IPFP and further experiments

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Abstract. Iterative Proportional Fitting Procedure is commonly used in probability theory for construction of a joint probability distribution from a system of its marginals. A similar idea can be used in case of belief functions thanks to special operators of composition defined in this framework. In this paper, a formerly designed IPF procedure is further studied. We propose a modification of composition operator (for the purpose of the procedure), compare the behavior of the modified procedure with the previous one and prove its convergence.

1 Introduction

The marginal problem, as one of the most challenging problem types in probability theory, addresses the question whether or not a common extension exists for a given system of marginals. The challenges lie not only in a wide range of the relevant theoretical problems, but also in its applicability to various problems of statistics [2], computer tomography [9], relational databases [12] and artificial intelligence [14]. In the last case, it is the problem how to obtain a global knowledge (represented by a multidimensional probability distribution) from pieces of local knowledge (represented by low-dimensional probability distributions).

To solve a discrete marginal problem, one can use Iterative Proportional Fitting Procedure (IPFP), introduced by Deming and Stephan already in 1940 [6]. Its convergence was finally proven by Csiszár [7] in 1975. Note that both the EM and the Newton-Raphson algorithms converge towards the same limit. However, in most cases, IPFP is preferred due to its computational speed, numerical stability and algebraic simplicity [15]. A possibilistic version of this procedure (parametrized by a continuous t-norm) was studied in [13].

A possible application of IPFP in the framework of belief functions was studied in [3]. Knowing that the probabilistic IPFP can be easily (and elegantly) expressed with the help of the so-called *operator of composition* [8], the same idea was applied in this framework. Two different composition operators for bpas were discussed in the above-mentioned paper: the first one has already been introduced in [4], the second one was based on *Dempster's combination rule* [10]. Let us note that the operator based Dempster's rule appeared as inappropriate for IPFP (for more details see [3]). That is why, in this paper, we focus on the original operator only. We illustrate one undesirable aspect of its behavior and suggest a possible modification to solve the problem.

The paper is organized as follows. After a brief overview of necessary concepts and notation (Section 2), in Section 3 we recall the concept of evidential IPF procedure and present its modification. Section 4 is devoted to the discussion of experimental results. The proof of convergence is given is Section 5.

2 Basic Concepts and Notation

In this section we will briefly recall basic concepts from evidence theory [10] concerning sets and set functions as well as the concept of the operator of composition [5].

2.1 Set Projections and Joins

In this paper $\mathbb{X}_N = \mathbb{X}_1 \times \mathbb{X}_2 \times \ldots \times \mathbb{X}_n$ denotes a finite multidimensional space, and its subspaces (for all $K \subseteq N$) are denoted by $\mathbb{X}_K = \times_{i \in K} \mathbb{X}_i$. For a point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{X}_N$, its *projection* into subspace $\mathbb{X}_K$ is denoted $x^{\downarrow K} = (x_i)_{i \in K}$, and for $A \subseteq \mathbb{X}_N$ $A^{\downarrow K} = \{y \in \mathbb{X}_K : \exists x \in A, x^{\downarrow K} = y\}$.

By a *join* of two sets $A \subseteq \mathbb{X}_K$ and $B \subseteq \mathbb{X}_L$ we understand a set $A \bowtie B = \{x \in \mathbb{X}_{K \cup L} : x^{\downarrow K} \in A \;\;\&\;\; x^{\downarrow L} \in B\}$. Let us note that if K and L are disjoint, then $A \bowtie B = A \times B$, if $K = L$ then $A \bowtie B = A \cap B$. Generally, for $C \subseteq \mathbb{X}_{K \cup L}$, C is a subset of $C^{\downarrow K} \bowtie C^{\downarrow L}$, which may be proper.

2.2 Basic Assigments

A *basic assignment* (bpa) m on $\mathbb{X}_K$ ($K \subseteq N$) is a real non-negative function on power set of $\mathbb{X}_K$, for which $\sum_{\emptyset \neq A \subseteq \mathbb{X}_K} m(A) = 1$. If $m(A) > 0$, then A is said to be a *focal element* of m.

A bpa is called vacuous, if it contains only one focal element, namely $\mathbb{X}_K$. In accordance with [3] we call a bpa *uniform* if $m(A) = 1/(2^{|\mathbb{X}_K|} - 1)$ for each $A \subseteq \mathbb{X}_K, A \neq \emptyset$.

Considering two bpas m_1, m_2 on the same space $\mathbb{X}_K$, we say that m_1 is *dominated* by m_2, if for all $A \subseteq \mathbb{X}_K$: $m_1(A) > 0 \Longrightarrow m_2(A) > 0$.

Having a bpa m on $\mathbb{X}_K$ one can consider its *marginal assignments.* On $\mathbb{X}_L$ (for $L \subseteq K$) it is defined (for each $\emptyset \neq B \subseteq \mathbb{X}_L$) as follows $m^{\downarrow L}(B) = \sum_{A \subseteq \mathbb{X}_K : A^{\downarrow L} = B} m(A)$.

Having two bpas m_1 and m_2 on $\mathbb{X}_K$ and $\mathbb{X}_L$, respectively ($K, L \subseteq N$), we say that these assignments are *projective* if $m_1^{\downarrow K \cap L} = m_2^{\downarrow K \cap L}$, which occurs if and only if there exists a bpa m on $\mathbf{X}_{K \cup L}$ such that both m_1 and m_2 are its marginal assignments.

2.3 Operator of Composition

Let us recall the definition of operator of composition $\triangleright$ introduced in [4].

Definition 1. *Consider two arbitrary basic assignments* m_1 *on* $\mathbb{X}_K$ *and* m_2 *on* $\mathbb{X}_L$ $(K \neq \emptyset \neq L)$*. A* composition $m_1 \triangleright m_2$ *is defined for each* $C \subseteq \mathbb{X}_{K\cup L}$ *by one of the following expressions:*

[a] *if* $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) > 0$ *and* $C = C^{\downarrow K} \bowtie C^{\downarrow L}$ *then*

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})};$$

[b] *if* $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) = 0$ *and* $C = C^{\downarrow K} \times \mathbb{X}_{L\setminus K}$ *then*

$$(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow K});$$

[c] *in all other cases* $(m_1 \triangleright m_2)(C) = 0$.

3 Iterative Proportional Fitting Procedure

Let us start this section by recalling the original design of evidential version of IPF procedure [3].

3.1 Original Design

Assume a system of n low-dimensional bpas $m_1, m_2, \ldots, m_n$ defined on $\mathbb{X}_{K_1}, \mathbb{X}_{K_2}, \ldots, \mathbb{X}_{K_n}$, respectively. During the computational process, an infinite sequence of bpas $\mu_0, \mu_1, \mu_2, \mu_3, \ldots$ is computed, each of them defined on $\mathbb{X}_{K_1\cup\ldots\cup K_n}$. If this sequence is convergent, its limit is the result of this process. For simplicity reason let us suppose that $K_1 \cup K_2 \cup \cdots \cup K_n = N$.

Algorithm IPFP
Define the starting bpa μ_0 on $\mathbb{X}_{K_1\cup K_2\cup\ldots\cup K_n}$. Then compute sequence $\{\mu_i\}_{i\in 1,2,3,\ldots}$ in the following way:

$$\begin{array}{llll}
\mu_1 = m_1 \triangleright \mu_0 & \mu_{n+1} = m_1 \triangleright \mu_n & \mu_{2n+1} = m_1 \triangleright \mu_{2n} & \\
\mu_2 = m_2 \triangleright \mu_1 & \mu_{n+2} = m_2 \triangleright \mu_{n+1} & \mu_{2n+2} = m_2 \triangleright \mu_{2n+1} & \vdots \\
\vdots & \vdots & \vdots & \\
\mu_n = m_n \triangleright \mu_{n-1} & \mu_{2n} = m_n \triangleright \mu_{2n-1} & \mu_{3n} = m_n \triangleright \mu_{3n-1} &
\end{array}$$

As already said in the Introduction, if this algorithm is applied to probability distributions, it has nice and useful properties, most of which were proven by Csiszár in his famous paper [7].

Based on the Csiszár's results, two nice properties on convergence were proven in [3].

Theorem 1. *Consider a system of bpas* $m_1, m_2, \ldots, m_n$ *defined on* $\mathbb{X}_{K_1}, \mathbb{X}_{K_2}, \ldots, \mathbb{X}_{K_n}$ *and a basic assignment* μ_0 *on* $\mathbb{X}_{K_1\cup\ldots\cup K_n}$*. If there exists a bpa* ν *on* $\mathbb{X}_{K_1\cup\ldots\cup K_n}$ *such that* ν *is dominated by* μ_0*, and* ν *is a common extension of all* $m_1, m_2, \ldots, m_n$*, then the sequence* $\mu_0, \mu_1, \mu_2, \mu_3, \ldots$ *computed by the Algorithm IPFP with* $\triangleright$ *converges.*

Theorem 2. *If the sequence $\mu_0, \mu_1, \mu_2, \mu_3, \ldots$ computed by the Algorithm IPFP converges then the bpa $\mu^* = \lim_{i \to +\infty} \mu_i$ is a common extension of all $m_1, m_2, \ldots, m_n$, i.e., $(\mu^*)^{\downarrow K_j} = m_j$ for all $j = 1, \ldots, n$.*

In experiments performed in [3], the uniform bpa was chosen to be μ_0. It seems to correspond to the probabilistic framework, where the uniform distribution is also used as the starting distribution. Moreover, uniform bpa dominates every other bpa on the same frame. Thus, if one starts the IPFP with uniform basic assignment, Theorem 1 guarantees its convergence whenever the common extension of the given assignments exists.

Nevertheless, there is a big difference between semantics of these two approaches. While in the probabilistic case uniform distribution is considered to be the least specific, nothing similar holds in the evidential framework. Here the vacuous bpa represents the least specific one. However, in this case the assumption of dominance of μ_0 is not valid, and the procedure need not converge (and it does not, in most cases).

3.2 Modification

If the composition operator is applied on projective marginals, part [b] of Definition 1 is never used. On the other hand, if it is not the case, then rule [b] is adding just one focal element of a very specific form. It is, in fact, cylindrical extension of the focal element on $\mathbb{X}_K$ to $\mathbb{X}_{K \cup L}$ (in case of m_1 on $\mathbb{X}_K$, m_2 on $\mathbb{X}_L$, and $m_1 \triangleright m_2$).

This led us to the following consideration. Let us start the IPFP procedure with vacuous bpa — reflecting total ignorance about the problem — and rewrite part [b] of the operator of composition in a way to be able to add more focal elements. We decided to add all focal elements that, being marginalized to $\mathbb{X}_K$ have a focal element in m_1. The respective mass is uniformly distributed among them. A lot of unnecessary focal elements may be (and really is) added, but they are left for future removal by rules [a] and [c] of Definition 1 — which remain unchanged.

Definition 2. *Consider two arbitrary bpas m_1 on $\mathbb{X}_K$ and m_2 on $\mathbb{X}_L$ ($K \neq \emptyset \neq L$) an* iterative composition *$m_1 \triangleright' m_2$ is defined for each $C \subseteq \mathbb{X}_{K \cup L}$ by one of the following expressions:*

[a] *if $m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L}) > 0$ and $C = C^{\downarrow K} \bowtie C^{\downarrow L}$ then*

$$(m_1 \triangleright' m_2)(C) = \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L})};$$

[b'] *if $m_2^{\downarrow K \cap L}(C^{\downarrow K \cap L}) = 0$ then $\forall D \in \mathcal{D} = \{D \in \mathbb{X}_{K \cup L} : D^{\downarrow K} = C^{\downarrow K}\}$*

$$(m_1 \triangleright' m_2)(D) = \frac{m_1(C^{\downarrow K})}{|\mathcal{D}|};$$

[c] *in all other cases* $(m_1 \triangleright' m_2)(C) = 0$.

In other words, in **[b']** instead of one focal element $C = C^{\downarrow K} \times X_{L\setminus K}$, a system of its subsets is added. The mass of $m_1(C^{\downarrow K})$ is uniformly distributed among them.

This approach has significant impact on the behavior of the IPFP procedure, which seems to behave better (or in the same way, in the worst case) as the one presented in [3] (cf. Section 4).

Now, let us summarize three observations concerning both original and modified IPFP, which will, hopefully, help the reader to understand not only the difference between them, but later also the idea of the proof.

Observation 1 *Note that in case of IPFP,* $K \subseteq L$ *in the definition of the operator of composition and therefore* $A = A^{\downarrow K} \bowtie A^{\downarrow L} = A^{\downarrow K} \bowtie A$ *for every* $A \subseteq \mathbb{X}_N$*.*

Note that in case of IPFP, there is a close connection between the notion of dominance and using of rule **[b]** in the definition of $\triangleright$:

Observation 2 *If* ν *is dominated by* μ_i*, then* ν *is dominated by* $\nu^{\downarrow K} \triangleright \mu_i$ *as well, and rule* ***[b]*** *from Definition 1 is never used.*

Proof. If it is not the case, then $\exists A \subseteq \mathbb{X}_N$ such that $\nu(A) > 0$ while $(\nu^{\downarrow K} \triangleright \mu_i)(A) = 0$. Since $\mu_i(A) > 0$ by dominance assumption, then, following Observation 1, rule **[a]** has to be used and therefore $(\nu^{\downarrow K} \triangleright \mu_i)(A) > 0$, which is a contradiction.

Having Observation 2 in mind, one can conclude:

Observation 3 *If* ν *is dominated by* μ_i *then* $\nu^{\downarrow K} \triangleright \mu_i = \nu^{\downarrow K} \triangleright' \mu_i$*.*

4 Experiments

Most experiments discussed in this section deal with cases of consistent bpas, i.e. bpas representing marginals of a multidimensional bpa. We will start with the original IPFP [3] to reveal the problems caused by its application.

4.1 Original Procedure

Let X, Y and Z be three binary variables with values in $\mathbb{X} = \mathbb{Y} = \mathbb{Z} = \{0, 1\}$. Joint basic assignment m on $\mathbb{X} \times \mathbb{Y} \times \mathbb{Z} = \{0, 1\}^3$ is defined in Table 1.

First, we calculate all three two-dimensional marginals of m — denoted by $m_1 = m^{\downarrow XY}$, $m_2 = m^{\downarrow YZ}$, and $m_3 = m^{\downarrow XZ}$ — and we apply them in this order on uniform μ_0 using IPFP. The computational process is illustrated by Table 2.

Notice, that the procedure converges to m' which is not in contradiction with results proven in [3] because both m' and m have the same two-dimensional marginals. This experiment has already been published in [3].

(focal) elements	m	m'	m_Ω	m'_Ω
$\{010, 100\}$	0.2	0.2	0.2	0.2
$\{001, 010\}$	0.3	0.3	0.3	0.3
$\{001, 011, 101, 110\}$	0.5	0.25	0.4	0.2
$\{001, 011, 101, 110, 111\}$	0	0.25	0	0.2
$\mathbb{X}$	0	0	0.1	0.1

Table 1. Three-dimensional assignments

focal elements	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_{100}	μ_{1000}
$\{010, 100\}$	0.156	0.200	0.166	0.166	0.200	0.172	0.195	0.199
$\{000, 010, 100, 110\}$	0.043	0.040	0.033	0.033	0.031	0.027	0.004	$4 \cdot 10^{-4}$
$\{001, 010\}$	0.146	0.146	0.300	0.211	0.211	0.300	0.293	0.299
$\{001, 010, 011\}$	0.153	0.153	0.124	0.088	0.085	0.079	0.006	$7 \cdot 10^{-4}$
$\{001, 011, 101, 110\}$	0.250	0.230	0.187	0.250	0.234	0.210	0.250	0.250
$\{001, 011, 101, 110, 111\}$	0.250	0.230	0.187	0.250	0.234	0.210	0.250	0.250

Table 2. IPFP with $\triangleright$, two-dimensional marginals of m, and uniform μ_0.

A problem appears if m_Ω is taken into account instead of m (in case of m_Ω — a non-zero mass has been put on the whole frame of discernment — $m_\Omega(\mathbb{X}) = 0.1$.) The computational process with respective marginals $m_1 = m_\Omega^{\downarrow XY}$, $m_2 = m_\Omega^{\downarrow YZ}$, and $m_3 = m_\Omega^{\downarrow XZ}$ is illustrated by Table 3.

focal elements	μ_3	μ_4	μ_5	μ_6	μ_{100}	μ_{1000}	μ_{10000}
$\{010, 100\}$	0.156	0.195	0.166	0.166	0.188	0.188	0.188
$\{000, 010, 100, 110\}$	0.043	0.040	0.034	0.034	0.011	0.011	0.011
$\{001, 010\}$	0.158	0.158	0.272	0.211	0.257	0.257	0.257
$\{001, 010, 011\}$	0.132	0.132	0.104	0.080	0.038	0.038	0.038
$\{001, 011, 101, 110\}$	0.181	0.167	0.132	0.172	0.170	0.170	0.170
$\{001, 011, 101, 110, 111\}$	0.181	0.167	0.132	0.172	0.170	0.170	0.170
$\mathbb{X}$	0.001	0.0009	0.001	0.001	0.0007	0.0007	0.0007
and 43 other elements	...	...	...	...	...	...	...

Table 3. IPFP with $\triangleright$, two-dimensional marginals of m_Ω, and uniform μ_0.

IPF procedure does not perform very well in this case. A stabilized state is achieved approximately in μ_{800} and it is far away from m_Ω, with 50 focal elements, although, according to Theorem 1, its two-dimensional marginals coincide with those of m_Ω.

It seems, that the problem consists in the fact that we start with too many focal elements in μ_0. It would be of a special interest to have a procedure that starts with vacuous μ_0. Because the ability of operator $\triangleright$ to add new focal elements is limited, operator $\triangleright'$ will be used instead.

4.2 Modified Procedure

operator	assignment	IPFP ordering	μ_0	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6
$\triangleright$	m	$m_1; m_2; m_3;$	255	99	15	6	6	6	6
		$m_2; m_1; m_3;$	255	45	15	6	6	6	6
		$m_3; m_1; m_2;$	255	45	17	6	6	6	6
	m_Ω	$m_1; m_2; m_3$	255	99	66	50	50	50	50
		$m_2; m_1; m_3$	255	126	66	50	50	50	50
		$m_3; m_1; m_2$	255	126	82	50	50	50	50
$\triangleright'$	m	$m_1; m_2; m_3$	1	19	6	30	11	5	5
		$m_1; m_3; m_2$	1	19	9	3	84	11	6
		$m_2; m_1; m_3$	1	45	15	6	6	6	6
	m_Ω	$m_1; m_2; m_3$	1	19	9	33	14	11	11
		$m_2; m_1; m_3$	1	46	16	11	11	11	11
		$m_3; m_1; m_2$	1	46	18	11	11	11	11

Table 4. Number of focal elements during IPFP

Note that in case of IPFP with $\triangleright'$ and vacuous μ_0, only eleven focal elements are taken into the account. Respective elements are depicted in Table 5. Note that the sequence $\{\mu_i\}_{i\to\infty}$ tends to m'_Ω from Table 1 which is not in conflict, because it has the same two-dimensional marginals as m_Ω.

Starting with vacuous assignment, potentially necessary focal elements have to be added. Operator $\triangleright'$ seems to be a reasonable choice. See Table 4 to compare the development of the number of focal elements for both operators $\triangleright$ and $\triangleright'$. Note that for $\triangleright$, the uniform assignment is solely used as μ_0 in respective IPFP. Similarly, $\triangleright'$ is associated with vacuous assignment as its starting point. This is highlighted in Table 4 — in the number of focal elements in column corresponding to μ_0. The number of focal elements stabilizes in μ_i for $i \geq 6$ in this case (some of them may disappear later by having mass converging to zero).

4.3 Inconsistent Marginals

In case of inconsistent marginals and IPFP based on $\triangleright'$ and vacuous μ_0, we observe the same behavior as for $\triangleright$ and uniform μ_0: After several cycles, the iteration process goes through cyclical changes. The length of the cycle corresponds to the number of basic assignments entering the computational process. The subsequences converge.

focal elements	μ_5	μ_{100}	μ_{1000}	μ_{10000}
$\{010, 100\}$	0.200	0.199	0.199	0.199
$\{001, 010\}$	0.155	0.292	0.299	0.299
$\{001, 010, 011\}$	0.143	0.007	10^{-4}	10^{-5}
$\{001, 011, 101, 110\}$	0.159	0.198	0.199	0.199
$\{001, 011, 101, 110, 111\}$	0.159	0.198	0.199	0.199
$\{010, 011, 100, 101\}$	0.001	0.001	10^{-4}	10^{-6}
$\{000, 001, 010, 011\}$	0.001	10^{-6}	10^{-9}	10^{-10}
$\{001, 011, 100, 101, 110\}$	0.026	0.001	10^{-4}	10^{-6}
$\{001, 011, 100, 110, 111\}$	0.026	0.001	10^{-4}	10^{-6}
$\{001, 011, 100, 101, 110, 111\}$	0.026	0.001	10^{-4}	10^{-6}
$\mathbb{X}$	0.098	0.099	0.099	0.099

Table 5. IPFP with $\triangleright'$, two-dimensional marginals of m_Ω, and vacuous μ_0.

5 Proof of convergence

To prove the convergence of the IPFP starting with vacuous bpa and using operator $\triangleright'$ from Definition 2 it is enough to show that the sequence $\mu_0, \mu_1, \mu_2, \ldots$ can be divided into two parts. In the first part, a bpa μ_k dominating ν (a common extension of given system of marginals) is found. Then, the second part converges because of Theorem 1 and Observations 2 and 3.

We work in a discrete space, therefore the number of focal elements is finite. We cope with a system of marginals $m_1, m_2, \ldots, m_n$ (of a joint (unknown) bpa on $\mathbb{X}_N = \mathbb{X}_{K_1 \cup \ldots \cup K_n}$) defined on $\mathbb{X}_{K_1}, \mathbb{X}_{K_2}, \ldots, \mathbb{X}_{K_n}$, respectively, and an infinite sequence of bpas $\mu_0, \mu_1, \mu_2, \mu_3, \ldots$ computed using IPFP algorithm and operator $\triangleright'$ from Definition 2, where μ_0 is vacuous bpa on $\mathbb{X}_N$.

Lemma 1. *Having a bpa on $\mathbb{X}_N$ and a system of its marginals $\{m_j\}_{j=1}^n$, create sequence $\mu_0, \mu_1, \mu_2, \ldots$ using IPFP starting with vacuous μ_0 and using $\triangleright'$. Let $A \subseteq \mathbb{X}_N$ be a focal element of μ_i such that $A^{\downarrow K_j}$ is a focal element of m_j $\forall j = 1, \ldots, n$, respectively. Then $\forall k \geq i$, A is a focal element in μ_k .*

Proof. Take an arbitrary $j = 1, \ldots, n$. Let $\mu_{i+1} = m_j \triangleright' \mu_i$. To prove the lemma, one has to realize that in case of IPFP, $K \subseteq L$ in the definition of the operator of composition and $A = A^{\downarrow K_j} \bowtie A$ for every A (Observation 1). Then, using lemma assumption, rule **[a]** from the definition of the composition operator is used in case of A. Because $A^{\downarrow K_j}$ is a focal element in m_j, then we multiply non-zero numbers and A is a focal element in μ_{i+1} as well. This reasoning can be iteratively repeated which finishes the proof.

Observation 4 *Let $K_j \subset N$, $A \subseteq \mathbb{X}_{K_j}$ and $\mathcal{B} = \{B \subseteq \mathbb{X}_N | B^{\downarrow K_j} = A\}$. If none $B \in \mathcal{B}$ is a focal element of μ_i and A is a focal element of m_j then $\mathcal{B}$ is a subset of focal elements of $\mu_{i+1} = m_j \triangleright' \mu_i$.*

Indeed, rule **[b']** from Definition 2 is used in this case.

Theorem 3. *Consider a system of bpas $m_1, m_2, \ldots, m_n$ defined on $\mathbb{X}_{K_1}, \mathbb{X}_{K_2}, \ldots, \mathbb{X}_{K_n}$, respectively, and a vacuous bpa μ_0 on $\mathbb{X}_{K_1 \cup \ldots \cup K_n}$. If a common extension of $\{m_j\}_{j=1}^n$ exists then the sequence $\mu_0, \mu_1, \mu_2, \mu_3, \ldots$ computed using IPFP with $\triangleright'$ converges to one of them.*

Proof. First, let us prove that in a finite number of steps we get bpa μ_i that dominates a common extension of $\{m_j\}_{j=1}^n$. To prove that, is is sufficient to realize three simple facts:

(i) following Lemma 1, once a focal element of a common extension is added, it cannot be removed,
(ii) focal elements are added if necessary (Observation 4) — note that at least of them has to be a focal element of a common extension and therefore it cannot be removed by Lemma 1, and
(iii) there is a finite number of focal elements.

Once a μ_i dominating a common extension is obtained, then, using Observation 3, Theorem 1 can be applied and such a sequence converges. Moreover, it converges to a common extension by Theorem 2 (using Observation 3, again).

6 Conclusions and Future Work

We studied recently designed IPF procedure for bpas based on the evidential composition operator in more detail and realized that its behavior is not satisfactory, especially in case of partial ignorance. Deeper study revealed the fact, that although starting from uniform distribution allows an elegant proof of convergence, the procedure produces a great number of focal elements.

We suggested an alternative approach starting with vacuous basic assignment and consisting in adding of potentially interesting focal elements and subsequent removing of the unimportant ones. Several experiments showed that this procedure behaves much better than the previous one.

Following Table 5, the computational complexity of the new approach seems to be lower. This is caused not only by the fact that the new approach is producing bpas with generally less focal elements, but also by the fact that it does not start with all possible focal elements in μ_0. This could be further improved by excluding the first part of the IPFP responsible for finding dominating bpa. Note that we are not interested in "probability" masses laid on focal elements in this part, but on the shape of focal elements only. This is a topic of further research.

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