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On the Lipschitz behavior of solution maps of a class of differential inclusions*

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Abstract We consider a general differential inclusion which is parameterized by a parameter. We perform time discretization and present conditions under which the discretized solution map is locally Lipschitz. Further, if the Lipschitzian modulus is bounded in some sense, we show that it is possible to obtain the local Lipschitzian property even for the original (not discretized) solution map. We conclude the paper with an example concerning stability analysis of nonregular electrical circuits with ideal diodes.

Keywords Differential inclusions · Lipschitzian continuity · Stability · Variational analysis · Electrical circuits

Mathematics Subject Classification (2000) 34A60 · 34H05 · 49K21 · 34A36

1 Introduction

We consider a general differential inclusion with known initial value

$$\begin{aligned} f(t,u,x(t),\dot{x}(t)) &\in \Omega(t,u,x(t),\dot{x}(t)), \quad t\in[0,T] \text{ a.e.} \\ x(0) &= a. \end{aligned} \tag{1}$$

This inclusion is parameterized by time independent control variable/parameter u and is to be solved for state variable x. Function f is single–valued and continuously differentiable in all but the time variable while multifunction Ω is only continuous in the time variable. The main goal is to investigate the stability properties of the so–called solution mapping $S: u \mapsto x$ which assigns an infinite–dimensional solution x of (1) to a finite–dimensional parameter u.

Even though it is simple to obtain local Lipschitzian continuity of $S: U \to W^{1,\infty}([0,T],\mathbb{R}^n)$ in case of an ordinary differential equation with sufficiently smooth data, which corresponds to $\Omega \equiv 0$ and f having a special form, to our best knowledge, there are not many results for differential inclusions with parameters entering the inclusion. However, for models with parameterized initial condition, numerous results exist, see [14, 15, 17, 24, 25].

Similar dependence was studied in a series of papers by N. S. Papageorgiou, see [10, 18, 19, 20], in which the following infinite–dimensional problem was considered

$$\begin{aligned} -\dot{x}(t) &\in \partial_x f(t,u,x(t)) + \Omega(t,u,x(t)), \quad t\in[0,T] \text{ a.e.} \\ x(0) &= a(u) \end{aligned} \tag{2}$$

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with $f(t,u,\cdot)$ being a convex lower semicontinuous function. In this series the continuity of $S : U \rightrightarrows \mathcal{C}([0,T],H)$ was proved for a Hilbert space H. This result was obtained for both Vietoris and Hausdorff topologies on the power set of $\mathcal{C}([0,T],H)$.

In the context of rate–independent processes and hysteresis models, the following differential inclusion was studied

$$\begin{aligned} -\dot{x}(t) + \dot{u}(t) &\in \mathrm{N}_Z(x(t)), \quad t \in [0,T] \text{ a.e.} \\ x(0) &= a \end{aligned} \tag{3}$$

by P. Krejčí, see [11,12,13]. In this case, the global Lipschitzian continuity of $S : \mathcal{C}([0,T],H) \to \mathcal{C}([0,T],H)$ and $S : W^{1,1}([0,T],H) \to W^{1,1}([0,T],H)$ was shown.

In the case of (2), a general model was considered but only the state variable and not its derivative entered the estimates. On the other hand, model (3) provided opposite results, rather specific model was considered but the estimates were sharper. We try to combine the strengths of both papers. Thus, we consider both general models and are able to obtain local Lipschitzian continuity of $S : U \rightrightarrows W^{1,2}([0,T],\mathbb{R}^n)$, which is a significantly stronger result than the continuity of $S : U \rightrightarrows \mathcal{C}([0,T],H)$ for $H = \mathbb{R}^n$ obtained for (2). However, this approach introduces some deficiencies as well. We cannot handle time dependent perturbation and are able to work only with finite dimensional values $x(t)$.

Instead of considering a fixed model, we consider rather a general one, perform its discretization and derive necessary conditions for the local Lipschitzian continuity of the discretized solution map. If the corresponding Lipschitzian modulus exhibits uniform behaviour in some sense, we are able to deduce the Lipschitzian behavior of the original solution map as well. The Lipschitzian modulus must exhibit some boundedness feature both in a neighborhood of the investigated parameter and upon the decrease of the time step. Even though the conditions for this boundedness may seem to be difficult to verify, we propose a class of models, for which these results are verifiable.

Similar stability results can be found in [7,8], in which a differential variational inequality, which is a differential inclusion of special form coupled with algebraic equation, is considered. Data in such models are approximated and if the approximated solutions converge in some sense, then this limit is a solution to the original problem.

Concerning the possible applicability of the obtained results, differential inclusion are nowadays an established field of research, see monographs [2,3,24]. We believe that our results can be used in postoptimal analysis of time–dependent models where some parameters are not known exactly or in Mathematical Programs with Evolutionary Equilibrium Constraints (MPEECs) where inclusion (1) is part of the constraint system. We also derive an estimate for the Lipschitzian modulus, providing not only a qualitative but also quantitative estimate. We have in mind one particular application, namely nonregular electrical circuits with ideal diodes [1] where the parameter u plays the role of parameters of various components in the circuit and the state variable x is the charge in the circuit.

The paper is organized as follows. It Section 2 a discretization of (1) is considered. First an upper estimate of a generalized derivative of the discretized solution map S^K is found. This estimate is stated via adjoint equations. Then we show that having some bound on the adjoint variables results immediately in the local Lipschitzian continuity of S^K and if this bound is uniform in a certain sense, then we can deduce the local Lipschitzian continuity for the original solution map S as well. Since this version uses some notions of modern variational analysis, selected basic facts from this field are summarized in the Appendix.

Since it may not be immediately clear how to use these results, in Section 3 we present two examples of their possible applications. In the first one, we apply these results to an ordinary differential equation and in the second one to a model arising in modeling of electrical circuits with ideal diodes [1]. We are fully aware that it is possible to obtain stronger results by simpler means for the first case, however, we decided to present this example because the remaining example uses very similar ideas as the first one, only its implementation is much more technically difficult.

We will make use of the following notation and simplifications. We will often omit the arguments of f. Partial derivatives $\nabla_u f$, $\nabla_x f$ and $\nabla_v f$ are taken with respect to u, x and $\dot{x}$. Upper index K

denoting the discretization level will be often omitted, especially in cases when K is fixed and no convergence analysis comes into play. For discretized problem (6) by x we understand $(x_1, \dots, x_K)$. Further, we define $|\cdot| := \|\cdot\|_2$. However, sometimes it will almost obligatory to emphasize which norm has been used, especially when both finite– and infinite–dimensional ones are present. In this case, we use $|\cdot|_{l^2}$ and $|\cdot|_{L^2}$, respectively. On product spaces we consider the standard Euclidean norm. Finally, by $W^{1,2}([0,T], \mathbb{R}^n)$ we understand Sobolev space on a time interval with values in $\mathbb{R}^n$.

2 Main result

In this section we will consider problem (1) and derive conditions under which the solution map $S : u \mapsto x$ is locally Lipschitz. To ease the notational burden, instead of problem (1) we will also consider the following problem

$$\begin{aligned} g(t, u, x(t), \dot{x}(t)) &\in \Lambda(t), \quad t \in [0,T] \text{ a.e.} \\ x(0) &= a, \end{aligned} \tag{4}$$

in which $u \in \mathbb{R}^d$ plays the role of a parameter or a control variable and the systems are to be solved for almost any time instant for $x(t) \in \mathbb{R}^n$ for which the initial value is given. Concerning the data, $g : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is a single–valued function which is continuously differentiable in all but the time variable and $\Lambda(t) \subset \mathbb{R}^m$ is a closed set.

Even though it seems that (1) is more general than (4), it is not entirely true. Having problem (1), we may state it in the form (4) by setting

$$\begin{aligned} g(t, u, x(t), \dot{x}(t)) &:= \begin{pmatrix} u \\ x(t) \\ \dot{x}(t) \\ f(t, u, x(t), \dot{x}(t)) \end{pmatrix} \\ \Lambda(t) &:= \operatorname{gph} \Omega(t, \cdot, \cdot, \cdot). \end{aligned} \tag{5}$$

By doing so, we add artificial functions, which means that the used constraint qualifications will become more difficult to verify. As an example, consider the implicit function theorem or its numerous generalizations from [5], in which one of the constraint qualifications states that ∇g has to have full row rank. This reformulation will be considered in examples in Section 3.

Consider now time discretization $0 = t_0^K < \cdots < t_K^K = T$ and together with the infinite–dimensional problem (4) also its finite–dimensional discretized counterpart

$$\begin{aligned} g_{k+1}^K(u^K, x_k^K, x_{k+1}^K) &\in \Lambda_{k+1}^K, \; k = 0, \dots, K-1 \\ x_0^K &= a \end{aligned} \tag{6}$$

in which $g_k : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is a continuously differentiable function and Λ_k is a closed set. For simplicity, the upper index K denoting discretization level will be often omitted. The exact discretization scheme is not specified but we require that for computation of x_{k+1} only the previous value x_k may be used.

As all the problems depend on a parameter u, we are interested in analysis of the so–called solution map, also known as the control–to–state mapping, which assigns a solution x of a system to a parameter u. This mapping will be denoted by S for continuous case (4) and S^K for discretized case (6). In this section we first derive an upper estimate for coderivative D^*S^K which is used to present conditions for verification of the Aubin property of S^K, a property which coincides with local Lipschitzian property under single–valuedness of S^K. Finally, we show that under single–valuedness of S^K and S and some boundedness of the Lipschitzian moduli, the Lipschitzian property can be transferred from S^K to S. These results will be used later in Section 3 where the local Lipschitzian continuity of

$$S : \mathbb{R}^d \to W^{1,2}([0,T], \mathbb{R}^n)$$

is shown.

Before stating the first lemma, we remind that $u \in \mathbb{R}^d$ and by $x := (x_1, \dots, x_K) \in \mathbb{R}^{Kn}$. The initial value x_0 is omitted because $x_0 = a$ is not a subject to change.

Lemma 1 *Consider problem (6) and fix any $\bar{x} \in S^K(\bar{u})$. Assume that for all $k = 1, \dots, K$, g_k is continuously differentiable around $(\bar{u}, \bar{x}_k, \bar{x}_{k+1})$ and Λ_k is closed. Assume further that the following constraint qualification holds: if (7)–(10) are satisfied with $x^* = 0$, then $u^* = 0$.*

Then for $D^*S^K : \mathbb{R}^{Kn} \to \mathbb{R}^d$ and for any element

$$u^* \in D^*S^K(\bar{u}, \bar{x})(x^*)$$

with $x^* = (x_1^*, \dots, x_K^*)$ there exist for $k = 0, \dots, K-1$ multipliers

$$p_{k+1} \in \mathrm{N}_{\Lambda_{k+1}}(g_{k+1}(\bar{u}, \bar{x}_k, \bar{x}_{k+1})) \tag{7}$$

such that

$$u^* = \sum_{k=1}^{K} (\nabla_u g_k(\bar{u}, \bar{x}_{k-1}, \bar{x}_k))^\top p_k. \tag{8}$$

Moreover, for $k = 1, \dots, K-1$ the adjoint equations

$$-x_k^* = (\nabla_v g_k(\bar{u}, \bar{x}_{k-1}, \bar{x}_k))^\top p_k + (\nabla_x g_{k+1}(\bar{u}, \bar{x}_k, \bar{x}_{k+1}))^\top p_{k+1} \tag{9}$$

and the terminal condition

$$-x_K^* = (\nabla_v g_K(\bar{u}, \bar{x}_{K-1}, \bar{x}_K))^\top p_K \tag{10}$$

are satisfied.

Proof It is simple to see that

$$\operatorname{gph} S^K = \left\{ (u, x) \middle| \begin{array}{c} g_1(u, x_0, x_1) \in \Lambda_1 \\ \dots \\ g_K(u, x_{K-1}, x_K) \in \Lambda_K \end{array} \right\} = \{(u, x) |\ G(u, x) \in \Sigma\}$$

where we have defined $\Sigma := \Lambda_1 \times \dots \Lambda_K$ and

$$G(u, x) := \begin{pmatrix} g_1(u, x_0, x_1) \\ \dots \\ g_K(u, x_{K-1}, x_K) \end{pmatrix}.$$

Using the multi–valued inverse, we can write $\operatorname{gph} S^K = G^{-1}(\Sigma)$, which due to the assumed constraint qualification by virtue of [23, Theorem 6.14] implies

$$\mathrm{N}_{\operatorname{gph} S^K}(\bar{u}, \bar{x}) \subset \nabla G(\bar{u}, \bar{x})^\top \mathrm{N}_\Sigma(G(\bar{u}, \bar{x})). \tag{11}$$

Using the definition of coderivative and (11), we obtain

$$\begin{aligned} D^*S^K(\bar{u}, \bar{x})(x^*) &= \left\{ u^* \middle|\ \begin{pmatrix} u^* \\ -x^* \end{pmatrix} \in \mathrm{N}_{\operatorname{gph} S^K}(\bar{u}, \bar{x}) \right\} \\ &\subset \left\{ u^* \middle|\ \begin{pmatrix} u^* \\ -x^* \end{pmatrix} \in \nabla G(\bar{u}, \bar{x})^\top \mathrm{N}_\Sigma(G(\bar{u}, \bar{x}) \right\}. \end{aligned} \tag{12}$$

The Jacobian of G has the following form

$$\nabla G(\bar{u}, \bar{x}) = \begin{pmatrix} \nabla_u g_1 & \nabla_v g_1 & & & \\ \nabla_u g_2 & \nabla_x g_2 & \nabla_v g_2 & & \\ \vdots & & \ddots & \ddots & \\ \nabla_u g_K & & & \nabla_x g_K & \nabla_v g_K \end{pmatrix} \tag{13}$$

From [23, Proposition 6.41] we know that

$$\mathrm{N}_\Sigma(G(\bar{u}, \bar{x})) = \bigtimes_{k=0}^{K-1} \mathrm{N}_{\Lambda_{k+1}}(g_{k+1}(\bar{u}, \bar{x}_k, \bar{x}_{k+1})). \tag{14}$$

Finally by plugging (13) and (14) into (12) we obtain statement of the lemma.

In the next few lines we will briefly comment on Lemma 1. As we have already said, the main goal is to perform the sensitivity analysis of solution map S^K. From Mordukhovich criterion [23, Theorem 9.40] we immediately see that S^K has the Aubin property if $\nabla_v g_{k+1}(\bar{u}, \bar{x}_k, \bar{x}_{k+1})$ has full row rank for all $k = 0, \dots, K-1$. Unfortunately, as in our desired application the sets Λ_k will mostly depend on state or control variables and will have to be rewritten via their graphs as shown in (5), additional artificial functions will be added and hence, this full row rank property cannot be expected and another approach has to be used.

It is certainly possible to relax the used constraint qualification. According to [9, Proposition 3.3] the constraint qualification from Lemma 1 corresponds to the Aubin property of multifunction

$$M(q) = \{(u, x) |\ g_{k+1}(u, x_k, x_{k+1}) + q_{k+1} \in \Lambda_{k+1},\ k = 0, \dots, K-1\}$$

around point $(0, \bar{u}, \bar{x})$. According to [9, Proposition 3.2], the same result would hold true if we assumed only calmness of M at the same point. However, since the main goal of this paper lies in the next theorem whose assumptions imply the Aubin property of M around the reference point, and thus the constraint qualification of Lemma 1 is satisfied, we keep the current state.

Having Lemma 1 at hand, the condition for fulfillment of the Aubin property of S^K, and thus of the local Lipschitzian property if S^K is single–valued, is well–known. To be able to deduce some some results for S, we need some uniformity boundedness of the Lipschitzian moduli both over time and on some neighborhood of $\bar{u}$. This condition is stated in (15).

Together with $x^K = (x_1^K, \dots, x_K^K) \in \mathbb{R}^{Kn}$, we will also consider its piecewise linear and piecewise constant extensions $x^K(\cdot)$. Both will satisfy $x^K(0) = a$ and $x^K(t_k) = x_k^K$ for all $k = 1, \dots, K$. The piecewise linear extension will be obtained by connecting these points while the piecewise constant extension will satisfy $x^K(t) = x^K(t_{k-1}^K)$ whenever $t \in [t_{k-1}^K, t_k^K)$ for all $k = 1, \dots, K$.

Theorem 1 *If in the setting of Lemma 1 there exists a constant $L(K, \bar{u})$ such that*

$$|u^{*K}| \leqslant L(K, \bar{u}) |x^{*K}|,$$

then S^K has the Aubin property around $(\bar{u}, \bar{x})$ with modulus not larger than $L(K, \bar{u})$.

Assume further that there exists a neighborhood V of $\bar{u}$ such that multifunctions S^K and S are single–valued on V. For all $u \in V$ find $L(K, u)$ as above, define

$$M(K, V) := \sup_{u \in V} L(K, u)$$

to be the upper bound for the Lipschitzian modulus of S^K on V and assume that

$$M(V) := \liminf_{K \to \infty} \frac{1}{\sqrt{K}} M(K, V) < \infty. \tag{15}$$

Finally, assume that for every $u \in V$ we have $x^K(\cdot) \rightharpoonup x$ in $L^2([0, T], \mathbb{R}^n)$, where $x = S(u)$, $x^K = S^K(u)$ and $x^K(\cdot)$ is the piecewise constant or piecewise linear extension of $S^K(u)$.

Then $S : \mathbb{R}^d \to L^2([0, T], \mathbb{R}^n)$ is locally Lipschitz on V with modulus less or equal to $\sqrt{T} M(V)$.

Proof The first statement follows immediately from [23, Theorem 9.40].

For the second part recall the already several times mentioned fact that the Aubin property coincides with the locally Lipschitzian property for single–valued mappings. Using the same theorem as in the first part, we obtain that S^K is locally Lipschitzian on V with modulus at most $M(K, V)$. Fix arbitrary parameters $u, \tilde{u} \in V$ and denote the corresponding state variables by x^K and $\tilde{x}^K$. Even though the Lipschitzian modulus is defined as infimum of all constants satisfying (50), having uniform bound for this modulus on V, we can deduce that

$$\frac{1}{\sqrt{K}} |\tilde{x}^K - x^K|_{l^2} \leqslant \frac{1}{\sqrt{K}} M(K, V) |\tilde{u} - u|. \tag{16}$$

For simplicity denote $z^K := \tilde{x}^K - x^K$ and consider first its piecewise constant extension denoted by $z^K(\cdot)$. For $k = 1, \dots, K$ and $j = 1, \dots, n$ we denote by $z_{k,j}^K$ the j–th component of z_k^K and similarly by $z_j^K(t)$ we understand the j–th component of its piecewise constant extension. Further recall that, as mentioned in the introduction, we consider the Euclidean norm on product spaces and hence

$$|z^K(\cdot)|_{L^2} = \sqrt{\sum_{j=1}^n |z_j^K(\cdot)|_{L^2}^2} = \sqrt{\sum_{j=1}^n \int_0^T (z_j^K(t))^2 dt}.$$

For the left–hand side of (16) due to $T = Kh$ we obtain

$$\frac{1}{\sqrt{K}}|z^K|_{l^2} = \sqrt{\frac{1}{Kh}\sum_{j=1}^n \sum_{k=1}^K h(z_{k,j}^K)^2} = \sqrt{\frac{1}{T}\sum_{j=1}^n \int_0^T (z_j^K(t))^2 dt} = \frac{1}{\sqrt{T}}|z^K(\cdot)|_{L^2}. \tag{17}$$

From the assumptions we know that for the solutions of the continuous problems $\tilde{x} := S(\tilde{u})$ and $x := S(u)$ we have $\tilde{x}^K - x^K \rightharpoonup \tilde{x} - x$ in $L^2([0,T],\mathbb{R}^n)$ and thus

$$\liminf_{K\to\infty} \frac{1}{\sqrt{K}}|\tilde{x}^K - x^K|_{l^2} = \liminf_{K\to\infty} \frac{1}{\sqrt{T}}|\tilde{x}^K(\cdot) - x^K(\cdot)|_{L^2} \geqslant \frac{1}{\sqrt{T}}|\tilde{x}(\cdot) - x(\cdot)|_{L^2} \tag{18}$$

and combination of (18), (16) and (15) yields

$$\begin{aligned}\frac{1}{\sqrt{T}}|\tilde{x}(\cdot) - x(\cdot)|_{L^2} &\leqslant \liminf_{K\to\infty} \frac{1}{\sqrt{K}}|\tilde{x}^K - x^K|_{l^2} \\ &\leqslant \liminf_{K\to\infty} \left[\frac{1}{\sqrt{K}}M(K,V)|\tilde{u} - u|\right] = M(V)|\tilde{u} - u|\end{aligned}$$

and the result for the case of piecewise constant extension has been proven.

If the extension is piecewise linear, the only thing which needs to be modified is (17). Again, we keep the same symbols for this extension as in the previous case. By simple computation it can be shown that

$$\begin{aligned}\frac{1}{\sqrt{T}}|z^K(\cdot)|_{L^2} &= \sqrt{\frac{h}{3T}\sum_{j=1}^n \sum_{k=1}^K (z_{k,j}^2 + z_{k,j}z_{k-1,j} + z_{k-1,j}^2)} \\ &\leqslant \sqrt{\frac{h}{3T}\sum_{j=1}^n \sum_{k=1}^K (3z_{k,j}^2 + 2z_{0,j}^2)} = \sqrt{\frac{1}{K}\sum_{j=1}^n \sum_{k=1}^K z_{k,j}^2} = \frac{1}{\sqrt{K}}|z^K|_{l^2}\end{aligned} \tag{19}$$

because $z_{0,j} = 0$. Finally, the theorem statement follows exactly from the same arguments as in the previous case.

3 Applications

In this final section we present applications of Theorem 1. Even though the main strength of this theorem lies in admitting the multi–valued part, first we set this part to zero and consider an ordinary differential equation. We are fully aware that for this case it is possible to obtain stronger results by simpler means, however, we have decided to include this example to illustrate the basic estimates on which the next example is based. In the second part we consider a differential inclusion called the sweeping process motivated by electrical circuits with ideal diodes [1] and show the local Lipschitzian continuity of the solution map.

In all cases we will need the following two lemmas, the first one being a discrete version of Gronwall's lemma.

Lemma 2 ([6, Proposition 3.1, Proposition 3.2]) *If* $1 - \lambda h > 0$ *and*

$$\frac{a_{k+1} - a_k}{h} \leqslant \lambda a_{k+1} + g_{k+1},$$

then

$$a_k \leqslant (1 - \lambda h)^{-k} \left(a_0 + h \sum_{j=1}^{k} g_k \right).$$

Similarly, if $1 + \lambda h > 0$ *and*

$$\frac{a_{k+1} - a_k}{h} \leqslant \lambda a_k + g_{k+1},$$

then

$$a_k \leqslant (1 + \lambda h)^{k} \left(a_0 + h \sum_{j=1}^{k} g_k \right).$$

Lemma 3 *Let* A *be a positive definite matrix. Fix any* r *and find any* p *and* q *solving the following system*

$$\begin{aligned} p - Aq &= r \\ p^\top q &\leqslant 0. \end{aligned} \tag{20}$$

Denoting

$$d := \min_{\|x\|=1} x^\top A x,$$

then one has

$$\|q\| \leqslant \frac{1}{d}\|r\|.$$

Finally, for $p^\top q \leqslant 0$ *it is sufficient that*

$$\begin{pmatrix} p \\ q \end{pmatrix} \in \mathrm{N}_{\mathrm{gph}\, \mathrm{N}_\Gamma}(x, y)$$

for any $y \in \mathrm{N}_\Gamma(x)$ *and convex* Γ.

Proof Constant d is positive because A is positive definite. For $q = 0$ the statement is obvious. In the opposite case, multiply equation (20) by q and perform simple algebraic operations to obtain

$$q^\top r = q^\top p - q^\top A q \leqslant -d\|q\|^2,$$

which implies

$$\|q\|^2 \leqslant -\frac{1}{d} q^\top r \leqslant \frac{1}{d}\|q\|\|r\|,$$

which in turn amounts to the first part of the lemma statement.

If Γ is convex, then by virtue of [22, Theorem 4] we know that $x \mapsto \mathrm{N}_\Gamma(x)$ is a maximal monotone operator, and thus [21, Theorem 2.1] implies $p^\top q \leqslant 0$.

3.1 Ordinary differential equation

Consider an ordinary differential equation

$$\dot{y}(t) = f(t, u, y(t)) \tag{21}$$

with initial condition $y(0) = a$. As we have already said, this is only an illustrative example with not satisfactory results and thus we impose rather strong assumptions on f to keep this subsection as short as possible. Specifically, we will assume that f is bounded, continuous in the time variable and continuously differentiable in last two variables.

First, we need to discretize (21). Since we are not interested in convergence analysis, we keep the discretization as simple as possible and use the forward Euler method. It is possible to use

$$y_{k+1}^K - y_k^K - h^K f(t_k^K, u, y_k^K) = 0, \tag{22}$$

however, in this case, we would obtain only local Lipschitzian continuity of the solution map of (21) when considered as $S : \mathbb{R}^d \to L^2([0, T], \mathbb{R}^n)$. Thus, we introduce an artificial variable and perform the following discretization

$$y_{k+1}^K - y_k^K - h^K z_{k+1}^K = 0 \tag{23a}$$

$$z_{k+1}^K - f(t_k^K, u, y_k^K) = 0 \tag{23b}$$

with $y_0 = a$ fixed. As we will see later, in this case, we will be able to prove local Lipschitzian continuity of the solution map of (21) when considered as $S : \mathbb{R}^d \to W^{1,2}([0, T], \mathbb{R}^n)$.

For discretized problem (23) we consider slightly redefined solution map, specifically we will consider $S^K : u \mapsto (y^K, z^K)$, hence to the parameter not only the state variable but also its derivate is assigned. It is clear that S^K is single–valued. Define now $f_k^K(u, y_k^K) := f(t_k^K, u, y_k^K)$, fix any $\bar{u}$ and some its bounded neighborhood V. Then the following constants are finite

$$\begin{aligned} c_1 &:= \sup\{\nabla_y f_k(u, y_k)|\ u \in V, (y^K, z^K) \in S^K(u), K \in \mathbb{N}, k = 1, \dots, K\} \\ c_2 &:= \sup\{\nabla_u f_k(u, y_k)|\ u \in V, (y^K, z^K) \in S^K(u), K \in \mathbb{N}, k = 1, \dots, K\}. \end{aligned} \tag{24}$$

Fix now any $u \in V$, compute $(y^K, z^K) = S^K(u)$ and set $(y^K(\cdot), z^K(\cdot)) \in L^2([0, T], \mathbb{R}^n) \times L^2([0, T], \mathbb{R}^n)$ to be the extension of $(y^K, z^K) \in \mathbb{R}^{Kn} \times \mathbb{R}^{Kn}$ which is piecewise linear for $y^K(\cdot)$ and piecewise constant for $z^K(\cdot)$. From the assumptions on f it is simple to deduce that $y^K(\cdot)$ and $z^K(\cdot)$ are uniformly bounded in $L^\infty([0, T], \mathbb{R}^n)$ and thus we may extract a subsequence such that $y^K(\cdot) \rightharpoonup y(\cdot)$ and $z^K(\cdot) \rightharpoonup z(\cdot)$, both in $L^2([0, T], \mathbb{R}^n)$. Moreover, it can be shown that $\dot{y}(\cdot) = z(\cdot)$ and that $y(\cdot)$ is the unique solution to (21). In the next several lines, we will consider fixed discretization level K and thus we omit it.

Denote the multiplier corresponding to (23a) by p_k and to (23b) by q_k. Then Lemma 1, with the not yet verified constraint qualification, yields that if $u^* \in D^*S^K(u, y, z)(y^*, z^*)$, then

$$u^* = -\sum_{k=1}^{K} (\nabla_u f_k(u, y_k))^\top q_k \tag{25}$$

and the terminal condition

$$-y_K^* = p_K \tag{26a}$$

$$-z_K^* = -hp_K + q_K \tag{26b}$$

and for $k = 1, \dots, K - 1$ the adjoint equations

$$-y_k^* = p_k - p_{k+1} - (\nabla_y f_{k+1}(u, y_{k+1}))^\top q_{k+1} \tag{27a}$$

$$-z_k^* = -hp_k + q_k. \tag{27b}$$

are satisfied. Moreover, from these expressions, it is simple to see that the constraint qualification of Lemma 1 indeed holds.

Plugging (27b) into (27a) yields

$$\frac{p_k - p_{k+1}}{h} = (\nabla_y f_{k+1}(u, y_{k+1}))^\top p_{k+1} - \frac{1}{h}(y_k^* + (\nabla_y f_{k+1}(u, y_{k+1}))^\top z_{k+1}^*), \tag{28}$$

which due to Lemma 2 results in

$$\|p_k\|_1 \leqslant e^{c_1 T}(\|y_K^*\|_1 + \sum_{j=k}^{K-1}(\|y_j^*\|_1 + c_1\|z_{j+1}^*\|_1)) \leqslant e^{c_1 T}\sum_{j=1}^{K}(\|y_j^*\|_1 + c_1\|z_j^*\|_1).$$

By plugging this estimate back to (27b) we have

$$\|q_k\|_1 \leqslant \|z_k^*\|_1 + he^{c_1 T}\sum_{j=1}^{K}(\|y_j^*\|_1 + c_1\|z_j^*\|_1) \tag{29}$$

and noting that $y^{*K} \in \mathbb{R}^{Kn}$ stands for vector $(y_1^*, \dots, y_K^*)$, we have

$$\|y^{*K}\|_1 = \sum_{k=1}^{K}\|y_k^*\|_1 \tag{30}$$

and thus from (25) due to (29) we have estimate

$$\begin{aligned} \|u^*\|_2 \leqslant \|u^*\|_1 &\leqslant c_2\sum_{k=1}^{K}\|q_k\|_1 \leqslant c_2\sum_{k=1}^{K}\left(\|z_k^*\|_1 + he^{c_1 T}\sum_{j=1}^{K}(\|y_j^*\|_1 + c_1\|z_j^*\|_1)\right) \\ &= c_2\left(\|z^{K*}\|_1 + Te^{c_1 T}(\|y^{K*}\|_1 + c_1\|z^{K*}\|_1)\right) \\ &\leqslant c_2\sqrt{Kn}\left(\|z^{K*}\|_2 + Te^{c_1 T}(\|y^{K*}\|_2 + c_1\|z^{K*}\|_2)\right). \end{aligned} \tag{31}$$

Since c_1 and c_2 are not dependent on the choice of $u \in V$, we may apply Theorem 2 to obtain that mapping $u \mapsto (y, \dot{y})$ solving (21) is locally Lipschitz continuous as $V \to L^2([0,T], \mathbb{R}^n) \times L^2([0,T], \mathbb{R}^n)$. This is equivalent to local Lipschitzian continuity of solution map $u \mapsto y$ to (21) when considered as $V \to W^{1,2}([0,T], \mathbb{R}^n)$. We again emphasize that this is only a toy example with simplified assumptions and not entirely satisfactory results.

3.2 Sweeping process

The purpose of the previous example was to give an insight of what needs to be done. In this second example we consider a proper differential inclusion with discontinuous multifunction and derive similar results as in the first example. The considered model is a version of the sweeping process from [1] which takes the following form

$$\begin{aligned} -A_1\dot{y}(t) - A_0 y(t) + f(t) &\in \mathrm{N}_{C(t)}(\dot{y}(t)), \quad t \in [0,T] \text{ a.e.} \\ y(0) &= a. \end{aligned} \tag{32}$$

Such systems arise in modeling electrical circuits with ideal diodes, matrices A_0 and A_1 accumulate information about components of the circuit and y stands for the charge in circuit. We add perturbation u to data and consider the following model

$$\begin{aligned} -A_1(u)\dot{y}(t) - A_0(u)y(t) + f(t,u) &\in \mathrm{N}_{C(t)}(\dot{y}(t)), \quad t \in [0,T] \text{ a.e.} \\ y(0) &= a. \end{aligned} \tag{33}$$

The goal is to perform sensitivity analysis of $S : u \mapsto x(\cdot)$ with $x(\cdot) := (y(\cdot), \dot{y}(\cdot))$ solving (33). Thus, we investigate how the change of various parameters of the model, such as resistances, influences the change of the charge in the circuit.

On the contrary to the original paper [1], instead of an arbitrary Hilbert space we restrict ourselves only to $y(t) \in \mathbb{R}^n$. Under certain assumptions, in [1, Theorem 5.6] the convergence of the discretized solutions is proved using a modified version of the catching up algorithm, in which one solves for $k = 0, \dots, K-1$ defines doubles $x_{k+1} := (y_{k+1}, z_{k+1})$ and solves iteratively the following system starting with $y_0 = a$

$$g_{k+1}(u, x_k, x_{k+1}) := \begin{pmatrix} z_{k+1} \\ -A_1(u)z_{k+1} - A_0(u)y_{k+1} + f_{k+1}(u) \\ \hdashline y_{k+1} - y_k - hz_{k+1} \end{pmatrix} \begin{matrix} \in \\ \\ = \end{matrix} \begin{pmatrix} \operatorname{gph} \mathrm{N}_{C_{k+1}} \\ \hdashline 0 \end{pmatrix} =: \Lambda_{k+1} \tag{34}$$

where $f_k^K(u) := f(t_k^K, u)$ and $C_k^K := C(t_k^K)$ are closed convex sets. It is shown in [1, Theorem 5.7] that S is single–valued under additional assumptions and similar result is shown for S^K in [1, Remark 2]. In the rest of this section we assume that this single–valuedness is satisfied.

From now on, we assume that the constraint qualification of Lemma 1 is satisfied. This constraint qualification will be verified later on. Fix any $\bar{u}$ and some its neighborhood V and assume further that for $i = 0, 1$ and $t \in [0, T]$ the mappings $u \mapsto A_i(u)$ and $u \mapsto f(t, u)$ are continuously differentiable for all $u \in V$.

First, fix any $u \in V$ and apply Lemma 1. It is well–known that $\operatorname{gph} \mathrm{N}_{C_k}$ is closed for any set C_k and hence, with the not–yet–verified constraint qualification, all the assumptions of this lemma are satisfied. For simplicity, for the corresponding multipliers from this lemma we omit their dependence on u, and hence write e. g. only p_k instead of $p_k(u)$. However, for various constants used as upper bounds, their dependence on u is emphasized.

Computing the partial derivatives of g_k, we obtain

$$\nabla_u g_k = \begin{pmatrix} 0 \\ B_k(u) \\ 0 \end{pmatrix}, \nabla_x g_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -I & 0 \end{pmatrix}, \nabla_v g_k = \begin{pmatrix} 0 & I \\ -A_0(u) & -A_1(u) \\ I & -hI \end{pmatrix}$$

with

$$B_{k+1}(u) := -\nabla_u A_1(u) z_{k+1} - \nabla_u A_0(u) y_{k+1} + \nabla_u f_{k+1}(u).$$

Lemma 1 then states that if $u^* \in D^*S(u, x)(x^*)$, then this element can be expressed as

$$u^* = \sum_{k=1}^{K} B_k^\top(u) q_k \tag{35}$$

where there are multipliers (p_k, q_k, r_k) satisfying for $k = 1, \dots, K$

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} \in \mathrm{N}_{\operatorname{gph} \mathrm{N}_{C_k}}(z_k, -A_1(u)z_k - A_0(u)y_k + f_k(u)). \tag{36}$$

Assuming that matrices $A_0(u)$ and $A_1(u)$ are symmetric, then the adjoint equations (9) read for $k = 1, \dots, K-1$

$$r_k = r_{k+1} + A_0(u) q_k - y_k^* \tag{37a}$$
$$p_k - A_1(u) q_k = h r_k - z_k^* \tag{37b}$$

and the terminal condition (10) is equal to

$$r_K = A_0(u) q_K - y_K^* \tag{38a}$$
$$p_K - A_1(u) q_K = h r_K - z_K^*. \tag{38b}$$

Define now the following constants

$$c(u) := \frac{1}{\min_{\|x\|_1 = 1} x^\top A_1(u) x}$$
$$d(K, u) := \frac{1}{1 - hc(u)\|A_0(u)\|_\infty} \left(1 - \frac{Tc(u)\|A(u)\|_\infty}{K}\right)^{-K}$$

If $A_1(u)$ is positive definite, we may employ Lemma 3 to (37b) and (38b) to obtain for $k = 1, \dots, K$

$$\|q_k\|_1 \leqslant c(u)\|hr_k - z_k^*\|_1. \tag{39}$$

Then by plugging estimate (39) into (38a) and (37a) we obtain

$$\begin{aligned} \|r_K\|_1 &\leqslant c(u)\|A_0(u)\|_\infty(h\|r_K\|_1 + \|z_K^*\|_1) + \|y_K^*\|_1 \\ \|r_K\|_1 &\leqslant \frac{1}{1 - hc(u)\|A_0(u)\|_\infty}(c(u)\|A_0(u)\|_\infty\|z_K^*\|_1 + \|y_K^*\|_1) \end{aligned} \tag{40}$$

and for $k = 1, \dots, K-1$

$$\|r_k\|_1 = \|r_{k+1} + A_0(u)q_k - y_k^*\|_1 \leqslant \|r_{k+1}\|_1 + c(u)\|A_0(u)\|_\infty(h\|r_k\|_1 + \|z_k^*\|_1) + \|y_k^*\|_1$$

or equivalently

$$\frac{\|r_k\|_1 - \|r_{k+1}\|_1}{h} \leqslant c(u)\|A_0(u)\|_\infty\|r_k\|_1 + \frac{1}{h}\|y_k^*\|_1 + \frac{1}{h}c(u)\|A_0(u)\|_\infty\|z_k^*\|_1.$$

This enables us to use Lemma 2 with, together with (40) and (30), to obtain

$$\begin{aligned} \|r_k\|_1 &\leqslant \left(1 - \frac{Tc(u)\|A(u)\|_\infty}{K}\right)^{-K}\left(\|r_K\|_1 + \sum_{l=k}^{K-1}[\|y_l^*\|_1 + c(u)\|A_0(u)\|_\infty\|z_l^*\|_1]\right) \\ &\leqslant d(K,u)\sum_{l=1}^{K}[\|y_l^*\|_1 + c(u)\|A_0(u)\|_\infty\|z_l^*\|_1] \\ &= d(K,u)[\|y^{K*}\|_1 + c(u)\|A_0(u)\|_\infty\|z^{K*}\|_1]. \end{aligned} \tag{41}$$

Assume further that there exist constants ρ_y and ρ_z such that for all $u \in V$, for all K and for the corresponding $(y^K, z^K) = S^K(u)$ we have

$$|y_k^K| \leqslant \rho_y, \ |z_k^K| \leqslant \rho_z$$

for all K and $k = 1, \dots, K$. Due to the structure of the considered model, this assumption is satisfied when all sets $C(t)$ are uniformly bounded. Then if $\|\nabla_u f(\cdot, u)\|_\infty$ is bounded on $[0, T]$, then we get the following estimate

$$\|B_k(u)\|_\infty \leqslant \|\nabla_u A_1(u)\|_\infty\rho_z + \|\nabla_u A_0(u)\|_\infty\rho_u + \sup_{t\in[0,T]}\|\nabla_u f(t,u)\|_\infty =: b(u). \tag{42}$$

Formulas (35), (42), (39) and (41) then imply

$$\begin{aligned} |u^*| &\leqslant \|u^*\|_1 \leqslant b(u)\sum_{k=1}^{K}\|q_k\|_1 \leqslant b(u)c(u)\sum_{k=1}^{K}(h\|r_k\|_1 + \|z_k^*\|_1) \\ &\leqslant b(u)c(u)[Td_1(K,u)d_2(K,u)[\|y^{K*}\|_1 + c(u)\|A_0(u)\|_\infty\|z^{K*}\|_1] + \|z^{K*}\|_1] \\ &\leqslant b(u)c(u)\max\{Td(K,u), Td(K,u)c(u)\|A_0(u)\|_\infty + 1\}\|(y^{K*}, z^{K*})\|_1 \\ &\leqslant \sqrt{2Kn}b(u)c(u)\max\{Td(K,u), Td(K,u)c(u)\|A_0(u)\|_\infty + 1\}|(y^{K*}, z^{K*})|. \end{aligned}$$

Moreover, from this formula it is clear that the constraint qualification of Lemma 1 is satisfied.

Theorem 1 then implies that S^K has the local Lipschitzian property around u with modulus less or equal to $L(K, u)$ where

$$L(K,u) := \sqrt{2Kn}b(u)c(u)\max\{Td(K,u), Td(K,u)c(u)\|A_0(u)\|_\infty + 1\}. \tag{43}$$

To use the second part of Theorem 1, we need to find $\sup_{u\in V} L(K,u)$. Assume furthermore that $u \mapsto \nabla_u f(t,u)$ is continuous on V uniformly in t and fix any $\varepsilon > 0$ such that $\varepsilon c(\bar{u}) < 1$. This, together with previously imposed assumptions, allows us to shrink the neighborhood V of $\bar u$ such that for all $u\in V$ and for all $t\in[0,T]$ we have

$$\begin{aligned}\|A_i(u)-A_i(\bar u)\|_\infty &\leqslant \varepsilon\\ \|\nabla_u A_i(u)-\nabla_u A_i(\bar u)\|_\infty &\leqslant \varepsilon\\ \|\nabla_u f(t,u)-\nabla_u f(t,\bar u)\|_\infty &\leqslant \varepsilon.\end{aligned}\tag{44}$$

Hence we obtain

$$\begin{aligned}b(u) &= \|\nabla_u A_1(u)\|_\infty\rho_z+\|\nabla_u A_0(u)\|_\infty\rho_u+\sup_{t\in[0,T]}\|\nabla_u f(t,u)\|_\infty\\ &\leqslant \|\nabla_u A_1(\bar u)\|_\infty\rho_z+\|\nabla_u A_0(\bar u)\|_\infty\rho_u+\sup_{t\in[0,T]}\|\nabla_u f(t,\bar u)\|_\infty+\varepsilon\rho_z+\varepsilon\rho_u+\varepsilon\end{aligned}\tag{45}$$

For an estimate for $c(u)$, fix first any $x\in\mathbb{R}^n$ with $\|x\|_1=1$. Then by (44) we have

$$|x^\top A_1(u)x-x^\top A_1(\bar u)x|\leqslant \|x\|_1\|A_1(u)-A_1(\bar u)\|_\infty\|x\|_1\leqslant\varepsilon,$$

which implies

$$c(u)=\frac{1}{\min_{\|x\|_1=1}x^\top A_1(u)x}\leqslant\frac{1}{\min_{\|x\|_1=1}x^\top A_1(\bar u)x-\varepsilon}=\frac{c(\bar u)}{1-\varepsilon c(\bar u)}.\tag{46}$$

Similarly, for K large enough, the following estimate, which is independent of the choice of $u\in V$, can be deduced

$$\begin{aligned}d(K,u)&\leqslant\left(\frac{1}{1-h(c(\bar u)+\varepsilon)(\|A(\bar u)\|_\infty+\varepsilon)}\right)\left(1-\frac{Tc(\bar u)(\|A_0(\bar u)\|_\infty+\varepsilon)}{K(1-\varepsilon c(\bar u))}\right)^{-K}\\ &\xrightarrow{K\to\infty}\exp\left(\frac{Tc(\bar u)(\|A_0(\bar u)\|_\infty+\varepsilon)}{1-\varepsilon c(\bar u)}\right).\end{aligned}\tag{47}$$

By plugging (45), (46) and (47) into (43) we find upper estimate of $\sup_{u\in V}L(K,u)$, which in accordance with Theorem 1 will be denoted by $M(K,V)$ and similarly for

$$M(V):=\limsup_{K\to\infty}\frac{1}{\sqrt{K}}M(K,V)$$

we have

$$M(V)\leqslant\sqrt{2n}\hat b\hat c\max\{T\hat d,T\hat d\hat c\hat A+1\}.$$

where

$$\begin{aligned}\hat A&:=\|A_0(\bar u)\|_\infty+\varepsilon\\ \hat b&:=\|\nabla_u A_1(\bar u)\|_\infty\rho_z+\|\nabla_u A_0(\bar u)\|_\infty\rho_u+\sup_{t\in[0,T]}\|\nabla_u f(t,\bar u)\|_\infty+\varepsilon\rho_u+\varepsilon\rho_z+\varepsilon\\ \hat c&:=\frac{c(\bar u)}{1-\varepsilon c(\bar u)}\\ \hat d&:=\exp\left(\frac{Tc(\bar u)(\|A_0(\bar u)\|_\infty+\varepsilon)}{1-\varepsilon c(\bar u)}\right).\end{aligned}$$

If $y^K(\cdot)\rightharpoonup y(\cdot)$ and $z^K(\cdot)\rightharpoonup\dot y(\cdot)$ both in $L^2([0,T],\mathbb{R}^n)$ with $y(\cdot)$ being the unique solution of (33) corresponding to u, then considering that ε may be arbitrarily small, Theorem 1 tells us that S is locally Lipschitz around $\bar u$ with modulus no more than

$$\sqrt{2n}bc\max\{Te^{Tc\|A_0(\bar u)\|_\infty},Te^{Tc\|A_0(\bar u)\|_\infty}c\|A_0(\bar u)\|_\infty+1\}.\tag{48}$$

where

$$\begin{aligned}b&:=\|\nabla_u A_1(\bar u)\|_\infty\rho_z+\|\nabla_u A_0(\bar u)\|_\infty\rho_u+\sup_{t\in[0,T]}\|\nabla_u f(t,\bar u)\|_\infty\\ c&:=\frac{1}{\min_{\|x\|_1=1}x^\top A_1(\bar u)x}.\end{aligned}$$

We summarize the result in the following theorem.

Theorem 2 *For $u \in \mathbb{R}^d$ define the set of solutions y of problem (33) by $S(u)$. Assume that for all $t \in [0,T]$ the sets $C(t)$ are closed and convex. Moreover, let $C(0)$ be a bounded set and $C(\cdot)$ have a continuous variation, which means that there exists a nondecreasing continuous function $\nu : [0,T] \to \mathbb{R}$ with $\nu(0) = 0$ such that*

$$|d(v, C(t)) - d(v, C(s)| \leqslant |\nu(t) - \nu(s)| \tag{49}$$

for all $v \in \mathbb{R}^n$ and $s, t \in [0,T]$.

Fix any $\bar{u}$ and assume that there exists its neighborhood V with the following properties

- *f is a continuous function on $[0,T] \times V$*
- *$f(t,\cdot)$ is differentiable on V for every $t \in [0,T]$ and $\nabla_u f(t,\cdot)$ is continuous at $\bar{u}$ uniformly in t by which we understand that for every $\varepsilon > 0$ there exists a neighborhood $\tilde{V}$ of $\bar{u}$ such that*

$$\sup_{t\in[0,T]} \sup_{u\in\tilde{V}} |\nabla_u f(t,u) - \nabla_u f(t,\bar{u})| \leqslant \varepsilon.$$

- *$A_i(u)$ is symmetric positive definite matrices such that $A_i(\cdot)$ is continuously differentiable at $\bar{u}$ for $i = 0, 1$ and $u \in V$.*

Then S^K and S are single–valued and there exist constants ρ_y and ρ_z such that for all $u \in V$ and all the corresponding $(y^K, z^K) = S^K(u)$ we have $|y_k^K| \leqslant \rho_y$, $|z_k^K| \leqslant \rho_z$ for all K and all $k = 1, \ldots, K$. Finally, $S : \mathbb{R}^d \to W^{1,2}([0,T], \mathbb{R}^n)$ is locally Lipschitz at $\bar{u}$ with modulus no more than (48).

Proof The proof has been basically performed in this section. We only fit the remaining blank spots concerning assumptions. Constant ρ_z exists due to the boundedness of sets $C(t)$. From this the existence of ρ_y follows immediately. The single–valuedness of S^K and of S follows from [1, Remark 2] and [1, Theorem 5.7]. The required convergence $y^K(\cdot) \rightharpoonup y(\cdot)$ and $z^K(\cdot) \rightharpoonup \dot{y}(\cdot)$ in $L^2([0,T], \mathbb{R}^n)$ is a result of the proof of [1, Theorem 5.6].

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A Basic tools of variational analysis

In this section we present basic notions of variational analysis, which are essential for this paper. More information can be found either in [23] for finite–dimensional setting or in [16] and [4] for the infinite–dimensional one.

All objects in this section are finite–dimensional. For sequence of sets $A_k \subset \mathbb{R}^n$ we define the Painlewé–Kuratowski upper limit as

$$\operatorname*{Limsup}_{k\to\infty} A_k = \{x \mid \exists x_k \in A_k, x \text{ is an accumulation point of } \{x_k\}\}.$$

For multifunction $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ we define this limit as

$$\operatorname*{Limsup}_{x\to\bar{x}} M(x) = \bigcup_{x_k\to\bar{x}} \operatorname*{Limsup}_{k\to\infty} M(x_k).$$

For $\bar{x} \in A$ we define the Fréchet and limiting normal cones as

$$\hat{\mathrm{N}}_A(\bar{x}) = \{x^* \mid \langle x^*, x - \bar{x}\rangle \leqslant o(\|x - \bar{x}\|) \text{ for all } x \in A\}$$
$$\mathrm{N}_A(\bar{x}) = \operatorname*{Limsup}_{x\xrightarrow{A}\bar{x}} \hat{\mathrm{N}}_A(x).$$

where by $x \xrightarrow{A} \bar{x}$ we understand standard convergence $x \to \bar{x}$ with $x \in A$.

To both normal cones, we can define the corresponding subdifferential of an extended single–valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ as

$$\hat{\partial} f(\bar{x}) = \{x^* \mid (x^*, -1) \in \hat{\mathrm{N}}_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))\}$$
$$\partial f(\bar{x}) = \{x^* \mid (x^*, -1) \in \mathrm{N}_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))\}.$$

If A happens to be convex, all the above cones coincide and are equal to the normal cone of convex analysis

$$\mathrm{N}_A(\bar{x}) = \{x^* | \ \langle x^*, x - \bar{x}\rangle \leqslant 0 \text{ for all } x \in A\}$$

and similarly, if f is differentiable at $\bar{x}$, then $\hat{\partial} f(\bar{x}) = \{\nabla f(\bar{x})\}$ but as the limiting subdifferential accumulates some information from the neighborhood, we have only $\partial f(\bar{x}) \supset \{\nabla f(\bar{x})\}$. We obtain equality in the previous relation if f is continuously differentiable at $\bar{x}$.

It is not possible to use this definition if f is multi– or vector–valued. In this case one usually works with the graph of the mapping instead of its epigraph. Hence, for a multifunction $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and for any $\bar{y} \in M(\bar{x})$ we define the coderivative $D^*M(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ at this point as

$$D^*M(\bar{x}, \bar{y})(y^*) = \{x^* | \ (x^*, -y^*) \in \mathrm{N}_{\mathrm{gph}\, M}(\bar{x}, \bar{y})\}$$

If M is single–valued, we write only $D^*M(\bar{x})(y^*)$ instead of $D^*M(\bar{x}, M(\bar{x}))(y^*)$. If M is single–valued and smooth, then its coderivative amounts to the adjoint Jacobian.

A multifunction $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has the Aubin property at $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M$ if there exist a nonnegative modulus L and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that for all $x, x' \in U$ the following inclusion holds true

$$M(x) \cap V \subset M(x') + L\|x - x'\| B(0, 1), \tag{50}$$

where $B(0, 1) \subset \mathbb{R}^m$ is the unit ball. If M is single–valued on some neighborhood of $\bar{x}$, then the Aubin property reduces to the local Lipschitzian property. Since this will be mostly the case in the paper, the reader may interchange these two properties. The infimum of all L satisfying (50) for some U and V will be called the modulus of Aubin property or Lipschitzian modulus for single–valued M.

For more information and properties of the abovementioned objects, we refer the reader again to [16] and [23].

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