

# PRIMAL-DUAL INTERIOR-POINT MULTIGRID METHOD FOR TOPOLOGY OPTIMIZATION\*

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**Abstract.** An interior point method for the structural topology optimization is proposed. The linear systems arising in the method are solved by the conjugate gradient method preconditioned by geometric multigrid. The resulting method is then compared with the so-called optimality condition method, an established technique in topology optimization. This method is also equipped with the multigrid preconditioned conjugate gradient algorithm. We conclude that, for large scale problems, the interior point method with an inexact iterative linear solver is superior to any other variant studied in the paper.

*Keywords:* topology optimization, multigrid methods, interior point methods, preconditioners for iterative methods

*MSC2010:* 65N55, 35Q93, 90C51, 65F08

**1. Introduction.** The discipline of topology optimization offers challenging problems to researchers working in large scale numerical optimization. The results are essentially colors of pixels in a 2d or 3d “pictures”. Hence, in order to obtain high-quality results, i.e., fine pictures capturing all details, a very large number of variables is essential. In this article we only consider the discretized, finite dimensional topology optimization problem. For its derivation and for general introduction to topology optimization, see, e.g., [5].

We will consider the basic problem of topology optimization: minimization of compliance under equilibrium equation constraints and the most basic linear constraints on the design variables:

$$\begin{aligned} & \min_{x \in \mathbb{R}^m, u \in \mathbb{R}^n} \frac{1}{2} f^T u \\ & \text{subject to} \\ & K(x)u = f \\ & \sum_{i=1}^m x_i = V \\ & x_i \geq 0, \quad i = 1, \dots, m \\ & x_i \leq \bar{x}, \quad i = 1, \dots, m \end{aligned} \tag{1.1}$$

where  $K(x) = \sum_{i=1}^m x_i K_i$ ,  $K_i \in \mathbb{R}^{n \times n}$  and  $f \in \mathbb{R}^n$ . We assume that  $K_i$  are symmetric and positive semidefinite and that  $\sum_{i=1}^m K_i$  is sparse and positive definite. We also assume that the data  $V \in \mathbb{R}$  and  $\bar{x} \in \mathbb{R}^m$  is chosen such that the problem is strictly feasible. For further reference, we will call the design variable  $x$  the *density*.

The most established and commonly used optimization methods to solve this problem are the Optimality Conditions (OC) method ([5, p.308]) and the Method of Moving Asymptotes (MMA) by Svanberg [24]. In both methods, the computational

\*This work has been partly supported by Iraqi Ministry of Higher Education and Scientific Research, Republic of Iraq and by the EU FP7 project no. 313781 AMAZE.

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bottleneck consists of the solution of a large scale linear system with a sparse symmetric positive definite matrix (the equilibrium equation). This is traditionally used by a direct solver, such as the Cholesky decomposition. Recently, several authors proposed the use of iterative solvers, mostly preconditioned Krylov subspace solvers, such as Conjugate Gradients (CG), MINRES or GMRES. These have one big advantage which is specific for their use within optimization algorithms: in the early (or even not-so-early) stages of the optimization method, only a very low accuracy of the linear solver is needed. They also have one big disadvantage: in the late stages of the optimization method, the linear solvers become very ill-conditioned and thus a vanilla iterative method can come into extreme difficulties.

It is therefore essential to use a good preconditioner for the Krylov subspace method. The difficulty lies in the fact that as we approach the optimal solution of the topology optimization problem, the condition number of the stiffness matrices increases significantly. In fact, it is only controlled by an artificial lower bound on the variable—if this bound was zero, the stiffness matrix would be singular. Wang et al. [26] studied the dependence of the condition number on the variables and concluded that it is a combination of the ratio of maximum and minimum density and the conditioning of a corresponding problem with constant density. Consequently, they proposed a rescaling of the stiffness matrix combined with incomplete Cholesky preconditioner. The rescaling results in constant order of condition number during the optimization iterations. For large scale example still hundreds of MINRES iterations are needed and hence the authors use recycling of certain Krylov subspaces from previous iterations of the optimization method. Recently, Amir et al. [2] proposed a multigrid preconditioner for the systems resulting from OC or MMA methods and demonstrated that the resulting linear system solver keeps its efficiency also for rapidly varying coefficient of the underlying PDE, i.e., rapidly varying  $x$  in (1.1).

While OC and MMA methods are the most popular methods in topology optimization, they may not be the most efficient. The basic problem (1.1) is convex (more precisely, it is equivalent to a convex problem) and we may thus expect interior point methods to be highly efficient (see, e.g., [21]). Indeed, Jarre et al. [16] proposed an interior point method for the truss topology optimization problem that is equivalent to the discretized problem (1.1), with the exception that the stiffness matrix may be dense. They reported high efficiency of the method and ability to solve large scale problems; they also proved convergence of the proposed method. Maar and Schulz [19] studied interior point methods for problem (1.1) with sparse stiffness matrices and proposed to use a multigrid preconditioner for the GMRES method to solve the arising indefinite linear systems.

A new comprehensive numerical study of optimization methods for topology optimization can be found in [22]. The authors compare the efficiency of different methods, including general purpose optimization solvers such as SNOPT [12].

In this article we follow the path outlined by Jarre et al. [16] and by Maar and Schulz [19]. We use the same interior point method as in [16] and, unlike in [19], reduce the linear systems to obtain positive definite matrices. This allows us to use standard conjugate gradient method preconditioned by standard V-cycle multigrid. We further use the same linear solver in the OC method (in the same way as suggested in [2]) to get a comparison with our interior point method. We will see that in both cases the inexact multigrid preconditioned CG method leads to a very efficient optimization solver. Most notably, in case of the interior point method we obtain an approximately constant number of CG iterations needed to solve the full problem

which is independent of the size of the problem. In case of the OC method, the total number of OC iterations is increasing with the problem size; however, for a given problem size, the number of CG steps per one linear systems remains almost constant, and very low, in all OC iterations, notwithstanding the condition number of the stiffness matrix.

In this paper, we primarily consider the so-called variable thickness sheet problem (1.1) and not its more popular cousin, the SIMP problem [5]. The reason is the (hidden) convexity and existence of solution of (1.1) (see [4] and [5, p.272–274]). The goal of the paper is to study and compare numerical methods for optimization problems. This can be done in a fair way if the problem is convex; by introducing non-convexity, as in the SIMP formulation, any such comparison is further influenced by many additional factors. To demonstrate these difficulties and the fact that the iterative solver is still a viable option in this context, we have added a brief section on the SIMP model.

Finally, the methodology proposed in this paper is fully based on (typically vectorizable and/or parallelizable) iterative schemes. It could thus be of benefit to the (re-)emerging methods of distributed optimization [7] and optimization on vector processors, namely GPU [23, 28], not only in the context of topology optimization.

**2. Newton systems for KKT conditions.** Let  $\mu \in \mathbb{R}^n$ ,  $\lambda \in \mathbb{R}$ ,  $\varphi \in \mathbb{R}^m$  and  $\psi \in \mathbb{R}^m$  denote the respective Lagrangian multipliers for constraints in (1.1). The Karush-Kuhn-Tucker (KKT) first order optimality conditions for (1.1) can be written as

$$-\text{Res}^{(1)} := K(x)u - f = 0 \quad (2.1)$$

$$-\text{Res}^{(2)} := \sum_{i=1}^m x_i - V = 0 \quad (2.2)$$

$$-\text{Res}^{(3)} := -\frac{1}{2}u^T K_i u - \lambda - \varphi_i + \psi_i = 0, \quad i = 1, \dots, m \quad (2.3)$$

$$\varphi_i x_i = 0, \quad i = 1, \dots, m \quad (2.4)$$

$$\psi_i(\bar{x} - x_i) = 0, \quad i = 1, \dots, m \quad (2.5)$$

$$x_i \geq 0, \quad \bar{x} - x_i \geq 0, \quad \varphi_i \geq 0, \quad \psi \geq 0 \quad (2.6)$$

We will perturb the complementarity constraints (2.4) and (2.5) by barrier parameters  $s, r > 0$ :

$$-\text{Res}^{(4)} := \varphi_i x_i - s = 0, \quad i = 1, \dots, m \quad (2.7)$$

$$-\text{Res}^{(5)} := \psi_i(\bar{x} - x_i) - r = 0, \quad i = 1, \dots, m \quad (2.8)$$

and apply Newton's method to the system of nonlinear equations (2.1), (2.2), (2.3), (2.7), (2.8). In every step of the Newton method, we have to solve the linear system

$$\begin{bmatrix} K(x) & 0 & B(u) & 0 & 0 \\ 0 & 0 & e^T & 0 & 0 \\ B(u)^T & e & 0 & I & -I \\ 0 & 0 & \Phi & X & 0 \\ 0 & 0 & -\Psi & 0 & \tilde{X} \end{bmatrix} \begin{bmatrix} d_u \\ d_\lambda \\ d_x \\ d_\varphi \\ d_\psi \end{bmatrix} = \begin{bmatrix} \text{Res}^{(1)} \\ \text{Res}^{(2)} \\ \text{Res}^{(3)} \\ \text{Res}^{(4)} \\ \text{Res}^{(5)} \end{bmatrix}. \quad (2.9)$$

Here  $B(u) = (K_1 u, K_2 u, \dots, K_m u)$ ,  $e$  is a vector of all ones and

$$X = \text{diag}(x), \quad \tilde{X} = \text{diag}(\bar{x} - x), \quad \Phi = \text{diag}(\varphi), \quad \Psi = \text{diag}(\psi)$$

are diagonal matrices with the corresponding vectors on the diagonal.

Because the last two equations only involve diagonal matrices, we can eliminate  $d_\varphi$  and  $d_\psi$ :

$$d_\varphi = X^{-1}(\text{Res}^{(4)} - \Phi d_t) \quad (2.10)$$

$$d_\psi = \tilde{X}^{-1}(\text{Res}^{(5)} - \Psi d_t). \quad (2.11)$$

This will reduce the system (2.9) to

$$\begin{bmatrix} K(x) & 0 & B(u) \\ 0 & 0 & e^T \\ B(u)^T & e & -(X^{-1}\Phi + \tilde{X}^{-1}\Psi) \end{bmatrix} \begin{bmatrix} d_u \\ d_\lambda \\ d_x \end{bmatrix} = \begin{bmatrix} \text{Res}^{(1)} \\ \text{Res}^{(2)} \\ \widetilde{\text{Res}}^{(3)} \end{bmatrix} \quad (2.12)$$

with

$$\widetilde{\text{Res}}^{(3)} = \text{Res}^{(3)} - X^{-1}\text{Res}^{(4)} + \tilde{X}^{-1}\text{Res}^{(5)}.$$

We can now follow two strategies. Firstly, we can solve the system (2.12) as it is, i.e., an indefinite system of dimension  $m + n + 1$ . To simplify things, we can still eliminate the multipliers  $\varphi$  and  $\psi$  as

$$\varphi_i = s/x_i, \quad \psi_i = r/(\bar{x} - x_i), \quad i = 1, \dots, m$$

to get

$$\begin{bmatrix} K(x) & 0 & B(u) \\ 0 & 0 & e^T \\ B(u)^T & e & -(sX^{-2} + r\tilde{X}^{-2}) \end{bmatrix} \begin{bmatrix} d_u \\ d_\lambda \\ d_x \end{bmatrix} = \begin{bmatrix} \text{Res}^{(1)} \\ \text{Res}^{(2)} \\ \widetilde{\text{Res}}^{(3)} \end{bmatrix}. \quad (2.13)$$

**Remark.** System (2.13) could be obtained directly as a Newton system for optimality conditions of the following “penalized” problem:

$$\begin{aligned} \min_u \quad & \frac{1}{2} f^T u + s \sum_{i=1}^m \log x_i + r \sum_{i=1}^m \log(\bar{x} - x_i) \\ \text{s.t.} \quad & K(x)u = f, \quad \sum_{i=1}^m x_i = V; \end{aligned}$$

see, e.g., [21, Ch.19.1].

Secondly, we can further reduce the Newton system (2.12). As the (3,3)-block matrix in (2.12) is diagonal, we will compute the Schur complement to the leading block to get

$$Z \begin{bmatrix} d_u \\ d_\lambda \end{bmatrix} = \text{Res}^{(Z)}, \quad (2.14)$$

with

$$Z = \begin{bmatrix} K(x) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B(u) \\ e^T \end{bmatrix} (X^{-1}\Phi + \tilde{X}^{-1}\Psi)^{-1} \begin{bmatrix} B(u)^T & e \end{bmatrix} \quad (2.15)$$

and

$$\text{Res}^{(Z)} = \begin{bmatrix} \text{Res}^{(1)} \\ \text{Res}^{(2)} \end{bmatrix} + \begin{bmatrix} B(u) \\ e^T \end{bmatrix} (X^{-1}\Phi + \tilde{X}^{-1}\Psi)^{-1} \widetilde{\text{Res}}^{(3)}. \quad (2.16)$$

The remaining part of the solution,  $d_x$ , is then computed by

$$d_x = (X^{-1}\Phi + \tilde{X}^{-1}\Psi)^{-1} \left( \widetilde{\text{Res}}^{(3)} - B^T \text{Res}^{(1)} - e \text{Res}^{(2)} \right). \quad (2.17)$$

**3. Interior point method.** Once we have derived the Newton systems, the interior point algorithm is straightforward (see, e.g., [21, Ch.19]). The details of the individual steps of the algorithm will be given in subsequent paragraphs.

**3.1. The algorithm.** Denote  $z = (u, \lambda, x, \varphi, \psi)^T$ . Set  $x_i = V/m$ ,  $i = 1, \dots, m$ ,  $u = K(x)^{-1}f$ ,  $\lambda = 1$ ,  $\varphi = e$ ,  $\psi = e$ . Set  $s = 1$ ,  $r = 1$ ,  $\sigma_s, \sigma_r \in (0, 1)$ . Do until convergence:

1. Solve either system (2.12) or (2.14) and compute the remaining components of vector  $d$  from (2.9).
2. Find the step length  $\alpha$ .
3. Update the solution

$$z = z + \alpha d.$$

4. If the stopping criterium for the Newton method is satisfied, update the barrier parameters

$$s = \sigma_s \cdot s, \quad r = \sigma_r \cdot r.$$

Otherwise, keep current values of  $s$  and  $r$ .

Return to Step 1.

**3.2. Barrier parameter update.** We use a fixed update of both parameters  $s$  and  $r$  with

$$\sigma_s = \sigma_r = 0.2.$$

This update leads to long steps and, consequently, small number of interior point iterations. The value of the update parameter is a result of testing and leads, in average, to the smallest overall number of Newton steps. A more sophisticated version of the algorithm, with an adaptive choice of the barrier parameters  $s$  and  $r$  can be found in [16].

**3.3. Step length.** We cannot take the full Newton step

$$z_{\text{new}} = z + d$$

because some variables could become infeasible with respect to the inequality constraints (2.6). We thus need to shorten the step in order to stay strictly feasible with some “buffer” to the boundary of the feasible domain. A simple step-length procedure is described below (see also [21, Ch.19.2]).

Find  $\alpha_l$  such that  $x_i + (d_x)_i > 0$  for  $i \in \{j : (d_x)_j < 0\}$  and  $\alpha_u$  such that  $x_i + (d_x)_i < \bar{x}$  for  $i \in \{j : (d_x)_j > 0\}$  using the following formulas:

$$\alpha_l = 0.9 \cdot \min_{i: (d_x)_i < 0} \left\{ -\frac{x_i}{(d_x)_i} \right\}, \quad \alpha_u = 0.9 \cdot \min_{i: (d_x)_i > 0} \left\{ \frac{\bar{x} - x_i}{(d_x)_i} \right\}.$$

The constant 0.9 guarantees the shortening of the step in the interior of the feasible domain. Now take the smaller of these numbers and, if applicable, reduce it to 1:

$$\alpha = \min\{\alpha_l, \alpha_u, 1\}.$$

A more sophisticated (and complicated) line-search procedure is described in [16].

It is worth noticing that for a properly chosen initial barrier parameter and its update, the step-length reduction is almost never needed; this was, at least, the case of our numerical examples and our choice of the parameters.

**3.4. Stopping rules.** Following [16], we terminate the Newton method whenever

$$\frac{\|\text{Res}^{(1)}\|}{\|f\|} + \frac{\|\widetilde{\text{Res}}^{(3)}\|}{\|\varphi\| + \|\psi\|} \leq \tau_{\text{NWT}}.$$

The full interior point method is stopped as soon as both parameters  $s$  and  $r$  are smaller than a prescribed tolerance:

$$\max\{s, r\} \leq \tau_{\text{IP}}. \quad (3.1)$$

In our numerical experiments, we have used the values  $\tau_{\text{NWT}} = 10^{-1}$  and  $\tau_{\text{IP}} = 10^{-8}$ .

REMARK 3.1. A more established criterium for terminating the interior point algorithm would be to stop whenever all (scaled) residua are below some tolerance, i.e.,

$$\frac{\|\text{Res}^{(1)}\|}{\|f\|} + \frac{\|\widetilde{\text{Res}}^{(3)}\|}{\|\varphi\| + \|\psi\|} + \frac{\varphi^T x}{\|\varphi\|\|x\|} + \frac{\psi^T(\bar{x} - x)}{\|\varphi\|\|x\|} \leq \tau_{\text{IP}}.$$

This criterium, however, leads to almost the same results as (3.1), hence we opted for the simpler and more predictable one.

REMARK 3.2. The parameter  $\tau_{\text{NWT}}$  is kept constant in our implementation, unlike in classic path-following methods. We will return to this point later in Section 8.1.

**4. Optimality Conditions method.** One of the goals of this paper is to compare the interior point method with the established and commonly used Optimality Condition (OC) method. We will therefore briefly introduce the basic algorithm and its new variant. For more details, see ([5, p.308]) and the references therein.

**4.1. OC algorithm.** Assume for the moment that the bound constraints in (1.1) are not present. Then the KKT condition (2.3) would read as

$$-u^T K_i u + \lambda = 0, \quad i = 1, \dots, m.$$

(For convenience, we multiplied  $\lambda$  from (2.3) by  $-\frac{1}{2}$ .) Multiplying both sides by  $x_i$ , we get

$$x_i \lambda = x_i u^T K_i u, \quad i = 1, \dots, m$$

which leads to the following iterative scheme:

$$x_i^{\text{NEW}} = \frac{1}{\lambda} x_i u^T K_i u, \quad i = 1, \dots, m.$$

The new value of  $x$  is then projected on the feasible set given by the bound constraints. The value of  $\lambda$  should be chosen such that  $\sum_{i=1}^m x_i^{\text{NEW}} = V$  and is obtained by a simple bisection algorithm. Hence we obtain the following algorithm called the OC method:

*Algorithm OC.* Let  $x \in \mathbb{R}^m$  be given such that  $\sum_{i=1}^m x_i = V$ ,  $x \geq 0$ . Repeat until convergence:

1.  $u = (K(x))^{-1}f$
2.  $\bar{\lambda} = 10000$ ,  $\underline{\lambda} = 0$
3. While  $\bar{\lambda} - \underline{\lambda} > \tau_\lambda$ 
  - (a)  $\lambda = (\bar{\lambda} + \underline{\lambda})/2$
  - (b)  $x_i^{\text{NEW}} = \min \left\{ x_i \frac{u^T K_i u}{\lambda}, \bar{x} \right\}, \quad i = 1, \dots, m$
  - (c)  $x = x^{\text{NEW}}$
  - (d) if  $\sum_{i=1}^m x_i > V$  then set  $\underline{\lambda} = \lambda$ ; else if  $\sum_{i=1}^m x_i \leq V$  then set  $\bar{\lambda} = \lambda$

The value of the bisection stopping criterium  $\tau_\lambda$  has been set to  $10^{-11}$ .

Notice that, due to positive semidefiniteness of  $K_i$ , the update in step 3(b) is always non-negative and thus the lower-bound constraint in the original problem (1.1) is automatically satisfied.

The basic version of the OC method converges (there are no known counter examples) but is extremely slow. The reason for this is that, from the very first iterations, the method is zig-zagging between two clusters of points. However, the following two modifications lead to a substantial improvement. To the best of our knowledge, the second modification called Averaged OC is new.

#### 4.2. Damped OC.

*Algorithm DOC.* Let  $x \in \mathbb{R}^m$  be given such that  $\sum x_i = V$ ,  $x \geq 0$ . Repeat:

1.  $u = (K(x))^{-1}f$
2.  $x_i^{\text{NEW}} = \min \left\{ x_i \frac{(u^T K_i u)^q}{\lambda}, \bar{x} \right\}, \quad i = 1, \dots, m$
3.  $x = x^{\text{NEW}}$

Here  $q$  is called the damping parameter; the typical choice is  $q = 1/2$ . This version of the method is widely used among the structural engineers.

**4.3. Averaged OC.** Let us define an operator  $OC(\cdot)$  as a result of one step of the standard OC algorithm.

*Algorithm AOC.* Let  $x \in \mathbb{R}^m$  be given such that  $\sum x_i = V$ ,  $x \geq 0$ . Repeat:

1.  $x^{(1)} = OC(x)$
2.  $x^{(2)} = OC(x^{(1)})$
3.  $x = \frac{1}{2}(x^{(1)} + x^{(2)})$

Numerical experiments suggest that Algorithm AOC is slightly faster than Algorithm DOC. This modification seems to be new, at least we did not find it in the existing literature.

**5. Multigrid conjugate gradient method.** In both optimization algorithms introduced above, we repeatedly need to solve systems of linear equations. In this section, we will introduce an efficient iterative method that seems to be most suitable for these problems. Throughout this section, we assume that we want to solve the problem

$$Az = b \tag{5.1}$$

where  $b \in \mathbb{R}^n$  and  $A$  is a  $n \times n$  symmetric positive definite matrix.

**5.1. Multigrid method for linear systems.** Recall first the Correction Scheme (CS) version of the multigrid algorithm (see, e.g., [15]). Let *opt* denote a (typically

but not necessarily) convergent iterative algorithm for (5.1):

$$z_{\text{new}} = \text{opt}(A, b; z, \epsilon, \nu),$$

where, on input,  $z$  is the initial approximation of the solution,  $\epsilon$  is the required precision and  $\nu$  the maximum number of iterations allowed. This will be called the *smoother*. A typical example is the Gauss-Seidel iterative method.

Assume that there exist  $\ell$  linear operators  $I_k^{k-1} : \mathbb{R}^{n_k} \rightarrow \mathbb{R}^{n_{k-1}}$ ,  $k = 2, \dots, \ell$ , with  $n := n_\ell > n_{\ell-1} > \dots > n_2 > n_1$  and let  $I_{k-1}^k := (I_k^{k-1})^T$ . These are either constructed from finite element or finite difference refinements of some original coarse grid (geometric multigrid) or from the matrix  $A$  (algebraic multigrid); see [8] for details.

Define the “coarse level” problems

$$A_k z_k = b_k, \quad k = 1, \dots, \ell - 1$$

with

$$A_{k-1} = I_k^{k-1}(A_k)I_{k-1}^k, \quad b_{k-1} = I_k^{k-1}(b_k), \quad k = 2, \dots, \ell.$$

*Algorithm MG.* (V-cycle correction scheme multigrid)

Set  $\epsilon, \epsilon_0$ . Initialize  $z^{(\ell)}$ .

for  $i = 1 : \text{niter}$

$$z^{(\ell)} := \text{mgm}(\ell, z^{(\ell)}, b_\ell)$$

test convergence

end

function  $z^{(k)} = \text{mgm}(k, z^{(k)}, r_k)$

if  $k = 0$

$$z^{(k)} := \text{opt}(A_1, b_1; z^{(k)}, \epsilon_0, \nu_0) \quad (\text{coarsest grid solution})$$

else

$$z^{(k)} := \text{opt}(A_k, b_k; z^{(k)}, \epsilon, \nu_1) \quad (\text{pre-smoothing})$$

$$r_{k-1} = I_k^{k-1}(r_k - A_k z^{(k)}) \quad (\text{restricted residuum})$$

$$v^{(k-1)} = \text{mgm}(k-1, 0_{n_{k-1}}, r_{k-1}) \quad (\text{coarse grid correction})$$

$$z^{(k)} := z^{(k)} + I_{k-1}^k v^{(k-1)} \quad (\text{solution update})$$

$$z^{(k)} := \text{opt}(A_k, b_k; z^{(k)}, \epsilon, \nu_2) \quad (\text{post-smoothing})$$

end

**5.2. Multigrid preconditioned conjugate gradient method.** Although the multigrid method described above is very efficient, an even more efficient tool for solving (5.1) may be the preconditioned conjugate gradient (CG) method, whereas the preconditioner consist of one step of the V-cycle multigrid method. The algorithm is described below (see, e.g., [13]).

*Algorithm PCG.*

Given initial  $z$ , set  $r := Az - b$

$$y := \text{mgm}(\ell, 0_n, r)$$

Set  $p := -y$

for  $i = 1 : \text{niter}$

$$\alpha := \frac{r^T y}{p^T A p}$$

$$z := z + \alpha p$$

$$\tilde{r} := r + \alpha A p$$

$$\tilde{y} := \text{mgm}(\ell, 0_n, \tilde{r})$$



$$\beta := \frac{\tilde{r}^T \tilde{y}}{r^T y}$$

$$p := -y + \beta p$$

$$r := \tilde{r}, y := \tilde{y}$$

test convergence

end

**6. Multigrid conjugate gradients for IP and OC methods.** The main goal of this section (and of the whole article) is to study the effect of the multigrid preconditioned CG method in the IP and OC algorithms. We will also compare them to their counterparts, IP and OC with direct solvers.

The details on discretization and the choice of prolongation and restriction operators will be given in Section 7.

**6.1. Multigrid conjugate gradients for IP.** Our goal is to solve the linear systems arising in the Newton method, by the conjugate gradient method preconditioned by one V-type multigrid step. We can choose one of the three equivalent systems to solve, namely the full system (2.9), the reduced saddle-point system (2.12) and the so-called augmented system (2.14). We prefer the last one for the following reasons.

- The matrix  $Z$  in (2.14) is positive definite and we can thus readily apply the standard conjugate gradient method together with the standard V-cycle as a preconditioner. We could, of course, use GMRES or MINRES for the indefinite systems in (2.9) and (2.12), however, the multigrid preconditioner, in particular the smoother, would become more complicated in this case; see [19], who used so-called transforming smoothers introduced by Wittum [27].
- In order to use the multigrid preconditioner, we have to define prolongation/restriction operators for the involved variables. This can be easily done in case of the system (2.14) that only involves the displacement variable  $u \in \mathbb{R}^n$  plus one additional variable  $\lambda$ , the Lagrangian multiplier associated with the volume constraint; see the next Section 7 for details.

If, on the other hand, we decided to solve (2.9) or (2.12), we would have to select an additional restriction operator for the variables associated with the finite elements; this operator should then be “compatible” with the nodal-based restriction operator. This is a rather non-trivial task and can be simply avoided by choosing system (2.14).

The matrix  $Z$  from (2.14) is positive definite, sparse and typically has an arrow-type sparsity structure: it is banded apart from the last full row and column; see Figure 6.1-left. The bandwidth grows, approximately, with the square root of the problem size. At the same time, the number of non-zeros in each row is always the same, notwithstanding the problem size.

*Stopping rule.* It is a big advantage of iterative methods, over direct solvers, that they allow us to control the precision of the approximate solution and stop whenever even a low required precision is reached. In our implementation, the PCG method is stopped whenever

$$\|\rho\| \|b\| \leq 10^{-2} \quad (6.1)$$

where  $\rho$  is residuum and  $b$  the right-hand side of the linear system, respectively. In this way we only compute an approximate Newton direction; it is shown, e.g., in [11] that the resulting method converges once the approximate Newton direction is “close

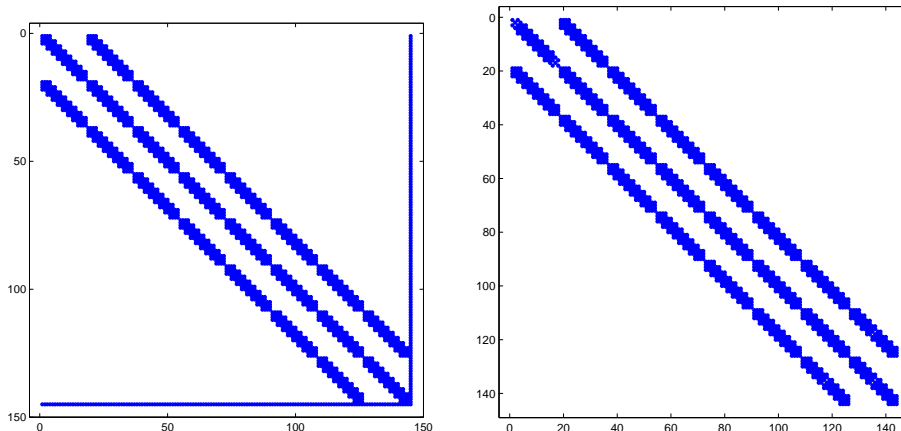


FIG. 6.1. Typical sparsity structure of matrix  $Z$  from the augmented system (2.14) (left) and of the stiffness matrix  $K$  (right)

enough” (though not infinitesimally close in the limit) to the exact solution of the Newton system. Furthermore, for convex quadratic programming problems, Gondzio [14] has shown that when the PCG method is stopped as soon as  $\|\rho\| \leq 0.05s$  ( $s$  being the barrier parameter), the theoretical complexity of the interior point method is the same as with the exact linear solver. Inexact iterative solvers in the context of other optimization problems and algorithms were further studied, e.g., in [10, 17, 20, 25].

In our case, the value of  $10^{-2}$  proved to be a good compromise between the overall number of Newton steps and the overall number of PCG iterations within the IP method. With this stopping criterium, the IP method requires, typically, 2–4 PCG iterations in the initial and in many subsequent IP steps. Only when we get close to the required accuracy, in the last 2–3 IP steps, the conditioning of the matrix  $Z$  increases significantly and so does the number of PCG steps, typically to 10–30; see the next section for detailed numerical results.

**6.2. Multigrid conjugate gradients for OC.** Within the OC algorithm, the multigrid CG method will be used to solve the discretized equilibrium equation  $Ku = f$ . Recall that  $K$  is assumed to be a positive definite matrix. Moreover  $K$  is very sparse and, if a reasonably good numbering of the nodes is used, banded. A typical non-zero structure of  $K$  is shown in Figure 6.1-right: it is exactly the same as for the matrix in (2.14) in the IP method, apart from the additional last column and row in the augmented matrix in (2.14).

The only degrees of freedom in the resulting algorithm are the stopping criteria for the OC method and for the multigrid CG method.

*The overall stopping criterium.* As the dual information (Lagrangian multipliers associated with the bound constraints) is not readily available, so far the only practical (and widely used) stopping criterium for the OC method is the difference in the objective function value in two subsequent iterations. Needless to say that, unless we have an estimate for the rate of convergence, this criterium can be misleading and may terminate the iteration process long before some expected approximation of the optimum has been reached. Nevertheless, many numerical experiments suggest that this criterium is not as bad as it seems and serves its purpose for the OC method.

Hence the OC method is typically stopped as soon as

$$|f^T u_k - f^T u_{k-1}| \leq \tau_{\text{OC}} \quad (6.2)$$

where  $k$  is the iteration index. In our numerical experiments we have used  $\tau_{\text{OC}} = 10^{-5}$ ; this value has been chosen such that the OC results are comparable to the IP results, in the number of valid digits both in the objective function and in the variables; see Section 8 for more details.

*Stopping criterium for the multigrid CG method.* As already mentioned above, one of the advantages of an iterative method is the fact that an exact solution to the linear system is not always needed. In such a case, we can stop the iterative method after reaching a relatively low accuracy solution. The required accuracy of these solutions (such that the overall convergence is maintained) is well documented and theoretically supported in case of the IP method; it is, however an unknown in case of the OC method; see [2] for detailed discussion. Clearly, if the linear systems in the OC method are solved too inaccurately, the whole method may diverge or just oscillate around a point away from the solution.

We have opted for the following heuristics that guarantees the (assumed) overall convergence of the OC method. Notice that the OC method is a feasible descent algorithm. That means that every iteration is feasible and the objective function value in the  $k$ -th iteration is smaller than that in the  $(k-1)$ -st iteration. Hence

- we start with  $\tau = 10^{-4}$ ;
- if  $f^T u_k > f^T u_{k-1}$ , we update  $\tau := 0.1 \tau$ .

In our numerical tests, the update had to be done only in few cases and the smallest value of  $\tau$  needed was  $\tau = 10^{-6}$ . Recall that this is due to our relatively mild overall stopping criterium (6.2). In the next section, we will see that this heuristics serves its purpose, as the number of OC iterations is almost always the same, whether we use an iterative or a direct solver for the linear systems.

**7. Numerical experiments.** This section contains detailed results of three numerical examples. All codes were written entirely in MATLAB. Notice, however, that when we refer to a direct solver for the solution of linear system, we mean the backslash operation in MATLAB which, for our symmetric positive definite systems, calls the CHOLMOD implementation of the Cholesky method [9]. This implementation is highly tuned, very efficient and written in the C language. So whenever we compare CPU times of the iterative solver with the direct solver, we should keep this in mind. These comparisons are given solely to show the tendency in the CPU time when increasing the problem size. All problems were solved on an Intel Core i5-3570 CPU at 3.4GHz with 8GB RAM, using MATLAB version 8.0.0 (2012b) running in 64 bit Windows 7.

In all examples, we use square finite elements with bilinear basis functions for the displacement variable  $u$  and constant basis functions for the thickness variable  $x$ , as it is standard in topology optimization. The prolongation operators  $I_{k-1}^k$  for the variable

$u$  are based on the nine-point interpolation scheme defined by the stencil  $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$ ;

see, e.g., [15]. When solving the linear system (2.14) in the interior point method, we also need to prolong and restrict the single additional variable  $\lambda$ ; here we simply use the identity.

The examples are solved with isotropic material with Young's modulus equal to 1 and Poisson's ratio 0.3. The physical dimensions of the computational domain are

given by the coarsest mesh, whereas the coarse level element has dimension  $1 \times 1$ . The upper bound on the variable  $x$  is set to  $\bar{x} = 2$ . The load is always defined on the finest discretization level on edges of two elements sharing a node on the boundary specified in each example. The load always acts in vertical direction. Thus the non-zero elements of the discretized load vector will be  $(-\frac{1}{2}, -1, -\frac{1}{2})$ , associated with the vertical components of the specified boundary nodes its two immediate neighbours on this boundary.

The meaning of the captions in the following tables:

**problem.**... the first two numbers describe the dimension of the computational domain, the last number is the number of mesh refinements

**variables.**... number of variables in the linear systems

**feval.**... total number of function evaluations (equal to the number of linear systems solved)

**total CG iters.**... total number of CG iterations in the optimization process

**solver CPU time.**... total CPU time spent in the solution of linear systems

**average CG iters.**... average number of CG iterations per one linear system

**7.1. Example 1.** We consider a square computational domain with the coarsest mesh consisting of  $2 \times 2$  elements. All nodes on the left-hand side are fixed, the right-hand middle node is subject to a vertical load; see Figure 7.1. We use up to nine refinements levels with the finest mesh having 262 144 elements and 525 312 nodal variables (after elimination of the fixed nodes).

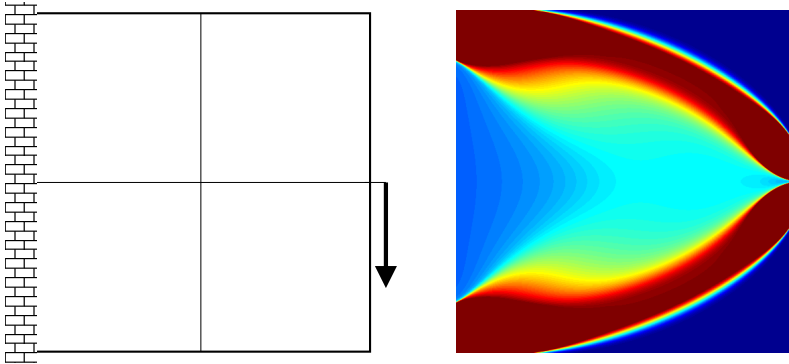


FIG. 7.1. *Example 1, initial setting with coarsest mesh and optimal solution.*

Table 7.1 presents the results of the interior point method. We can see that, with increasing size of the problem, the total number of CG iterations is actually decreasing. This is due to our specific stopping criterium explained in the previous section. We also observe that the average number of CG iterations per linear system is very low and, in particular, *is not increasing with the problem size*, the result of the multigrid preconditioner.

Let us now compare these results with those for the OC method where the linear system is just the equilibrium equation; see Table 7.2. As expected, the number of OC iterations (and thus the number of linear systems and the total number of CG iterations) *grows* with the size of the problem. Also in this case the average number of CG iterations is almost constant, notwithstanding the size of the problem.

The comparison of the interior point method with the OC method is graphically presented in Figure 7.2 (left). Here we can see, in the log-log scale, the total CPU time

TABLE 7.1  
Example 1, interior point method with iterative solver

problem	variables	feval	total CG iters	solver CPU time	average CG iters
223	145	31	253	0.18	8.16
224	545	30	281	0.44	9.37
225	2 113	29	197	0.91	6.79
226	8 321	28	139	2.79	4.96
227	33 025	27	119	12.7	4.41
228	131 585	25	104	45.8	4.16
229	525 313	27	85	156.0	3.15

TABLE 7.2  
Example 1, OC method with iterative solver

problem	variables	feval	total CG iters	solver CPU time	average CG iters
223	144	19	56	0.04	2.95
224	544	33	100	0.14	3.03
225	2 112	55	164	0.65	2.98
226	8 320	85	254	4.84	2.99
227	33 024	111	332	30.8	2.99
228	131 584	119	362	133.0	3.04
229	525 312	123	368	636.0	2.99

spent in the linear solver, growing with the size of the problem. While initially worse than the OC method, the interior point method grows slower and soon catches up and overtakes the OC method. For both methods, the growth is almost linear for the larger problems, so that we can estimate the growth in the CPU time as a polynomial function  $cn^d$  of the problem dimension  $n$ . For the interior point method, the degree  $d = 0.907$  while for the OC method  $d = 1.09$ . This means that the overall computational complexity of the IP method with inexact Newton and inexact multigrid CG methods is slightly *sublinear*. For the OC method, it is just a bit worse than linear.

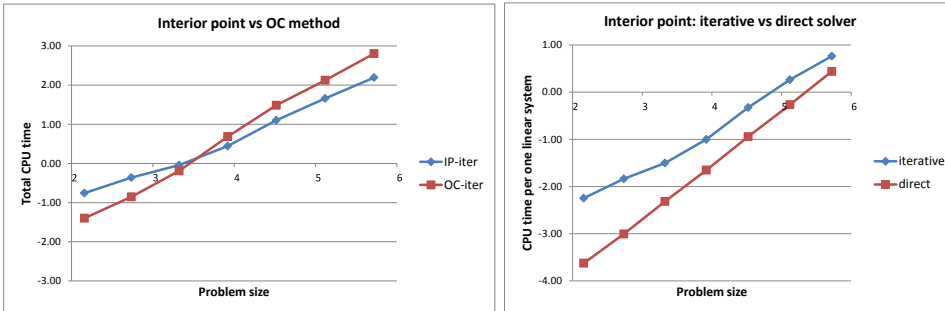


FIG. 7.2. Example 1, left: total CPU time spent in the iterative linear solver for the interior point and the OC method; right: interior point method, total CPU time spent in the iterative linear solver and in the direct solver.

In Figure 7.2 (right) we compare the iterative solver used in the interior point method with a direct Cholesky solver (see the warning at beginning of this section!). We can clearly see that the time for the (C coded) direct solver grows quicker than for the (MATLAB coded) iterative solver.

**7.2. Example 2.** The next example is similar to the previous one, only the computational domain is “longer” in the horizontal direction; the coarsest mesh consists of  $4 \times 2$  elements. It is well known that the conditioning of this kind of examples grows with the slenderness of the domain. As before, all nodes on the left-hand side are fixed, the right-hand middle node is subject to a vertical load; see Figure 7.3. Again, we use up to nine refinements levels with the finest mesh having 524 288 elements and 1 050 624 nodal variables (after elimination of the fixed nodes).

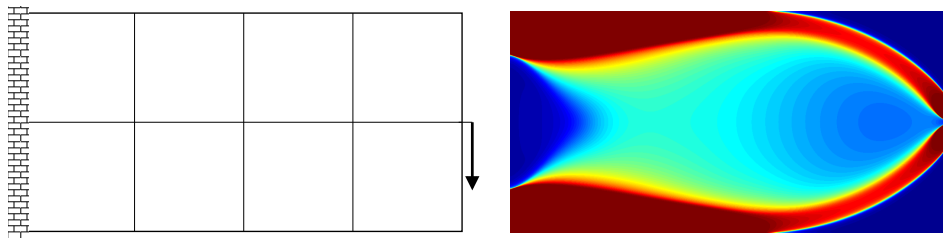


FIG. 7.3. *Example 2, initial setting with coarsest mesh and optimal solution.*

We first show the results of the interior point method in Table 7.3. Just as in the previous example, the total number of CG iterations is decreasing with the increasing size of the problem. Again, the average number of CG iterations per linear system is very low and not increasing. Compare this with the OC solver results in Table 7.4.

TABLE 7.3  
*Example 2, IP method with iterative solver*

problem	variables	feval	total CG iters	solver CPU time	average CG iters
423	288	33	265	0.24	8.03
424	1 088	32	342	0.87	10.69
425	4 224	31	207	1.89	6.68
426	16 640	30	160	7.77	5.33
427	66 048	29	139	31.1	4.79
428	263 168	27	123	119.0	4.56
429	1 050 624	27	101	385.0	3.74

In this case, we only consider eight refinement levels, as the largest problem would take too much time on our computer. Contrary to the previous example, the average number of CG iterations is slightly increasing due to the worse conditioning.

Figure 7.4 (left) gives the comparison of the interior point with the OC method. We can see even more clearly than in the previous example the faster growth of the OC method. When we calculate the degree of the assumed polynomial function  $cn^d$  of the problem dimension  $n$  from the larger examples, we will obtain  $d = 0.944$  for the interior point method (so a linear growth) and  $d = 1.28$  for the OC method.

TABLE 7.4  
Example 2, OC method with iterative solver

problem	variables	feval	total CG iters	solver CPU time	average CG iters
423	288	39	117	0.13	3.00
424	1 088	45	144	0.34	3.20
425	4 224	77	262	2.10	3.40
426	16 640	123	423	16.2	3.44
427	66 048	157	542	97.1	3.45
428	263 168	165	739	552	4.48

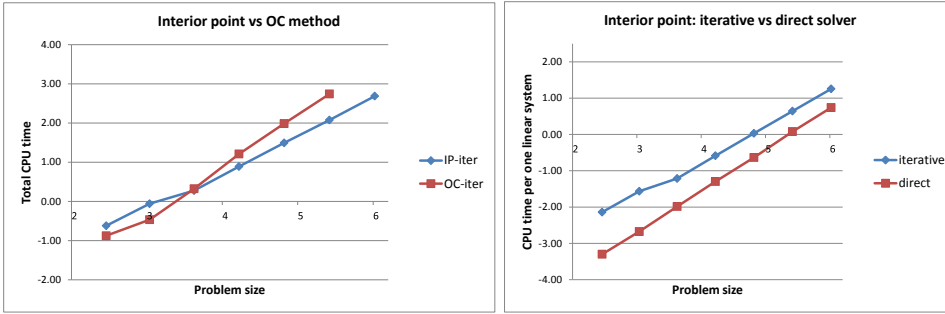


FIG. 7.4. Example 2, left: total CPU time spent in the iterative linear solver for the interior point and the OC method; right: interior point method, total CPU time spent in the iterative linear solver and in the direct solver.

Figure 7.4 (right) compares the iterative solver used in the interior point method with the Cholesky solver (see the beginning of this section), giving the same picture as in the previous example.

Finally in Figure 7.5 we compare the average number of CG steps per linear system in the interior point and the OC solver. We can see that while the graph is decreasing for the IP method, it is slowly increasing in case of the OC method. The reason for that is that, in this example, we had to decrease the stopping criterium for the CG solver in the OC method, in order to guarantee its convergence (see Section 6.2 for explanation).

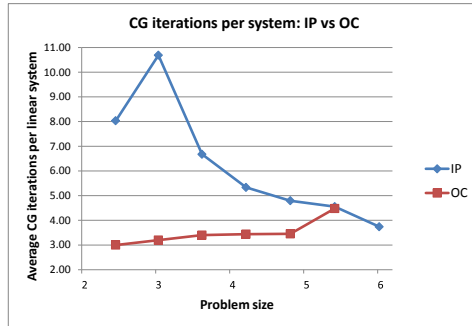


FIG. 7.5. Example 2, average number of CG iterations per linear system for the interior point and the OC method.

**7.3. Example 3.** The computational domain for our final example is a rectangle, initially discretized by  $8 \times 2$  finite elements. The two corner points on the lower edge are fixed and a vertical load is applied in the middle point of this edge; see Figure 7.6. We use up to eight refinement levels with the finest mesh having 262 144 elements and 568 850 nodal variables (after elimination of the fixed nodes).

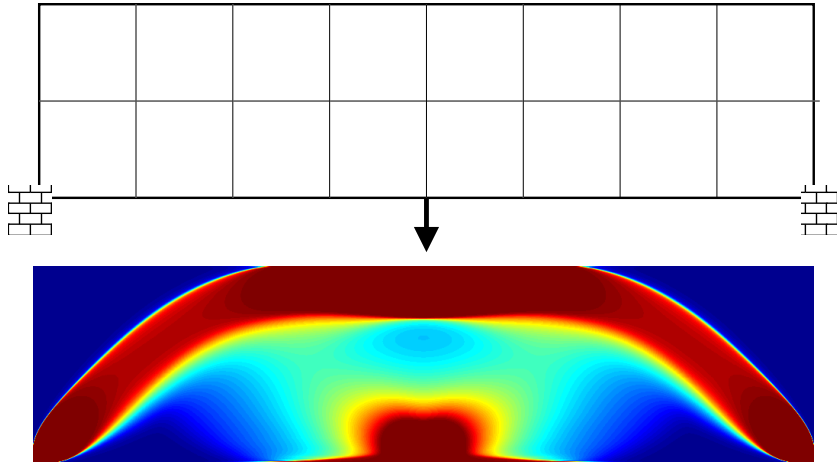


FIG. 7.6. *Example 3, initial setting with coarsest mesh and optimal solution.*

The results of the interior point method are shown in Table 7.5. Yet again, the total number of CG iterations is decreasing with the increasing size of the problem and the average number of CG iterations per linear system is very low and not increasing. The negative complexity factor is caused by the exceptional difficulties of the CG method in the last interior point step in problem 823.

TABLE 7.5  
*Example 3, IP method with iterative solver*

problem	variables	feval	total CG iters	solver CPU time	average CG iters
822	170	33	284	0.21	8.61
823	594	31	383	0.78	12.35
824	2210	32	121	0.60	3.78
825	8514	31	166	3.41	5.35
826	33410	26	140	14.8	5.38
827	132354	26	133	78.8	5.12
828	526850	25	121	217.0	4.84

Table 7.3 presents the results of the OC method. As in Example 2, the average number of CG iterations is increasing due to the worse conditioning.

Figure 7.7 (left) compares of the interior point with the OC method. Yet again, the interior point method is a clear winner, both in the absolute timing as in the growth tendency. Calculating the degree of the assumed polynomial function  $cn^d$  of the problem dimension  $n$  from the larger examples, we get  $d = 1.09$  for the interior point method and  $d = 1.24$  for the OC method.



TABLE 7.6  
Example 3, OC method with iterative solver

problem	variables	feval	total CG iters	solver CPU time	average CG iters
822	170	23	69	0.04	3.00
823	594	37	147	0.21	3.97
824	2210	57	267	1.16	4.68
825	8514	75	374	7.40	4.99
826	33410	99	495	51.8	5.00
827	132354	111	665	290.0	5.99
828	526850	113	677	1250.0	5.99

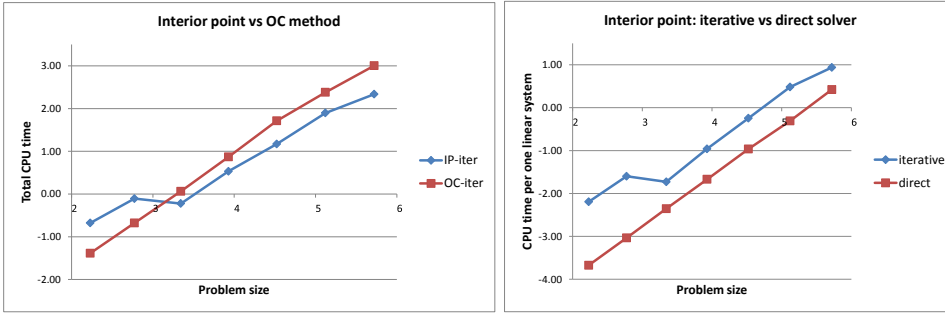


FIG. 7.7. Example 3, left: total CPU time spent in the iterative linear solver for the interior point and the OC method; right: interior point method, total CPU time spent in the iterative linear solver and in the direct solver.

In Figure 7.7 (right) we compare the iterative solver used in the interior point method with the Cholesky solver (see the beginning of this section). Finally in Figure 7.8 we compare the average number of CG steps per linear system in the interior point and the OC solver. We can see that while the graph for the IP method has a decreasing tendency, it is increasing in case of the OC method. As before, the reason for that is that we had to decrease the stopping criterium for the CG solver, in order to guarantee its convergence (see Section 6.2).

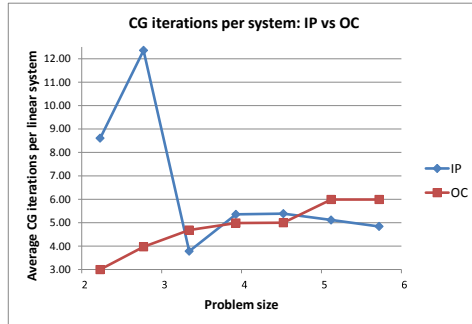


FIG. 7.8. Example 3, average number of CG iterations per linear system for the interior point and the OC method.

## 8. How exact is ‘exact’?

**8.1. Interior point method.** In this article we are using slightly nonstandard stopping criteria within the interior point method. In particular, with the decreasing barrier parameters  $r, s$  we do *not* decrease the stopping tolerances  $\tau_{\text{NWT}}$  and  $\tau_{\text{CG}}$  for the Newton method and for the conjugate gradients, respectively, although both are required for the theoretical convergence proof. In Figure 8.1 we try to give a schematic explanation. Here we depict the feasible region and three points  $x_1, x_2, x_3$  on the central path, corresponding to three values of the barrier parameter  $r_1 > r_2 > r_3$ . The exact solution lies in the corner of the feasible region. The circle around each of these points depicts the region of stopping tolerance of the Newton method, once we get within, the Newton method will stop. The radius of these circles is decreasing, even though  $\tau_{\text{NWT}}$  is kept constant. The idea is now obvious: it is “better” to stay within

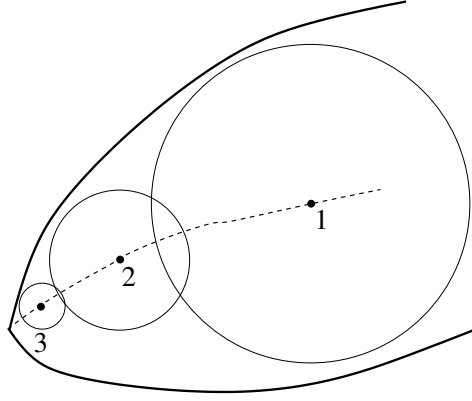


FIG. 8.1. *Illustration of interior-point method.*

the tolerance circle of  $x_3$  rather than to get very close to  $x_2$ .

In the lemma below,  $x^*$  is a point on the central path corresponding to a barrier parameter  $s$  and  $x$  an approximation of  $x^*$  resulting from inexact Newton method. We will show that, even with a fixed stopping criterium for the Newton method,  $x$  must converge to  $x^*$  with  $s$  going to zero. For simplicity of notation, we will just verify it for the lower bound complementarity part of  $\widetilde{\text{Res}}^{(3)}$ .

LEMMA 8.1. *Let  $x^* > 0$  satisfies the perturbed scaled complementary condition*

$$\frac{\varphi_i x_i^* - s}{x_i} = 0, \quad i = 1, \dots, m \quad (8.1)$$

*and let  $x > 0$  be an approximation of  $x^*$  satisfying*

$$\|z\| \leq \tau, \quad z_i = \frac{\varphi_i x_i - s}{x_i} \quad (8.2)$$

*with some  $\tau > 0$ . Then there is an  $\varepsilon > 0$  depending on  $s$  and  $\tau$  such that  $\|x^* - x\| \leq \varepsilon$ . Moreover, if  $s$  tends to zero then also  $\varepsilon$  tends to zero.*

*Proof.* From (8.1) we have that  $\varphi_i = \frac{s}{x_i^*}$  and thus (8.2) can be written as

$$\sum_{i=1}^m \left( \frac{s}{x_i^*} - \frac{s}{x_i} \right)^2 \leq \tau^2$$

which is, in particular, means that

$$\left| \frac{s}{x_i^*} - \frac{s}{x_i} \right| \leq \tau, \quad i = 1, \dots, m,$$

i.e.,

$$\frac{|x_i^* - x_i|}{x_i^* x_i} \leq s\tau, \quad i = 1, \dots, m.$$

Clearly, when  $s$  tends to zero,  $x$  must tend to  $x^*$ .  $\square$

How good solution can we get when replacing the (“exact”) direct solver by an inexact iterative method for the solution of the Newton systems? We may expect that, with the ever decreasing barrier parameter, the inexact version will get into numerical difficulties sooner than the exact one. Table 8.1 answers this question. In topology optimization, the important variable is  $x$ , the “density”. With lower bound equal to zero, the quality of the solution may be characterized by the closeness of components of  $x$  to this lower bound (that is, in examples where the lower bound is expected to be reached, such as in Example 1 with sufficiently fine discretization). In Table 8.1 we display the smallest component of  $x$ , denoted by  $x_{\min}$  for Example 1 with 6 refinements levels, i.e., example 226 from Table 7.2. The meaning of other columns in Table 8.1 is the following:

- barrier**... the smallest value of the barrier parameters  $s, r$  before the interior point algorithm was terminated;
- IP,NWT,CG**... the total number of iterations of the interior point method, the Newton method and conjugate gradients, respectively;
- Cholesky**... the linear system was solved by the CHOLMOD implementation of the Cholesky method;
- CG tol fixed**... the linear system was solved by the multigrid preconditioned conjugate gradient method with a fixed stopping criterium  $\|r\| \|b\| \leq 10^{-2}$ ; see (6.1);
- CG tol decreasing**... as above but with a variable stopping criterium  $\|r\| \|b\| \leq \tau_{\text{CG}}$ , where  $\tau_{\text{CG}}$  is initially equal to  $10^{-2}$  and is then multiplied by 0.5 after each major iteration of the interior point method.

TABLE 8.1  
Number of iterations and error in the IP solution for different values of  $\tau_{\text{IP}}$  and three different linear solvers.

		Cholesky		CG tol fixed			CG tol decreasing		
$\tau_{\text{IP}}$	IP	NWT	$x_{\min}$	NWT	CG	$x_{\min}$	NWT	CG	$x_{\min}$
$10^{-8}$	12	28	$1.6 \cdot 10^{-5}$	28	139	$1.8 \cdot 10^{-5}$	28	587	$1.6 \cdot 10^{-5}$
$10^{-10}$	15	34	$1.0 \cdot 10^{-7}$	35	291	$2.7 \cdot 10^{-7}$	34	4285	$1.0 \cdot 10^{-7}$
$10^{-12}$	18	40	$1.0 \cdot 10^{-9}$	72	2832	$2.4 \cdot 10^{-9}$	40	10042	$1.5 \cdot 10^{-9}$
$10^{-14}$	21	46	$6.4 \cdot 10^{-12}$	296	63674	$1.9 \cdot 10^{-11}$	53	23042	$1.9 \cdot 10^{-11}$
$10^{-16}$	24	52	$6.2 \cdot 10^{-14}$	489	88684	$1.4 \cdot 10^{-13}$	82	52042	$1.4 \cdot 10^{-13}$

We can see that all three algorithms were able to solve the problem to very high accuracy. However, both versions of the CG method had problems with very low values of the barrier parameter. The “CG tol fixed” version needed very high number

of the Newton steps, while the “CG tol decreasing” version needed very high number of the CG steps to reach the increased accuracy. (Notice that the maximum number of CG iterations for one system was limited to 1000.) On the other hand, for barrier parameter equal to  $10^{-8}$  (our choice in the numerical examples above), both inexact solvers were on par with the exact one and, due to the lower accuracy required and thus lower number of CG steps, the “CG tol fixed” version is the method of choice.

**8.2. OC method.** In the OC method, we have to solve the equilibrium problem with the stiffness matrix  $K(x)$ ; that means,  $K(x)$  must not be singular. A common way how to approach this is to assume that  $x$  is strictly positive, though very small. Typically, one would modify the lower bound constraint to  $0 < \underline{x} \leq x_i, i = 1, \dots, m$  with  $\underline{x} = 10^{-6}$ , for instance. Once the OC method is terminated, all values of  $x$  with  $x_i = \underline{x}$  are set to zero. This is usually considered a weakness of the OC method, because we do not exactly solve the original problem, only its approximation (see [1]). Somewhat surprisingly, in the examples we solved using our MATLAB code, the value of  $\underline{x}$  could be actually very low, such as  $\underline{x} = 10^{-30}$ . The stiffness matrix  $K(x)$  will, consequently, become extremely ill-conditioned (in the above case the condition number will be of order  $10^{30}$ ), nevertheless, CHOLMOD does not seem to have a problem with that and the OC method converges in about the same number of iterations as if we set  $\underline{x} = 10^{-6}$ .

The main question is how does the quality of the solution depends on the heuristic stopping criterium (6.2). Our next Table 8.2 sheds some light on this. We solve the example 226 from Table 7.2 for various values of the stopping criterium  $\tau_{OC}$  and two different values of the lower bound  $\underline{x}$ . We then compute, pair-wise, the norm of the difference of these solution. The notation  $\tau_{OC} = 10^{-\text{inf}}$  is used for the case when the stopping criterion (6.2) is ignored and the OC method is terminated after a very high number of iterations, in this case 5000 (i.e., 10000 solutions of the linear system). The resulting solution serves as the best approximation of the exact solution that can be obtained within the computational framework. So looking at Table 8.2, we can see that, for instance, the maximum norm of the difference between the solutions with  $\tau_{OC} = 10^{-5}$  and  $\tau_{OC} = 10^{-\text{inf}}$  is  $\|x_{-5} - x_{-\text{inf}}\|_{\infty} = 0.126$ , while  $\|x_{-9} - x_{-\text{inf}}\|_{\infty} = 0.016$ . Notice that the norm is not scaled, e.g., by the dimension of  $x$ , hence the numbers are relatively large. Also, to get a clearer picture, we used a direct linear system solver.

TABLE 8.2

*The norm of difference of two OC solutions  $x$  for various values of the stopping criterium  $\tau_{OC} = 10^{-5}, 10^{-7}, 10^{-9}, 10^{-\text{inf}}$ , and for two values of the lower bound  $\underline{x} = 10^{-7}$  and  $\underline{x} = 10^{-17}$ . Upper triangle shows the 2-norm, lower triangle the infinity norm.*

lower bound		$10^{-7}$				$10^{-17}$			
	$\tau_{OC}$	-5	-7	-9	-inf	-5	-7	-9	-inf
$10^{-7}$	-5	0	1.09	1.19	1.26	$2 \cdot 10^{-6}$	1.10	1.18	1.26
	-7	0.114	0	0.116	0.281	1.09	0.013	0.110	0.281
	-9	0.123	0.01	0	0.190	1.19	0.103	0.009	0.190
	-inf	0.126	0.025	0.016	0	1.26	0.271	0.198	$4 \cdot 10^{-6}$
$10^{-17}$	-5	2e-7	0.114	0.123	0.126	0	1.10	1.19	1.26
	-7	0.116	0.001	0.009	0.024	0.116	0	0.094	0.271
	-9	0.123	0.009	$8 \cdot 10^{-4}$	0.017	0.123	0.008	0	0.198
	-inf	0.126	0.025	0.016	$3 \cdot 10^{-7}$	0.126	0.024	0.017	0

**8.3. Interior point versus OC method.** We again solve example 226 from Table 7.2, this time by the interior point method with an exact linear solver and various stopping parameters  $\tau_{\text{IP}}$ . In Table 8.3, these solutions are compared (in two different norms), to the ‘exact’ solution obtained in the previous section by 5000 iterations of the OC method with  $\underline{x} = 10^{-17}$ . Comparing these numbers to those in Table 8.2, we can see that the IP method delivers very good solution already for our standard value  $\tau_{\text{IP}} = 10^{-8}$ ; this is comparable to OC solution with  $\tau_{\text{OC}} = 10^{-7}$ . Moreover, decrease of  $\tau_{\text{IP}}$  leads to a rapid decrease of the error, unlike in the OC method.

TABLE 8.3

*Two different norms of the error of the IP method in variable  $x$  for different values of the stopping parameter  $\tau_{\text{IP}}$ . As an ‘exact’ solution  $x^*$  we take the OC solution after 5000 iterations with lower bound  $\underline{x} = 10^{-17}$ .*

$\tau_{\text{IP}}$	$\ x - x^*\ _2$	$\ x - x^*\ _\infty$
$10^{-8}$	$2.47 \cdot 10^{-1}$	$2.49 \cdot 10^{-2}$
$10^{-10}$	$9.60 \cdot 10^{-3}$	$1.50 \cdot 10^{-3}$
$10^{-12}$	$2.07 \cdot 10^{-4}$	$4.95 \cdot 10^{-5}$
$10^{-14}$	$1.70 \cdot 10^{-6}$	$4.66 \cdot 10^{-7}$

**9. The SIMP model of topology optimization.** A natural question arises about the applicability of the presented approach to a more popular model of topology optimization, namely, the Solid Isotropic Material with Penalisation (SIMP) [5]. When used with a suitable filtering, one can guarantee at least the existence of a solution of the infinite-dimensional problem and convergence of the finite element discretization to this solution [6]. However, the problem is non-convex and exhibits many local minima, as demonstrated in Figure 9.1. There we show solutions obtained by the code `top88` [3] from various initial points (with a load vector modified to the original code). The calling sequence of the code was `top88(512,64,1,3,1.5,2)`. The ordering of the plots in Figure 9.1 is  $\begin{bmatrix} (a) & (b) \\ (c) & (d) \end{bmatrix}$  and the respective values of the computed optimal compliance are (a) 128.402 (from the default starting point), (b) 126.773, (c) 134.359, (d) 129.002. Notice that we strengthened the stopping criterion of `top88` from  $10^{-2}$  to  $10^{-3}$ .

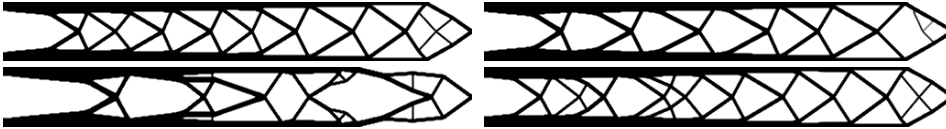


FIG. 9.1. Optimal result SIMP of code `top88` starting from the default initial point and from three other initial points.

It is thus difficult to compare the efficiency of various algorithms, as each may converge to a different local solution (see also [22]). Moreover, starting from two different initial points, an algorithm may converge to two different solutions and the number of iterations needed to find the respective solutions can differ substantially. For these reasons, we only give a brief comparison of the IP method with an exact solver and with an iterative solver, demonstrating that the iterative method is still a

viable and efficient option. (A similar comparison for the OC method can be found in [2].)

The SIMP model with the so-called density filter consists in a modification of our original problem (1.1) where we replace the equilibrium equation  $K(x)u = f$  by

$$\widehat{K}(\tilde{x})u = f \quad \text{with} \quad \widehat{K}(\tilde{x}) = \sum_{i=1}^m \tilde{x}_i^p K_i,$$

where  $\tilde{x}_i$  is computed as a weighted average of the values of  $x$  in a close neighborhood of the  $i$ -th element. More precisely,

$$\tilde{x} = Wx \quad \text{with} \quad W_{:i} = \frac{1}{\sum_{j=1}^m \widehat{W}_{ij}} \widehat{W}_{:i} \quad \text{and} \quad \widehat{W}_{ij} = \max(0, r_{\min} - \text{dist}(i, j)),$$

where  $W_{ij}$  is the  $(i, j)$ -th element of matrix  $W$  and  $W_{:i}$  denotes the  $i$ -th column of  $W$ . Here  $\text{dist}(i, j)$  is a function measuring the distance between the  $i$ -th and  $j$ -th element (e.g. Manhattan distance or Euclidean distance of element centers), and  $r_{\min}$  is a given radius of the density filter. The typical choice of  $p$  is  $p = 3$ .

The interior point method from Sections 2 and 3 has to be adapted to the SIMP model. In particular, the KKT condition (2.3) will change to

$$-\frac{1}{2}u^T \left[ p \sum_{j=1}^m W_{ji} (Wx)_j^{p-1} K_j \right] u - \lambda - \varphi_i + \psi_i = 0, \quad i = 1, \dots, m$$

which means that the matrix  $B(u)$  in the linear system (2.12) (and thus (2.14)) will be replaced by

$$\widehat{B}(u) = \left( p \sum_{j=1}^m W_{j1} (Wx)_j^{p-1} K_j u, \dots, p \sum_{j=1}^m W_{jm} (Wx)_j^{p-1} K_j u \right).$$

The consequence of using the density filter is that the matrix  $\widehat{B}(u)\widehat{B}(u)^T$  (and thus the matrix  $Z$  in (2.14)) will have bigger band-width and thus the Cholesky factors will have more non-zero elements.

The purpose of the next example is merely to show that the iterative solver is still a viable alternative to the direct one, even for the non-convex problem. The reader should not be too concerned with the behaviour of the interior point method, as it still uses the vanilla algorithm and explicit choice of step length that were suitable for the convex problem but should be upgraded for the non-convex one. However, this was not the goal of this paper.

**9.1. Example 4.** Consider again the problem from Example 2 with an upper bound  $\bar{x} = 3$ , this time solved using the SIMP model with density filter. The filter uses Manhattan distance with  $r_{\min} = 2$ . Figure 9.2 shows the optimal results for 8 refinement levels (problem 428) when using the iterative solver (left) and the Cholesky method (right). We can see that also in this example the two versions of the code converge to different local minima with almost identical objective function values (see Table 9.1).

The numbers presented in Table 9.1 for problems 427 and 428 show the total number of Newton steps (feval), the CPU time only needed by the linear solver and the optimal objective function value (obj). The number of Newton steps needed by the IP method is not significantly influenced by the choice of the solver. Also the

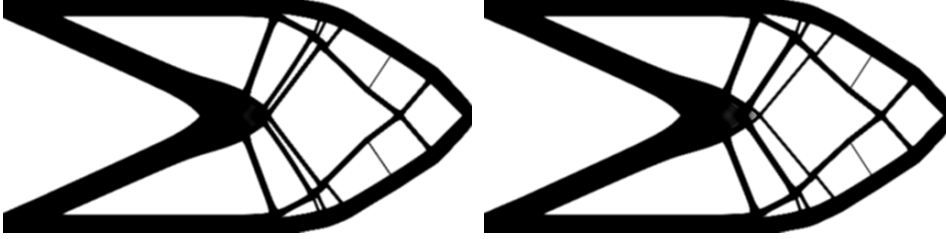


FIG. 9.2. Example 4, optimal solution  $\tilde{x}^*$  using the iterative (left) and the Cholesky (right) solver.

TABLE 9.1  
Example 4, IP method with iterative and Cholesky solver

problem	variables	CG				Cholesky		
		feval	CG iters	time	obj	feval	time	obj
427	66 048	151	514	316	3.5659	128	146	3.5681
428	263 168	262	934	2410	3.4709	231	1500	3.4762

number of CG iterations is still kept very low with average 3.5 per linear system. To demonstrate that was the purpose of this example.

Finally, in Table 9.2 we present the number of Newton steps needed in every major IP iteration. The first row shows the value of the barrier parameter  $s$  (reduced in every iteration by factor 0.2). The next two rows refer to problem 427 and show the number of Newton steps first when using Cholesky method and then for the iterative solver. The final two rows are for problem 428. As we can see, most effort is spent in the last iterations, unlike in the convex case, where the number of Newton steps was almost constant. As mentioned before, a more sophisticated version of the IP method would be needed for the non-convex case to avoid this behaviour.

TABLE 9.2  
Example 4, the barrier parameter  $s$  and the corresponding number of Newton steps needed at every major IP iteration with the Cholesky and the CG solver, respectively.

	$s$	$5^0$	$5^{-1}$	$5^{-2}$	$5^{-3}$	$5^{-4}$	$5^{-5}$	$5^{-6}$	$5^{-7}$	$5^{-8}$	$5^{-9}$	$5^{-10}$	$5^{-11}$
427	Chol	2	2	2	4	4	6	9	11	24	22	21	21
	CG	2	2	3	3	4	6	9	12	24	23	25	38
428	Chol	2	2	2	3	3	4	7	11	21	44	65	67
	CG	2	2	2	3	3	4	7	11	21	53	68	86

**10. Conclusions.** Based on the results of our numerical experiments, we make the following conclusions. These conclusions only concern the convex problem.

- The interior point method clearly outperforms the OC method on large-scale problems. The larger the problem, the bigger the difference. This is independent of the fact whether direct or iterative solver is used for the linear system. It is also independent of the fact whether the linear systems (in both methods) are solve exactly or inexactly.
- The inexact multigrid preconditioned CG method outperforms even a very sophisticated direct solver, at least for large to very-large scale problems.

This holds for both, the interior point and the OC method.

- The behaviour of the interior point method is very predictable. More surprisingly, also the behaviour of the chosen iterative method, the multigrid preconditioned conjugate gradients, is also very predictable and *independent on the size of the problem*.
- Also in the OC method, the multigrid preconditioned CG algorithm is predictable and very stable, both with respect to the size of the problem and of the OC iteration (and thus of the condition number of the stiffness matrix). Perhaps rather surprisingly, not more than 10 CG iterations are needed, even when high precision of the OC method is required. This is the effect of the multigrid preconditioner: notice that in [26] the authors report about 100–200 CG steps needed (with a different preconditioner) and thus propose to use so-called recycling of the Krylov subspaces, in order to accelerate CG convergence speed. This is just not needed here, given the very low number of CG steps.
- The OC method has one noticeable advantage over the interior point method. It can quickly identify the “very strongly” active constraints, those with large Lagrangian multiplier. Due to the projection of variables on the feasible set, the active variables are then exactly equal to the bounds. Contrary to that, the interior point method only approaches the boundary. This may be particularly significant in case of lower bounds, when the user has to decide which values are cut off and considered zero (and thus interpreted as void). Clearly, the lower bound for the OC method has to be positive but it can be set very low (e.g.,  $10^{-17}$ ) and is then exactly reached.

From the above, it seems to be obvious to recommend the interior point method with multigrid preconditioned CG solver as the method of choice for large scale topology optimization problem. However, we should keep in mind that the use of multigrid is rather restricted by the assumed existence of regularly refined finite element meshes. This is easily accomplished when using “academic” examples with regular computational domains such as squares, rectangles, prisms and unions of these. For geometrically complex domains appearing in practical examples, multigrid may not be so suitable or may even be unusable. In these cases, we can resort to domain decomposition preconditioners. In [18] it was shown that, in connection with the interior point method, they also lead to very efficient techniques for topology optimization problems.

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