frac{U_1-L_1}{2}\right)^2 & \frac{U_1+L_1}{2}\frac{U_2+L_2}{2} & \dots & \frac{U_1+L_1}{2}x_n \\ \frac{U_1+L_1}{2}\frac{U_2+L_2}{2} & \left(\frac{U_2+L_2}{2}\right)^2 + \frac{1}{3}\left(\frac{U_2-L_2}{2}\right)^2 & \dots & \frac{U_2+L_2}{2}x_n \\ \vdots & \vdots & & \vdots \\ \frac{U_1+L_1}{2}x_n & \frac{U_2+L_2}{2}x_n & \dots & x_n^2 \end{bmatrix} \mathrm{d}x_n = \\
&= \prod_{k=1}^{n-1}(U_k-L_k)\int_{L_n}^{U_n} x_n^\nu \left(\begin{bmatrix} \frac{U_1+L_1}{2} \\ \frac{U_2+L_2}{2} \\ \vdots \\ x_n \end{bmatrix} \left[\frac{U_1+L_1}{2}, \frac{U_2+L_2}{2}, \dots, x_n\right] + \frac{1}{3}D_0D_0' \right) \mathrm{d}x_n, \qquad (31)
\end{aligned}$$

where

$$D_0 = \begin{bmatrix} \frac{U_1-L_1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{U_2-L_2}{2} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \frac{U_{n-1}-L_{n-1}}{2} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{32}$$

To continue,

$$\begin{aligned} A &= \prod_{k=1}^{n-1}(U_k-L_k)\int_{L_n}^{U_n} x_n^\nu\left(\begin{bmatrix} \mathbb{I}_{n-1,n}\frac{U+L}{2} \\ x_n \end{bmatrix}\left[\frac{U'+L'}{2}\mathbb{I}_{n,n-1}, x_n\right]+\frac{1}{3}D_0D_0'\right)\mathrm{d}x_n = \\ &= \prod_{k=1}^{n-1}(U_k-L_k)\int_{L_n}^{U_n} x_n^\nu\left(\begin{bmatrix} \mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & \mathbb{I}_{n-1,n}\frac{U+L}{2}x_n \\ x_n\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & x_n^2 \end{bmatrix}+\frac{1}{3}D_0D_0'\right)\mathrm{d}x_n = \\ &= \prod_{k=1}^{n-1}(U_k-L_k)\left(\begin{bmatrix} \mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1}\frac{U_n^{\nu+1}-L_n^{\nu+1}}{\nu+1} & \mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{U_n^{\nu+2}-L_n^{\nu+2}}{\nu+2} \\ \frac{U_n^{\nu+2}-L_n^{\nu+2}}{\nu+2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & \frac{U_n^{\nu+3}-L_n^{\nu+3}}{\nu+3} \end{bmatrix}+\right. \\ &\quad\left.+\frac{1}{3}D_0D_0'\frac{U_n^{\nu+1}-L_n^{\nu+1}}{\nu+1}\right) = \\ &= \frac{U_n^{\nu+1}-L_n^{\nu+1}}{\nu+1}\prod_{k=1}^{n-1}(U_k-L_k)\left(\begin{bmatrix} \mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & \mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{U_n^{\nu+2}-L_n^{\nu+2}}{U_n^{\nu+1}-L_n^{\nu+1}}\frac{\nu+1}{\nu+2} \\ \frac{U_n^{\nu+2}-L_n^{\nu+2}}{U_n^{\nu+1}-L_n^{\nu+1}}\frac{\nu+1}{\nu+2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & \frac{U_n^{\nu+3}-L_n^{\nu+3}}{U_n^{\nu+1}-L_n^{\nu+1}}\frac{\nu+1}{\nu+3} \end{bmatrix}+\frac{1}{3}D_0D_0'\right) = \\ &= U_n^{\nu+1}\frac{1-\gamma^{\nu+1}}{\nu+1}\prod_{k=1}^{n-1}(U_k-L_k)\left(\underbrace{\begin{bmatrix} \mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & U_n\mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{1-\gamma^{\nu+2}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu+2} \\ U_n\frac{1-\gamma^{\nu+2}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu+2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & U_n^2\frac{1-\gamma^{\nu+3}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu+3} \end{bmatrix}}_{\tilde{A}}+\frac{1}{3}D_0D_0'\right) \end{aligned} \tag{33}$$

Let us attempt to express the term $\tilde{A}_{nn}$ so that the matrix $\tilde{A}$ in (33) is an outer product, similarly as in (31), i.e.

$$\tilde{A}_{nn} = U_n^2\,\frac{1-\gamma^{\nu+3}}{1-\gamma^{\nu+1}}\,\frac{\nu+1}{\nu+3} = \left(U_n\,\frac{1-\gamma^{\nu+2}}{1-\gamma^{\nu+1}}\,\frac{\nu+1}{\nu+2}\right)^2+\frac{1}{3}D_{nn}^2. \tag{34}$$

Hence,

$$\begin{aligned} D_{nn} &= U_n\sqrt{3}\sqrt{\frac{1-\gamma^{\nu+3}}{1-\gamma^{\nu+1}}\,\frac{\nu+1}{\nu+3}-\left(\frac{1-\gamma^{\nu+2}}{1-\gamma^{\nu+1}}\,\frac{\nu+1}{\nu+2}\right)^2} = \\ &= U_n\sqrt{3}\,\frac{\nu+1}{1-\gamma^{\nu+1}}\sqrt{\frac{1-\gamma^{\nu+3}}{\nu+3}\,\frac{1-\gamma^{\nu+1}}{\nu+1}-\left(\frac{1-\gamma^{\nu+2}}{\nu+2}\right)^2}, \end{aligned} \tag{35}$$

which is not a very pretty formula, but its arrangement, ideally removing the square root, is left for someone else. Some expressions: $\frac{1-\gamma^\nu}{1-\gamma} = \sum_{k=0}^{\nu-1}\gamma^k$, where ν is integer. $1-\gamma^{\nu+3} = \gamma^2(1-\gamma^{\nu+1})+1-\gamma^2$. Another: $1-\gamma^{\nu+2} = \gamma(1-\gamma^{\nu+1})+1-\gamma$. May be helpful.

We define a diagonal matrix D, where $D_{ii} = D_{0,ii}$, for $i = 1, \dots, n-1$, and D_{nn} is defined in (35), i.e.

$$D = \begin{bmatrix} \frac{U_1-L_1}{2} & 0 & \dots & 0 & 0 \\ 0 & \frac{U_2-L_2}{2} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \frac{U_{n-1}-L_{n-1}}{2} & 0 \\ 0 & 0 & \dots & 0 & D_{nn} \end{bmatrix}. \tag{36}$$

Then, if we substitute (35) and (33) into (28), we get

$$\begin{aligned} \mathcal{E}[\Theta\Theta'] &= V^{-1}\left(\begin{bmatrix} \mathbb{I}_{n-1,n}\frac{U+L}{2} \\ U_n\frac{1-\gamma^{\nu+2}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu+2} \end{bmatrix}\left[\frac{U'+L'}{2}\mathbb{I}_{n,n-1},\ U_n\frac{1-\gamma^{\nu+2}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu+2}\right] + \frac{1}{3}DD'\right)V^{-1'} = \\ &= \mathcal{E}[\Theta](\mathcal{E}[\Theta])' + \frac{1}{3}V^{-1}DD'V^{-1'}. \end{aligned} \tag{37}$$

Covariance, by its definition, is then

$$\operatorname{cov}\Theta = \mathcal{E}[\Theta\Theta'] - \mathcal{E}[\Theta](\mathcal{E}[\Theta])' = \frac{1}{3}V^{-1}DD'V^{-1'}. \tag{38}$$

3.4 Mean value of r

According to the definition, the mean value of r is

$$\hat{r} = \frac{1}{J_\nu}\int r\,\Theta_n^\nu \chi(L_t \le V_t\Theta \le U_t)\,\mathrm{d}\Theta = \left|\ r = \frac{1}{\Theta_n}\ \right| = \frac{1}{J_\nu}\int \Theta_n^{\nu-1}\chi(L_t \le V_t\Theta \le U_t)\,\mathrm{d}\Theta = \frac{J_{\nu-1}}{J_\nu}. \tag{39}$$

Using (26) or (27),

$$\hat{r} = \mathcal{E}[r] = V_{nn}\frac{\nu+1}{\nu}\frac{U_n^\nu - L_n^\nu}{U_n^{\nu+1} - L_n^{\nu+1}} = \frac{V_{nn}}{U_n}\frac{\nu+1}{\nu}\frac{1-\gamma^\nu}{1-\gamma^{\nu+1}}, \tag{40}$$

where $\gamma = \frac{L_n}{U_n}$.

3.5 Mean value of ϑ

The minus sign in (41) coming from (24) remains in front of the integral. It works in the code. But in two papers, including [1], it was omitted. The paper is still under revision (at this moment), so this mistake will be corrected.

$$\begin{aligned} \hat{\vartheta} = \mathcal{E}[\vartheta] &= \frac{1}{J_\nu}\int \vartheta\,\Theta_n^\nu \chi\left(L \le V\Theta \le U\right)\,\mathrm{d}\Theta = \left|\begin{array}{l} \Theta = \left[-\frac{\vartheta'}{r}, \frac{1}{r}\right]' \\ \vartheta = -\frac{1}{\Theta_n}\mathbb{I}_{n-1,n}\Theta \end{array}\right| = \\ &= -\frac{1}{J_\nu}\int \mathbb{I}_{n-1,n}\Theta\Theta_n^{\nu-1}\chi\left(L \le V\Theta \le U\right)\,\mathrm{d}\Theta = \left|\begin{array}{ll} x = V\Theta, & \mathrm{d}x = |\det V|\,\mathrm{d}\Theta \\ \Theta = V^{-1}x, & \mathrm{d}\Theta = \frac{1}{|\det V|}\,\mathrm{d}x \\ x_n = V_{nn}\Theta_n & \end{array}\right| = \\ &= -\frac{1}{J_\nu\,|\det V|\,V_{nn}^{\nu-1}}\int \mathbb{I}_{n-1,n}V^{-1}x\,x_n^{\nu-1}\ \chi\left(L \le x \le U\right)\,\mathrm{d}x. \end{aligned} \tag{41}$$

For a matrix $V = \begin{bmatrix} V_\psi & V_y \\ 0 & V_{nn} \end{bmatrix}$, where $V_{nn} > 0$ and, generally, $V_{nn} \neq 1$, the inversion is

$$V^{-1} = \begin{bmatrix} V_\psi^{-1} & -\frac{V_\psi^{-1}V_y}{V_{nn}} \\ 0 & \frac{1}{V_{nn}} \end{bmatrix}. \tag{42}$$

Using the block form (42),

$$
\begin{aligned}
\hat{\vartheta} &= -\frac{1}{J_\nu\,|\det V|\,V_{nn}^{\nu-1}}\int\limits_L^U \left(V_\psi^{-1}\mathbb{I}_{n-1,n}x - \frac{V_\psi^{-1}V_y}{V_{nn}}x_n\right)x_n^{\nu-1}\,\mathrm{d}x = \\
&= -\frac{1}{J_\nu\,|\det V|\,V_{nn}^{\nu-1}}\left(V_\psi^{-1}\int\limits_{L_n}^{U_n}x_n^{\nu-1}\int\limits_{L_{n-1}}^{U_{n-1}}\dots\int\limits_{L_1}^{U_1}\begin{bmatrix}x_1\\ \vdots\\ x_{n-1}\end{bmatrix}\mathrm{d}x_1\dots\mathrm{d}x_{n-1}\,\mathrm{d}x_n - \right. \\
&\quad \left. -\frac{V_\psi^{-1}V_y}{V_{nn}}\int\limits_{L_n}^{U_n}x_n^{\nu}\int\limits_{L_{n-1}}^{U_{n-1}}\dots\int\limits_{L_1}^{U_1}\mathrm{d}x_1\,\mathrm{d}x_{n-1}\dots\mathrm{d}x_n\right) = \\
&= -\frac{1}{J_\nu\,|\det V|\,V_{nn}^{\nu-1}}\left(V_\psi^{-1}\frac{U_n^\nu - L_n^\nu}{\nu}\begin{bmatrix}\frac{U_1^2-L_1^2}{2}\prod_{i=2}^{n-1}(U_i-L_i)\\ \vdots\\ \frac{U_{n-1}^2-L_{n-1}^2}{2}\prod_{i=1}^{n-2}(U_i-L_i)\end{bmatrix} - \frac{V_\psi^{-1}V_y}{V_{nn}}\frac{U_n^{\nu+1}-L_n^{\nu+1}}{\nu+1}\prod_{i=1}^{n-1}(U_i-L_i)\right) = \\
&= -\frac{1}{J_\nu\,|\det V|\,V_{nn}^{\nu-1}}\prod_{i=1}^{n-1}(U_i-L_i)\left(V_\psi^{-1}\frac{U_n^\nu-L_n^\nu}{\nu}\begin{bmatrix}\frac{U_1+L_1}{2}\\ \vdots\\ \frac{U_{n-1}+L_{n-1}}{2}\end{bmatrix} - \frac{V_\psi^{-1}V_y}{V_{nn}}\frac{U_n^{\nu+1}-L_n^{\nu+1}}{\nu+1}\right) \\
&= -\frac{1}{J_\nu}\underbrace{\frac{1}{|\det V|\,V_{nn}^{\nu}}\frac{U_n^{\nu+1}-L_n^{\nu+1}}{\nu+1}\prod_{i=1}^{n-1}(U_i-L_i)}_{J_\nu}\left(\underbrace{V_{nn}\frac{\nu+1}{\nu}\frac{U_n^\nu-L_n^\nu}{U_n^{\nu+1}-L_n^{\nu+1}}}_{\hat{r}}V_\psi^{-1}\mathbb{I}_{n-1,n}\frac{U+L}{2} - V_\psi^{-1}V_y\right) = \\
&= V_\psi^{-1}V_y - \hat{r}V_\psi^{-1}\mathbb{I}_{n-1,n}\frac{U+L}{2}.
\end{aligned} \tag{43}
$$

The formula for $\hat{\vartheta}$ is explicitly independent of V_{nn}, which is hidden in $\hat{r}$. The symbol $\mathbb{I}_{n-1,n}$ is $(n-1)\times(n-1)$ identity matrix with added trailing zero column.

3.6 Variance of r

Similarly as in (39),

$$
\begin{aligned}
\mathcal{E}[r^2] &= \frac{J_{\nu-2}}{J_\nu} = \frac{V_{nn}^2}{U_n^2}\frac{\nu+1}{\nu-1}\frac{1-\gamma^{\nu-1}}{1-\gamma^{\nu+1}}, \\
\operatorname{var} r &= \mathcal{E}[r^2] - (\mathcal{E}[r])^2 = \frac{V_{nn}^2}{U_n^2}\frac{\nu+1}{1-\gamma^{\nu+1}}\left(\frac{1-\gamma^{\nu-1}}{\nu-1} - \frac{\nu+1}{\nu^2}\frac{(1-\gamma^\nu)^2}{1-\gamma^{\nu+1}}\right),
\end{aligned}
$$

where $\gamma = \frac{L_n}{U_n}$.

3.7 Covariance of ϑ

As in Section 3.6, covariance leads to a formula that is not simple. In Section 3.3, where we deal with Θ, even if the final formula (38) seems simple, the last diagonal element of D, see (35), has a complicated form because of non-symmetry of the pdf in Θ_n. In the case of ϑ, the situation is even worse because $\vartheta = -\mathbb{I}_{n-1,n}\Theta/\Theta_n$ and, therefore, the k-th non-central moment will contain the term $\Theta_n^{\nu-k}$ in the integral. Each moment will have a different power of Θ_n which will lead to a different result in powers of γ etc. As a consequence, the trick with the subtraction, as in (38), cannot be used effectively and the result will not be so simple.

To demonstrate this effect, we will express (43) without (42)

$$\mathcal{E}[\vartheta] = \underbrace{\frac{V_{nn}}{U_n}\frac{1-\gamma^{\nu}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu}}_{\mathcal{E}[r]}\mathbb{I}_{n-1,n}V^{-1}\begin{bmatrix}\mathbb{I}_{n-1,n}\frac{U+L}{2}\\ U_n\frac{1-\gamma^{\nu+1}}{1-\gamma^{\nu}}\frac{\nu}{\nu+1}\end{bmatrix}, \tag{44}$$

which will be used later.

The second non-central moment of ϑ is

$$\begin{aligned}
\mathcal{E}[\vartheta\vartheta'] &= \frac{1}{J_\nu}\int \vartheta\vartheta'\Theta_n^{\nu}\chi\left(L \le V\Theta \le U\right)\,\mathrm{d}\Theta = \left|\begin{array}{l}\Theta = \left[-\frac{\vartheta'}{r}, \frac{1}{r}\right]' \\ \vartheta = -\frac{1}{\Theta_n}\mathbb{I}_{n-1,n}\Theta\end{array}\right| = \\
&= \frac{1}{J_\nu}\int \mathbb{I}_{n-1,n}\Theta\Theta'\mathbb{I}_{n,n-1}\Theta_n^{\nu-2}\chi\left(L \le V\Theta \le U\right)\,\mathrm{d}\Theta = \left|\begin{array}{ll}x = V\Theta, & \mathrm{d}x = |\det V|\,\mathrm{d}\Theta \\ \Theta = V^{-1}x, & \mathrm{d}\Theta = \frac{1}{|\det V|}\,\mathrm{d}x \\ x_n = V_{nn}\Theta_n & \end{array}\right| = \\
&= \frac{1}{J_\nu\,|\det V|\,V_{nn}^{\nu-2}}\int \mathbb{I}_{n-1,n}V^{-1}xx'V^{-1\prime}\mathbb{I}_{n,n-1}x_n^{\nu-2}\;\chi\left(L \le x \le U\right)\,\mathrm{d}x = \\
&= \frac{\mathbb{I}_{n-1,n}V^{-1}}{J_\nu\,|\det V|\,V_{nn}^{\nu-2}}\underbrace{\left(\int_L^U xx'x_n^{\nu-2}\,\mathrm{d}x\right)}_{B}V^{-1\prime}\mathbb{I}_{n,n-1}.
\end{aligned} \tag{45}$$

The matrix-integral B in (45) is equal to A in (28) except of $\nu-2$ instead of ν. Therefore, we can use the ready solution (33) with the adaptation $\nu \longrightarrow (\nu-2)$. Hence,

$$B = U_n^{\nu-1}\frac{1-\gamma^{\nu-1}}{\nu-1}\prod_{k=1}^{n-1}(U_k-L_k)\left(\begin{bmatrix}\mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & U_n\mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{1-\gamma^{\nu}}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu}\\ U_n\frac{1-\gamma^{\nu}}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & U_n^2\frac{1-\gamma^{\nu+1}}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu+1}\end{bmatrix}+\frac{1}{3}D_0D_0'\right), \tag{46}$$

where D_0 is defined in (32).

If we consider (44), it is obvious that the matrix in (46) can be hardly expressed by outer product of $\mathcal{E}[\vartheta]$ with a help of D, as in (37). But, maybe, someone will make it.

To finish the second moment, we will substitute (46) and (26) to (45).

$$\mathcal{E}[\vartheta\vartheta'] = \left(\frac{V_{nn}}{U_n}\right)^2\frac{1-\gamma^{\nu-1}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu-1}\mathbb{I}_{n-1,n}V^{-1}\;C\;V^{-1\prime}\mathbb{I}_{n,n-1}, \tag{47}$$

$$C = \begin{bmatrix}\mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & U_n\mathbb{I}_{n-1,n}\frac{U+L}{2}\frac{1-\gamma^{\nu}}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu}\\ U_n\frac{1-\gamma^{\nu}}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu}\frac{U'+L'}{2}\mathbb{I}_{n,n-1} & U_n^2\frac{1-\gamma^{\nu+1}}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu+1}\end{bmatrix}+\frac{1}{3}D_0D_0'. \tag{48}$$

The formula (47) can be adapted using the block form (42) but it will be even more complicated. The adaptation is suitable for computation purposes (instead of $n\times n$, matrix $(n-1)\times(n-1)$ is to be inverted). Similarly as in Section 3.3, we can write

$$\begin{aligned}
C &= \begin{bmatrix}\mathbb{I}_{n-1,n}\frac{U+L}{2}\\ U_n\frac{1-\gamma^{\nu}}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu}\end{bmatrix}\underbrace{\left[\frac{U'+L'}{2}\mathbb{I}_{n,n-1},\;U_n\frac{1-\gamma^{\nu}}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu}\right]}_{F'}+\frac{1}{3}\tilde{D}\tilde{D}' = \\
&= FF'+\frac{1}{3}\tilde{D}\tilde{D}',
\end{aligned} \tag{49}$$

where

$$\tilde{D}_{nn} = U_n\sqrt{3}\;\frac{\nu-1}{1-\gamma^{\nu-1}}\sqrt{\frac{1-\gamma^{\nu+1}}{\nu+1}\;\frac{1-\gamma^{\nu-1}}{\nu-1}-\left(\frac{1-\gamma^{\nu}}{\nu}\right)^2} \tag{50}$$

and $\tilde{D}$ is defined similarly like in (36).

To get the block form, we compute several auxiliary formulae:

$$\begin{aligned}\mathbb{I}_{n-1,n}V^{-1}F &= V_\psi^{-1}\mathbb{I}_{n-1,n}\frac{U+L}{2} - \frac{V_\psi^{-1}V_y}{V_{nn}}U_n\frac{1-\gamma_\nu}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu} = \\ &= R_1 - R_2\frac{U_n}{V_{nn}}\frac{1-\gamma_\nu}{1-\gamma^{\nu-1}}\frac{\nu-1}{\nu},\end{aligned} \tag{51}$$

$$\mathbb{I}_{n-1,n}V^{-1}\tilde{D} = V_\psi^{-1}\mathbb{I}_{n-1,n}\frac{U-L}{2} - \frac{V_\psi^{-1}V_y}{V_{nn}}\tilde{D}_{nn} = R_3 - R_2\frac{\tilde{D}_{nn}}{V_{nn}} \tag{52}$$

with the vectors R_1, R_2 and R_3 defined as

$$\begin{aligned}R_1 &= V_\psi^{-1}\mathbb{I}_{n-1,n}\frac{U+L}{2}, \\ R_2 &= V_\psi^{-1}V_y, \\ R_3 &= V_\psi^{-1}\mathbb{I}_{n-1,n}\frac{U-L}{2}.\end{aligned} \tag{53}$$

Substituting (52), (51), (50) and (49) into (47), we get

$$\begin{aligned}\mathcal{E}[\vartheta\vartheta'] &= \left(\frac{V_{nn}}{U_n}\right)^2\frac{1-\gamma^{\nu-1}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu-1}\left(R_1R_1' + \frac{1}{3}R_3R_3'\right) - \\ &\quad - \hat{r}\left[R_1R_2' + R_2R_1' + (R_3R_2' + R_2R_3')\frac{\sqrt{3}}{3}\sqrt{\frac{(1-\gamma^{\nu-1})(1-\gamma^{\nu+1})}{(\nu-1)(\nu+1)}\left(\frac{\nu}{1-\gamma^\nu}\right)^2 - 1}\right] + \\ &\quad + R_2R_2'.\end{aligned} \tag{54}$$

If we express the mean value (43) using R_1 and R_2

$$\begin{aligned}\mathcal{E}[\vartheta] &= R_2 - \hat{r}R_1, \\ \mathcal{E}[\vartheta](\mathcal{E}[\vartheta])' &= \hat{r}^2R_1R_1' - \hat{r}\left(R_1R_2' + R_2R_1'\right) + R_2R_2'.\end{aligned} \tag{55}$$

and if we denote

$$\xi = \frac{(1-\gamma^{\nu-1})(1-\gamma^{\nu+1})}{(\nu-1)(\nu+1)}\left(\frac{\nu}{1-\gamma^\nu}\right)^2, \tag{56}$$

then $\left(\frac{V_{nn}}{U_n}\right)^2\frac{1-\gamma^{\nu-1}}{1-\gamma^{\nu+1}}\frac{\nu+1}{\nu-1} = \hat{r}^2\xi$ and the formula for covariance is

$$\begin{aligned}\mathrm{cov}\,\vartheta &= \mathcal{E}[\vartheta\vartheta'] - \mathcal{E}[\vartheta](\mathcal{E}[\vartheta])' = \\ &= \hat{r}^2\left(R_1R_1'(\xi-1) + \frac{1}{3}R_3R_3'\xi\right) - \hat{r}(R_3R_2' + R_2R_3')\frac{\sqrt{3}}{3}\sqrt{\xi-1}.\end{aligned} \tag{57}$$

The condition $\xi \geq 1$ should hold. I hope this is all. The formulae have not been verified but they seem reasonable. Note: ξ is, in some sense, an indicator of nonsymmetry and "troubles". If $\xi = 1$ (which would be valid approximately for $\nu \gg 1$), the formula is simple as mentioned above.

4 Conclusion

The procedures to derive the requested formulae have been presented in detail. The report may serve, except its primary purpose, as a material for inspiration, education etc.

5 Acknowledgement

I would like to thank Dr. Lenka Pavelková for initial impulse, encouragement and valuable remarks till the end.

References

[1] L. Pavelková and L. Jirsa. Recursive Bayesian estimation of autoregressive model with uniform noise using approximation by parallelotopes. *ACASP*, 2016. Submitted.