Löwenheim–Skolem theorems for non-classical first-order algebraizable logics*

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Abstract

This paper is a contribution to the model theory of non-classical first-order predicate logics. In a wide framework of first-order systems based on algebraizable logics, we study several notions of homomorphisms between models and find suitable definitions of elementary homomorphism, elementary substructure and elementary equivalence. Then we obtain (downward and upward) Löwenheim–Skolem theorems for these non-classical logics, by direct proofs and by describing their models as classical 2-sorted models.

*We would like to thank the anonymous referees that reviewed the submitted version of the paper for their useful corrections and remarks.

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Keywords: Löwenheim–Skolem theorems, first-order predicate logics, non-classical logics, algebraizable logics, model theory

1 Introduction

Classical model theory (see e.g. [3, 18]) studies mathematical structures as first-order models, that is, structures that can be described in terms of classical first-order predicate formulas. These models are sets (universes, domains) of elements, where monadic relational symbols are interpreted as subsets of the domain and n-ary relational symbols are interpreted as sets of n-tuples of elements of the domain. One of the main goals of model theory is the classification of such first-order structures according to the properties that can be expressed in the language, that is, two models are put in the same class if they have the same first-order theory, i.e. if they are elementarily equivalent. In this context, Löwenheim–Skolem theorems are crucial results in classical model theory that allow to find models of each infinite cardinal for a consistent theory.

Motivated from diverse points of view, many non-classical logics have been introduced and studied in the literature, usually endowing them with first-order predicate formalisms to guarantee a sufficient expressive power. The semantics of such logics needs some notion of non-classical first-order structure. A useful approach, inspired by the study of predicate intuitionistic logic by Rasiowa and Sikorski [28] and some many-valued predicate logics by Mostowski [23], takes non-classical first-order structures as domains in which the relational symbols are not interpreted as subsets (or sets of tuples) of the universe, but as mappings from the universe (or Cartesian products of the universe) into some algebra of truth-values. These algebras are those naturally connected to the underlying propositional calculus and are usually endowed with an order relation. Then the existential (resp. universal) quantification of a formula can be interpreted as the supremum (resp. infimum) of the values of its instances with respect to that order relation.

Already in her monograph [27] Rasiowa generalized this treatment of first-order logics to the wider class of *implicative logics*. Such approach has been particularly fruitful in the field of fuzzy logics (see e.g. [12, 5]) in the study of first-order systems with a Hilbert-style axiomatization and a corresponding sound and complete semantics of first-order fuzzy models (where, naturally, predicates are interpreted as fuzzy sets in the sense of Zadeh [30]), giving rise to several works on model theory of fuzzy logics (see e.g. [25, 24, 17, 4, 8, 9]). Finally, the recent paper [7] takes the mentioned approach to first-order logics to a much broader setting by allowing any algebraizable logic (in the sense of Blok and Pigozzi [2]) in the place of the underlying propositional logic, providing axiomatization and semantics and corresponding completeness theorems for the resulting first-order logics by a suitable modification and generalization of the Henkin-style proof for classical logic.

The evolution we have briefly described calls for a systematic development of a non-classical model theory. After the few initial steps done in the field of fuzzy logics, one can consider the ambitious endeavour of studying non-classical first-order models in an as wide as possible setting. Among the obvious first items of its agenda there should be, at least, the essential question of understanding a suitable notion of elementary equivalence that would allow to classify non-classical models according to their first-order theories, and the possibility of finding some kind of Löwenheim–Skolem theorems that would show the existence of elementarily equivalent models of different cardinals. The main goal of this paper is the latter of these two items, which can be seen as well as a contribution to the former.

We choose to work in the framework of non-classical first-order algebraizable logics of [7] because it provides a very wide setting for first-order calculi with a sound and complete semantics of non-classical models that generalize the classical ones in a natural way. We will inspect the classical proofs of Löwenheim–Skolem theorems and realize that, not surprisingly, they rely on very specific properties of classical logic (such as compactness and existence of witnesses for existential quantifiers) that in general are lost in non-classical logics. We will need to go around such difficulties by constructing different direct proofs (partly taking inspiration in the pioneering work of G. Gerla in [11] for first-order fuzzy models) and, alternatively, by means of translation into the classical many-sorted setting. This twofold strategy, as it will be argued, will yield different (dis)advantages. Moreover, the construction of elementarily equivalent models of bigger and smaller cardinalities will need a previous development of several usual notions of classical model theory, now in the non-classical framework: elementary equivalence itself, elementary embeddings, substructures, and several notions of homomorphism between non-classical structures that will be carefully defined and mutually separated. We take this preliminary study as a valuable by-product of the main results of this paper.

The paper is organized as follows: after this introduction, Section 2 presents the necessary preliminaries for the proposed framework for non-classical first-order logics; then Section 3 contains a step-by-step study of suitable notions of homomorphisms for non-classical structures from three complementary points of view: structure-preserving morphisms, categorical morphisms and the model-theoretic approach as formula-preserving morphisms. Section 4 introduces the necessary notions of substructure and elementary substructure and also a non-classical version of the Tarksi–Vaught test. Based on all these auxiliary notions, the central part of the paper is Section 6, which briefly recalls the classical Löwenheim–Skolem theorems and their usual proofs, discusses the difficulties one meets when trying to extend them to non-classical logics, and offers direct new general proofs. Finally, Section 7 explores the alternative route of rendering non-classical structures as classical two-sorted structures, obtaining other forms of Löwenheim–Skolem theorems with different pros and cons.

2 Preliminaries

The framework chosen for the paper is that of non-classical first-order algebraizable logics proposed in [7]. The underlying propositional logics are *algebraizable logics* in the sense of Blok and Pigozzi in [2], which provide a suitable paradigm, arguably the best in abstract algebraic logic, for a class of logics with a strong link with an algebraic semantics, staying as close as possible to the connection between classical propositional calculus and Boolean algebras.

We follow the same presentation as in [7], which highlights the role of implication. Given a set $\rightarrow(p, q)$ of formulas in two variables p and q, a set $\mathcal{E}(p)$ of equations in one variable p, formulas φ and ψ, and two sets T and S of formulas:

$\varphi \to \psi$ denotes the set $\to(\varphi, \psi)$,

$\varphi \leftrightarrow \psi$ denotes the set $(\varphi \to \psi) \cup (\psi \to \varphi)$,

$T \vdash S$ means that $T \vdash \varphi$ for each $\varphi \in S$,

$T \dashv\vdash S$ means that $T \vdash S$ and $S \vdash T$,

$\leftrightarrow[\mathcal{E}(\varphi)]$ denotes the set $\bigcup\{\alpha(\varphi) \leftrightarrow \beta(\varphi) \mid \alpha \approx \beta \in \mathcal{E}\}$.

Let $\mathcal{L}$ be any propositional language containing at least a truth constant $\overline{1}$. Let $\to(p, q)$ be a finite set of formulas in two variables and $\mathcal{E}(p)$ a finite set of equations in one variable in the language $\mathcal{L}$. We assume that a propositional logic L in $\mathcal{L}$ is given by the provability relation $\vdash_{\mathrm{L}}$ on $Fm_{\mathcal{L}}$ given by a finitary Hilbert-style system such that:

$$\vdash_{\mathrm{L}} \varphi \to \varphi, \qquad \varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi, \qquad \varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi,$$

$$\varphi \dashv\vdash_{\mathrm{L}} \overline{1} \to \varphi, \qquad \varphi \dashv\vdash_{\mathrm{L}} \leftrightarrow[\mathcal{E}(\varphi)],$$

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \circ(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \leftrightarrow \circ(\chi_1, \dots \chi_i, \psi, \dots, \chi_n),$$

for every n-ary $\circ \in \mathcal{L}$ and $i < n$.

For any algebra $\boldsymbol{A}$ for the language $\mathcal{L}$, we define the following set and binary relation:

$$\mathcal{F}^{\boldsymbol{A}} = \{a \mid \boldsymbol{A} \models \mathcal{E}^{\boldsymbol{A}}(a)\}, \qquad a \leq_{\boldsymbol{A}} b \quad \text{if and only if} \quad (a \to^{\boldsymbol{A}} b) \subseteq \mathcal{F}^{\boldsymbol{A}}.$$

$\boldsymbol{A}$ is an L-algebra, in symbols: $\boldsymbol{A} \in \mathbb{L}$, if for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and $a, b \in A$ hold:

1. $\Gamma \vdash_{\mathrm{L}} \varphi$ implies that for each $\boldsymbol{A}$-evaluation e we have $e(\varphi) \in \mathcal{F}_{\boldsymbol{A}}$ whenever $e[\Gamma] \subseteq \mathcal{F}^{\boldsymbol{A}}$,
2. $a \leq_{\boldsymbol{A}} b$ and $b \leq_{\boldsymbol{A}} a$ implies $a = b$.

Observe that in each L-algebra $\boldsymbol{A}$, $\leq_{\boldsymbol{A}}$ is actually an order. On the other hand, $\mathcal{F}^{\boldsymbol{A}} = \{a \mid \overline{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} a\}$, which is an upper set with respect to $\leq_{\boldsymbol{A}}$ and it is usually called the *filter* of $\boldsymbol{A}$.

This propositional framework is wide enough to contain most of the usual propositional systems considered in the literature. Indeed, it encompasses in particular classical logic, intuitionistic logic, usual substructural logics, relevant logics with unit, and fuzzy logics.

Typical examples of fuzzy logics are Gödel–Dummett and Łukasiewicz logics (see e.g. [5]). Let us briefly recall them, so we can they be used in the paper to provide some counterexamples. Both logics can be given in a language with three binary connectives $(\to, \wedge, \vee)$ and one constant $(\overline{0})$. The constant $\overline{1}$ is defined as $\overline{0} \to \overline{0}$. The semantics in both cases is defined over the real unit interval $[0, 1]$ and the algebraic interpretation of the connectives tries to keep them close to the Boolean operations, giving rise to two different algebras: $[0, 1]_{\mathrm{G}}$ and $[0, 1]_{\text{Ł}}$. In both cases $\wedge$, $\vee$ and $\overline{0}$ are interpreted respectively as the minimum, the maximum and the number 0, while the interpretation of implication differs:

$$a \to^{[0,1]_G} b = \begin{cases} 1, & \text{if } a \leq b, \\ b, & \text{otherwise.} \end{cases}$$

$$a \to^{[0,1]_Ł} b = \begin{cases} 1, & \text{if } a \leq b, \\ 1-a+b, & \text{otherwise.} \end{cases}$$

for every $a, b \in [0,1]$. Note that the defined constant $\overline{1}$ is then interpreted as 1. Then the logics are semantically obtained by the following definition. Given a formula φ and of set of formulas Γ:

1. $\Gamma \vdash_Ł \varphi$ if and only if there is a finite $\Gamma_0 \subseteq \Gamma$ such that for every $[0,1]_Ł$-evaluation e, if $e(\gamma) = 1$ for every $\gamma \in \Gamma_0$, then $e(\varphi) = 1$.

2. $\Gamma \vdash_G \varphi$ if and only if for every $[0,1]_G$-evaluation e, if $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$.

An important definable connective in Łukasiewicz logic is the *strong conjunction*, semantically interpreted as $a \mathbin{\&}^{[0,1]_Ł} b = \max\{a + b - 1, 0\}$, for every $a, b \in [0,1]$. From this conjunction, one can define powers as $a^1 = a$ and $a^{n+1} = a^n \mathbin{\&}^{[0,1]_Ł} a$ for every $n \geq 1$. It is usual in the fuzzy logic literature to extend the language with a unary connective $\triangle$ always semantically defined as $\triangle(a) = 1$ if $a = 1$ and $\triangle(a) = 0$ otherwise.

A *predicate language* $\mathcal{P}$ is a triple $\langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$, where $\mathbf{P}$ is a non-empty set of predicate symbols, $\mathbf{F}$ is a set of function symbols, and $\mathbf{ar}$ is a function assigning to each symbol a natural number called the *arity* of the symbol. Let us further fix a denumerable set V whose elements are called *object variables*. The sets of $\mathcal{P}$-*terms*, *atomic* $\mathcal{P}$-*formulas*, and $\langle \mathcal{L}, \mathcal{P} \rangle$-*formulas* are defined as in classical logic. A $\mathcal{P}$-*structure* $\mathbf{M}$ is a pair $\langle \boldsymbol{A}, \mathbf{M} \rangle$ where $\boldsymbol{A} \in \mathbb{L}$ and $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle F_{\mathbf{M}} \rangle_{F \in \mathbf{F}} \rangle$, where M is a non-empty domain; $P_{\mathbf{M}}$ is a function $M^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$; and $F_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary function symbol $F \in \mathbf{F}$. An $\mathbf{M}$-*evaluation* of the object variables is a mapping $\mathrm{v} \colon V \to M$; by $\mathrm{v}[x{\to}a]$ we denote the $\mathbf{M}$-evaluation where $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each object variable $y \neq x$. We define the *values* of the terms and the *truth values* of the formulas as:

$$\begin{array}{rcll}
\|x\|^{\mathbf{M}}_{\mathrm{v}} & = & \mathrm{v}(x), & \\
\|F(t_1, \dots, t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & F_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}), & \text{for } F \in \mathbf{F}, \\
\|P(t_1, \dots, t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & P_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathbf{M}}_{\mathrm{v}}), & \text{for } P \in \mathbf{P}, \\
\|\circ(\varphi_1, \dots, \varphi_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & \circ^{\boldsymbol{A}}(\|\varphi_1\|^{\mathbf{M}}_{\mathrm{v}}, \dots, \|\varphi_n\|^{\mathbf{M}}_{\mathrm{v}}), & \text{for } \circ \in \mathcal{L}, \\
\|(\forall x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} & = & \inf_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}, & \\
\|(\exists x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} & = & \sup_{\leq_{\boldsymbol{A}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}. &
\end{array}$$

If the infimum or supremum does not exist, the corresponding value is undefined. We say that $\mathbf{M}$ is *safe* if and only if $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}}$ is defined for each $\mathcal{P}$-formula φ and each $\mathbf{M}$-evaluation v. We say that $\mathbf{M}$ is a *model* of φ if it is safe and $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} \in \mathcal{F}^{\boldsymbol{A}}$ for each evaluation v. A $\mathcal{P}$-formula φ is a *semantical consequence* of a set of formulas T, in symbols $T \models_{\mathbb{L}} \varphi$, if each model of T is also a model of φ.

Observe that in this general presentation we do not require the presence of an equality symbol in the language. One can add it and force its interpretation to be crisp equality in the models (see [5]). In the paper, we will not assume (and not exclude either) the presence of equality, unless stated otherwise.

Definition 1 ([7])**.** *Let* L *be a logic in* $\mathcal{L}$ *presented by an axiomatic system* $\mathcal{AS}$*. The* minimal predicate logic over L *(in a predicate language* $\mathcal{P}$*), denoted as* L$\forall$*, is given by the following axiomatic system:*

(P) *the axioms and rules resulting from those of* $\mathcal{AS}$ *by substituting propositional variables by* $\mathcal{P}$*-formulas,*

($\forall$1) $\vdash_{\mathrm{L}\forall} (\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$*, where* t *is substitutable for* x *in* φ*,*

($\exists$1) $\vdash_{\mathrm{L}\forall} \varphi(t, \vec{z}) \to (\exists x)\varphi(x, \vec{z})$*, where* t *is substitutable for* x *in* φ*,*

($\forall$2) $\chi \to \varphi \vdash_{\mathrm{L}\forall} \chi \to (\forall x)\varphi$*, where* x *is not free in* χ*,*

($\exists$2) $\varphi \to \chi \vdash_{\mathrm{L}\forall} (\exists x)\varphi \to \chi$*, where* x *is not free in* χ*.*

Theorem 2 ([7])**.** *Let* L *be a logic and* $T \cup \{\varphi\}$ *a* $\mathcal{P}$*-theory. Then we have:* $T \vdash_{\mathrm{L}\forall} \varphi$ *if and only if* $T \models_{\mathbb{L}} \varphi$*.*

Therefore, these logics have indeed a completeness theorem with respect to the class of non-classical models defined over their algebraic counterpart and it makes sense to develop a general model theory to study them. In some non-classical logics, however, the intended semantics is restricted to a specific subclass of these models; for instance, in fuzzy logics, typically one considers only models based on linearly ordered algebras. To this end, the minimal predicate logic from the previous definition is extended to a stronger one by adding the following axiom:

($\forall$3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, *where* x *is not free in* χ*.*

This strengthened axiomatization corresponds indeed to the semantics of models based on chains. More generally, one can give an axiomatization for the logic of the models based on relatively finitely subdirectly irreducible algebras, which in the case of fuzzy logics are exactly the linearly ordered algebras (see [7] for details). If we consider, for instance, the two particular examples of propositional fuzzy logics mentioned above, we will denote by Ł$\forall$ and G$\forall$ their first-order versions complete with respect to models over linearly ordered algebras.

Finally, another variant of the semantics is that based on witnessed models (see [13, 14, 15, 16, 19]). A $\mathcal{P}$-model $\mathbf{M}$ is *witnessed* if for each $\mathcal{P}$-formula $\varphi(x, \vec{y})$ and for each $\vec{a} \in M$ there are $b_s, b_i \in M$ such that: $\|(\forall x)\varphi(x, \vec{a})\|^{\mathbf{M}} = \|\varphi(b_i, \vec{a})\|^{\mathbf{M}}$ and $\|(\exists x)\varphi(x, \vec{a})\|^{\mathbf{M}} = \|\varphi(b_s, \vec{a})\|^{\mathbf{M}}$. Not all logics are complete with respect to witnessed models. In fact, it has been shown (in the context of fuzzy logics, see e.g. [5]) that a logic is complete with respect to witnessed models if and only if it has the following theorems:

$$(\exists x)((\exists y)\psi(y, \vec{z}) \to \psi(x, \vec{z})) \text{ and } (\exists x)(\psi(x, \vec{z}) \to (\forall y)\psi(y, \vec{z})).$$

3 Homomorphisms

In this section we start by discussing several notions of homomorphism for non-classical structures aiming to find a suitable definition upon which we can base a non-classical model theory. Different definitions have been introduced so far in the literature for homomorphisms of structures. In particular, the notion of elementary morphism of fuzzy models in [10], elementary embeddings in [17], elementary fuzzy submodel and isomorphism of structures of first-order logic with graded syntax in [25], complete morphism in languages with a similarity predicate in [1], the notion of σ-embedding in [4]. Finally, taking all these previous works as starting point, the notions of weak homomorphism, homomorphism and strong homomorphism are introduced in [8], where the challenge was to encompass the most commonly used definitions in the literature and extend, still in a fuzzy logic framework, the corresponding notions of classical predicate logics in satisfactory way.

The present paper can be seen as a continuation of [8] by generalizing it to homomorphisms of models of a wide class of non-classical logics, beyond the context of fuzzy logics. We will propose three notions of homomorphism for non-classical logics: one in a structure-preserving fashion, another one of a categorical kind and, finally, another one of model-theoretic nature. As we will see, these three notions coincide in the case of classical logic, when homomorphisms are strict. In the next three subsections we formulate and discuss such notions.

3.1 Structure-Preserving Homomorphisms

Firstly we introduce the notion of homomorphism as a mapping that preserves the structure, that is, all operations and relations. Originating in the notion of homomorphism in Abstract Algebra, in classical logic homomorphisms are introduced as structure-preserving mappings (see [3, 18]). Let us recall this notion as it is defined in classical model theory.

Definition 3. (Cf. [18, Section 1.2]) *Let $\mathcal{P}$ be a first-order language and let $\mathbf{M}$ and $\mathbf{N}$ be $\mathcal{P}$-structures. A* homomorphism *from $\mathbf{M}$ to $\mathbf{N}$ is a mapping $g\colon M \to N$ such that:*

1. *For every n-ary functional symbol $F \in \mathcal{P}$, and elements $d_1, \ldots, d_n \in M$,*

$$g(F_{\mathbf{M}}(d_1, \ldots, d_n)) = F_{\mathbf{N}}(g(d_1), \ldots, g(d_n)).$$

2. *For every n-ary predicate symbol $P \in \mathcal{P}$, and elements $d_1, \ldots, d_n \in M$,*

$$P_{\mathbf{M}}(d_1, \ldots, d_n) = 1 \Rightarrow P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) = 1.$$

Observation 4. *Note that in Definition 3 we write $P_{\mathbf{M}}(d_1, \ldots, d_n) = 1$ instead of the usual notation for classical logic $\mathbf{M} \models P_{\mathbf{M}}(d_1, \ldots, d_n)$. Our purpose is to highlight that classical homomorphisms are a special case of the non-classical definition that will be formulated later with that notation. Although this notation is not the usual one, we can find it in the work of Mal'cev [20] and in Hájek's book [12, Section 1.3] when they introduce classical first-order logic.*

Now we introduce the notions of *mapping* and *homomorphism* between non-classical structures.

Definition 5 (Mapping)**.** *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be $\mathcal{P}$-structures. Let f be a mapping from A to B, and g be a mapping from M to N. The pair $\langle f, g\rangle$ is said to be a* mapping *from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$.*

Definition 6 (Homomorphism)**.** *Let $\langle f, g\rangle$ be a mapping from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$. We say that $\langle f, g\rangle$ is a* homomorphism *if and only if*

1. *f is a homomorphism of* L*-algebras.*

2. g is a homomorphism between the algebraic reducts of the first-order structures, that is, for every n-ary function symbol $F \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,

$$g(F_{\mathbf{M}}(d_1, \ldots, d_n)) = F_{\mathbf{N}}(g(d_1), \ldots, g(d_n)).$$

3. For every n-ary predicate symbol $P \in \mathcal{P}$, and every $d_1, \ldots, d_n \in M$,

$$P_{\mathbf{M}}(d_1, \ldots, d_n) \in \mathcal{F}^{\boldsymbol{A}} \Rightarrow P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) \in \mathcal{F}^{\boldsymbol{B}}.$$

We say that a homomorphism $\langle f, g\rangle$ is strict *if instead of 3 it satisfies the stronger condition:*

3s. For every n-ary predicate symbol $P \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,

$$P_{\mathbf{M}}(d_1, \ldots, d_n) \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) \in \mathcal{F}^{\boldsymbol{B}}.$$

We say that the pair $\langle f, g\rangle$ is an embedding *if it is a strict homomorphism and both functions f and g are injective. We say that an embedding $\langle f, g\rangle$ is an* isomorphism *if both functions f and g are onto. If f preserves all the existing infima and suprema, then $\langle f, g\rangle$ is called a σ-*homomorphism.

Note that this definition extends the classical notion of homomorphism in Definition 3, where the Boolean notion of truth (the element 1) has been replaced by the corresponding filters $\mathcal{F}^{\boldsymbol{A}}$ and $\mathcal{F}^{\boldsymbol{B}}$.

Example 7. *Recall the three-valued algebra $\boldsymbol{Ł}_3$ for the corresponding three-valued Łukasiewicz logic, in the language $\{\to, \wedge, \vee, \overline{0}\}$, whose universe is $\{0, \frac{1}{2}, 1\}$ and the operations are interpreted as $a \to^{\boldsymbol{Ł}_3} b = \min\{1, 1-a+b\}$, $a \wedge^{\boldsymbol{Ł}_3} b = \min\{a, b\}$, $a \vee^{\boldsymbol{Ł}_3} b = \max\{a, b\}$, and $\overline{0}^{\boldsymbol{Ł}_3} = 0$. The truth-constant $\overline{1}$ is defined as $\overline{0} \to \overline{0}$ and, hence, its interpretation is $\overline{1}^{\boldsymbol{Ł}_3} = 1$. Let $\mathcal{P}$ be a language with a binary predicate symbol P and a binary functional symbol F. Consider the $\mathcal{P}$-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$, and $\langle \boldsymbol{A}, \mathbf{N}\rangle$, where $\boldsymbol{A} = \boldsymbol{Ł}_3$, $M = N = \{0, 1, 2\}$, and the interpretations of P and F in both structures are defined in the following way:*

$P_{\mathbf{M}}$ is the relation given by the matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$$

$P_{\mathbf{N}}$ is the relation given by the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, $P_{\mathbf{N}}$ is the crisp relation $\{\langle 0, 1\rangle, \langle 1, 0\rangle\}$. The operators $F_{\mathbf{M}}$ and $F_{\mathbf{N}}$ are equally defined, $F_{\mathbf{M}} = F_{\mathbf{N}} = \oplus$, by the following table:

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Now consider the mapping $\langle f, g\rangle$ from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{A}, \mathbf{N}\rangle$, where $f = \mathrm{Id}_{Ł_3}$ and $g\colon \{0, 1, 2\} \to \{0, 1, 2\}$ is defined by $g(x) = x \oplus x$. The identity on $\boldsymbol{Ł}_3$ is obviously a homomorphism, and it is easy to check that g is a homomorphism from the algebraic reduct of the first structure to the algebraic reduct of the second one. Now we show that $\langle f, g\rangle$ is a strict homomorphism. Since $\mathcal{F}^{\boldsymbol{Ł}_3} = \{1\}$, we must prove that, for every $a, b \in \{0, 1, 2\}$,

$$P_{\mathbf{M}}(a, b) = 1 \quad \Leftrightarrow \quad P_{\mathbf{N}}(g(a), g(b)) = 1.$$

Indeed, $P_{\mathbf{M}}(a, b) = 1 \Leftrightarrow a = 0$ and $b = 2$ or $a = 2$ and $b = 0$, and this is equivalent to say that $P_{\mathbf{N}}(g(a), g(b)) = 1$. Thus $\langle f, g\rangle$ is a strict homomorphism from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{A}, \mathbf{N}\rangle$. Moreover, since f and g are bijective, we have that $\langle f, g\rangle$ is an isomorphism.

The next example shows that there are homomorphisms $\langle f, g\rangle$ where both f and g bijective and yet $\langle f, g\rangle$ is not an isomorphism.

Example 8. *Let $\mathcal{P}$ be as in Example 7. Consider now a $\mathcal{P}$-interpretation $\mathbf{N}'$ with the same domain than $\mathbf{M}$, the same interpretation of F, but such that $P_{\mathbf{N}'}$ is defined by the matrix:*

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, $P_{\mathbf{N}'}$ is the crisp relation $\{\langle 0, 1\rangle, \langle 1, 0\rangle, \langle 0, 2\rangle\}$. Now, take the mapping $\langle f, g\rangle$ from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{A}, \mathbf{N}'\rangle$ defined as in the previous example. This mapping satisfies conditions 1 and 2 in Definition 6. Moreover, we have that $P_{\mathbf{M}}(a, b) = 1$ is equivalent to $a = 0$ and $b = 2$ or $a = 2$ and $b = 0$ and this implies that $P_{\mathbf{N}}(g(a), g(b)) = 1$. However, the converse is not true since we have $P_{\mathbf{N}'}(g(0), g(1)) = P_{\mathbf{N}'}(0, 2) = 1$, but $P_{\mathbf{M}}(0, 1) = \frac{1}{2}$. Thus $\langle f, g\rangle$ is a homomorphism from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{A}, \mathbf{N}'\rangle$ which is not strict.

3.2 A Categorical Definition of Homomorphism

Di Nola and Gerla introduced in [10] the notions of *valuation structure* and *fuzzy model* of a given first-order language in a categorial setting (see also [11]). They worked with models where each quantifier is definable by a formula of the classical first-order language with equality and a unique monadic predicate P [11, Definition 8.1]. In [10] they presented also a notion of homomorphism for this kind of models from a categorical perspective. Let us recast it in our notation: given two structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ a homomorphism in the sense of [10] is a pair $\langle f, g\rangle$, where f is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, and g is a homomorphism between the algebraic reducts of $\mathbf{M}$ and $\mathbf{N}$ in such a way that, for every k-ary relational symbol P the following diagram commutes:

$$\begin{tikzcd} M^k \arrow[r, "G"] \arrow[d, "P^{\mathbf{M}}"'] & N^k \arrow[d, "P^{\mathbf{N}}"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where $G\colon M^k \longrightarrow N^k$ is the mapping $\langle d_1, \ldots, d_k\rangle \mapsto \langle g(d_1), \ldots, g(d_n)\rangle$.

We encompass this categorical notion of homomorphism introduced by Gerla and Di Nola in the present paper by means of the following notion of *strong homomorphism*.

Definition 9 (Strong Homomorphism). *Let $\langle f, g\rangle$ be a mapping from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$. We say that $\langle f, g\rangle$ is a* strong homomorphism *if and only if*

1) f is a homomorphism of L*-algebras.*

2) g is a homomorphism of the algebraic reducts of the first-order structures.

3) For every n-ary predicate symbol $P \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,

$$f(P_{\mathbf{M}}(d_1, \ldots, d_n)) = P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)).$$

Observe that every strong homomorphism is a homomorphism in the sense of Definition 6. Next we give an example of strong homomorphism:

Example 10. *Let $\mathcal{P}$ be the predicate language considered in Example 7. Consider the $\mathcal{P}$-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$, and $\langle \boldsymbol{A}, \mathbf{N}\rangle$, where again $M = N = \{0, 1, 2\}$, and $\boldsymbol{A}$ is the linearly ordered commutative residuated lattice defined on $\{0, \frac{1}{2}, 1\}$ by the uninorm given by the table:*

$*$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	0	1	1

The interpretation of F in both structures is the sum $\oplus$ defined in Example 7. $P_{\mathbf{M}}$ is now defined by the matrix:

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and $P_{\mathbf{N}}$ is now defined by the matrix:

$$\begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 1 \\ \frac{1}{2} & 1 & 0 \end{pmatrix}$$

Now consider the mapping $\langle f, g\rangle$, with $f = \mathrm{Id}_A$, and g is the function defined by $g(x) = x \oplus x$. We have that $\mathcal{F}^{\mathbf{A}} = \{\frac{1}{2}, 1\}$ (because in this uninorm-based logic the constant $\overline{1}$ is interpreted as $\frac{1}{2}$). This mapping is a strict homomorphism because

$$P_{\mathbf{M}}(a, b) \in \{\frac{1}{2}, 1\} \quad \Leftrightarrow \quad a \neq b \quad \Leftrightarrow \quad P_{\mathbf{N}}(a, b) \in \{\frac{1}{2}, 1\}.$$

An easy computation shows that $f(P_{\mathbf{M}}(a, b)) = P_{\mathbf{M}}(a, b) = P_{\mathbf{N}}(g(a), g(b))$, for every $a, b \in \{0, 1, 2\}$. Consequently, $\langle f, g\rangle$ is a strong homomorphism.

Next example demonstrates that not all strict homomorphisms are strong.

Example 11. *We take two structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$, and $\langle \boldsymbol{A}, \mathbf{N}\rangle$ as in Example 10 with the only difference that the definition of $P_{\mathbf{N}}$ is given by the matrix:*

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

We take the same mapping $\langle f, g\rangle$. On the one hand, this mapping is a strict homomorphism because

$$P_{\mathbf{M}}(a,b) \in \{\frac{1}{2}, 1\} \quad \Leftrightarrow \quad a \neq b \quad \Leftrightarrow \quad P_{\mathbf{N}}(a,b) \in \{1\} \subseteq \{\frac{1}{2}, 1\}.$$

On the other hand we have that, for instance, $P_{\mathbf{M}}(0,1) = \frac{1}{2}$ and $P_{\mathbf{N}}(g(0), g(1)) = P_{\mathbf{N}}(0,2) = 1$. Thus, we have $f(P_{\mathbf{M}}(0,1)) \neq P_{\mathbf{N}}(g(0), g(1))$. Therefore, $\langle f, g\rangle$ is not a strong homomorphism.

We also have that there are strong homomorphisms that are not strict, as the following example shows.

Example 12. *Let $\mathcal{P}$ be a predicate language with only one predicate symbol P of arity 1. We take two $\mathcal{P}$-structures $\langle \boldsymbol{B}, \mathbf{M}\rangle$, and $\langle \boldsymbol{B}, \mathbf{N}\rangle$, where $M = N = \{a, b\}$, and where $\boldsymbol{B}$ is the standard Gödel algebra $[0,1]_{\boldsymbol{G}}$. Thus now the filter is $\mathcal{F}^{\boldsymbol{B}} = \{1\}$. The interpretations for P in both structures are as follows: $P_{\mathbf{M}}(a) = 1, P_{\mathbf{M}}(b) = \frac{2}{3}$, and $P_{\mathbf{N}}(a) = P_{\mathbf{N}}(b) = 1$. We take now a mapping $\langle f, g\rangle$, where g is Id_M, and f is the algebraic homomorphism $f\colon [0,1] \longrightarrow [0,1]$ defined as follows:*

$$f(x) = \begin{cases} 1, & \text{if } x \in (\frac{1}{2}, 1], \\ x, & \text{otherwise.} \end{cases}$$

It is easy to check that $\langle f, g\rangle$ satisfies, for every $d \in M$, that $f(P_{\mathbf{M}}(d)) = P_{\mathbf{N}}(d)$ and hence it is a strong homomorphism. Nevertheless, we have that $P_{\mathbf{N}}(b) = 1$ but $P_{\mathbf{M}}(b) = \frac{2}{3} \neq 1$.

Remark that in the classical case, the notions of homomorphism and strong homomorphism do not coincide (if the homomorphism is not strict). Indeed, consider two first-order structures $\mathbf{M}$ and $\mathbf{N}$ with only two elements a and b in their domain and take f and g be the identity mappings. If the language has only one monadic predicate symbol P interpreted as $P_{\mathbf{M}}(a) = 1 = P_{\mathbf{N}}(a) = P_{\mathbf{N}}(b)$ and $P_{\mathbf{M}}(b) = 0$, then $\langle f, g\rangle$ is a homomorphism, though not strong.

Now we introduce an intermediate categorical notion of homomorphism, that we will call *filter-strong homomorphism.*

Definition 13 (Filter-strong Homomorphism)**.** *Let $\langle f, g\rangle$ be a mapping from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$. We say that $\langle f, g\rangle$ is a* filter-strong homomorphism *if and only if it satisfies Conditions 1) and 2) in Definition 9 and the following condition instead of 3):*

3') For every n-ary predicate symbol $P \in \mathcal{P}$, and every $d_1, \ldots, d_n \in M$,

$$P_{\mathbf{M}}(d_1, \ldots, d_n) \in \mathcal{F}^{\boldsymbol{A}} \Rightarrow f(P_{\mathbf{M}}(d_1, \ldots, d_n)) = P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)).$$

Notice that every filter-strong homomorphism is a homomorphism in the sense of Definition 6. Remark that in the classical case the notion of homomorphism and filter-strong homomorphism coincide. Example 11 shows a homomorphism which is not a filter-strong homomorphism since $P_{\mathbf{M}}(0,1) = \frac{1}{2} \in \mathcal{F}^{\boldsymbol{A}}$ but $f(P_{\mathbf{M}}(0,1)) \neq P_{\mathbf{N}}(g(0), g(1))$. Observe also that every strong homomorphism is a filter-strong homomorphism. The next example shows that the converse does not hold:

Example 14. *We take two structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$, and $\langle \boldsymbol{A}, \mathbf{N}\rangle$ as in Example 10 with the only difference that the definition of $P_{\mathbf{N}}$ is given by the matrix:*

$$\begin{pmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 1 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}$$

We take the same mapping $\langle f, g\rangle$. Recall that $\mathcal{F}^{\mathbf{A}} = \{\frac{1}{2}, 1\}$.

- *$\langle f, g\rangle$ is a homomorphism since for every $a, b \in \{0, 1, 2\}$ we have:*

$$P_{\mathbf{M}}(a, b) \in \{\frac{1}{2}, 1\} \Rightarrow a \neq b \Rightarrow g(a) \neq g(b) \Rightarrow P_{\mathbf{N}}(g(a), g(b)) \in \{\frac{1}{2}, 1\}$$

- *An easy computation shows that, for every $a, b \in \{0, 1, 2\}$, we have that $P_{\mathbf{M}}(a, b) = P_{\mathbf{N}}(g(a), g(b))$ whenever $P_{\mathbf{M}}(a, b) \in \{\frac{1}{2}, 1\}$.*
- *However, $f(P_{\mathbf{M}}(1, 1)) = P_{\mathbf{M}}(1, 1) = 0$, but $P_{\mathbf{N}}(g(1), g(1)) = P_{\mathbf{N}}(2, 2) = 1$.*

Thus, $\langle f, g\rangle$ is a filter-strong homomorphism but not a strong homomorphism.

3.3 A Model-theoretic Definition of Homomorphism

There are several works in the literature of fuzzy logics in which the notion of homomorphism and other related notions have been defined from a model-theoretic perspective as mappings preserving certain classes of first-order formulas, in the sense that if a formula (of the corresponding class) is true in the first structure then so it is its image in the second one. In [10] the authors define *elementary homomorphism* as a homomorphism (a strong homomorphism in our sense) that preserves all the formulas. In [17] an *elementary embedding* is defined as a pair $\langle f, g\rangle$ where f is an embedding of L-algebras and g is an injection such that all formulas are preserved. In the papers [8, 9], *weak homomorphism* and *homomorphism* are required to preserve quantifier-free formulas. Similarly, we now introduce a model-theoretic characterization of the notion of homomorphism given in Section 3.1.

Definition 15. *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be $\mathcal{P}$-structures and let $\langle f, g\rangle$ be a mapping from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$. Let Φ be a set of $\mathcal{P}$-formulas. We say that the mapping $\langle f, g\rangle$* preserves the formulas in Φ *if and only if, for every $\varphi(x_1, \ldots, x_n) \in \Phi$, and $d_1, \ldots, d_n \in M$,*

$$\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \ \Rightarrow\ \|\varphi(g(d_1), \ldots, g(d_n))\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}.$$

Theorem 16. *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be $\mathcal{P}$-structures and let $\langle f, g\rangle$ be a mapping from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$. The pair $\langle f, g\rangle$ is a homomorphism from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$ if and only if*

1. *f is a homomorphism of* L*-algebras.*
2. *g is a homomorphism between the algebraic reducts of the first-order structures.*
3. *$\langle f, g\rangle$ preserves all the atomic formulas.*

Proof. If (f, g) preserves all the atomic formulas, then for every n-ary predicate symbol $P \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$, if we consider the formula $P(x_1, \ldots, x_n)$ we have that

$$\|P(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \quad \Rightarrow \quad \|P(g(d_1), \ldots, g(d_n))\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}},$$

and thus we have condition 3 of Definition 6:

$$P_{\mathbf{M}}(d_1, \ldots, d_n) \in \mathcal{F}^{\boldsymbol{A}} \quad \Rightarrow \quad P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) \in \mathcal{F}^{\boldsymbol{B}}.$$

Now suppose that $\langle f, g\rangle$ is a homomorphism. To show that under this hypothesis condition 3) is also satisfied, we first prove that, for every $\mathcal{P}$-term $t(x_1, \ldots, x_n)$, and every $d_1, \ldots, d_n$,

$$g(\|t(d_1, \ldots, d_n)\|_{\mathbf{M}}) = \|t(g(d_1), \ldots, g(d_n))\|_{\mathbf{N}}. \tag{1}$$

We proceed by induction over the complexity of the term. If t is a variable x, then $g(\|x(d)\|_{\mathbf{M}}) = g(d) = \|x(g(d))\|_{\mathbf{N}}$. If $t = F(t_1, \ldots, t_k)$, and the variables of the terms t_i are in $\{x_1, \ldots, x_n\}$ we have:

$$\begin{aligned} g(\|F(t_1, \ldots, t_k)(d_1, \ldots, d_n)\|_{\mathbf{M}}) &= \\ g(F_{\mathbf{M}}(\|t_1(d_1, \ldots, d_n)\|_{\mathbf{M}}, \ldots, \|t_k(d_1, \ldots, d_n)\|_{\mathbf{M}})) &= \\ F_{\mathbf{N}}(g(\|t_1(d_1, \ldots, d_n)\|_{\mathbf{M}}), \ldots, g(\|t_k(d_1, \ldots, d_n)\|_{\mathbf{M}})) &= \\ F_{\mathbf{N}}(\|t_1(g(d_1), \ldots, g(d_n))\|_{\mathbf{N}}, \ldots, \|t_k(g(d_1), \ldots, g(d_n))\|_{\mathbf{N}}) &= \\ \|F(t_1, \ldots, t_k)(g(d_1), \ldots, g(d_n))\|_{\mathbf{N}}. \end{aligned}$$

The second equality is by Condition 2), and the third one by applying the induction hypothesis.

In order to prove 3), let $\varphi(x_1, \ldots, x_n)$ be an atomic formula. Suppose that $\varphi = P(t_1, \ldots, t_k)(x_1, \ldots, x_n)$. We have:

$$\begin{aligned} \|P(t_1, \ldots, t_k)(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} &\quad \Leftrightarrow \\ P_{\mathbf{M}}(\|t_1(d_1, \ldots, d_n)\|_{\mathbf{M}}, \ldots, \|t_k(d_1, \ldots, d_n)\|_{\mathbf{M}}) \in \mathcal{F}^{\boldsymbol{A}} &\quad \Rightarrow \\ P_{\mathbf{N}}(g(\|t_1(d_1, \ldots, d_n)\|_{\mathbf{M}}), \ldots, g(\|t_k(d_1, \ldots, d_n)\|_{\mathbf{M}})) \in \mathcal{F}^{\boldsymbol{B}} &\quad \Leftrightarrow \\ P_{\mathbf{N}}(\|t_1(g(d_1), \ldots, g(d_n))\|_{\mathbf{N}}, \ldots, \|t_k(g(d_1), \ldots, g(d_n))\|_{\mathbf{N}}) \in \mathcal{F}^{\boldsymbol{B}} &\quad \Leftrightarrow \\ \|P(t_1, \ldots, t_k)(g(d_1), \ldots, g(d_n))\|_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}. & \end{aligned}$$

The implication between second and third lines is by condition 3 in Definition 6; the biconditional between third and forth lines is justified by applying (1). $\square$

Table 1: Relationship among three different notions of homomorphism

Strong Homomorphism
$\Downarrow \qquad \not\Uparrow$
Filter-Strong Homomorphism
$\Downarrow \qquad \not\Uparrow$
Homomorphism
$\Uparrow \qquad \not\Downarrow$
Quantifier-free Preserving Homomorphism

Let us compare the characterization in Theorem 16 with the corresponding result in classical logic.

Theorem 17. (Cf. [18, Theorem 1.3.1]). *Let $\mathcal{P}$ be a first-order language, $\mathbf{M}$ and $\mathbf{N}$ be $\mathcal{P}$-structures, and g a mapping of M into N. The following conditions are equivalent:*

1. *g is a homomorphism of $\mathbf{M}$ into $\mathbf{N}$.*
2. *For every atomic $\mathcal{P}$-formula $\varphi(x_1, \ldots, x_n)$ and $d_1, \ldots, d_n \in M$,*

$$\|\varphi(d_1, \ldots, d_n)\|_{\mathbf{M}} = 1 \Rightarrow \|\varphi(g(d_1), \ldots, g(d_n))\|_{\mathbf{N}} = 1.$$

Moreover, if the homomorphism is strict, by induction on the complexity of the quantifier-free formulas, using Theorem 17 and the fact that in classical logic any quantifier-free formula is logically equivalent to one in normal form, we can prove that strict homomorphisms preserve all quantifier-free formulas.

On the other hand, by Theorem 16, if $\langle f, g\rangle$ preserves all quantifier-free formulas and f and g are homomorphisms, then $\langle f, g\rangle$ is a homomorphism. But the converse is not true: take any non-strict classical homomorphism g such that for an n-ary predicate symbol P and $d_1, \ldots, d_n \in M$, $P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) = 1$ but $P_{\mathbf{M}}(d_1, \ldots, d_n) = 0$, then the formula $\neg P(x_1, \ldots, x_n)$ is not preserved.

In Table 1 we summarize the relationships between the notions of homomorphism considered in this section. We conclude by adding yet a stronger notion of homomorphism that will be essential for the rest of the paper.

Definition 18 (Elementary Homomorphism)**.** *Let $\langle f, g\rangle$ be a homorphism from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$. We say that $\langle f, g\rangle$ is an* elementary homomorphism *if and only if, for every formula $\varphi(x_1, \ldots, x_n)$, and $d_1, \ldots, d_n \in M$,*

$$f(\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}) = \|\varphi(g(d_1), \ldots, g(d_n))\|^{\boldsymbol{B}}_{\mathbf{N}}.$$

Observe that this last notion combines, in a way, the three perspectives (structure-preserving, categorical and formula-preserving) considered in this section. More precisely, an elementary homomorphism is a strong homomorphism in which the defining condition 3 is required, not only for atomic formulas, but for all formulas, and hence it also preserves all formulas in the sense of Definition 15. This observation naturally brings us to the notion of elementarily equivalent structures.

Definition 19. *We say that two $\mathcal{P}$-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ are* elementarily equivalent *if for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in F^{\boldsymbol{A}}$ if and only if $\|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in F^{\boldsymbol{B}}$.*

Clearly, if there is a strict elementary homomorphism between two structures, then they are elementarily equivalent.

4 Substructures

As in classical logic, we want to define a substructure of a bigger one in such a way that quantifier-free formulas take the same truth-value in both structures.

Definition 20 (Substructure)**.** *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be $\mathcal{P}$-structures. We will say that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a* substructure *of $\langle \boldsymbol{B}, \mathbf{N}\rangle$ if the following conditions are satisfied:*

1. *$M \subseteq N$.*
2. *For each n-ary function symbol $F \in \mathcal{P}$, and elements $d_1, \dots, d_n \in M$,*
$$F_{\mathbf{M}}(d_1, \dots, d_n) = F_{\mathbf{N}}(d_1, \dots, d_n).$$
3. *$\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}$.*
4. *For every quantifier-free formula $\varphi(x_1, \dots, x_n)$, and $d_1, \dots, d_n \in M$,*
$$\|\varphi(d_1, \dots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi(d_1, \dots, d_n)\|^{\boldsymbol{B}}_{\mathbf{N}}.$$

Moreover, if both structures are safe, it is said that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is an *elementary substructure* of $\langle \boldsymbol{B}, \mathbf{N}\rangle$ if and only if conditions 1, 2 and 3 (of substructure) are satisfied and condition 4 holds for arbitrary formulas. When $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is an elementary substructure of $\langle \boldsymbol{B}, \mathbf{N}\rangle$ we say that $\langle \boldsymbol{B}, \mathbf{N}\rangle$ is an *elementary extension* of $\langle \boldsymbol{A}, \mathbf{M}\rangle$.

Observe that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a substructure of $\langle \boldsymbol{B}, \mathbf{N}\rangle$ if and only if $M \subseteq N$ and $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $\langle \mathrm{Id}_A, \mathrm{Id}_M\rangle$ is a quantifier-free preserving strong homomorphism, in the sense of the previous section. Similarly, $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a elementary substructure of $\langle \boldsymbol{B}, \mathbf{N}\rangle$ if and only if $\langle \mathrm{Id}_A, \mathrm{Id}_M\rangle$ is an elementary homomorphism; in this case, obviously, $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ are elementarily equivalent.

One of the first important results on classical model theory regarding elementary substructures is the Tarski–Vaught Test [29], which give us necessary and sufficient criteria for a structure to be an elementary substructure of another one. Here we prove a non-classical version, using the notion of definable set of elements of the algebra of the structure:

Definition 21 (Definable set of the algebra with parameters)**.** *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ be a $\mathcal{P}$-structure, $K \subseteq M$, $e_1, \dots, e_n \in K$, and $\varphi(x, y_1, \dots, y_n)$ a $\mathcal{P}$-formula. We denote by $X^{\langle \boldsymbol{A}, \mathbf{M}\rangle}_{\varphi, e_1, \dots, e_n, K}$ the following subset of A:*
$$\{\|\varphi(d, e_1, \dots, e_n)\|^{\boldsymbol{A}}_{\mathbf{M}} \mid d \in K\}.$$
It is said that a subset Y of A is definable with parameters in $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *if there are $K \subseteq M$, $e_1, \dots, e_n$, and a $\mathcal{P}$-formula $\varphi(x, y_1, \dots, y_n)$ such that*
$$Y = X^{\langle \boldsymbol{A}, \mathbf{M}\rangle}_{\varphi, e_1, \dots, e_n, K}.$$

Proposition 22 (Non-Classical Tarski–Vaught Test)**.** *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be safe $\mathcal{P}$-structures. Then the following are equivalent:*

1) $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is an elementary substructure of $\langle \boldsymbol{B}, \mathbf{N}\rangle$.

2) $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a substructure of $\langle \boldsymbol{B}, \mathbf{N}\rangle$ and, for every formula $\varphi(x, y_1, \dots, y_n)$, and elements $e_1, \dots, e_n \in M$, the sets $X^{\langle \boldsymbol{A},\mathbf{M}\rangle}_{\varphi,e_1,\dots,e_n,M}$ and $X^{\langle \boldsymbol{B},\mathbf{N}\rangle}_{\varphi,e_1,\dots,e_n,N}$ have the same infimum and supremum in $\boldsymbol{A}$.

Proof. 1) ⇒ 2): Since $\langle \boldsymbol{B}, \mathbf{M}\rangle$ is an elementary substructure of $\langle \boldsymbol{A}, \mathbf{N}\rangle$, given a $\mathcal{P}$-formula
$\varphi(x, y_1, \dots, y_n)$, and elements $e_1, \dots, e_n \in M$, we have:

$$\|(\forall x)\varphi(x, e_1, \dots, e_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|(\forall x)\varphi(x, e_1, \dots, e_n)\|^{\boldsymbol{B}}_{\mathbf{N}}, \quad \text{and}$$

$$\|(\exists x)\varphi(x, e_1, \dots, e_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|(\exists x)\varphi(x, e_1, \dots, e_n)\|^{\boldsymbol{B}}_{\mathbf{N}}.$$

This implies that the sets $X^{\langle \boldsymbol{A},\mathbf{M}\rangle}_{\varphi,e_1,\dots,e_n,M}$ and $X^{\langle \boldsymbol{B},\mathbf{N}\rangle}_{\varphi,e_1,\dots,e_n,N}$ have the same infimum and the same supremum.

2) ⇒ 1): Since $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a substructure of $\langle \boldsymbol{B}, \mathbf{N}\rangle$, we have that, for every quantifier-free formula $\varphi(y_1, \dots, y_n)$, and elements $e_1, \dots, e_n \in M$,

$$\|\varphi(e_1, \dots, e_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi(e_1, \dots, e_n)\|^{\boldsymbol{B}}_{\mathbf{N}}.$$

We must prove that this identity holds for every $\mathcal{P}$-formula. Suppose that $\varphi = (\forall x)\psi(x, y_1, \dots, y_n)$ (the case for existential formulas is analougous). Now, by using 2), we have:

$$\|(\forall x)\psi(x, e_1, \dots, e_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = \inf X^{\langle \boldsymbol{A},\mathbf{M}\rangle}_{\psi,e_1,\dots,e_n,M} =$$

$$= \inf X^{\langle \boldsymbol{B},\mathbf{N}\rangle}_{\psi,e_1,\dots,e_n,N} = \|(\forall x)\psi(x, e_1, \dots, e_n)\|^{\boldsymbol{B}}_{\mathbf{N}}.$$

□

5 Löwenheim–Skolem theorems for classical logic

Before we present our versions of Löwenheim–Skolem theorems for a wide class of non-classical logics, let us recall their classical formulations and their usual proofs identifying the use of classical properties and the difficulties to generalize the results to a non-classical framework.

Theorem 23 (Classical Downward Löwenheim–Skolem theorem)**.** *Let $\mathcal{P}$ be a predicate language and $\mathbf{M}$ a $\mathcal{P}$-structure. For each subset $A \subseteq M$, and κ a cardinal such that* $\max\{\omega, |\mathcal{P}|, |A|\} \leq \kappa \leq |M|$*, there is an elementary substructure $\mathbf{N}$ of $\mathbf{M}$ such that $|N| = \kappa$ and $A \subseteq N$.*

Let us recall the main ideas behind the usual proof of this result. Given a $\mathcal{P}$-formula $\varphi(\vec{x}, y)$ and a tuple $\overrightarrow{a}$ of elements of M such that $\mathbf{M} \models (\exists y)\varphi(\vec{a}, y)$, we can always choose a witness for that satisfied existential formula, i.e. an element $b_{\varphi,\vec{a}} \in M$ such that $\mathbf{M} \models \varphi(\vec{a}, b_{\varphi,\vec{a}})$. For any $X \subseteq M$, define

$$X' = \bigcup_{n\in\omega} \{b_{\varphi,\vec{a}} \mid \vec{a} \in X^n, \varphi \text{ a } \mathcal{P}\text{-formula with } n+1 \text{ free variables}\}.$$

Now, extend A to a subset $X_0 \subseteq M$ of cardinality κ. For each $n \geq 1$, define $X_{n+1} = X_n \cup (X_n)'$ and $N = \bigcup_{n\in\omega} X_n$. Clearly, for each $n \in \omega$, $|X_n| = \kappa$ and, hence, $|N| = \kappa$. By the Tarski–Vaught Test, N is the domain of an elementary substructure of $\mathbf{M}$.

This reasoning cannot be repeated in non-classical logics, because in general it is not guaranteed that the satisfaction of an existential formula is witnessed by a particular element of the domain.

Example 24 (Failure of witnessing in non-classical logics)**.** *Consider the Gödel-Dummett first-order logic* $\mathrm{G}\forall$*,* $\mathcal{P}$ *with a unary predicate* P *and* $\langle [0,1]_{\mathrm{G}}, \mathbf{M}\rangle$ *with* $M = \mathsf{N}$ *and* $P_{\mathbf{M}}(n) = \frac{n-1}{n}$ *for each* $n \in \mathsf{N}$*. Then:* $\|(\exists x)P(x)\|^{[0,1]_{\mathrm{G}}}_{\mathbf{M}} = 1$*, while for each* $n \in \mathsf{N}$*,* $\|P(n)\|^{[0,1]_{\mathrm{G}}}_{\mathbf{M}} < 1$*.* $\mathrm{G}\forall$ *is not even complete with respect to witnessed models (one can easily check that* $\nvdash_{\mathrm{G}\forall} (\exists x)(\varphi(x) \to (\forall y)\varphi(y))$*).*

Theorem 25 (Classical Upward Löwenheim–Skolem theorem)**.** *Let* $\mathcal{P}$ *be a predicate language,* $\mathbf{M}$ *an infinite* $\mathcal{P}$*-structure, and* $\kappa \geq \max\{|\mathcal{P}|, |M|\}$ *a cardinal. Then there is an elementary extension* $\mathbf{N}$ *of* $\mathbf{M}$ *such that* $|N| = \kappa$*.*

To prove this theorem, one first extends the language with κ-many new constants, that is, we define $\mathcal{P}' = \mathcal{P} \cup \{c_i \mid i < \kappa\}$. Then, we consider the $\mathcal{P}'$-theory $\Sigma = D(\mathbf{M}) \cup \{\neg(c_i \approx c_j) \mid i < j < \kappa\}$, containing the diagram of $\mathbf{M}$ and forcing all new constants to be interpreted in pairwise different elements. Observe that the hypothesis that $|M| \leq \kappa$ allows us to assume that the diagram language is included in $\mathcal{P}'$, i.e. $\{\bar{a} \mid a \in M\} \subseteq \{c_i \mid i < \kappa\}$. Let $\mathbf{M}'$ be the expansion of $\mathbf{M}$ where for each $a \in M$, $\bar{a}_{\mathbf{M}'} = a$. It is clear that $\mathbf{M}'$ satisfies any finite subset of Σ. Therefore, Σ is finitely consistent and, hence, by compactness, there is a $\mathcal{P}'$-structure $\mathbf{N}' \models \Sigma$. Because of the definition of Σ, $|N'| \geq \kappa$; let $\mathbf{N}$ be its $\mathcal{P}$-reduct. Since $\mathbf{N}'$ is a model of the diagram of $\mathbf{M}$, we know that $\mathbf{N}$ is an elementary extension of $\mathbf{M}$. By the Downward Löwenheim–Skolem theorem we can assume that $|N| = \kappa$.

Again, we have used a crucial property of classical logic, namely compactness, that cannot be taken for granted in non-classical logics.

Example 26 ([6], Failure of compactness in non-classical logics)**.** *Consider the first-order Łukasiewicz logic with* $\triangle$ *in a predicate language with two unary predicates* R *and* P*, and take the theory*

$$\Gamma = \{\neg\triangle\neg R(c), \neg\triangle P(c)\} \cup \{\triangle(R(c) \to P(c)^i) \mid i \geq 1\}.$$

Given an arbitrary finite $\Gamma_0 \subseteq \Gamma$*, let* j *be the maximum exponent* i *in* Γ_0*. Take* $\langle [0,1]_{\text{Ł}_\triangle}, \mathbf{M}\rangle$ *such that* $\|R(c)\|^{[0,1]_{\text{Ł}_\triangle}}_{\mathbf{M}} = \frac{1}{2}$ *and* $\|P(c)\|^{[0,1]_{\text{Ł}_\triangle}}_{\mathbf{M}} < 1$ *and* $\left\|P(c)^j\right\|^{[0,1]_{\text{Ł}_\triangle}}_{\mathbf{M}} \geq \frac{1}{2}$*. This is a model of* Γ_0*. Therefore,* Γ *is finitely satisfiable. Assume now that* Γ *has a model* $\langle [0,1]_{\text{Ł}_\triangle}, \mathbf{M}\rangle$*. Then* $\|P(c)\|^{[0,1]_{\text{Ł}_\triangle}}_{\mathbf{M}} < 1$ *and* $\|R(c)\|^{[0,1]_{\text{Ł}_\triangle}}_{\mathbf{M}} > 0$*, and thus there is* j *such that* $\left\|P(c)^j\right\|^{[0,1]_{\text{Ł}_\triangle}}_{\mathbf{M}} < \|R(c)\|^{[0,1]_{\text{Ł}_\triangle}}_{\mathbf{M}}$*; a contradiction!*

6 Löwenheim–Skolem theorems for non-classical logics

In this section we present direct proofs of Löwenheim–Skolem theorems for non-classical predicate logics. An important precedent is the work of G. Gerla in [11], where he proposed an interesting approach to the study of first-order fuzzy models. He defined the notions of d-filter, of reduced product and of ultraproduct of a family of fuzzy models with definable quantifiers, that is, models such that for each quantifier there is a formula of the classical first-order language with equality with a unique monadic predicate that defines it (see [11, Definition (8.1)]). By using these constructions he showed analogues to the Löwenheim–Skolem–Tarski Theorems for fuzzy models. Here we present new proofs or these theorems without making use of the ultraproduct construction and, in the case of the Upward Löwenheim–Skolem Theorem, we improve the theorem obtaining a model over the same L-algebra.

Definition 27 (Cardinality of a structure)**.** *Given a structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$, we say that its* cardinality *is the cardinality of the domain M, denoted by $|M|$.*

Definition 28. *Given a structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$, we denote by $p(\boldsymbol{A})$ the minimum cardinal γ such that, for every $X \subseteq A$ definable with parameters in $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that its infimum and supremum exist, there is a $Y \subseteq X$ of cardinality $\leq \gamma$, which also has infimum and supremum and such that* $\inf X = \inf Y$ *and* $\sup X = \sup Y$.

Definition 29 (Generated substructure)**.** *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ be a $\mathcal{P}$-structure and take two sets $A_0 \subseteq A$ and $M_0 \subseteq M$. The* substructure of *$\langle \boldsymbol{A}, \mathbf{M}\rangle$ generated by A_0 and M_0 is the intersection of all the substructures $\langle \boldsymbol{B}, \mathbf{N}\rangle$ of $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that $A_0 \subseteq B$ and $M_0 \subseteq N$.*

Theorem 30 (Non-classical Downward Löwenheim–Skolem Theorem)**.** *Take a safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and assume that every subset of A definable with parameters in $\langle \boldsymbol{A}, \mathbf{M}\rangle$ has infimum and supremum. Then, for every $Z \subseteq M$ and every cardinal κ such that*

$$\max\{|\mathcal{P}|, \omega, |Z|, p(\boldsymbol{A})\} \leq \kappa \leq |M|,$$

there is a safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{N}\rangle$ which is an elementary substructure of $\langle \boldsymbol{A}, \mathbf{M}\rangle$ such that $|N| \leq \kappa$ and $Z \subseteq N$.

Proof. Given $Z \subseteq M$ and κ such that $\max\{|\mathcal{P}|, \omega, |Z|, p(\boldsymbol{A})\} \leq \kappa \leq |M|$, we define inductively a chain $\langle Z_n \mid n \in \omega\rangle$ of subsets of M such that, for every $n \in \omega$, $Z \subseteq Z_n$ and $|Z_n| \leq \kappa$. We start by choosing $Z_0 = Z$. Now, given $Z_n \subseteq M$, Z_{n+1} is defined in the following form: for every $\mathcal{P}$-formula $\varphi(x, y_1, \ldots, y_k)$ and $e_1, \ldots, e_k \in Z_n$, we take a subset of $X^{\langle \boldsymbol{A}, \mathbf{M}\rangle}_{\varphi, e_1, \ldots, e_k, M}$ with the same supremum and infimum, say $Y_{\varphi, e_1, \ldots, e_k, n}$, with cardinality $\leq p(\boldsymbol{A})$. Now, for each element $b \in Y_{\varphi, e_1, \ldots, e_k, n}$, we choose $d_b \in M$ such that

$$\|\varphi(d_b, e_1, \ldots, e_k)\|^{\boldsymbol{A}}_{\mathbf{M}} = b$$

and then take Z_{n+1} to be the domain of the substructure generated by

$$Z_n \cup \{d_b \mid b \in Y_{\varphi, e_1, \ldots, e_k, n}, \varphi \text{ a } \mathcal{P}\text{-formula, } e_1, \ldots, e_k \in Z_n\}.$$

From $\max\{|\mathcal{P}|, \omega, p(\boldsymbol{A})\} \leq \kappa$, an easy induction on n shows that for every $n \in \omega$, $|Z_n| \leq \kappa$. Now, let $\langle \boldsymbol{A}, \mathbf{N}\rangle$ the substructure with domain $N = \bigcup_{n\in\omega} Z_n$. Observe that the hypothesis about existence of suprema and infima of definable sets implies that $\langle \boldsymbol{A}, \mathbf{N}\rangle$ is safe. Moreover, it is clear that $|N| \leq \kappa$ and $Z \subseteq N$.

Finally we must prove that $\langle \boldsymbol{A}, \mathbf{N}\rangle$ is an elementary substructure of $\langle \boldsymbol{A}, \mathbf{M}\rangle$. We do it by induction on the complexity of a formula $\varphi(y_1, \ldots, y_k)$. The induction base and steps for quantifier-free $\varphi(y_1, \ldots, y_k)$ follow directly from the fact that $\langle \boldsymbol{A}, \mathbf{N}\rangle$ is a substructure of $\langle \boldsymbol{A}, \mathbf{M}\rangle$. Assume that the property holds for $\varphi(y_1, \ldots, y_k)$ and we have to show $\|(\exists x)\varphi(x, e_1, \ldots, e_k)\|^{\boldsymbol{A}}_{\mathbf{N}} = \|(\exists x)\varphi(x, e_1, \ldots, e_k)\|^{\boldsymbol{A}}_{\mathbf{M}}$, where $e_1, \ldots, e_k \in N$. By the induction hypothesis, $X^{\langle \boldsymbol{A},\mathbf{N}\rangle}_{\varphi,e_1,\ldots,e_k,N} = X^{\langle \boldsymbol{A},\mathbf{M}\rangle}_{\varphi,e_1,\ldots,e_k,N}$. Take the minimum $n \in \omega$ such that $e_1, \ldots, e_k \in Z_n$. By the construction above $Y_{\varphi,e_1,\ldots,e_k,n}$ and $X^{\langle \boldsymbol{A},\mathbf{M}\rangle}_{\varphi,e_1,\ldots,e_k,M}$ have the same supremum. We show that $Y_{\varphi,e_1,\ldots,e_k,n} \subseteq X^{\langle \boldsymbol{A},\mathbf{N}\rangle}_{\varphi,e_1,\ldots,e_k,N}$. Indeed, for each $b \in Y_{\varphi,e_1,\ldots,e_k,n}$, by the construction we know that there is $d_b \in Z_{n+1}$ such that $b = \|\varphi(d_b, e_1, \ldots, e_k)\|^{\boldsymbol{A}}_{\mathbf{M}}$; since $d_b, e_1, \ldots, e_k \in N$, and using the induction hypothesis, we have $b \in X^{\langle \boldsymbol{A},\mathbf{N}\rangle}_{\varphi,e_1,\ldots,e_k,N}$. Therefore, we can write:

$$Y_{\varphi,e_1,\ldots,e_k,n} \subseteq X^{\langle \boldsymbol{A},\mathbf{N}\rangle}_{\varphi,e_1,\ldots,e_k,N} = X^{\langle \boldsymbol{A},\mathbf{M}\rangle}_{\varphi,e_1,\ldots,e_k,N} \subseteq X^{\langle \boldsymbol{A},\mathbf{M}\rangle}_{\varphi,e_1,\ldots,e_k,M},$$

and hence $X^{\langle \boldsymbol{A},\mathbf{N}\rangle}_{\varphi,e_1,\ldots,e_k,N}$ and $X^{\langle \boldsymbol{A},\mathbf{M}\rangle}_{\varphi,e_1,\ldots,e_k,M}$ also have the same supremum as we wanted. The induction step for the universal quantifier is shown analogously. □

We can also obtain a non-classical Upward Löwenheim–Skolem Theorem, but only assuming that the language does not include a symbol for crisp equality.

Theorem 31 (Non-classical Upward Löwenheim–Skolem Theorem)**.** *Let $\mathcal{P}$ be an equality-free language. For every infinite safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and every cardinal κ with $\max\{|M|, |\mathcal{P}|\} \leq \kappa$, there is a safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{N}\rangle$ of cardinality κ and an elementary embedding from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{A}, \mathbf{N}\rangle$.*

Proof. Take an enumeration $M = \{d_j \mid j \in \kappa\}$ and a set of new (pairwise different) variables $V_\kappa = \{v_j \mid j \in \kappa\}$. Let N be the set of all $\mathcal{P}$-terms built from variables in V_κ. Since κ is infinite, it is clear that $|N| = \kappa$.

We will define a structure $\langle \boldsymbol{A}, \mathbf{N}\rangle$. First, for every n-ary functional symbol F and $t_1, \ldots, t_n \in N$, we define: $F_{\mathbf{N}}(t_1, \ldots, t_n) = F(t_1, \ldots, t_n)$. Now consider the mapping $g_0 \colon V_\kappa \to M$ defined as $g_0(v_j) = d_j$ for each $j \in \kappa$ and extend it, in the obvious way, to a homomorphism $g \colon N \to M$. For every n-ary relational symbol P and $t_1, \ldots, t_n \in N$, we define: $P_{\mathbf{N}}(t_1, \ldots, t_n) = P_{\mathbf{M}}(g(t_1), \ldots, g(t_n))$.

First, let us show that, for every $\mathcal{P}$-term $r(x_1, \ldots, x_n)$, and $t_1, \ldots, t_n \in N$, we have that

$$g(\|r(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}}) = \|r(g(t_1), \ldots, g(t_n))\|^{\boldsymbol{A}}_{\mathbf{M}}. \tag{2}$$

Indeed, it can be seen by an easy induction: the base case when r is a variable is obvious and the induction step follows from the fact that g is a homomorphism with respect to all functional symbols. Now, we prove that, for each $\mathcal{P}$-formula $\varphi(x_1, \ldots, x_n)$, and each $t_1, \ldots, t_n \in N$, we have that

$$\|\varphi(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}} = \|\varphi(g(t_1), \ldots, g(t_n))\|^{\boldsymbol{A}}_{\mathbf{M}}.$$

Assume first that φ is an atomic formula $P(r_1, \ldots, r_k)$. Then, by using definitions and the equation (2), we have:

$$\begin{aligned}
& \|P(r_1, \ldots, r_k)(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}} = \\
= \quad & P_{\mathbf{N}}(\|r_1(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}}, \ldots, \|r_k(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}}) = \\
= \quad & P_{\mathbf{M}}(g(\|r_1(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}}), \ldots, g(\|r_k(t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}})) = \\
= \quad & P_{\mathbf{M}}(\|r_1(g(t_1), \ldots, g(t_n))\|^{\boldsymbol{A}}_{\mathbf{M}}, \ldots, \|r_k(g(t_1), \ldots, g(t_n))\|^{\boldsymbol{A}}_{\mathbf{M}}) = \\
= \quad & \|P(r_1, \ldots, r_k)(g(t_1), \ldots, g(t_n))\|^{\boldsymbol{A}}_{\mathbf{M}}.
\end{aligned}$$

The induction step for propositional connectives is clear. Finally, assume that $\varphi(t_1, \ldots, t_n) = (\forall x)\psi(x, t_1, \ldots, t_n)$ (the case of existential formulas is completely analogous). We have:

$$\begin{aligned}
\|(\forall x)\psi(x, t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}} &= \inf\{\|\psi(t, t_1, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{N}} \mid t \in N\} = \\
&= \inf\{\|\psi(g(t), g(t_1), \ldots, g(t_n))\|^{\boldsymbol{A}}_{\mathbf{M}} \mid t \in N\} = \\
&= \inf\{\|\psi(d, g(t_1), \ldots, g(t_n))\|^{\boldsymbol{A}}_{\mathbf{M}} \mid d \in M\} = \\
&= \|(\forall x)\psi(x, g(t_1), \ldots, g(t_n))\|^{\boldsymbol{A}}_{\mathbf{M}}.
\end{aligned}$$

(The second equality holds by induction hypothesis, the third one because g is onto, the rest by definition; observe that the infimum exists because $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is safe.)

Therefore, we have obtained that $\langle \boldsymbol{A}, \mathbf{N}\rangle$ is a safe $\mathcal{P}$-structure and, moreover, $\langle \mathrm{Id}_A, g\rangle$ is an elementary homomorphism. Now we define a mapping $\langle \mathrm{Id}_A, h\rangle \colon \langle \boldsymbol{A}, \mathbf{M}\rangle \to \langle \boldsymbol{A}, \mathbf{N}\rangle$. For every $d \in M$ take the minimum $j \in \kappa$ such that $d = d_j$, and define $h(d_j) = v_j$. Clearly, h is an injective mapping. It only remains to show that $\langle \mathrm{Id}_A, h\rangle$ is an elementary homomorphism too. Take an arbitrary $\mathcal{P}$-formula φ with n free variables and take $d_{i_1}, \ldots, d_{i_n} \in M$. Then we have (using that $\langle \mathrm{Id}_A, g\rangle$ is an elementary homomorphism):

$$\begin{aligned}
\mathrm{Id}_{\boldsymbol{A}}(\|\varphi(d_{i_1}, \ldots, d_{i_n})\|^{\boldsymbol{A}}_{\mathbf{M}}) &= \|\varphi(d_{i_1}, \ldots, d_{i_n})\|^{\boldsymbol{A}}_{\mathbf{M}} = \\
&= \|\varphi(g(v_{i_1}), \ldots, g(v_{i_n}))\|^{\boldsymbol{A}}_{\mathbf{M}} = \\
&= \|\varphi(v_{i_1}, \ldots, v_{i_n})\|^{\boldsymbol{A}}_{\mathbf{N}} = \\
&= \|\varphi(h(d_{i_1}), \ldots, h(d_{i_n}))\|^{\boldsymbol{A}}_{\mathbf{N}}.
\end{aligned}$$

□

In case the language contains no functional symbols (and still no crisp equality), we can improve the previous result and obtain an upward theorem in which, instead of using a mapping, the initial structure is directly extended to a bigger one in such a way that inclusion is the desired elementary embedding.

Theorem 32 (Non-classical Upward Löwenheim–Skolem Theorem for relational languages)**.** *If $\mathcal{P}$ is an equality-free purely relational predicate language. For every safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and every cardinal κ with $\max\{|M|, |\mathcal{P}|\} \leq \kappa$, there is a safe structure $\langle \boldsymbol{A}, \mathbf{N}\rangle$ of cardinality κ such that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is an elementary substructure of $\langle \boldsymbol{A}, \mathbf{N}\rangle$.*

Proof. Since $|M| \leq \kappa$, we can extend it to a superset N of cardinal κ. Let us take an enumeration $N = \{e_j \mid j \in \kappa\}$. We use the same cardinal to enumerate the smaller set M, possibly with repetitions, as follows. We define $j_0 = \min\{j < k \mid e_j \in M\}$ and then we state $M = \{d_j \mid j \in \kappa\}$, where:

$$d_j = \begin{cases} e_j, & \text{if } e_j \in M, \\ e_{j_0}, & \text{otherwise.} \end{cases}$$

Now we define the function:

$$\begin{aligned} g\colon N &\to M \\ e_j &\mapsto d_j \end{aligned}$$

Thus, g so defined, we have that $g{\restriction}M = \mathrm{Id}_M$ and $g{\restriction}(N \backslash M) = (N \backslash M) \times \{e_{j_0}\}$. Now, we define a structure $\langle \boldsymbol{A}, \mathbf{N} \rangle$ by establishing that for every n-ary relational symbol P, and $e_{j_1}, \ldots, e_{j_n} \in N$,

$$P_{\mathbf{N}}(e_{j_1}, \ldots, e_{j_n}) = P_{\mathbf{M}}(g(e_{j_1}), \ldots, g(e_{j_n})).$$

From the previous definition, it is obvious that $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is a substructure of $\langle \boldsymbol{A}, \mathbf{N} \rangle$. In order to check that $\langle \boldsymbol{A}, \mathbf{N} \rangle$ is safe, we will prove by induction that for every formula $\varphi(x_1, \ldots, x_n)$, and elements $e_{j_1}, \ldots, e_{j_n} \in N$,

$$\|\varphi(e_{j_1}, \ldots, e_{j_n})\|^{\boldsymbol{A}}_{\mathbf{N}} = \|\varphi(g(e_{j_1}), \ldots, g(e_{j_n}))\|^{\boldsymbol{A}}_{\mathbf{M}}. \tag{3}$$

Indeed, suppose that φ is an atomic formula given by an n-ary predicate P. Since the language is purely relational, it must be of the form $P(x_1, \ldots, x_n)$. Then, using the definition of $P_{\mathbf{N}}$, we have:

$$\|\varphi(e_{j_1}, \ldots, e_{j_n})\|^{\boldsymbol{A}}_{\mathbf{N}} = P_{\mathbf{N}}(e_{j_1}, \ldots, e_{j_n}) =$$

$$= P_{\mathbf{M}}(g(d_{j_1}), \ldots, g(d_{j_n})) = \|\varphi(g(e_{j_1}), \ldots, g(e_{j_n}))\|^{\boldsymbol{A}}_{\mathbf{M}}.$$

The induction step for formulas built by using the connectives of the logic is clear; the step for quantified formulas is proved analogously as in the proof of Theorem 31 essentially using that g is onto.

Having condition (3) for every $\mathcal{P}$-formula φ, we have obtained, a fortiori, that $\langle \mathrm{Id}_A, g \rangle$ is an elementary homomorphism. We can use this fact to prove that $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is an elementary substructure of $\langle \boldsymbol{A}, \mathbf{N} \rangle$. Take $d_{j_1}, \ldots, d_{j_n} \in M$. Then,

$$\|\varphi(d_{j_1}, \ldots, d_{j_n})\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi(g(d_{j_1}), \ldots, g(d_{j_n}))\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi(d_{j_1}, \ldots, d_{j_n})\|^{\boldsymbol{A}}_{\mathbf{N}}.$$

(The first equality holds by the definition of g and the second one is (3).) □

None of these two non-classical versions of the Upward Löwenheim–Skolem Theorem can be proved in general for logics with equality, as the following example shows.

Example 33 (Failure of the Upward Löwenheim–Skolem Theorem for logics with equality)**.** *Consider the Gödel-Dummett first-order logic with the projection connective* $\triangle$ *and assume that the language contains a unary predicate* P *and an equality symbol* $\approx$*. Let us take a semantics of models* $\langle [0,1]_{\mathrm{G}}, \mathbf{M} \rangle$ *over the standard* G*-algebra, where* $\approx$ *is interpreted as classical equality, i.e.*

for each $a, b \in M$, $\|a \approx b\|_{\mathbf{M}}^{[0,1]_{\mathrm{G}}} = 1$ if and only if $a = b$. A counterexample to the upwards theorem can be obtained by considering the formula[1] $\chi = (\forall x)(\forall y)(\neg\triangle(x \approx y) \to \neg\triangle(P(x) \leftrightarrow P(y)))$ *that codifies the fact that P is interpreted as an injective mapping from the domain to the algebra of truth-values. Indeed, if $\langle [0,1]_{\mathrm{G}}, \mathbf{M}\rangle$ is a model of χ, then for every $a, b \in M$, we have $\|\neg\triangle(a \approx b) \to \neg\triangle(P(a) \leftrightarrow P(b))\|_{\mathbf{M}}^{[0,1]_{\mathrm{G}}} = 1$, i.e. if $a \neq b$, then $P_{\mathbf{M}}(a) \neq P_{\mathbf{M}}(b)$. Therefore, $\langle [0,1]_{\mathrm{G}}, \mathbf{M}\rangle$ is a model of χ if and only if $|M| \leq 2^{\aleph_0}$, and hence the upward theorem does not hold.*

7 A many-sorted approach

As it is well known, classical many-sorted models also enjoy their own versions of Löwenheim–Skolem theorems. Therefore, if we manage to describe our non-classical structures in the framework of classical many-sorted models, we will obtain an alternative approach to the results we have just proved. Let us first formally recall the corresponding definitions and theorems. As references of many-sorted languages and structures see [21, 22].

Definition 34. *A* many-sorted predicate language $\mathcal{P}$ *is a tuple*

$$\langle \mathcal{S}, \mathit{Pred}_{\mathcal{P}}, \mathit{Func}_{\mathcal{P}}, \mathit{Ar}_{\mathcal{P}}, \mathit{Sort}_{\mathcal{P}}\rangle,$$

where $\mathcal{S}$ is a non-empty set of sorts*, $\mathit{Pred}_{\mathcal{P}}$ is a non-empty set of* sorted predicate symbols*, $\mathit{Func}_{\mathcal{P}}$ is a set (disjoint with $\mathit{Pred}_{\mathcal{P}}$) of* sorted function symbols*, $\mathit{Ar}_{\mathcal{P}}$ is the* arity function*, assigning to each predicate or function symbol a natural number called the* arity *of the symbol, and $\mathit{Sort}_{\mathcal{P}}$ is a function that maps each n-ary $R \in \mathit{Pred}_{\mathcal{P}}$ to a sequence of n sorts and each n-ary $F \in \mathit{Pred}_{\mathcal{P}}$ to a sequence of $n+1$ sorts.*

Definition 35. *Given a many-sorted predicate language $\mathcal{P}$, we define a $\mathcal{P}$-structure as a tuple $\mathbf{M} = \langle M, \langle R^{\mathbf{M}}\rangle_{R\in \mathit{Pred}_{\mathcal{P}}}, \langle F^{\mathbf{M}}\rangle_{f\in \mathit{Func}_{\mathcal{P}}}\rangle$, where M is a family of non-empty domains $\{S(M) \mid S \in \mathcal{S}\}$; for each n-ary $R \in \mathit{Pred}_{\mathcal{P}}$, if $\mathit{Sort}_{\mathcal{P}}(R) = \langle S_1, \dots, S_n\rangle$, $R^{\mathbf{M}} \subseteq S_1(M) \times \ldots \times S_n(M)$; for each n-ary $F \in \mathit{Func}_{\mathcal{P}}$, if $\mathit{Sort}_{\mathcal{P}}(F) = \langle S_1, \dots, S_n, S\rangle$, $F^{\mathbf{M}}$ is a function from $S_1(M) \times \ldots \times S_n(M)$ to $S(M)$. By the cardinality $|M|$ of $\mathbf{M}$ we mean the sum of the cardinalities of the sets $\{S(M) \mid S \in \mathcal{S}\}$.*

Definition 36. *Let $\mathcal{P}$ be a many-sorted predicate language and let $\mathbf{M}$ and $\mathbf{N}$ be $\mathcal{P}$-structures. We say that $\mathbf{M}$ is a* substructure *of $\mathbf{N}$ if*

1. *for each $S \in \mathcal{S}$, $S(M) \subseteq S(N)$,*
2. *for each $F \in \mathit{Func}_{\mathcal{P}}$ with $\mathit{Sort}_{\mathcal{P}}(F) = \langle S_1, \dots, S_n, S\rangle$, and for each element $a_i \in S_i(M)$, we have $F^{\mathbf{M}}(a_1, \dots, a_n) = F^{\mathbf{N}}(a_1, \dots, a_n)$,*
3. *for each $R \in \mathit{Pred}_{\mathcal{P}}$ with $\mathit{Sort}_{\mathcal{P}}(R) = \langle S_1, \dots, S_n\rangle$, we have that $R^{\mathbf{M}} = R^{\mathbf{N}} \cap (S_1(M) \times \ldots \times S_n(M))$.*

$\mathbf{M}$ *is an* elementary substructure *of* $\mathbf{N}$ *(and* $\mathbf{N}$ *is an* elementary extension *of* $\mathbf{M}$*) if moreover for each $\mathcal{P}$-formula $\varphi(x_1, \dots, x_n)$ and each $a_1, \dots, a_n$ in the sorts of M corresponding to the variables of φ, we have $\mathbf{M} \models \varphi(a_1, \dots, a_n)$ if and only if $\mathbf{N} \models \varphi(a_1, \dots, a_n)$.*

[1] We have learned this formula from Matthias Baaz.

Now let us recall the classical downward and upward Löwenheim–Skolem theorem for many-sorted structures (for the proofs see [26, Propositions 1.27 and 1.31]).

Theorem 37 (Classical Downward Löwenheim–Skolem Theorem for many–sorted structures)**.** *Let $\mathcal{P}$ be a many-sorted predicate language, $\mathbf{M}$ a $\mathcal{P}$-structure, for each $S \in \mathcal{S}$ a subset $Z_S \subseteq S(M)$, and κ a cardinal such that for each $S \in \mathcal{S}$, $\max\{\omega, |\mathcal{P}|, |Z_S|\} \leq \kappa \leq |M|$. Then there is an elementary substructure $\mathbf{N}$ of $\mathbf{M}$ such that $|N| = \kappa$ and for each $S \in \mathcal{S}$, $Z_S \subseteq S(N)$.*

Theorem 38 (Classical Upward Löwenheim–Skolem Theorem for classical many–sorted structures)**.** *Let $\mathcal{P}$ be a many-sorted predicate language, $\mathbf{M}$ an infinite $\mathcal{P}$-structure, and κ a cardinal such that $\max\{|\mathcal{P}|, |M|\} \leq \kappa$. Then there is an elementary extension $\mathbf{N}$ of $\mathbf{M}$ such that $|N| = \kappa$.*

Let us show now how we can translate the predicate language $\mathcal{P}$ into a classical 2-sorted language $\mathcal{P}_2$:

- For each sort $i \in \{1, 2\}$, we take quantifiers $\forall_i$ and $\exists_i$.
- Variables of sort 1 are denoted as $x, y, z, x_1, \ldots, x_n, \ldots$, and of sort 2 as $v, w, v_1, \ldots, v_n, \ldots$
- For each sort $i \in \{1, 2\}$, we take an equality symbol $\approx_i$.
- For each propositional n-ary connective λ, we take the same symbol λ as a functional of type $\langle 1, \overset{(n)}{\ldots}, 1, 1\rangle$.
- For each n-ary functional symbol $F \in \mathit{Func}_{\mathcal{P}}$, we take the same symbol F as a functional of type $\langle 2, \overset{(n)}{\ldots}, 2, 2\rangle$.
- For each n-ary relational symbol $R \in \mathit{Pred}_{\mathcal{P}}$, we take the same symbol R as a functional of type $\langle 2, \overset{(n)}{\ldots}, 2, 1\rangle$.

Now, given a $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M}\rangle$, we build a 2-sorted $\mathcal{P}_2$-structure $\boldsymbol{B}_{\mathbf{M}}$:

- The universe of sort 1 is B and the universe of sort 2 is M.
- The symbols $\approx_i$ are interpreted as crisp equality in the corresponding sorts.
- For each propositional n-ary connective λ, define $\lambda^{\boldsymbol{B}_{\mathbf{M}}}$ as $\lambda^{\boldsymbol{B}}$.
- For each n-ary functional symbol $F \in \mathit{Func}_{\mathcal{P}}$, define $F^{\boldsymbol{B}_{\mathbf{M}}}$ as $F_{\mathbf{M}}$.
- For each n-ary relational symbol $P \in \mathit{Pred}_{\mathcal{P}}$, define $P^{\boldsymbol{B}_{\mathbf{M}}}$ as $P_{\mathbf{M}}$.

Lemma 39. *For each $\mathcal{P}$-formula $\varphi(v_1, \ldots, v_n)$, there is a $\mathcal{P}_2$-formula $E_\varphi(v_1, \ldots, v_n, x)$ such that, for every $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M}\rangle$, and $d_1, \ldots, d_n \in M$,*

$$\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{B}}_{\mathbf{M}} = b \quad \Leftrightarrow \quad \boldsymbol{B}_{\mathbf{M}} \models E_\varphi(d_1, \ldots, d_n, b).$$

Proof. We proceed by induction on the complexity of $\varphi(v_1, \ldots, v_n)$. If φ is an atomic formula of the form $P(t_1, \ldots, t_k)$, where $P \in \mathcal{P}$ is an n-ary predicate, and $t_1, \ldots, t_k$ are $\mathcal{P}$-terms with their free variables among $v_1, \ldots, v_n$, we take $E_\varphi(v_1, \ldots, v_n, x)$ to be the formula

$$P(t_1, \ldots, t_k) \approx_1 x.$$

Let $\varphi(v_1, \ldots, v_n) = \lambda(\psi_1, \ldots, \psi_k)(v_1, \ldots, v_n)$, where λ is an n-ary connective, and we assume inductively that, for each formula $\psi_i(v_1, \ldots, v_n)$, the property holds for the $\mathcal{P}_2$-formula $E_{\psi_i}(v_1, \ldots, v_n, y_i)$, where $i \in \{1, \ldots, k\}$. In this case, take $E_\varphi(v_1, \ldots, v_n, x)$ to be

$$(\forall y_1, \ldots, y_k)\big(\wedge_{i=1}^{k} E_{\psi_i}(v_1, \ldots, v_n, y_i) \to \lambda(y_1, \ldots, y_k) \approx_1 x\big).$$

For the universal quantifier case $\varphi(v_1, \ldots, v_n) = (\forall w)\psi(v_1, \ldots, v_n, w)$, we assume inductively that for the $\mathcal{P}$-formula $\psi(v_1, \ldots, v_n, w)$ the property holds for the $\mathcal{P}_2$-formula $E_\psi(v_1, \ldots, v_n, w, y)$. Now take $E_\varphi(v_1, \ldots, v_n, x)$ to be

$$(\forall z)(z \leq x \leftrightarrow (\forall w, y)(E_\psi(v_1, \ldots, v_n, w, y) \to z \leq y)).$$

The existential quantifier step has an analogous proof. $\square$

Corollary 40. *A $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M} \rangle$ is safe if and only if, for every $\mathcal{P}$-formula $\varphi(v_1, \ldots, v_n)$,*

$$\boldsymbol{B}_{\mathbf{M}} \models (\forall v_1, \ldots, v_n)(\exists! x)E_\varphi(v_1, \ldots, v_n, x).$$

This gives us all the necessary ingredients to obtain, via classical 2-sorted structures, an alternative form the non-classical Downward Löwenheim–Skolem Theorem (cf. Theorem 30).

Theorem 41 (Non-classical Downward Löwenheim–Skolem Theorem - 2nd version)**.** *Let $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be a safe $\mathcal{P}$-structure. Then, for every $Z \subseteq N$, every $X \subseteq B$ and every cardinal κ such that*

$$\max\{|\mathcal{P}|, \omega, |Z|, |X|\} \leq \kappa \leq \max\{|B|, |N|\},$$

there is a safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$ which is an elementary substructure of $\langle \boldsymbol{B}, \mathbf{N} \rangle$ such that $|A| + |M| = \kappa$, $Z \subseteq M$, and $X \subseteq A$.

Proof. First we build from $\langle \boldsymbol{B}, \mathbf{N} \rangle$ the 2-sorted $\mathcal{P}_2$-structure $\boldsymbol{B}_{\mathbf{N}}$ as described above. Observe that Z and X are subsets of the domains of the corresponding sorts of $\boldsymbol{B}_{\mathbf{N}}$ and all the hypotheses of Theorem 37 are satisfied, so we can apply it and obtain a 2-sorted elementary substructure $\mathbf{O}$ such that $X \subseteq S_1(O)$, $Z \subseteq S_2(O)$, and $|S_1(O)| + |S_2(O)| = \kappa$. Define $A = S_1(O)$ and $M = S_2(O)$. Since the interpretation of each functional symbol from $\mathcal{P}_2$ in $\mathbf{M}$ is the restriction of its interpretation in $\boldsymbol{B}_{\mathbf{N}}$, we obtain that A is the universe of a subalgebra $\boldsymbol{A}$ of $\boldsymbol{B}$ and we have a $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$. Since $\langle \boldsymbol{B}, \mathbf{N} \rangle$ is safe, it satisfies the formulas from Corollary 40 and so it does $\mathbf{M}$ (because it is an elementary substructure); therefore $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is also safe. Finally, the fact that $\mathbf{N}$ is an elementary substructure of $\boldsymbol{B}_{\mathbf{M}}$ in the sense of $\mathcal{P}_2$ clearly entails that $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is an elementary substructure of $\langle \boldsymbol{B}, \mathbf{N} \rangle$ in the sense of $\mathcal{P}$. $\square$

Using classical 2-sorted structures we can obtain also a new version of the non-classical upward Löwenheim–Skolem theorem (cf. Theorem 31).

Theorem 42 (Non-classical Upward Löwenheim–Skolem Theorem - 2nd version)**.** *Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ be a safe infinite $\mathcal{P}$-structure and κ a cardinal such that $\max\{|\mathcal{P}|, |A|, |M|\} \leq \kappa$. Then there is a safe $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{N} \rangle$ such that $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is an elementary substructure of $\langle \boldsymbol{B}, \mathbf{N} \rangle$ and $|B| + |N| = \kappa$.*

Proof. As in the previous proof, we first build the 2-sorted $\mathcal{P}_2$-structure $\boldsymbol{A}_{\mathbf{M}}$ from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ as described above. Applying Theorem 38 we obtain a 2-sorted elementary extension $\mathbf{O}$ of $\boldsymbol{A}_{\mathbf{M}}$ such that $|O| = |S_1(O)| + |S_2(O)| = \kappa$. Define $B = S_1(O)$ and $N = S_2(O)$. Since $\boldsymbol{A}_{\mathbf{M}}$ is an elementary substructure of $\mathbf{O}$, it is clear that $\boldsymbol{A}$ is a subalgebra of the algebra $\boldsymbol{B}$ defined over B with the operations determined by the interpretation in $\mathbf{O}$ of the corresponding functional symbols and, hence, we have obtained a $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{N} \rangle$ which has $\langle \boldsymbol{A}, \mathbf{M} \rangle$ as elementary substructure. The preservation of safeness is justified as in the previous proof. □

On the one hand, the 2-sorted approach has allowed us to obtain a better form of the Downward Löwenheim–Skolem Theorem. Indeed, Theorem 41 is more general than Theorem 30, because it does not use the rather complicated cardinal $p(\boldsymbol{A})$ as an extra lower bound for κ and, moreover, it allows to select a subset X in the algebra to obtain a smaller elementary substructure, whose algebraic reduct contains the selected X.

On the other hand, the situation is quite different when it comes to the Upward theorems. Although Theorem 42 can be seen as an improvement of Theorem 31 because it can accommodate languages with equality (the counterexample shown before in Example 33 does not apply here, because the theorem would provide models over big enough algebras), when restricted to equality-free languages, the new result is not as good as the previous one. Indeed, Theorem 31 does not include the lower bound $|A|$ for κ. This is due to the fact that the notion of cardinality of a 2-sorted structure $\boldsymbol{A}_{\mathbf{M}}$ is different from the one used in the non-classical structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$, where only the first-order domain, not the algebra, is taken into account when the cardinality is calculated. Since in first-order many-valued logics one often wants to keep the same algebra of truth-values for all models in a given context (as they may be considered essential in the inteded semantics), our previous version of the upward theorem would be more appropriate. These different pros and cons of the two theorems justify the twofold approach we have proposed for Löwenheim–Skolem theorems for non-classical first-order algebraizable logics.

8 Funding

Dellunde and García-Cerdaña are supported by EdeTRI (TIN2012-39348-C02-01); Dellunde is also supported by the Spanish project MILESS (TIN2013-45039-P) and the grant 2014SGR-118 from the Generalitat de Catalunya; García-Cerdaña is also supported by the Spanish MICINN project *Lógica algebraica y lógicas no clásicas* (MTM2011-25747) and the grant 2014SGR-788 from the Generalitat de Catalunya; Noguera is supported by the project GA13-14654S of the Czech Science Foundation and the FP7-PEOPLE-2009-IRSES project MaToMUVI (PIRSES-GA-2009-247584).

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