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ON LIPSCHITZIAN PROPERTIES OF IMPLICIT MULTIFUNCTIONS*

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Abstract. This paper is devoted to the development of new sufficient conditions for the calmness and the Aubin property of implicit multifunctions. As the basic tool we employ the directional limiting coderivative which, together with the graphical derivative, enables a fine analysis of the local behavior of the investigated multifunction along relevant directions. For verification of the calmness property, in addition, a new condition has been discovered which parallels the missing implicit function paradigm and permits us to replace the original multifunction by a substantially simpler one. Moreover, as an auxiliary tool, a handy formula for the computation of the directional limiting coderivative of the normal-cone map with a polyhedral set has been derived which perfectly matches the framework of [A. L. Dontchev and R. T. Rockafellar, *SIAM J. Optim.*, 6 (1996), pp. 1087–1105]. All important statements are illustrated by examples.

Key words. solution map, calmness, Aubin property, directional limiting coderivative

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1. Introduction.

Given a multifunction M of two variables, say p, x, define the associated *implicit multifunction* S by

$$S(p) := \{x \mid 0 \in M(p,x)\}. \tag{1.1}$$

The aim of this paper is to derive new conditions ensuring the *calmness* and the *Aubin (Lipschitz-like)* property of S at or around the given reference point $(\bar{p}, \bar{x})$, respectively. The definitions of these stability properties, together with several other notions that are important for this development, are collected in section 2.1. Starting with the principal work of Dini [7], there is a large number of works dealing with the classical variant of (1.1), where M is a (mostly smooth) single-valued map. The rapid development of modern variational analysis, having started in the 1970s, has enabled, however, a step by step weakening of the assumptions imposed on M and eventually leads to general multifunctional formulation (1.1). This modern framework has a lot of advantages and allows us to capture, for instance, various types of parameter-dependent constraint and variational systems; cf. [33, sections 4.3 and 4.4]. From the long list of relevant references let us mention the papers [35], [37], [8], [10], [11], [5], [25], where the authors consider various special (frequently arising) classes of multifunctions M and derive conditions ensuring a Lipschitzian behavior of S. The recent monograph [12] contains a comprehensive presentation of currently available results accompanied by a detailed explanation of the so-called *implicit function paradigm*. This paradigm substantially facilitates the derivation of conditions,

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ensuring various stability properties of S, but, as pointed out in [12, p. 200], it does not work in the case of calmness. To overcome this hurdle, we have significantly improved an approach from [24, Lemma 1] concerning parameterized constraint systems. Our result (Theorem 3.3) works for general multifunctions M and enables us to ensure the calmness of S via the *metric subregularity* of the mapping

$$M_{\bar{p}}(x) := M(\bar{p}, x)$$

at $(\bar{x}, 0)$ and a special relaxed calmness condition imposed on a mapping associated with M. Our result has the same structure as [9, Proposition 2.3], where the authors state a criterion for the Aubin property of S around $(\bar{p}, \bar{x})$.

To ensure the metric subregularity of $M_{\bar{p}}$ we can employ one of the various approaches developed in the literature; see, e.g., [27], [41], [14], [15], [29]. However, the derivative-like objects used in these papers do not possess a decent calculus, and so it is difficult to compute them, e.g., in the case of parameterized constraint or variational systems.

The approach used in this paper is related to the techniques from [16],[18]. It is based on the notion of the *directional limiting coderivative* introduced in [17] (for a slightly different version, see [22]) which provides us with a convenient description of the local behavior of considered multifunctions *along* specified directions. Moreover, the directional coderivatives do possess a considerable calculus. The usage of this tool enables us not only to prove the calmness of S (which was our main intention) but also, in some cases, to ensure at the same time the nonemptiness of the sets $S(p)$ for p close to $\bar{p}$.

In contrast to the property of calmness, there already exists an efficient characterization of the Aubin property of S in terms of a derivative-like object associated with M. Herewith we mean the *Mordukhovich criterion*, expressed via the limiting coderivative; cf. [31], [39, Theorem 9.40] and [28] for a preceding result of this sort. Further efficient characterizations can be found, e.g., in [12, Chapter 4.2]. Nevertheless, in some situations we are not able to precisely compute the limiting coderivative of the implicitly given mapping S, and thus resulting sufficient conditions can be far from necessity. This difficulty arises, e.g., when

$$M(p, x) = G(p, x) + Q(x),$$

where G is a continuously differentiable function with a nonsurjective partial Jacobian $\nabla_p G$ at the reference point $(\bar{p}, \bar{x})$. We compute thus only an upper estimate of the limiting coderivative of S, which makes the resulting condition too rough. Being motivated by this type of problem, we have again employed the directional limiting coderivative to construct a new, substantially weaker (less restrictive) criterion which is able to detect the Aubin property even if the existing criteria based on the standard limiting coderivative fail. In such cases it suffices, namely, to examine the (local) behavior of M only with respect to directions for which the graphical derivative of M at $(\bar{p}, \bar{x}, 0)$ vanishes.

Both investigated properties, namely the calmness and the Aubin property of S, belong to the basic stability properties of multifunctions. The generalized implicit multifunction model (1.1) is amenable to a large class of parametric models ranging from constraint systems over variational inequalities up to complicated optimization and equilibrium problems. For all of these problems the obtained conditions can be used as an efficient tool of postoptimal analysis.

The plan of the paper is as follows. The next "preliminary" section is divided into three parts. The first contains the basic definitions, whereas the second is devoted to the metric subregularity and its relationship with other notions such as the directional metric (sub)regularity and the directional limiting coderivative. In the third part we present several auxiliary results which are extensively used in what follows. Some of them are interesting in their own right and also could be used in a different context. In particular, in Theorem 2.12 we present an easy-to-apply formula for the directional limiting normal cone to the graph of the normal-cone map associated with a convex polyhedron. Sections 3 and 4, containing our main results, are then devoted to the new criteria of the calmness and the Aubin property of S, respectively. The obtained results are illustrated by examples.

Our notation is basically standard. In a Euclidean space, $\|\cdot\|$ is the (Euclidean) norm, and $\mathrm{d}(x,\Omega)$ denotes the distance from a point x to the set Ω. Further, $\mathcal{B}_{\mathbb{R}^n}$ and $\mathcal{S}_{\mathbb{R}^n}$ denote the closed unit ball and the unit sphere in $\mathbb{R}^n$, respectively, and $\mathcal{B}(x,r) := \{u \,|\, \|u-x\| \le r\}$. Given a metric space X, $\rho_X(\cdot,\cdot)$ stands for the corresponding metric, dist_X denotes the respective point-to-set distance function and $\mathcal{B}_X(x,r) := \{u \in X \,|\, \rho_X(u,x) \le r\}$. Given the product $X \times Y$ of two (metric, Euclidean) spaces, we use the "Euclidean" metric

$$\rho_{X\times Y}((x,y),(x',y')) := \sqrt{(\rho_X(x,x'))^2 + (\rho_Y(y,y'))^2}.$$

For a multifunction F, $\mathrm{gph}\, F := \{(x,y) | y \in F(x)\}$ is its graph, $\mathrm{dom}\, F := \{x \,|\, F(x) \neq \emptyset\}$ stands for its domain, $\mathrm{rge}\, F := \{y \,|\, \exists x \text{ with } y \in F(x)\}$ denotes its range, and F^{-1} means the respective inverse mapping. Finally, K° is the (negative) polar cone to a cone K, and the notation of the objects from variational analysis, together with the respective definitions, is introduced in the next section.

2. Preliminaries.

2.1. Basic notions. Consider general closed-graph multifunctions $\mathcal{M} : X \rightrightarrows Y$ and $F : Y \rightrightarrows X$, where X, Y are metric spaces.

DEFINITION 2.1.

(i) *We say that* F *has the* Aubin property *around* $(\bar{y},\bar{x}) \in \mathrm{gph}\, F$*, provided there are reals* $\kappa \ge 0$ *and* $\varepsilon > 0$ *such that*

$$\mathrm{d}_X(x, F(y)) \le \kappa \rho_Y(v,y) \quad \textit{provided} \quad \rho_Y(y,\bar{y}) < \varepsilon, \rho_X(x,\bar{x}) < \varepsilon, x \in F(v).$$

(ii) $\mathcal{M}$ *is said to be* metrically regular *around* $(\bar{x},\bar{y}) \in \mathrm{gph}\, \mathcal{M}$*, provided there are* $\kappa \ge 0$ *and* $\varepsilon > 0$ *such that*

$$\mathrm{d}_X(x, \mathcal{M}^{-1}(y)) \le \kappa \mathrm{d}_Y(y, \mathcal{M}(x)) \quad \textit{provided} \quad \rho_X(x,\bar{x}) < \varepsilon, \rho_Y(y,\bar{y}) < \varepsilon.$$

It is easy to see that F has the Aubin property around $(\bar{y},\bar{x})$ if and only if F^{-1} is metrically regular around $(\bar{x},\bar{y})$. The Aubin property has been introduced in [3] (under a different name), and since that time it has been widely used in variational analysis both as a desired local stability property and in various qualification conditions in the nonsmooth calculus. It also has a close connection with the conclusions of the theorems of Graves and Lyusternik.

DEFINITION 2.2. *In the setting of Definition* 2.1 *we say that*

(i) F *is* calm *at* $(\bar{y},\bar{x})$*, provided there are reals* $\kappa \ge 0$ *and* $\varepsilon > 0$ *such that*

$$\mathrm{d}_X(x, F(\bar{y})) \le \kappa \rho_Y(y,\bar{y}) \;\; \textit{provided} \;\; \rho_Y(y,\bar{y}) < \varepsilon, \rho_X(x,\bar{x}) < \varepsilon, \;\; \textit{and } x \in F(y).$$

(ii) $\mathcal{M}$ *is* metrically subregular *at* $(\bar{x},\bar{y})$*, provided there are* $\kappa \ge 0$ *and* $\varepsilon > 0$ *such that*

$$\mathrm{d}_X(x, \mathcal{M}^{-1}(\bar{y})) \le \kappa \mathrm{d}_Y(\bar{y}, \mathcal{M}(x)) \quad \textit{provided} \quad \rho_X(x,\bar{x}) < \varepsilon.$$

Again, F is calm at $(\bar{y}, \bar{x})$ if and only if F^{-1} is metrically subregular at $(\bar{x}, \bar{y})$. Further, we observe that the pair of properties from Definition 2.2 is strictly weaker (less restrictive) than their counterparts from Definition 2.1 and that the calmness of F at $(\bar{y}, \bar{x})$ does not entail the nonemptiness of $F(y)$ for y close to $\bar{y}$. To the best of our knowledge, the metric subregularity has been introduced in [26] (under a different name), whereas the calmness arose for the first time in the context of optimal value functions in [6]. Later, in [40] it was then generalized to the form arising in Definition 2.2 (i) and used as a weak constraint qualification (again under a different name). From the point of view of local postoptimal analysis it is, however, also a valuable property, in particular when one proves in addition that $F(y) \neq \emptyset$ on a neighborhood of $\bar{y}$.

The above defined stability properties will be central in our development. To be able to conduct their thorough analysis in the investigated model, we will make use of several basic notions of the nonsmooth calculus stated below. Since we will be working with them only in finite dimensions, we will present their definitions below in the finite-dimensional setting.

Let A be a closed set in $\mathbb{R}^s$, and let $\mathcal{M}$ now be a closed-graph multifunction mapping $\mathbb{R}^s$ into (sets of) $\mathbb{R}^d$.

Definition 2.3. *Assume that $\bar{x} \in A$. Then*

(i)
$$T_A(\bar{x}) := \operatorname*{Lim\,sup}_{t \searrow 0} \frac{A - \bar{x}}{t}$$

is the tangent (contingent) cone *to A at $\bar{x}$;*

(ii)
$$\hat{N}_A(\bar{x}) := (T_A(\bar{x}))^\circ$$

is the regular normal cone *to A at $\bar{x}$;*

(iii)
$$N_A(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \hat{N}_A(x)$$

is the limiting normal cone *to A at $\bar{x}$, and,*

(iv) *given a direction $u \in \mathbb{R}^s$,*

$$N_A(\bar{x}; u) := \operatorname*{Lim\,sup}_{\substack{t \searrow 0 \\ u' \to u}} \hat{N}_A(\bar{x} + tu')$$

is the directional limiting normal cone *to A at $\bar{x}$ in direction u.*

The symbol "Lim sup" in (i), (iii), and (iv) stands for the outer (upper) set limit in the sense of Painlevé and Kuratowski; cf. [39, Chapter 4B]. If A is convex, then both the regular and the limiting normal cones coincide with the normal cone in the sense of convex analysis. Therefore, in this case we will use the notation N_A.

We say that a tangent $u \in T_A(\bar{x})$ is *derivable* if there exists a mapping $\xi : [0, \varepsilon] \to A$ such that $\xi(0) = \bar{x}$ and $\xi(t) - (\bar{x} + tu) = o(t)$; cf. [39, Definition 6.1]. This notion also arises in the definition of the tangent cone in [13].

The above listed cones enable us to describe the local behavior of multifunctions via various generalized derivatives.

DEFINITION 2.4. *Consider a point* $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{M}$. *Then*

(i) *the multifunction* $D\mathcal{M}(\bar{x}, \bar{y}) : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$, *defined by*

$$D\mathcal{M}(\bar{x}, \bar{y})(u) := \{v \in \mathbb{R}^d | (u, v) \in T_{\operatorname{gph} \mathcal{M}}(\bar{x}, \bar{y})\}, \ u \in \mathbb{R}^s,$$

is called the graphical derivative *of* $\mathcal{M}$ *at* $(\bar{x}, \bar{y})$;

(ii) *the multifunction* $\hat{D}^*\mathcal{M}(\bar{x}, \bar{y}) : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, *defined by*

$$\hat{D}^*\mathcal{M}(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^s | (x^*, -y^*) \in \hat{N}_{\operatorname{gph} \mathcal{M}}(\bar{x}, \bar{y})\}, \ y^* \in \mathbb{R}^d,$$

is called the regular coderivative *of* $\mathcal{M}$ *at* $(\bar{x}, \bar{y})$;

(iii) *the multifunction* $D^*\mathcal{M}(\bar{x}, \bar{y}) : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, *defined by*

$$D^*\mathcal{M}(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^s | (x^*, -y^*) \in N_{\operatorname{gph} \mathcal{M}}(\bar{x}, \bar{y})\}, \ y^* \in \mathbb{R}^d,$$

is called the limiting coderivative *of* $\mathcal{M}$ *at* $(\bar{x}, \bar{y})$;

(iv) *finally, given a pair of directions* $(u, v) \in \mathbb{R}^s \times \mathbb{R}^d$, *the multifunction* $D^*\mathcal{M}((\bar{x}, \bar{y}); (u, v)) : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, *defined by*

$$D^*\mathcal{M}((\bar{x}, \bar{y}); (u, v))(y^*) := \{x^* \in \mathbb{R}^s | (x^*, -y^*) \in N_{\operatorname{gph} \mathcal{M}}((\bar{x}, \bar{y}); (u, v))\}, \quad y^* \in \mathbb{R}^d, \tag{2.1}$$

is called the directional limiting coderivative *of* $\mathcal{M}$ *at* $(\bar{x}, \bar{y})$ *in direction* (u, v).

For the properties of the cones (i)–(iii) from Definition 2.3 and generalized derivatives (i)–(iii) from Definition 2.4 we refer the interested reader to the monographs [39] and [33]. Various properties of the directional limiting normal cone and coderivative can be found in [17], [18], [19], [20], [21]. In what follows, we will make use of the fact that for a multifunction $\mathcal{M} : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ and a smooth mapping $h : \mathbb{R}^s \to \mathbb{R}^d$ one has (cf. [32])

$$T_{\operatorname{gph}(h+\mathcal{M})}(\bar{x}, \bar{y}) = \{(u, \nabla h(\bar{x})u + v)) \,|\, (u, v) \in T_{\operatorname{gph} \mathcal{M}}(\bar{x}, \bar{y} - h(\bar{x}))\}, \tag{2.2}$$

$$\widehat{N}_{\operatorname{gph}(h+\mathcal{M})}(\bar{x}, \bar{y}) = \{(x^* - \nabla h(\bar{x})^T y^*, y^*) \,|\, (x^*, y^*) \in \widehat{N}_{\operatorname{gph} \mathcal{M}}(\bar{x}, \bar{y} - h(\bar{x}))\}, \tag{2.3}$$

and consequently,

$$D^*(h + \mathcal{M})((\bar{x}, \bar{y}); (u, v))(y^*) = \nabla h(\bar{x})^T y^* + D^*\mathcal{M}((\bar{x}, \bar{y} - h(\bar{x})); (u, v - \nabla h(\bar{x})u))(y^*). \tag{2.4}$$

2.2. Coderivative criteria for metric subregularity and calmness. In this subsection we will summarize some conditions for metric subregularity given by the first author which are used in what follows. In addition, this subsection provides the reader with some geometrical insight essential for the results presented in the last section.

One can find numerous sufficient conditions for metric subregularity in the literature; see, e.g., [14], [15], [19], [23], [27], [29], [30], [41]); However, these sufficient conditions are often very difficult to verify. The reason is because the property of metric subregularity is in general unstable under small perturbations, (see, e.g.,[12]), and this instability is reflected by the sufficient conditions. However, in applications it is important to have workable criteria, and thus we are looking for some sufficient conditions for metric subregularity which are not as weak as possible but stable with respect to certain perturbations.

Consider the following definition.

DEFINITION 2.5. *Let $\mathcal{M} : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ be a multifunction, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{M}$. The* limit set critical for metric subregularity*, denoted by* $\operatorname{Cr}_0 \mathcal{M}(\bar{x}, \bar{y})$*, is the collection of all elements $(v, u^*) \in \mathbb{R}^d \times \mathbb{R}^s$ such that there are sequences $t_k \searrow 0$, $(u_k, v_k^*) \in \mathcal{S}_{\mathbb{R}^s} \times \mathcal{S}_{\mathbb{R}^d}$, $(v_k, u_k^*) \to (v, u^*)$ with $(-u_k^*, v_k^*) \in \widehat{N}_{\operatorname{gph} \mathcal{M}}(\bar{x} + t_k u_k, \bar{y} + t_k v_k)$.*

In [16, Theorem 3.2] it was shown that the condition $(0, 0) \notin \operatorname{Cr}_0 \mathcal{M}(\bar{x}, \bar{y})$ is sufficient for metric subregularity of $\mathcal{M}$ at $(\bar{x}, \bar{y})$. We will now show that this criterion for metric subregularity is stable under small C^1 perturbations. Moreover, we reformulate it in terms of directional limiting coderivatives to obtain the condition (2.5) below, which is very congenial to the Mordukhovich criterion for metric regularity; cf. [31], [39, Theorem 9.40].

Remark 1. In [16, Theorem 3.2] an infinite-dimensional setting was considered, and therefore a further limit set critical for metric subregularity, denoted by $\operatorname{Cr} \mathcal{M}(\bar{x}, \bar{y})$, appears. However, in finite dimensions both limit sets coincide: $\operatorname{Cr} \mathcal{M}(\bar{x}, \bar{y}) = \operatorname{Cr}_0 \mathcal{M}(\bar{x}, \bar{y})$; cf. [16, p. 1450].

THEOREM 2.6. *Let $\mathcal{M} : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ be a multifunction, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{M}$. Then the following statements are equivalent:*

(i) $(0, 0) \notin \operatorname{Cr}_0 \mathcal{M}(\bar{x}, \bar{y})$.

(ii) *There exists $r > 0$ such that for every $h \in C^1(\mathbb{R}^s, \mathbb{R}^d)$ with $h(\bar{x}) = 0$ and $\|\nabla h(\bar{x})\| \leq r$ the mapping $\mathcal{M} + h$ is metrically subregular at $(\bar{x}, \bar{y})$.*

(iii)

$$\forall 0 \neq u \in \mathbb{R}^s : \ 0 \in D^* \mathcal{M}((\bar{x}, \bar{y}); (u, 0))(v^*) \Rightarrow v^* = 0. \tag{2.5}$$

Proof. By the second part of [16, Theorem 3.2] we have $\neg$(i) $\Rightarrow$ $\neg$(ii), and therefore (ii) $\Rightarrow$ (i) follows. We prove the reverse implication by contraposition. Assume that (ii) does not hold, i.e., there exists a sequence of functions $h_j \in C^1(\mathbb{R}^s, \mathbb{R}^d)$ with $h_j(\bar{x}) = 0$ and $\|\nabla h_j(\bar{x})\| \leq 1/j$ such that $\mathcal{M} + h_j$ is not metrically subregular at $(\bar{x}, \bar{y})$. By [16, Theorem 3.2], for every j we have $(0, 0) \in \operatorname{Cr}_0(\mathcal{M} + h_j)(\bar{x}, \bar{y})$, and hence for every j there exist sequences $t_{jk} \searrow 0$, $(u_{jk}, v_{jk}^*) \in \mathcal{S}_{\mathbb{R}^s} \times \mathcal{S}_{\mathbb{R}^d}$, $(v_{jk}, u_{jk}^*) \to (0, 0)$ with $(-u_{jk}^*, v_{jk}^*) \in \widehat{N}_{\operatorname{gph}(\mathcal{M}+h)}(\bar{x} + t_{jk} u_{jk}, \bar{y} + t_{jk} v_{jk})$. For every j we can find some index $k(j)$ such that $t_{jk(j)} \leq 1/j$, $\|v_{jk(j)}\| \leq 1/j$, $\|u_{jk(j)}^*\| \leq 1/j$, $\|\nabla h_j(\bar{x} + t_{jk(j)} u_{jk(j)}) - \nabla h_j(\bar{x})\| \leq 1/j$, and $\|h_j(\bar{x} + t_{jk(j)} u_{jk(j)}) - h_j(\bar{x}) - t_{jk(j)} \nabla h_j(\bar{x}) u_{jk(j)}\| \leq t_{jk(j)}/j$. Putting $t_j := t_{jk(j)}$, $u_j := u_{jk(j)}$, $v_j := v_{jk(j)} - h(\bar{x} + t_j u_j)/t_j$, $v_j^* := v_{jk(j)}^*$, and $u_j^* = u_{jk(j)}^* - \nabla h_j(\bar{x} + t_j u_j)^T v_j^*$ we obtain $(u_j, v_j^*) \in \mathcal{S}_{\mathbb{R}^s} \times \mathcal{S}_{\mathbb{R}^d}$, $\|u_j^*\| \leq \|u_{jk(j)}^*\| + \|\nabla h_j(\bar{x} + t_j u_j)\| \|v_j^*\| \leq 3/j$ and

$$\|v_j\| \leq \|v_{jk(j)}\| + \|h(\bar{x} + t_j u_j)\|/t_j \leq 1/j + (t_j/j + \|h_j(\bar{x}) + t_j \nabla h_j(\bar{x}) u_j\|)/t_j \leq 3/j.$$

Since $(-u_{jk(j)}^*, v_j^*) \in \widehat{N}_{\operatorname{gph}(\mathcal{M}+h)}(\bar{x} + t_j u_j, \bar{y} + t_j v_{jk(j)})$, by (2.3) we have

$$\begin{aligned}(-u_j^*, v_j^*) &= (-u_{jk(j)}^* + \nabla h_j(\bar{x} + t_j u_j)^T v_j^*, v_j^*) \\ &\in \widehat{N}_{\operatorname{gph} \mathcal{M}}(\bar{x} + t_j u_j, \bar{y} + t_j v_{jk(j)} - h_j(\bar{x} + t_j u_j)) = \widehat{N}_{\operatorname{gph} \mathcal{M}}(\bar{x} + t_j u_j, \bar{y} + t_j v_j),\end{aligned}$$

and $(0, 0) \in \operatorname{Cr}_0 \mathcal{M}(\bar{x}, \bar{y})$ follows. Hence (i) $\Rightarrow$ (ii) also holds, and the equivalence between (i) and (ii) is established. The equivalence between (i) and (iii) follows from the definitions and the fact that any sequence $(u_k, v_k^*) \in \mathcal{S}_{\mathbb{R}^s} \times \mathcal{S}_{\mathbb{R}^d}$ has a convergent subsequence. ∎

One easily concludes from Theorem 2.6 that condition (2.5) implies the metric subregularity of $\mathcal{M}$ at $(\bar{x}, \bar{y})$. In what follows we will call this condition a *first-order sufficient condition for metric subregularity* and use the acronym FOSCMS.

Conditions (2.5) examines the limiting coderivative only in directions of the form $(u, 0)$, $u \neq 0$, and therefore we have to look into normals to the graph of $\mathcal{M}$ at points (x, y) with $\|y - \bar{y}\| = o(\|x - \bar{x}\|)$.

Remark 2. Note that condition (2.5) is, in particular, fulfilled if either there is no direction $u \neq 0$ with $0 \in D\mathcal{M}(\bar{x}, \bar{y})(u)$ or

$$0 \in D^*\mathcal{M}(\bar{x}, \bar{y})(v^*) \Rightarrow v^* = 0.$$

The former of these two special cases is equivalent to the so-called *strong metric subregularity*; cf. [12, Theorem 4E.1], whereas the latter is equivalent to metric regularity of $\mathcal{M}$ by the Mordukhovich criterion; cf. [31], [39, Theorem 9.40]. However, condition (2.5) is by far not restricted to these two special cases. A simple multifunction $\mathcal{M}$, where condition (2.5) is fulfilled but the respective S is neither strong metrically subregular nor metrically regular, can be found in Example 3.

To gain more insight into the equivalences of Theorem 2.6, consider the following definitions; cf. [17].

DEFINITION 2.7. *Let $\mathcal{M} : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ be a multifunction, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{M}$.*

(i) *Given $(u, v) \in \mathbb{R}^s \times \mathbb{R}^d$, $\mathcal{M}$ is called* metrically regular in direction (u, v) *at $(\bar{x}, \bar{y})$, provided there exist positive reals $\delta > 0$ and $\kappa > 0$ such that*

$$\mathrm{d}(\bar{x} + tu', \mathcal{M}^{-1}(\bar{y} + tv')) \leq \kappa \mathrm{d}(\bar{y} + tv', \mathcal{M}(\bar{x} + tu')) \tag{2.6}$$

holds for all $t \in [0, \delta]$ and all $(u', v') \in \mathcal{B}((u, v), \delta)$ with $\mathrm{d}((\bar{x} + tu', \bar{y} + tv'), \operatorname{gph} \mathcal{M}) \leq \delta t$.

(ii) *For given $u \in \mathbb{R}^s$, $\mathcal{M}$ is said to be* metrically subregular in direction u *at $(\bar{x}, \bar{y})$ if there are positive reals $\delta > 0$ and $\kappa' > 0$ such that*

$$\mathrm{d}(\bar{x} + tu', \mathcal{M}^{-1}(\bar{y})) \leq \kappa' \mathrm{d}(\bar{y}, \mathcal{M}(\bar{x} + tu')) \tag{2.7}$$

holds for all $t \in [0, \delta]$ and $u' \in \mathcal{B}(u, \delta)$.

The infimum of κ and κ', respectively, over all such combinations of δ, κ, and κ', respectively, is called the modulus of the respective property.

Note that these definitions imply that a multifunction $\mathcal{M}$ is automatically metrically regular in direction (u, v) when $(u, v) \notin T_{\operatorname{gph} \mathcal{M}}(\bar{x}, \bar{y})$ and that $\mathcal{M}$ is metrically subregular in direction u when $(u, 0) \notin T_{\operatorname{gph} \mathcal{M}}(\bar{x}, \bar{y})$.

Metric subregularity in direction u was introduced by Penot [34] (under the name *directional metric* regularity). The above definition of directional metric regularity is due to Gfrerer [17]. Note that in [1],[2] Arutyunov et al. introduced and studied another notion of directional metric regularity which is an extension of an earlier notion used in [4].

LEMMA 2.8. *Let $\mathcal{M} : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ be a multifunction, and let $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{M}$.*

(i) *Consider the following statements:*
   (a) *$\mathcal{M}$ is metrically regular around $(\bar{x}, \bar{y})$.*
   (b) *$\mathcal{M}$ is metrically regular in direction $(0, 0)$ at $(\bar{x}, \bar{y})$.*
   (c) *$\mathcal{M}$ is metrically regular in every direction $(u, v) \neq (0, 0)$.*

   Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c).

(ii) *Consider the following statements:*
   (a) $\mathcal{M}$ *is metrically subregular at* $(\bar{x}, \bar{y})$.
   (b) $\mathcal{M}$ *is metrically subregular in direction* $(0,0)$ *at* $(\bar{x}, \bar{y})$.
   (c) $\mathcal{M}$ *is metrically subregular in every direction* $u \neq 0$.
   Then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(iii) *If* $\mathcal{M}$ *is metrically regular in direction* $(u,0)$ *at* $(\bar{x}, \bar{y})$, *then it is also metrically subregular in direction* u.

Proof. Statement (i) follows immediately from the definition, statement (ii) follows from the definition and [20, Lemma 2.7], and statement (iii) was shown in [17, Lemma 1]. ∎

The following theorem is a directional extension of the Mordukhovich criterion [31], [39, Theorem 9.40] for metric regularity.

THEOREM 2.9. *Let* $\mathcal{M} : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ *be a multifunction with closed graph, and let* $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{M}$. *Then* $\mathcal{M}$ *is metrically regular in direction* $(u,v) \in \mathbb{R}^s \times \mathbb{R}^d$ *at* $(\bar{x}, \bar{y})$ *if and only if*

$$0 \in D^*\mathcal{M}((\bar{x}, \bar{y}); (u,v))(v^*) \Rightarrow v^* = 0.$$

Hence, condition (2.5) holds if and only if $\mathcal{M}$ is metrically regular in every direction $(u,0)$ with $u \neq 0$. However, by Lemma 2.8 we see that for verifying metric subregularity of $\mathcal{M}$, we only need metric subregularity of $\mathcal{M}$ in every direction $u \neq 0$. Assuming some special structure of the multifunction $\mathcal{M}$, directional metric subregularity can be ensured by a second-order sufficient condition. This is done in the following theorem, which is a specialized version of [18, Theorem 4.3(2)].

THEOREM 2.10. *Let* $(\bar{x}, \bar{y})$ *belong to the graph of the mapping* $\mathcal{M}(x) = G(x) + Q(x)$, *where* $G : \mathbb{R}^s \to \mathbb{R}^d$ *is strictly differentiable at* $\bar{x}$ *and* $Q : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ *is a polyhedral multifunction, i.e., its graph is the union of finitely many convex polyhedra. Further, let* $u \neq 0$, *and assume that the limit*

$$G''(\bar{x}; u) := \lim_{\substack{t \searrow 0 \\ u' \to u}} \frac{G(\bar{x} + tu') - G(\bar{x}) - t\nabla G(\bar{x})u'}{t^2/2}$$

exists. If the inequality

$$\langle v^*, G''(\bar{x}; u) \rangle < 0$$

holds for every nonzero element $0 \neq v^* \in \mathbb{R}^d$ *satisfying*

$$0 \in D^*\mathcal{M}((\bar{x}, \bar{y}); (u,0))(v^*) = \nabla G(\bar{x})^T v^* + D^*Q((\bar{x}, \bar{y} - G(\bar{x})); (u, -\nabla G(\bar{x})u))(v^*),$$

then $\mathcal{M}$ *is metrically subregular in direction* u *at* $(\bar{x}, \bar{y})$.

Note that the criterion of Theorem 2.10 is stable under perturbations $h \in C^2(\mathbb{R}^s, \mathbb{R}^d)$ with $h(\bar{x}) = 0$, $\nabla h(\bar{x}) = 0$, and $\|\nabla^2 h(\bar{x})\|$ sufficiently small.

In the following corollary we summarize the preceding results for the special case of constraint systems; cf. also [21, Corollary 1].

COROLLARY 2.11. *Let the multifunction* $\mathcal{M} : \mathbb{R}^s \rightrightarrows R^d$ *be given by* $\mathcal{M}(x) := G(x) - D$, *where* $G : R^s \to \mathbb{R}^d$ *is continuously differentiable and* $D \subset \mathbb{R}^d$ *is a closed set. Then* $\mathcal{M}$ *is metrically subregular at* $(\bar{x}, 0)$ *if one of the following conditions is fulfilled:*

1. First-order sufficient condition for metric subregularity (FOSCMS): *For every* $0 \neq u \in \mathbb{R}^s$ *with* $\nabla G(\bar{x})u \in T_D(G(\bar{x}))$ *one has*

$$\nabla G(\bar{x})^T v^* = 0,\ v^* \in N_D(G(\bar{x}); \nabla G(\bar{x})u) \implies v^* = 0.$$

2. Second-order sufficient condition for metric subregularity (SOSCMS): *G is twice Fréchet differentiable at $\bar{x}$, D is the union of finitely many convex polyhedra, and for every $0 \neq u \in \mathbb{R}^s$ with $\nabla G(\bar{x})u \in T_D(G(\bar{x}))$ one has*

$$\nabla G(x)^T v^* = 0,\ v^* \in N_D\big(G(\bar{x}); \nabla G(\bar{x})u\big),\ u^T \nabla^2 ({v^*}^T G)(\bar{x}) u \geq 0 \implies v^* = 0.$$

For a SOSCMS of constraint systems $0 \in G(x) - D$ when D is convex but not necessarily polyhedral, we refer the reader to [16].

2.3. Auxiliary results. In this section we consider the computation of the directional limiting normal cone to the graph of the normal-cone mapping N_Γ with a convex polyhedral set Γ. To this end, we introduce for each $(y, y^*) \in \operatorname{gph} N_\Gamma$ the *critical cone*

$$K_\Gamma(y, y^*) := T_\Gamma(y) \cap [y^*]^\perp,$$

where $[u]$ denotes the subspace $\{\alpha u \,|\, \alpha \in \mathbb{R}\}$ for any vector u. Further, we set $K_\Gamma(y, y^*) := \emptyset$ if $(y, y^*) \notin \operatorname{gph} N_\Gamma$.

Now consider a fixed pair $(\bar{y}, \bar{y}^*) \in \operatorname{gph} N_\Gamma$. Then it is well known that the geometry of the normal-cone mapping N_Γ near $(\bar{y}, \bar{y}^*)$ coincides with the geometry of the normal-cone mapping to $K_\Gamma(\bar{y}, \bar{y}^*)$ near $(0, 0)$. In particular, we have (cf. [12, Lemma 2E.4])

$$(\bar{y}+v, \bar{y}^*+v^*) \in \operatorname{gph} N_\Gamma \;\Leftrightarrow\; (v, v^*) \in \operatorname{gph} N_{K_\Gamma(\bar{y}, \bar{y}^*)} \text{ for } (v, v^*) \text{ sufficiently near } (0, 0), \tag{2.8}$$

and therefore,

$$T_{\operatorname{gph} N_\Gamma}(\bar{y}, \bar{y}^*) = \operatorname{gph} N_{K_\Gamma(\bar{y}, \bar{y}^*)}. \tag{2.9}$$

Further, it was shown in [10, Proof of Theorem 2] that

$$\widehat{N}_{\operatorname{gph} N_\Gamma}(\bar{y}, \bar{y}^*) = (K_\Gamma(\bar{y}, \bar{y}^*))^\circ \times K_\Gamma(\bar{y}, \bar{y}^*) \tag{2.10}$$

and that $N_{\operatorname{gph} N_\Gamma}(\bar{y}, \bar{y}^*)$ is the union of all product sets $K^\circ \times K$ associated with cones K of the form $F_1 - F_2$, where F_1 and F_2 are closed faces of the critical cone $K_\Gamma(\bar{y}, \bar{y}^*)$ satisfying $F_2 \subset F_1$.

Thanks to the definition of a face of a convex set (see [38, Chapter 18]), the closed faces F of any polyhedral convex cone K are the polyhedral convex cones of the form

$$F = K \cap [z^*]^\perp \text{ for some } z^* \in K^\circ.$$

We will denote the collection of all closed faces of a polyhedral convex cone K by $\mathcal{F}(K)$.

In the following proposition we state a similar description of the directional limiting normal cone in terms of selected faces of the critical cone $K_\Gamma(\bar{y}, \bar{y}^*)$.

THEOREM 2.12. *Let Γ be a convex polyhedral set in $\mathbb{R}^m$, and let $(\bar{y}, \bar{y}^*) \in \operatorname{gph} N_\Gamma$ and $(v, v^*) \in T_{\operatorname{gph} N_\Gamma}(\bar{y}, \bar{y}^*)$ be given. Then $N_{\operatorname{gph} N_\Gamma}((\bar{y}, \bar{y}^*); (v, v^*))$ is the union of all product sets $K^\circ \times K$ associated with cones K of the form $F_1 - F_2$, where F_1 and F_2 are closed faces of the critical cone $K_\Gamma(\bar{y}, \bar{y}^*)$ satisfying $v \in F_2 \subset F_1 \subset [v^*]^\perp$.*

In order to prove this theorem we need two preparatory lemmas.

LEMMA 2.13. *Let Γ be a convex polyhedral set in $\mathbb{R}^m$, and let $(\bar{y}, \bar{y}^*) \in \operatorname{gph} N_\Gamma$. Then there exists some radius $\rho > 0$ such that for every $(y, y^*) \in \operatorname{gph} N_\Gamma \cap \mathcal{B}((\bar{y}, \bar{y}^*), \rho)$ one has*

$$K_\Gamma(y, y^*) = (K_\Gamma(\bar{y}, \bar{y}^*) \cap [y^* - \bar{y}^*]^\perp) + [y - \bar{y}]. \tag{2.11}$$

Proof. From [10, Proof of Theorem 2] we can distill that there exists some radius $\rho > 0$ such that for every $(y, y^*) \in \operatorname{gph} N_\Gamma \cap \mathcal{B}((\bar{y}, \bar{y}^*), \rho)$ one has

$$K_\Gamma(y, y^*) = (T_\Gamma(\bar{y}) \cap [y^*]^\perp) + [y - \bar{y}], \tag{2.12}$$

$$T_\Gamma(\bar{y}) \cap [y^*]^\perp \subset K_\Gamma(\bar{y}, \bar{y}^*). \tag{2.13}$$

Now consider $(y, y^*) \in \operatorname{gph} N_\Gamma \cap \mathcal{B}((\bar{y}, \bar{y}^*), \rho)$. We will show that $T_\Gamma(\bar{y}) \cap [y^*]^\perp = K_\Gamma(\bar{y}, \bar{y}^*) \cap [y^* - \bar{y}^*]^\perp$. Fix any $w \in T_\Gamma(\bar{y}) \cap [y^*]^\perp$. By (2.13) we have $w \in K_\Gamma(\bar{y}, \bar{y}^*)$ and thus $w \in [\bar{y}^*]^\perp$. Since $w \in [y^*]^\perp$ we obtain

$$0 = \langle y^*, w\rangle = \langle \bar{y}^* + (y^* - \bar{y}^*), w\rangle = \langle y^* - \bar{y}^*, w\rangle,$$

and $w \in T_\Gamma(\bar{y}) \cap [\bar{y}^*]^\perp \cap [y^* - \bar{y}^*]^\perp = K_\Gamma(\bar{y}, \bar{y}^*) \cap [y^* - \bar{y}^*]^\perp$ follows. This shows $T_\Gamma(\bar{y}) \cap [y^*]^\perp \subset K_\Gamma(\bar{y}, \bar{y}^*) \cap [y^* - \bar{y}^*]^\perp$. On the other hand, we always have $[y^*]^\perp \supset [\bar{y}^*]^\perp \cap [y^* - \bar{y}^*]^\perp$, yielding $T_\Gamma(\bar{y}) \cap [y^*]^\perp \supset T_\Gamma(\bar{y}) \cap [\bar{y}^*]^\perp \cap [y^* - \bar{y}^*]^\perp = K_\Gamma(\bar{y}, \bar{y}^*) \cap [y^* - \bar{y}^*]^\perp$. Hence, the claimed relation $T_\Gamma(\bar{y}) \cap [y^*]^\perp = K_\Gamma(\bar{y}, \bar{y}^*) \cap [y^* - \bar{y}^*]^\perp$ holds, and the statement of the lemma follows from (2.12). ∎

LEMMA 2.14. *Let $K \subset \mathbb{R}^d$ be a convex polyhedral cone, and let $(v, v^*) \in \operatorname{gph} N_K$. Then*

$$\mathcal{F}((K \cap [v^*]^\perp) + [v]) = \{F + [v] \,|\, F \in \mathcal{F}(K), v \in F \subset [v^*]^\perp\} \tag{2.14}$$

and

$$\begin{aligned}\{\tilde{F}_1 - \tilde{F}_2 \,|\, \tilde{F}_1, \tilde{F}_2 \in \mathcal{F}((K \cap [v^*]^\perp) + [v]), \tilde{F}_2 \subset \tilde{F}_1\} = \{F_1 - F_2 \,|\, F_1, F_2 \in \mathcal{F}(K),\\ v \in F_2 \subset F_1 \subset [v^*]^\perp\}.\end{aligned} \tag{2.15}$$

Proof. Note that for two convex polyhedral cones K_1, K_2 their polar cones K_1°, K_2° and their sum $K_1 + K_2$ are also convex polyhedral by [38, Corollaries 19.2.2 and 19.3.2] and therefore closed. This implies $(K_1 \cap K_2)^\circ = K_1^\circ + K_2^\circ$ and $(K_1 + K_2)^\circ = K_1^\circ \cap K_2^\circ$ by [38, Corollary 16.4.2]. Hence, $\tilde{K}^\circ = (K^\circ + [v^*]) \cap [v]^\perp$, where $\tilde{K} := (K \cap [v^*]^\perp) + [v]$. Let $\tilde{F} \in \mathcal{F}(\tilde{K})$ and consider $z^* \in \tilde{K}^\circ$ with $\tilde{F} = \tilde{K} \cap [z^*]^\perp$. From $z^* \in \tilde{K}^\circ$ we conclude that $z^* \in [v]^\perp$, and $\tilde{F} = K \cap [v^*]^\perp \cap [z^*]^\perp + [v]$ follows. Further, $z^* = w^* + \alpha v^*$ with $w^* \in K^\circ$ and $\alpha \in \mathbb{R}$, implying $[z^*]^\perp \cap [v^*]^\perp = [w^*]^\perp \cap [v^*]^\perp$. Since $v^* \in N_K(v) = K^\circ \cap [v]^\perp$, we obtain $w^* + v^* \in K^\circ$, $v \in [w^* + v^*]^\perp$, and $K \cap [w^* + v^*]^\perp = K \cap [w^*]^\perp \cap [v^*]^\perp$. This shows that $\tilde{F} = F + [v]$, where $F = K \cap [v^*]^\perp \cap [z^*] = K \cap [w^* + v^*]^\perp$ is a face of K verifying $v \in F \subset [v^*]^\perp$. Conversely, let $F \in \mathcal{F}(K)$ satisfying $v \in F \subset [v^*]^\perp$, and choose $w^* \in K^\circ$ with $F = K \cap [w^*]^\perp$. Then $w^* + v^* \in K^\circ$ and $v \in [w^*]^\perp \cap [v^*]^\perp = [w^* + v^*]^\perp \cap [v^*]^\perp$. Hence, $w^* + v^* \in (\tilde{K}^\circ + [v^*]) \cap [v]^\perp \cap [v]^\perp = \tilde{K}^\circ$ and $F + [v] = F \cap [v^*]^\perp + [v] = K \cap [w^* + v^*]^\perp \cap [v^*]^\perp + [v] = ((K \cap [v^*]^\perp) + [v]) \cap [w^* + v^*]^\circ \in \mathcal{F}(\tilde{K})$ follows.

In order to prove (2.15), consider $\tilde{F}_1, \tilde{F}_2 \in \mathcal{F}((K \cap [v^*]^\perp) + [v])$ with $\tilde{F}_2 \subset \tilde{F}_1$ and, following (2.14), some corresponding faces $F_1, F_2 \in \mathcal{F}(K)$ with $v \in F_i \subset [v^*]^\perp$ such that $\tilde{F}_i = F_i + [v]$, $i = 1, 2$. Now consider an arbitrary element $f_2 \in F_2$. Then, due to $\tilde{F}_2 \subset \tilde{F}_1$, there are $f_1 \in F_1$ and $\alpha \in \mathbb{R}$ such that $f_2 = f_1 + \alpha v$. Expressing α as the difference of two nonnegative numbers α_1 and α_2, we obtain $f_2 + \alpha_2 v = f_1 + \alpha_1 v \in F_2 \cap F_1$. Hence, for all reals β we have $f_2 + \beta v = f_2 + \alpha_2 v + (\beta - \alpha_2) v \in (F_2 \cap F_1) + [v]$ showing $\tilde{F}_2 \subset (F_2 \cap F_1) + [v]$. Since we obviously have $\tilde{F}_2 \supset (F_2 \cap F_1) + [v]$, the equality $\tilde{F}_2 = F_2 \cap F_1 + [v]$ holds. The intersection $F_2' = F_2 \cap F_1$ of the closed faces

F_1, F_2 of K is again a closed face of K, the inclusion $v \in F_2' \subset F_1 \subset [v^*]^\perp$ obviously holds, and we obtain

$$\tilde{F}_1 - \tilde{F}_2 = (F_1 + [v]) - (F_2' + [v]) = F_1 - F_2' + [\tilde{v}] = (F_1 + \mathbb{R}^+\{v\}) - (F_2' + \mathbb{R}_+\{v\}) = F_1 - F_2'.$$

On the other hand, for faces F_1, F_2 of K with $v \in F_2 \subset F_1 \subset [v^*]^\perp$ we have $F_2 + [v] \subset F_1 + [v]$ and

$$F_1 - F_2 = (F_1 + \mathbb{R}^+\{v\}) - (F_2 + \mathbb{R}_+\{v\}) = F_1 - F_2 + [\tilde{v}] = (F_1 + [v]) - (F_2 + [v]).$$

Relation (2.15) now follows from this relation and (2.14). ◻

Proof of Theorem 2.12. Let $\bar{K}$ denote the critical cone $K_\Gamma(\bar{y}, \bar{y}^*)$. Note that the requirement $(v, v^*) \in T_{\mathrm{gph}\, N_\Gamma}(\bar{y}, \bar{y}^*)$ is equivalent to $(v, v^*) \in \mathrm{gph}\, N_{\bar{K}}$ by virtue of (2.9). This means that $v \in \bar{K}$ and $v^* \in N_{\bar{K}}(v) = \bar{K}^\circ \cap [v]^\perp$. Since $\mathrm{gph}\, N_\Gamma$ is the union of finitely many convex polyhedrons, we can apply [18, Lemma 3.4] together with (2.10) to obtain

$$N_{\mathrm{gph}\, N_\Gamma}((\bar{y}, \bar{y}^*); (v, v^*)) = \bigcup_{\substack{t \in (0, \bar{t}] \\ (w, w^*) \in \mathcal{B}((v, v^*), \delta)}} (K_\Gamma(\bar{y} + tw, \bar{y}^* + tw^*))^\circ \times K_\Gamma(\bar{y} + tw, \bar{y}^* + tw^*)$$

for all $\delta, \bar{t} > 0$ sufficiently small. By virtue of (2.8), we can choose δ and $\bar{t}$ small enough such that for every $t \in (0, \bar{t})$ and every $(w, w^*) \in \mathcal{B}((v, v^*), \delta)$ the condition $(\bar{y} + tw, \bar{y}^* + tw^*) \in \mathrm{gph}\, N_\Gamma$ is equivalent to $(w, w^*) \in \mathrm{gph}\, N_{\bar{K}}$. By taking into account that $K_\Gamma(\bar{y} + tw, \bar{y}^* + tw^*) = \emptyset$ when $(\bar{y} + tw, \bar{y}^* + tw^*) \notin \mathrm{gph}\, N_\Gamma$, we arrive at the more precise statement

$$N_{\mathrm{gph}\, N_\Gamma}((\bar{y}, \bar{y}^*); (v, v^*)) = \bigcup_{\substack{t \in (0, \bar{t}) \\ (w, w^*) \in \mathcal{B}((v, v^*), \delta) \cap \mathrm{gph}\, N_{\bar{K}}}} (K_\Gamma(\bar{y} + tw, \bar{y}^* + tw^*))^\circ \times K_\Gamma(\bar{y} + tw, \bar{y}^* + tw^*). \tag{2.16}$$

Further, by decreasing $\bar{t}$ if necessary, we can also assume that $(\bar{y} + tw, \bar{y}^* + tw^*) \subset \mathrm{gph}\, N_\Gamma \cap \mathcal{B}((\bar{y}, \bar{y}^*), \rho)$ holds for all $t \in (0, \bar{t})$ and all $(w, w^*) \in \mathcal{B}((v, v^*), \delta) \cap \mathrm{gph}\, N_{\bar{K}}$, where ρ is given by Lemma 2.13, implying $K_\Gamma(\bar{y} + tw, \bar{y}^* + tw^*) = (\bar{K} \cap [tw^*]^\perp) + [tw] = (\bar{K} \cap [w^*]^\perp) + [w] = K_\Gamma(\bar{y} + \bar{t}w, \bar{y}^* + \bar{t}w^*)$ and

$$N_{\mathrm{gph}\, N_\Gamma}((\bar{y}, \bar{y}^*); (v, v^*)) = \bigcup_{(w, w^*) \in \mathcal{B}((v, v^*), \delta) \cap \mathrm{gph}\, N_{\bar{K}}} (K_\Gamma(\bar{y} + \bar{t}w, \bar{y}^* + \bar{t}w^*))^\circ \times K_\Gamma(\bar{y} + \bar{t}w, \bar{y}^* + \bar{t}w^*). \tag{2.17}$$

By the critical superface lemma [12, Lemma 4H.2] we obtain

$$\begin{aligned} &\{K_\Gamma(\bar{y} + \bar{t}w, \bar{y}^* + \bar{t}w^*) \,|\, (w, w^*) \in \mathcal{B}((v, v^*), \delta) \cap \mathrm{gph}\, N_{\bar{K}}\} \\ &\qquad = \{F_1 - F_2 \,|\, F_1, F_2 \in \mathcal{F}(K_\Gamma(\bar{y} + \bar{t}v, \bar{y}^* + \bar{t}v^*)),\ F_2 \subset F_1\} \end{aligned}$$

for every $\delta > 0$ sufficiently small, and the statement of the theorem follows from (2.17) and Lemmas 2.13, and 2.14. ◻

3. Calmness of implicit multifunctions. This section is divided into two parts. In the first we prove that the calmness of F is ensured by the two conditions imposed on M, mentioned already in the introduction. In contrast to the remainder of the paper, subsection 3.1 is formulated in the setting of general metric spaces. Subsection 3.2 is then focused on the question of how these two conditions can be verified by using the tools of variational analysis.

3.1. General theory. Let P, X, Y be metric spaces. With respect to this general setting we will analyze now, instead of (1.1), the multifunction

$$S(p) := \{x \in X | \bar{y} \in M(p, x)\}, \tag{3.1}$$

where $M : P \times X \rightrightarrows Y$ is a given multifunction and $\bar{y}$ is a given element of Y.

In [24, Lemma 1] the authors considered the special case

$$M(p, x) = G(p, x) - D, \tag{3.2}$$

where the function $G : P \times X \to Y$ is Lipschitz near the reference pair $(\bar{p}, \bar{x})$ and D is a closed subset of Y. When P, X, Y are normed spaces, it has been shown therein that the calmness of the respective multifunction S at $(\bar{p}, \bar{x})$ is implied by the metric subregularity of the (simpler) mapping $M_{\bar{p}} : X \rightrightarrows Y$ defined by

$$M_{\bar{p}}(x) := M(\bar{p}, x) = G(\bar{p}, x) - D$$

at $(\bar{x}, 0)$.

Next, we present a deep generalization of this result which is valid even in our general setting in metric spaces and in which the structural assumption (3.2) is abandoned. We associate with M the multifunctions $H_M : P \rightrightarrows X \times Y$ and $M_p : X \rightrightarrows Y$ defined by

$$H_M(p) := \{(x, y) \,|\, y \in M(p, x)\} \tag{3.3}$$

and

$$M_p(x) := \{y \,|\, y \in M(p, x)\} \quad \text{for each } p \in P. \tag{3.4}$$

Note that $\operatorname{gph} H_M = \operatorname{gph} M$. The following auxiliary notion will be crucial for our analysis.

Definition 3.1. *Let* $(\bar{p}, \bar{x}, \bar{y}) \in \operatorname{gph} M$. *We say that* M *has the* restricted calmness property with respect to p at $(\bar{p}, \bar{x}, \bar{y})$ *if there are reals* $L \geq 0$ *and* $\epsilon > 0$ *such that*

$$\mathrm{d}_{X\times Y}((x, \bar{y}), H_M(\bar{p})) \leq L\rho_P(p, \bar{p}) \text{ provided } \rho_P(p, \bar{p}) < \varepsilon, \rho_X(x, \bar{x}) < \varepsilon \text{ and } (x, \bar{y}) \in H_M(p). \tag{3.5}$$

Remark 3. It is easy to see that M has the restricted calmness property with respect to p at $(\bar{p}, \bar{x}, \bar{y})$ if H_M is calm at $(\bar{p}, (\bar{x}, \bar{y}))$. In particular, in the setting of normed spaces this condition is fulfilled for multifunctions of the form $M(p, x) = G(p, x) + Q(x)$, where G is a Lipschitz continuous function and Q is set-valued.

The following lemma states that the restricted calmness property with respect to p is necessary for the calmness of the solution mapping S.

Lemma 3.2. *If* S *is calm at* $(\bar{p}, \bar{x})$, *then* M *has the restricted calmness property with respect to* p *at* $(\bar{p}, \bar{x}, \bar{y})$.

Proof. According to the definition of calmness we choose reals $L \geq 0$ and $\epsilon > 0$ such that

$$\mathrm{d}_X(x, S(\bar{p})) \leq L\rho_P(p, \bar{p}) \text{ provided } p \in \mathcal{B}_P(\bar{p}, \varepsilon) \text{ and } x \in S(p) \cap \mathcal{B}_X(\bar{x}, \varepsilon).$$

Next, for every $p \in \mathcal{B}_P(\bar{p}, \epsilon)$ and every $(x, \bar{y}) \in H_M(p)$ with $x \in \mathcal{B}_X(\bar{x}, \varepsilon)$ we have $x \in S(p) \cap \mathcal{B}_X(\bar{x}, \varepsilon)$. Clearly, for each $\alpha > 0$ there is some $\tilde{x} \in S(\bar{p})$ with $\rho_X(x, \tilde{x}) \leq \mathrm{d}_X(x, S(\bar{p})) + \alpha$, and so it follows that $\rho_X(x, \tilde{x}) \leq L\rho_P(p, \bar{p}) + \alpha$. Note that $(\tilde{x}, \bar{y}) \in H_M(\bar{p})$, whence

$$\mathrm{d}_{X \times Y}((x, \bar{y}), H_M(\bar{p})) \leq \rho_X(x, \tilde{x}) \leq L\rho_P(p, \bar{p}) + \alpha.$$

Since a suitable point $\tilde{x}$ can be found for any arbitrarily small α, one can conclude that

$$\mathrm{d}_{X \times Y}((x, \bar{y}), H_M(\bar{p})) \leq L\rho_P(p, \bar{p}),$$

which amounts to the restricted calmness property with respect to p of M at $(\bar{p}, \bar{x}, \bar{y})$. ∎

We state now a sufficient criterion for the calmness of S.

THEOREM 3.3. *Let $\bar{y} \in M(\bar{p}, \bar{x})$, and assume that M has the restricted calmness property with respect to p at $(\bar{p}, \bar{x}, \bar{y})$ and that $M_{\bar{p}}$ is metrically subregular at $(\bar{x}, \bar{y})$. Then S is calm at $(\bar{p}, \bar{x})$.*

Proof. By virtue of the restricted calmness property with respect to p of M at $(\bar{p}, \bar{x}, \bar{y})$ we can find moduli L and κ along with some radii $r_p, r_x, \sigma > 0$ such that

(3.6) $\quad \mathrm{d}_{X \times Y}((x, \bar{y}), H_M(\bar{p})) \leq L\rho_P(p, \bar{p})$ provided $p \in \mathcal{B}_P(\bar{p}, r_p), x \in \mathcal{B}_X(\bar{x}, r_x)$ and $(x, \bar{y}) \in H_M(p)$,

and, by the metric subregularity of $M_{\bar{p}}$ at $(\bar{x}, \bar{y})$, one has

$$\mathrm{d}_X(x, M_{\bar{p}}^{-1}(\bar{y})) \leq \kappa \mathrm{d}_Y(\bar{y}, M_{\bar{p}}(x)) \text{ provided } x \in \mathcal{B}_X(\bar{x}, \sigma). \tag{3.7}$$

By decreasing the radii r_p and r_x if necessary we can assume $r_x + Lr_p < \sigma$. Now fix $p \in \mathcal{B}_P(\bar{p}, r_p)$ and consider $x \in S(p) \cap \mathcal{B}_X(\bar{x}, r_x)$ so that $(x, \bar{y}) \in H_M(p) \cap (\mathcal{B}_X(\bar{x}, r_x) \times \{\bar{y}\})$. Further, observe that for each $\beta > 0$ there is some $(\tilde{x}, \tilde{y}) \in H_M(\bar{p})$ satisfying

$$\rho_{X \times Y}((x, \bar{y}), (\tilde{x}, \tilde{y})) \leq \mathrm{d}_{X \times Y}((x, \bar{y}), H_M(\bar{p})) + \beta. \tag{3.8}$$

Consequently, by virtue of (3.6),

$$\rho_{X \times Y}((x, \bar{y}), (\tilde{x}, \tilde{y})) \leq L\rho_P(p, \bar{p}) + \beta. \tag{3.9}$$

It follows from the triangle inequality and (3.9) that

$$\rho_X(\tilde{x}, \bar{x}) \leq \rho_X(\tilde{x}, x) + \rho_X(x, \bar{x}) \leq L\rho_P(p, \bar{p}) + \beta + \rho_X(x, \bar{x}) \leq Lr_p + \beta + r_x.$$

Since $Lr_p + r_x < \sigma$, β can be chosen sufficiently small to obtain that $\rho_X(\tilde{x}, \bar{x}) < \sigma$ as well. Further, we note that for each $\gamma > 0$ there is some $\hat{x} \in S(\bar{p}) = M_{\bar{p}}^{-1}(\bar{y})$ such that

$$\rho_X(\tilde{x}, \hat{x}) \leq \mathrm{d}_X(\tilde{x}, M_{\bar{p}}^{-1}(\bar{y})) + \gamma.$$

This implies, by virtue of (3.7), that

$$\rho_X(\tilde{x}, \hat{x}) \leq \kappa \mathrm{d}_Y(\bar{y}, M_{\bar{p}}(\tilde{x})) + \gamma \leq \kappa \rho_Y(\bar{y}, \tilde{y}) + \gamma, \tag{3.10}$$

where the last inequality follows from the fact that $\tilde{y} \in M_{\bar{p}}(\tilde{x})$.

Hence, by successively using the triangle inequality, estimate (3.10), the Cauchy–Schwarz inequality, and estimates (3.9) and (3.6), we obtain

$$
\begin{aligned}
\mathrm{d}_X(x, S(\bar{p})) &\leq \rho_X(x, \hat{x}) \leq \rho_X(x, \tilde{x}) + \rho_X(\tilde{x}, \hat{x}) \leq \rho(x, \tilde{x}) + \kappa \rho_Y(\bar{y}, \tilde{y}) + \gamma \\
&\leq \sqrt{1+\kappa^2}\sqrt{(\rho_X(x, \tilde{x}))^2 + (\rho_Y(\bar{y}, \tilde{y}))^2} + \gamma \\
&= \sqrt{1+\kappa^2}\rho_{X\times Y}((x, \bar{y}), (\tilde{x}, \tilde{y})) + \gamma \\
&\leq \sqrt{1+\kappa^2}(L\rho_P(p, \bar{p}) + \beta) + \gamma.
\end{aligned}
$$

It remains to notice again that suitable points $(\tilde{x}, \tilde{y})$ and $\hat{x}$ can be found for arbitrarily small values of β and γ, whence

$$\mathrm{d}_X(x, S(\bar{p})) \leq \sqrt{1+\kappa^2} L\rho_P(p, \bar{p}).$$

Since $p \in \mathcal{B}_P(\bar{p}, r_p)$ was arbitrarily fixed, the claimed calmness of S at $(\bar{p}, \bar{x})$ follows. □

Remark 4. Note that the restricted calmness property with respect to p of M is strictly less stringent than condition (3.3) in [5, Theorem 3.1]. This condition was used there, together with a sufficient condition for metric subregularity of $\mathcal{M}_{\bar{p}}$, to show the calmness of S.

COROLLARY 3.4. *Consider the implicit multifunction S given by* (3.1) *with $\bar{y} = 0$, and consider*

$$M(p, x) = A(p)x + b(p) + Q(x)$$

around the reference point $(\bar{p}, \bar{x}) \in \operatorname{gph} S$. Assume that the mappings $A : \mathbb{R}^l \to \mathbb{R}^{m\times n}$ and $b : \mathbb{R}^l \to \mathbb{R}^m$ are Lipschitz near $\bar{p}$ and that the graph of $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a union of finitely many convex polyhedra. Then S is calm at $(\bar{p}, \bar{x})$.

Proof. As mentioned in Remark 1, such a mapping M has the restricted calmness property with respect to p at $(\bar{p}, \bar{x}, 0)$. Furthermore, $M_{\bar{p}}$ is polyhedral and hence metrically subregular at $(\bar{x}, 0)$ by virtue of [36, Proposition 1]. The statement thus directly follows from Theorem 3.3. □

In the above way one can model solution maps to parameterized quadratic programs with the parameter arising in the objective.

The next subsection is devoted to workable conditions ensuring that the assumptions of Theorem 3.3 are fulfilled.

3.2. Calmness criteria. The next theorem states a sufficient condition for the restricted calmness property with respect to p of M based on generalized differentiation. From now on $P = \mathbb{R}^l$, $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and $\bar{y} = 0$.

THEOREM 3.5. *Let $0 \in M(\bar{p}, \bar{x})$ and assume that there do not exist elements $u \in \mathcal{S}_{\mathbb{R}^n}$, $q^* \in \mathcal{S}_{\mathbb{R}^l}$ and sequences $(q_k, u_k, v_k) \to (0, u, 0)$, $(q_k^*, u_k^*, v_k^*) \to (q^*, 0, 0)$, $t_k \searrow 0$ such that*

$$(q_k^*, u_k^*) \in \hat{D}^* M((\bar{p}, \bar{x}, 0) + t_k(q_k, u_k, v_k))(v_k^*) \ \forall k, \tag{3.11}$$

$$q_k \neq 0 \ \forall k, \quad \lim_{k\to\infty} \left\langle q_k^*, \frac{q_k}{\|q_k\|} \right\rangle = -1. \tag{3.12}$$

Then M has the restricted calmness property with respect to p at $(\bar{p}, \bar{x}, 0)$.

Proof. The proof is obtained by contraposition. Assume on the contrary that M does not have the restricted calmness property with respect to p at $(\bar p,\bar x,0)$. Then there are sequences $(p_k,x_k)\to(\bar p,\bar x)$ such that $(x_k,0)\in H_M(p_k)$ and $\|x_k-\bar x\|\geq \mathrm{d}((x_k,0),H_M(\bar p))>k\|p_k-\bar p\|$. Next we denote by $(\bar p_k,\bar x_k,\bar y_k)$ for each k a global solution of the program

$$\min_{p,x,y}\phi_k(p,x,y):=\|p-\bar p\|+\frac{2}{k}\|x-x_k\|+\frac{1}{\sqrt{k}}\|y\|\quad\text{subject to}\quad (p,x,y)\in\operatorname{gph}M.$$

Then we must have $\bar p_k\neq\bar p$, since otherwise we would obtain

$$\begin{aligned}\frac{1}{k}\mathrm{d}((x_k,0),H_M(\bar p))&\leq\frac{1}{k}(\|(x_k-\bar x_k\|+\|\bar y_k\|)\leq\frac{2}{k}\|\bar x_k-x_k\|+\frac{1}{\sqrt{k}}\|\bar y_k\|\\&=\phi_k(\bar p,\bar x_k,\bar y_k)=\phi_k(\bar p_k,\bar x_k,\bar y_k)\leq\phi_k(p_k,x_k,0)=\|p_k-\bar p\|,\end{aligned}$$

contradicting $\mathrm{d}((x_k,0),H_M(\bar p))>k\|p_k-\bar p\|$. Hence $t_k:=\|(\bar p_k,\bar x_k,\bar y_k)-(\bar p,\bar x,0)\|>0$, and by passing to a subsequence if necessary we can assume that $((\bar p_k,\bar x_k,\bar y_k)-(\bar p,\bar x,0))/t_k$ converges to some element $(q,u,v)\in\mathcal{S}_{\mathbb{R}^l\times\mathbb{R}^n\times\mathbb{R}^m}$. Since $\phi_k(\bar p_k,\bar x_k,\bar y_k)\leq\phi_k(p_k,x_k,0)=\|p_k-\bar p\|$, we can conclude that $\|\bar p_k-\bar p\|\leq\|p_k-\bar p\|$, $\frac{2}{k}\|\bar x_k-x_k\|\leq\|p_k-\bar p\|$ and $\frac{1}{\sqrt{k}}\|\bar y_k\|\leq\|p_k-\bar p\|$, yielding, together with $\|p_k-\bar p\|<\frac{1}{k}\|x_k-\bar x\|$, the relations

$$\begin{aligned}\|\bar x_k-\bar x\|&\geq\|x_k-\bar x\|-\|\bar x_k-\bar x\|\geq\|x_k-\bar x\|-\frac{k}{2}\|p_k-\bar p\|>\frac{1}{2}\|x_k-\bar x\|>\frac{k}{2}\|p_k-\bar p\|\\&\geq\frac{k}{2}\|\bar p_k-\bar p\|\end{aligned}$$

and $\|\bar y_k\|\leq\sqrt{k}\|p_k-\bar p\|<\frac{1}{\sqrt{k}}\|x_k-\bar x\|<\frac{2}{\sqrt{k}}\|\bar x_k-\bar x\|$. Hence, we can conclude that $\|\bar p_k-\bar p\|/\|\bar x_k-\bar x\|\to 0$, $\|\bar y_k\|/\|\bar x_k-\bar x\|\to 0$, and $q=0$, $v=0$ follows. Since

$$\|\bar x_k-\bar x\|\leq\|x_k-\bar x\|+\|\bar x_k-\bar x\|\leq\|x_k-\bar x\|+\frac{k}{2}\|p_k-\bar p\|<\frac{3}{2}\|x_k-\bar x\|\to 0,$$

it also follows that $t_k\searrow 0$.

Next we utilize the optimality condition $0\in\partial\phi_k(\bar p_k,\bar x_k,\bar y_k)+N_{\operatorname{gph}M}(\bar p_k,\bar x_k,\bar y_k)$ (see [39, Theorem 8.15]), where $\partial\phi_k$ can be taken as the subdifferential of convex analysis since ϕ_k is convex. Let $(\alpha_k^*,\beta_k^*,\gamma_k^*)\in(-\partial\phi_k(\bar p_k,\bar x_k,\bar y_k))\cap N_{\operatorname{gph}M}(\bar p_k,\bar x_k,\bar y_k)$. Then, standard arguments from convex analysis yield $\alpha_k^*=-(\bar p_k-\bar p)/\|\bar p_k-\bar p\|$, and $\beta_k^*\in\frac{2}{k}\mathcal{B}_{\mathbb{R}^n}$, $\gamma_k^*\in\frac{1}{\sqrt{k}}\mathcal{B}_{\mathbb{R}^m}$, and we deduce $(\beta_k^*,\gamma_k^*)\to(0,0)$ as $k\to\infty$. By the definition of the limiting normal cone we can find for each k some elements $(\bar p_k',\bar x_k',\bar y_k')$ and $(q_k^*,u_k^*,-v_k^*)\in\hat N_{\operatorname{gph}M}(\bar p_k',\bar x_k',\bar y_k')$, verifying

$$\|(\bar p_k',\bar x_k',\bar y_k')-(\bar p_k,\bar x_k,\bar y_k)\|\leq\|\bar p_k-\bar p\|/k,\quad \|(q_k^*,u_k^*,-v_k^*)-(\alpha_k^*,\beta_k^*,\gamma_k^*)\|\leq\frac{1}{k}.$$

Then we obviously have $(u_k^*,v_k^*)\to(0,0)$. Since $\alpha_k^*\in\mathcal{S}_{\mathbb{R}^l}$ for all k and $q_k^*-\alpha_k^*\to 0$, by passing to a subsequence if necessary we can assume that q_k^* converges to some $q^*\in\mathcal{S}_{\mathbb{R}^l}$. Now set $(q_k,u_k,v_k):=((\bar p_k',\bar x_k',\bar y_k')-(\bar p,\bar x,\bar y))/t_k$. Since $\|\bar p_k-\bar p\|\leq t_k$, the choice of $(\bar p_k',\bar x_k',\bar y_k')$ guarantees that (q_k,u_k,v_k) converges to $(0,u,0)$. Further, we have

$$\begin{aligned}\left\|\frac{\bar p_k-\bar p}{\|\bar p_k-\bar p\|}-\frac{\bar p_k'-\bar p}{\|\bar p_k'-\bar p\|}\right\|&=\left\|\frac{(\bar p_k'-\bar p)(\|\bar p_k'-\bar p\|-\|\bar p_k-\bar p\|)-(\bar p_k'-\bar p_k)\|\bar p_k'-\bar p\|}{\|\bar p_k'-\bar p\|\cdot\|\bar p_k-\bar p\|}\right\|\\&\leq\frac{2\|\bar p_k'-\bar p_k\|\cdot\|\bar p_k'-\bar p\|}{\|\bar p_k'-\bar p\|\cdot\|\bar p_k-\bar p\|}\leq\frac{2}{k},\end{aligned}$$

implying

$$\lim_{k\to\infty}\left\langle q_k^*, \frac{q_k}{\|q_k\|}\right\rangle = \lim_{k\to\infty}\left\langle q_k^*, \frac{\bar{p}_k'-\bar{p}}{\|\bar{p}_k'-\bar{p}\|}\right\rangle = \lim_{k\to\infty}\left\langle q_k^*, \frac{\bar{p}_k-\bar{p}}{\|\bar{p}_k-\bar{p}\|}\right\rangle$$
$$= \lim_{k\to\infty}\left\langle \alpha_k^*, \frac{\bar{p}_k-\bar{p}}{\|\bar{p}_k-\bar{p}\|}\right\rangle = -1.$$

Finally, note that we have $(q_k^*, u_k^*) \in \hat{D}^* M((\bar{p},\bar{x},0)+t_k(q_k,u_k,v_k))(v_k^*)$ by our construction. Hence, we see that q^*, u together with the sequences t_k, (q_k,u_k,v_k) and (q_k^*,u_k^*,v_k^*) violate the assumptions of the theorem, yielding the desired contradiction. This finishes the proof. ∎

Condition (3.11) suggests the following definition.

DEFINITION 3.6. *The* outer coderivative of M with respect to p in direction u at $(\bar{p},\bar{x},0)$ *is the multifunction* $D^*_{>_p}M((\bar{p},\bar{x},0);u) : \mathbb{R}^m \rightrightarrows \mathbb{R}^l\times\mathbb{R}^n$*, where* $D^*_{>_p}M((\bar{p},\bar{x},0);u)(v^*)$ *consists of all pairs* (q^*,u^*) *such that there are sequences* $t_k \searrow 0$*,* $(q_k,u_k,v_k)\to(0,u,0)$*,* $(q_k^*,u_k^*,v_k^*)\to(q^*,u^*,v^*)$ *verifying*

$$q_k \neq 0 \quad \textit{and} \quad (q_k^*,u_k^*,-v_k^*) \in \widehat{N}_{\operatorname{gph} M}(\bar{p}+t_kq_k, \bar{x}+t_ku_k, t_kv_k)\ \forall k.$$

By the definition of the directional limiting coderivative we see that

$$D^*_{>_p}M((\bar{p},\bar{x},0);u)(v^*) \subset D^*M((\bar{p},\bar{x},0);(0,u,0))(v^*)\ \forall v^*\in\mathbb{R}^m. \tag{3.13}$$

Further, we have $D^*_{>_p}M((\bar{p},\bar{x},0);u) \equiv \emptyset$ whenever $(0,u,0)\notin T_{\operatorname{gph} M}(\bar{p},\bar{x},0)$, i.e., $0\notin DM(\bar{p},\bar{x},0)(0,u)$. These observations yield the following point based sufficient condition required for the restricted calmness property with respect to p to hold.

COROLLARY 3.7. *Let* $0\in M(\bar{p},\bar{x})$*, and assume that there do not exist elements* $u\in\mathcal{S}_{\mathbb{R}^n}$ *and* $q\in\mathcal{S}_{\mathbb{R}^l}\cap T_{\operatorname{dom} H_M}(\bar{p})$ *satisfying*

$$0\in DM(\bar{p},\bar{x},0)(0,u), \tag{3.14}$$
$$(-q,0)\in D^*_{>_p}M((\bar{p},\bar{x},0);u)(0). \tag{3.15}$$

Then M *has the restricted calmness property with respect to* p *at* $(\bar{p},\bar{x},0)$*.*

Proof. Consider the sequences specified in Theorem 3.5 which satisfy, in particular, the relations (3.11), (3.12). By passing to a subsequence if necessary we can assume that $q_k/\|q_k\|$ converges to some $q\in\mathcal{S}_{\mathbb{R}^l}$. Since we also have $\bar{p}+(t_k\|q_k\|)(q_k/\|q_k\|) \in \operatorname{dom} H_M$ and $t_k\|q_k\|\to 0$, the inclusion $q\in T_{\operatorname{dom} H_M}(\bar{p})$ follows. From the second condition in (3.12) it follows that

$$\langle q^*, q\rangle = -1 \text{ with } q^*, q\in S_{\mathbb{R}^l}.$$

However, this is possible only when $q^*=-q$, and we are done. ∎

In the following example we demonstrate the application of Theorem 3.5 in a situation when M is of the form $M(x,y)=G(x,y)+Q(y)$ with G being non-Lipschitzian.

Example 1. Consider the multifunction $M:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $M(p,x)=\sqrt{|px|}+x+\mathbb{R}_+$ at $(\bar{p},\bar{x})=(0,0)$. Straightforward calculations yield

$$T_{\operatorname{gph} M}(0,0,0)=\{(q,u,v)\,|\,v\geq\sqrt{|qu|}+u\},$$

and in the case when $px \neq 0$,

$$\hat{D}^* M(p,x,y)(v^*) = \begin{cases} \{(0,0)\} & \text{if } y > \sqrt{|px|} + x,\ v^* = 0, \\ \{(\sqrt{|x/p|}v^* \operatorname{sign} p,\ \sqrt{|p/x|}v^* \operatorname{sign} x + v^*)\} & \text{if } y = \sqrt{|px|} + x,\ v^* \geq 0, \\ \emptyset & \text{else.} \end{cases}$$

Now assume that there are sequences $t_k \searrow 0$, $(q_k, u_k, v_k) \to (0, u, 0) \in T_{\operatorname{gph} M}(0,0,0)$ and $(q_k^*, u_k^*, v_k^*) \to (q^*, 0, 0)$ such that $u \in \mathcal{S}_{\mathbb{R}}$, $q^* \in \mathcal{S}_{\mathbb{R}}$ and (3.11), (3.12) hold. The condition $(0, u, 0) \in T_{\operatorname{gph} M}(0,0,0)$ amounts to $u \leq 0$, this condition together with $u \in \mathcal{S}_{\mathbb{R}}$ gives us $u = -1$, and therefore $u_k \neq 0$ holds for all k sufficiently large. Since $q_k \neq 0$, we obtain $v_k^* \geq 0$,

$$q_k^* = \sqrt{\left|\frac{t_k u_k}{t_k q_k}\right|} v_k^* \operatorname{sign} t_k q_k = \sqrt{\left|\frac{u_k}{q_k}\right|} v_k^* \operatorname{sign} q_k,$$

and, consequently,

$$\left\langle q_k^*, \frac{q_k}{|q_k|} \right\rangle = \sqrt{\left|\frac{u_k}{q_k}\right|} v_k^* \geq 0,$$

contradicting (3.12). Hence, it follows from Theorem 3.5 that M has the restricted calmness property with respect to p at $(\bar{p}, \bar{x}, 0)$.

Corollary 3.7 is illustrated by the following example.

Example 2. Let $M(p,x) = \hat{N}_{\Gamma(p)}(x)$, where $\Gamma(p) = \{x \in \mathbb{R} \,|\, px \leq 0\}$ for $p \in \mathbb{R}$, and let $(\bar{p}, \bar{x}) = (0,0)$. Then

$$\Gamma(p) = \begin{cases} \mathbb{R} & \text{if } p = 0, \\ \mathbb{R}_- & \text{if } p > 0, \\ \mathbb{R}_+ & \text{if } p < 0, \end{cases}$$

and therefore,

$$M(p,x) = \begin{cases} \{0\} & \text{if } p = 0 \text{ or } px < 0, \\ \mathbb{R}_+ & \text{if } p > 0,\ x = 0, \\ \mathbb{R}_- & \text{if } p < 0,\ x = 0, \\ \emptyset & \text{else,} \end{cases}$$

$$H_M(p) = \begin{cases} \mathbb{R} \times \{0\} & \text{if } p = 0, \\ (\{0\} \times \mathbb{R}_+) \cup (\mathbb{R}_- \times \{0\}) & \text{if } p > 0, \\ (\{0\} \times \mathbb{R}_-) \cup (\mathbb{R}_+ \times \{0\}) & \text{if } p < 0. \end{cases}$$

Hence, M has the restricted calmness property with respect to p at $(\bar{p}, \bar{x}, 0)$, but H_M is not calm at $(\bar{p}, (\bar{x}, 0))$. Now let us consider the conditions of Corollary 3.7. Since (3.15) involves the outer directional coderivative of M with respect to p only in directions $u \neq 0$, we have to compute the regular normal cone to $\operatorname{gph} M$ at points (p, x, y, z) with $px \neq 0$. Straightforward calculations yield

$$\widehat{N}_{\operatorname{gph} M}(p,x,y) = \begin{cases} \{0\} \times \{0\} \times \mathbb{R} & \text{if } y = 0 \text{ and } px < 0, \\ \emptyset & \text{else if } px \neq 0, \end{cases}$$

showing $D^*_{>_p} M((\bar{p},\bar{x},0);u)(v^*) = \{(0,0)\}$ for all $v^* \in \mathbb{R}$ provided $u \neq 0$. Hence, we can also conclude from Corollary 3.7 that M has the restricted calmness property with respect to p. Note that in this example the inclusion (3.13) is proper, and one could not detect the restricted calmness property with respect to p of M by using the standard directional limiting coderivative.

To verify the metric subregularity of the mapping $M_{\bar{p}}$ at $(\bar{x},0)$ one can use the criteria presented in section 2. In the following theorem we apply these conditions to the frequently arising case of the so-called parameterized constraint systems. We prove that we obtain not only calmness of the solution mapping S but also, under some suitable assumptions, the nonemptiness of $S(p)$ near $\bar{p}$.

THEOREM 3.8. *Let*

$$M(p,x) = G(p,x) - D, \tag{3.16}$$

where $D \subset \mathbb{R}^m$ is closed and G maps $\mathbb{R}^l \times \mathbb{R}^n$ into $\mathbb{R}^m$, and consider the reference point $(\bar{p},\bar{x}) \in G^{-1}(D)$. Assume that $G(\bar{p},\cdot)$ is strictly differentiable at $\bar{x}$ and there are neighborhoods W of $\bar{p}$, and, U of $\bar{x}$ and there is a real L' such that

$$\|G(p,x) - G(\bar{p},x)\| \leq L'\|p-\bar{p}\| \ \forall (p,x) \in W \times U. \tag{3.17}$$

If there do not exist vectors $0 \neq u \in \mathbb{R}^n$, $0 \neq v^* \in \mathbb{R}^m$ such that

$$\nabla_x G(\bar{p},\bar{x})u \in T_D(G(\bar{p},\bar{x})), \tag{3.18}$$

$$0 = \nabla_x G(\bar{p},\bar{x})^T v^*, \tag{3.19}$$

$$v^* \in N_D(G(\bar{p},\bar{x}); \nabla_x G(\bar{p},\bar{x})u), \tag{3.20}$$

then S is calm at $(\bar{p},\bar{x})$.

If in addition G is partially differentiable with respect to x on $W \times U$, if the partial derivative $\nabla_x G$ is continuous at $(\bar{p},\bar{x})$, if for every $p \in W$ the mapping $\nabla_x G(p,\cdot)$ is continuous on U, and if there exists some $\tilde{u} \in \mathcal{S}_{\mathbb{R}^n}$ such that $\nabla_x G(\bar{p},\bar{x})\tilde{u} \in T_D(G(\bar{p},\bar{x}))$ and $\nabla_x G(\bar{p},\bar{x})\tilde{u}$ is derivable, then there exists a neighborhood $\tilde{W}$ of $\bar{p}$ and a real $\tilde{L}$ such that $S(p) \neq \emptyset$ for all $p \in \tilde{W}$ and

$$\mathrm{d}(\bar{x}, S(p)) \leq \tilde{L}\|p-\bar{p}\|, \ p \in \tilde{W}. \tag{3.21}$$

Proof. We first show that M has the restricted calmness property with respect to p at $(\bar{p},\bar{x},0)$. Indeed, if $p \in W$ and $(x,0) \in H_M(p) \cap U \times \{0\}$, then $0 \in M(p,x) = G(p,x) - D$, and, consequently, $G(\bar{p},x) - G(p,x) \in G(\bar{p},x) - D$, i.e., $(x, G(\bar{p},x) - G(p,x)) \in H_M(\bar{p})$. Hence,

$$\mathrm{d}((x,0), H_M(\bar{p})) \leq \|(x,0) - (x, G(\bar{p},x) - G(p,x))\| \leq \|G(\bar{p},x) - G(p,x)\| \leq L'\|p-\bar{p}\|,$$

and the restricted calmness property with respect to p for M at $(\bar{p},\bar{x},\bar{y})$ follows. By using FOSCMS of Corollary 2.11 we see that the imposed assumptions guarantee metric subregularity of $M_{\bar{p}}$ at $(\bar{x},0)$. Thus, calmness of S follows from Theorem 3.3. It remains to show the nonemptiness of S near $\bar{p}$ and the bound (3.21). This is done by contraposition. Assume on the contrary that there is some sequence p_k converging to $\bar{p}$ such that $p_k \neq \bar{p}$ and

$$\mathrm{d}(\bar{x}, S(p_k)) > k\|p_k - \bar{p}\|. \tag{3.22}$$

Since the tangent vector $\nabla_x G(\bar p,\bar x)\tilde u$ is assumed to be derivable, there exists a mapping $\xi:\mathbb{R}_+\to D$ such that $\xi(0)=G(\bar p,\bar x)$ and $\xi(t)-(G(\bar p,\bar x)+t\nabla_x G(\bar p,\bar x)\tilde u)=o(t)$ as $t\searrow 0$. Since $G(\bar p,\cdot)$ is assumed to be continuously differentiable, $\eta(t):=\|G(\bar p,\bar x+t\tilde u)-\xi(t)\|=o(t)$ follows. By passing to a subsequence if necessary we can assume that $\|p_k-\bar p\|\le k^{-2}$ and that $\eta(t_k)\le\|p_k-\bar p\|$ holds for all k, where $t_k:=\frac{k\|p_k-\bar p\|}{2}$. Then $t_k\searrow 0$, and we can find some $L>0$ such that $\|G(p_k,\bar x+t_k\tilde u)-G(\bar p,\bar x+t_k\tilde u)\|\le L\|p_k-\bar p\|$ holds for all k sufficiently large, without loss of generality for all k.

Next, consider for each k a solution $(\bar x_k,\bar y_k)$ of the program

$$\min_{x,y}\phi_k(x,y):=\frac{4(L+1)}{k}\|x-(\bar x+t_k\tilde u)\|+\|y\|\quad\text{subject to}\quad (x,y)\in\operatorname{gph}M_{p_k}.$$

Because of $(\bar x+t_k\tilde u,G(p_k,\bar x+t_k\tilde u)-\xi(t_k))\in\operatorname{gph}M_{p_k}$ we obtain

$$\begin{aligned}\phi_k(\bar x_k,\bar y_k)&\le\|G(p_k,\bar x+t_k\tilde u)-\xi(t_k)\|\\&\le\|G(p_k,\bar x+t_k\tilde u)-G(\bar p,\bar x+t_k\tilde u)\|+\|G(\bar p,\bar x+t_k\tilde u)-\xi(t_k)\|\\&\le L\|p_k-\bar p\|+\eta(t_k)\le(L+1)\|p_k-\bar p\|.\end{aligned}$$

Further, we must have $\bar y_k\neq 0$, since otherwise we would have $\frac{4(L+1)}{k}\|\bar x_k-(\bar x+t_k\tilde u)\|=\phi_k(\bar x_k,\bar y_k)\le(L+1)\|p_k-\bar p\|$ and $0=\bar y_k\in M(p_k,\bar x_k)$, implying $\bar x_k\in S(p_k)$ and

$$\mathrm{d}(\bar x,S(p_k))\le\|\bar x_k-\bar x\|\le\|\bar x_k-(\bar x+t_k\tilde u)\|+t_k\|\tilde u\|\le\frac{k}{4}\|p_k-\bar p\|+\frac{k}{2}\|p_k-\bar p\|,$$

which clearly contradicts the inequality (3.22).

By the first-order optimality conditions [39, Theorem 8.15] we have $0\in\partial\phi_k(\bar x_k,\bar y_k)+N_{\operatorname{gph}M_{p_k}}(\bar x_k,\bar y_k)$, where $\partial\phi_k$ stands for the subdifferential in the sense of convex analysis; cf. [39, Proposition 8.12]. Hence, there are elements $(x_k^*,y_k^*)\in-\partial\phi_k(\bar x_k,\bar y_k)\cap N_{\operatorname{gph}M_{p_k}}(\bar x_k,\bar y_k)$. Then $x_k^*\in\frac{4(L+1)}{k}\mathcal{B}_{\mathbb{R}^n}$ and $y_k^*\in\mathcal{S}_{\mathbb{R}^m}$ because of $\bar y_k\neq 0$. By the definition of the limiting normal cone we can find elements $(\bar x_k',\bar y_k')\in\operatorname{gph}M_{p_k}$ and $(u_k^*,v_k^*)\in\widehat N_{\operatorname{gph}M_{p_k}}(\bar x_k',\bar y_k')$ such that $\|(\bar x_k,\bar y_k)-(\bar x_k',\bar y_k')\|\le t_k/k$ and $\|(x_k^*,y_k^*)-(u_k^*,v_k^*)\|\le 1/k$. From $M_{p_k}(\cdot)=G(p_k,\cdot)-D$ we conclude that $-v_k^*\in\widehat N_D(G(p_k,\bar x_k')-\bar y_k')$ and $u_k^*=-\nabla_x G(p_k,\bar x_k')^T v_k^*\to 0$. Since

$$\begin{aligned}\|\bar x_k'-\bar x\|\ge\|\bar x_k-\bar x\|-\frac{t_k}{k}\ge t_k\|\tilde u\|-\|\bar x_k-(\bar x+t_k\tilde u)\|-\frac{t_k}{k}\ge t_k-\frac{k}{4}\|p_k-\bar p\|-\frac{t_k}{k}\\=\left(\frac{k}{4}-\frac{1}{2}\right)\|p_k-\bar p\|\end{aligned}$$

and

$$\begin{aligned}\|\bar x_k'-\bar x\|\le\|\bar x_k-\bar x\|+\frac{t_k}{k}\le t_k\|\tilde u\|+\|\bar x_k-(\bar x+t_k\tilde u)\|+\frac{t_k}{k}\le t_k+\frac{k}{4}\|p_k-\bar p\|+\frac{t_k}{k}\\=\left(\frac{3k}{4}+\frac{1}{2}\right)\|p_k-\bar p\|,\end{aligned}$$

it follows that $\tau_k\to 0$ and $(p_k-\bar p)/\tau_k\to 0$, where $\tau_k:=\|\bar x_k'-\bar x\|$. By passing to a subsequence we may assume that the sequence $-v_k^*$ converges to v^* and the sequence $(\bar x_k'-\bar x)/\tau_k$ converges to some $u\in\mathcal{S}_{\mathbb{R}^n}$. These conditions together with

$$\|\bar y_k'\|\le\|\bar y_k\|+\frac{t_k}{k}\le\phi_k(\bar x_k,\bar y_k)+\frac{t_k}{k}\le\left(L+\frac{3}{2}\right)\|p_k-\bar p\|=o(\tau_k)$$

give us

$$\lim_{k\to\infty} \frac{G(p_k, \bar{x}'_k) - \bar{y}'_k - G(\bar{p}, \bar{x})}{\tau_k} = \lim_{k\to\infty} \left(\frac{G(p_k, \bar{x}'_k) - G(\bar{p}, \bar{x}'_k)}{\tau_k} + \frac{G(\bar{p}, \bar{x}'_k) - G(\bar{p}, \bar{x})}{\tau_k} \right)$$
$$= \nabla_x G(\bar{p}, \bar{x})u.$$

This shows $v^* \in N_D(G(\bar{p}, \bar{x}), \nabla_x G(\bar{p}, \bar{x})u)$ and $\nabla_x G(\bar{p}, \bar{x})u \in T_D(G(\bar{p}, \bar{x}))$. Since we also have

$$0 = \lim_{k\to\infty} \nabla_x G(p_k, \bar{x}'_k)^T(-v^*_k) = \nabla_x G(\bar{p}, \bar{x})^T v^*,$$

we obtain a contradiction to (3.18)–(3.20), and this completes the proof. ◻

The next statement concerns the frequently arising case when D is a union of convex polyhedrons, which occurs, e.g., in the case of parameterized complementarity problems.

THEOREM 3.9. *In the setting of Theorem* 3.8 *consider the situation that* $D \subset \mathbb{R}^m$ *is the union of finitely many convex polyhedra,* $G(\bar{p}, \cdot)$ *is twice Fréchet differentiable at* $\bar{x}$*, and there are neighborhoods* W *of* $\bar{p}$ *and* U *of* $\bar{x}$ *and there is a real* L' *such that* (3.17) *holds.*

If there do not exist vectors $0 \neq u \in \mathbb{R}^n$, $0 \neq v^* \in \mathbb{R}^m$ *verifying* (3.18)–(3.20) *and*

$$u^T \nabla^2_{xx}({v^*}^T G)(\bar{p}, \bar{x})u \geq 0, \tag{3.23}$$

then S *is calm at* $(\bar{p}, \bar{x})$.

Moreover, if in addition G *is twice partially differentiable with respect to* x *on* $W \times U$*, if* G*,* ∇G*, and* $\nabla^2_{xx} G$ *are continuous on* $W \times U$*, if*

$$\|\nabla_x G(p, \bar{x}) - \nabla_x G(\bar{p}, \bar{x})\| \leq L' \|p - \bar{p}\|$$

holds for all p *near* $\bar{p}$*, and if there exists some nonzero* $\tilde{u}$ *with* $\nabla_x G(\bar{p}, \bar{x})u \in T_D(G(\bar{p}, \bar{x}))$*, then there exist a neighborhood* $\tilde{W}$ *of* $\bar{p}$ *and a real* $\tilde{L}$ *such that* $S(p) \neq \emptyset$ *for all* $p \in \tilde{W}$ *and*

$$\mathrm{d}(\bar{x}, S(p)) \leq \tilde{L} \|p - \bar{p}\|^{1/2}, \ p \in \tilde{W}. \tag{3.24}$$

Proof. The same arguments as in the proof of Theorem 3.8 show that M has the restricted calmness property with respect to p at $(\bar{p}, \bar{x}, 0)$. By our assumptions, SOSCMS of Corollary 2.11 is fulfilled for $M_{\bar{p}}$ at $(\bar{x}, 0)$, and therefore the calmness of S at $(\bar{p}, \bar{x})$ follows from Theorem 3.3. Further, the nonemptiness of $S(p)$ and the bound (3.24) follow from [21, Proposition 2(2)]. ◻

The situation of Theorem 3.8 is illustrated in the following example.

Example 3. Let $p \in \mathbb{R}^2, x \in \mathbb{R}^2$, and S be implicitly given by the complementarity problem

$$0 \leq x_1 - p_1 \perp x_2 - p_2 \geq 0$$

combined with the (nonlinear) inequality constraints $-x_1 - x_1^2 \leq x_2 \leq x_1 + x_1^2$. Let $(\bar{p}, \bar{x}) = (0_{\mathbb{R}^2}, 0_{\mathbb{R}^2})$. This problem attains the form (3.16) with

$$G(p, x) = \begin{bmatrix} -p_1 + x_1 \\ -p_2 + x_2 \\ -x_1 - x_1^2 - x_2 \\ -x_1 - x_1^2 + x_2 \end{bmatrix} \tag{3.25}$$

and $D = \mathcal{K} \times \mathbb{R}^2_-$, where $\mathcal{K}$ denotes the "complementarity angle," i.e.,

$$\mathcal{K} := \{z \in \mathbb{R}^2_+ | z_1 z_2 = 0\}.$$

G clearly fulfills all the assumptions of Theorem 3.8. It follows from (3.18) that we have to analyze the directions $u \in \mathbb{R}^2$ satisfying the conditions

$$(u_1, u_2) \in \mathcal{K},\ -u_1 - u_2 \leq 0,\ -u_1 + u_2 \leq 0,$$

which amount to $u_1 \geq 0, u_2 = 0$. As to condition (3.19), we obtain the equalities

$$\begin{aligned} v_1^* - v_3^* - v_4^* &= 0, \\ v_2^* - v_3^* + v_4^* &= 0. \end{aligned} \tag{3.26}$$

Finally, we observe that for any sequence of vectors $u^{(k)} \to (\bar{u}_1, 0)$ with $\bar{u}_1 > 0$ and for any sequence of reals $t^{(k)} \searrow 0$ such that

$$G(\bar{p}, \bar{x}) + t^{(k)} \nabla_x G(\bar{p}, \bar{x}) u^{(k)} = t^{(k)} \begin{bmatrix} u_1^{(k)} \\ u_2^{(k)} \\ -u_1^{(k)} - u_2^{(k)} \\ -u_1^{(k)} + u_2^{(k)} \end{bmatrix} \in \mathcal{K} \times \mathbb{R}^2_-,$$

one has $u_2^{(k)} = 0$, and therefore

$$N_{\mathcal{K} \times \mathbb{R}^2_-}(G(\bar{p}, \bar{x}) + t^k \nabla_x G(\bar{p}, \bar{x}) u^{(k)}) = \{v^* \in \mathbb{R}^4 | v_1^* = v_3^* = v_4^* = 0\}. \tag{3.27}$$

Thus, by combining (3.26) and (3.27) we conclude that $v_2^* = 0$ as well and, due to (3.20), the corresponding implicit multifunction S is calm at $(\bar{p}, \bar{x})$. Since the direction $\tilde{u} = (1, 0)$ fulfills $\nabla_x G(\bar{p}, \bar{x}) \tilde{u} = (1, 0, -1 - 1) \in D = T_D(G(\bar{p}, \bar{x}))$, and $\nabla_x G(\bar{p}, \bar{x}) \tilde{u}$ is derivable, we also have $S(p) \neq \emptyset$ and $\mathrm{d}(\bar{x}, S(p)) \leq \tilde{L} \|p - \bar{p}\|$ for some real $\tilde{L}$ and all p near $\bar{p}$.

Note that in the above example the implicit multifunction S does not possess the Aubin property around $(\bar{p}, \bar{x})$, because we have

$$(0,0) \in S(0, p_2)\ \forall p_2 < 0, \qquad (0,0) \notin S(p_1, p_2)\ \forall p_1 > 0,\ p_2 < 0.$$

Further, S does not have the isolated calmness property at $(\bar{p}, \bar{x})$ as well because $S(\bar{p}) = \mathbb{R}_+ \times \{0\}$. Moreover, $M_{\bar{p}}$ fulfills FOSCMS but is neither strongly metrically subregular nor metrically regular.

The next example illustrates the situation of Theorem 3.9.

Example 4. For $p \in \mathbb{R}$ let $S(p)$ be given by the solutions $x \in \mathbb{R}^2$ of the nonlinear inequalities

$$p - \frac{1}{2} x_1^2 + x_2 \leq 0, \qquad p - \frac{1}{2} x_1^2 - x_2 \leq 0.$$

Again this system can be written in the form (3.16) with

$$G(p, x) = \begin{pmatrix} p - \frac{1}{2} x_1^2 + x_2 \\ \\ p - \frac{1}{2} x_1^2 - x_2 \end{pmatrix}, \qquad D = \mathbb{R}^2_-.$$

Let $\bar{p} = 0$, $\bar{x} = 0_{\mathbb{R}^2}$. D is a convex polyhedron, and we will apply Theorem 3.9. The conditions (3.18)–(3.23) amount to

$$u_2 \le 0,\ -u_2 \le 0,\ v_1^* - v_2^* = 0,\ v_1^* \ge 0,\ v_2^* \ge 0,\ -(v_1^* + v_2^*)u_1^2 \ge 0,$$

which cannot be fulfilled with $u = (u_1, u_2) \neq (0,0)$, $v^* = (v_1^*, v_2^*) \neq (0,0)$. Hence, S is calm at $(\bar{p}, \bar{x})$, and since the direction $\tilde{u} = (1,0)$ fulfills $\nabla_x G(\bar{p}, \bar{x})\tilde{u} = (0,0) \in T_D(G(\bar{p}))$, we conclude that $S(p) \neq \emptyset$, and an estimate of the form $\mathrm{d}(\bar{x}, S(p)) = O(\sqrt{|p|})$ holds for p near 0. Indeed, we have $\mathrm{d}(\bar{x}, S(p)) = \|(0, \pm\sqrt{-p})\| = \sqrt{-p}$ for all $p < 0$. This example demonstrates the antisymmetry of the calmness property. Although the points $x \in S(p)$ near $\bar{x}$ are close to $S(\bar{p})$ up to the order $O(\|p - \bar{p}\|)$, the point $\bar{x} \in S(\bar{p})$ is not close to $S(p)$ with this order.

Since every inclusion $0 \in M(p,x)$ can be written equivalently in the form (3.16) by

$$0 \in \tilde{M}(p,x) := (p,x,0) - \operatorname{gph} M, \tag{3.28}$$

one can combine Corollary 3.7 and Theorems 3.3 and 3.8 to obtain point based conditions for the calmness of solution mappings of general inclusions.

COROLLARY 3.10. *Let $S(p) := \{x \mid 0 \in M(p,x)\}$, where $M : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^m$ is a closed multifunction, and let $\bar{x} \in S(\bar{p})$.*

(i) *Assume that there do not exist directions $u \neq 0$, $\hat{u} \neq 0$, and $q \in T_{\operatorname{dom} H_M}$ and elements $q^* \in \mathcal{S}_{\mathbb{R}^l}$, $v^* \in \mathcal{S}_{\mathbb{R}^m}$ such that*

$$0 \in DM(\bar{p}, \bar{x}, 0)(0,u), \quad (q^*, 0) \in D^*_{>_p} M((\bar{p}, \bar{x}, 0); u)(0), \quad \langle q^*, q\rangle = -1, \tag{3.29}$$

$$0 \in DM_{\bar{p}}(\bar{x}, 0)(\hat{u}), \quad 0 \in D^* M_{\bar{p}}((\bar{x}, 0); (\hat{u}, 0))(v^*). \tag{3.30}$$

Then S is calm at $(\bar{p}, \bar{x})$.

(ii) *If there do not exist a direction $u \neq 0$ and elements $(q^*, v^*) \neq (0,0)$ such that*

$$0 \in DM(\bar{p}, \bar{x}, 0)(0,u), \quad (q^*, 0) \in D^* M((\bar{p}, \bar{x}, 0); (0,u,0))(v^*), \tag{3.31}$$

and if there exists a direction $\tilde{u}$ such that $(0, \tilde{u}, 0) \in T_{\operatorname{gph} M}(\bar{p}, \bar{x}, 0)$ and $(0, \tilde{u}, 0)$ is derivable, then S is calm at $(\bar{p}, \bar{x})$ and there exist a real $\tilde{L}$ and a neighborhood $\tilde{W}$ of $\bar{p}$ such that

$$S(p) \neq \emptyset, \quad \mathrm{d}(\bar{x}, S(p)) \le \tilde{L}\|p - \bar{p}\|, \forall p \in \tilde{W}.$$

Proof. By Corollary 3.7, condition (3.29) ensures that M has the restricted calmness property with respect to p at $(\bar{p}, \bar{x}, 0)$. Further, $M_{\bar{p}}$ is metrically subregular at $(\bar{x}, 0)$ due to (3.30) and FOSCMS (2.5). Hence, calmness of S follows from Theorem 3.3. The second statement follows from Theorem 3.8 together with the observation that conditions (3.18)–(3.20) applied to $\tilde{M}$ given by (3.28) amount to (3.31). ∎

4. Aubin property of implicit multifunctions. The aim of this section is to investigate the Aubin property of S given by (1.1) with a closed-graph mapping $M : \mathbb{R}^l \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. We start with the following proposition.

PROPOSITION 4.1. *Let $\mathcal{M} : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ be a multifunction with closed graph. Given $(\bar{x}, \bar{y}) \in \operatorname{gph} \mathcal{M}$ and a direction $u \in \mathbb{R}^s$, assume that $\mathcal{M}$ is metrically subregular in direction u at $(\bar{x}, \bar{y})$ with modulus κ. Then*

$$u \in T_{\mathcal{M}^{-1}(\bar{y})}(\bar{x}) \quad \Leftrightarrow \quad 0 \in D\mathcal{M}(\bar{x}, \bar{y})(u) \tag{4.1}$$

and

$$N_{\mathcal{M}^{-1}(\bar y)}(\bar x;u)\subset\{x^*\,|\,\exists y^*\in\kappa\|x^*\|\mathcal{B}_{\mathbb{R}^d}:\ (x^*,y^*)\in N_{\operatorname{gph}\mathcal{M}}((\bar x,\bar y);(u,0))\}\\ \subset \operatorname{rge}\,D^*\mathcal{M}((\bar x,\bar y);(u,0)). \tag{4.2}$$

Proof. In order to prove the forward implication in (4.1), assume $u\in T_{\mathcal{M}^{-1}(\bar y)}(\bar x)$. Then there are sequences $t_k\searrow 0$ and $u_k\to u$ such that $\bar x+t_ku_k\in\mathcal{M}^{-1}(\bar y)$ or, equivalently, $\bar y=\bar y+t_k0\in\mathcal{M}(\bar x+t_ku_k)$, implying $(u,0)\in T_{\operatorname{gph}\mathcal{M}}(\bar x,\bar y)$, which is the same as $0\in D\mathcal{M}(\bar x,\bar y)(u)$. Conversely, if $0\in D\mathcal{M}(\bar x,\bar y)(u)$, then there are sequences $t_k\searrow 0$ and $(u_k,v_k)\to(u,0)$ such that $\bar y+t_kv_k\in\mathcal{M}(\bar x+t_ku_k)$. By virtue of the assumed metric subregularity of $\mathcal{M}$ in direction u and choosing $\kappa'>\kappa$ we have

$$\mathrm{d}(\bar x+t_ku_k,\mathcal{M}^{-1}(\bar y))\le\kappa'\mathrm{d}(\bar y,\mathcal{M}(\bar x+t_ku_k))\le\kappa't_k\|v_k\|$$

for all k sufficiently large, and therefore we can find a sequence $x_k\in\mathcal{M}^{-1}(\bar y)$ verifying $\|x_k-(\bar x+t_ku_k)\|\le\kappa't_k\|v_k\|$. Thus,

$$\lim_{k\to\infty}\left\|\frac{x_k-\bar x}{t_k}-u\right\|\le\lim_{k\to\infty}\left\|\frac{x_k-\bar x}{t_k}-u_k\right\|+\|u_k-u\|\le\lim_{k\to\infty}\kappa'\|v_k\|+\|u_k-u\|=0,$$

and we conclude $u\in T_{\mathcal{M}^{-1}(\bar y)}(\bar x)$. Hence the relation (4.1) is shown.

To prove inclusion (4.2), consider $x^*\in N_{\mathcal{M}^{-1}(\bar y)}(\bar x;u)$. Then there are sequences $t_k\searrow 0$, $u_k\to u$, and $x_k^*\to x^*$ such that $x_k^*\in\widehat N_{\mathcal{M}^{-1}(\bar y)}(x_k)$, where $x_k:=\bar x+t_ku_k$. Hence, for each k we can find some radius $r_k>0$ such that

$$\langle x_k^*,x-x_k\rangle\le\frac1k\|x-x_k\|\quad\forall x\in\mathcal{M}^{-1}(\bar y)\cap\mathcal{B}(x_k,r_k).$$

By decreasing the radii r_k if necessary we can assume $r_k/t_k\le 1/k$. Further, by the supposed metric subregularity of $\mathcal{M}$ in direction u with modulus κ and by passing to a subsequence if necessary, we have

$$\mathrm{d}(x,\mathcal{M}^{-1}(\bar y))\le\left(\kappa+\frac1k\right)\mathrm{d}(\bar y,\mathcal{M}(x))\quad\forall x\in\mathcal{B}(x_k,r_k)\quad\forall k.$$

Next fix k, consider $x\in\mathcal{B}(x_k,r_k/2)$, and let $\tilde x$ denote the projection of x onto $\mathcal{M}^{-1}(\bar y)$. Then $\|\tilde x-x_k\|\le\|x-\tilde x\|+\|x-x_k\|\le 2\|x-x_k\|\le r_k$ and $\|x-\tilde x\|=\mathrm{d}(x,\mathcal{M}^{-1}(\bar y))\le(\kappa+\frac1k)\mathrm{d}(\bar y,\mathcal{M}(x))$, and therefore

$$\begin{aligned}\langle x_k^*,x-x_k\rangle&=\langle x_k^*,\tilde x-x_k\rangle+\langle x_k^*,x-\tilde x\rangle\le\frac1k\|\tilde x-x_k\|+\|x_k^*\|\|x-\tilde x\|\\&\le\frac2k\|x-x_k\|+\left(\kappa+\frac1k\right)\|x_k^*\|\inf_{y\in\mathcal{M}(x)}\|y-\bar y\|.\end{aligned}$$

Since $x\in\mathcal{B}(x_k,r_k/2)$ was arbitrary, we conclude

$$\left(\kappa+\frac1k\right)\|x_k^*\|\|y-\bar y\|-\langle x_k^*,x-x_k\rangle+\frac2k\|x-x_k\|\ge 0\ \ \forall x\in\mathcal{B}(x_k,r_k/2)\ \forall(x,y)\in\operatorname{gph}M.$$

Taking into account that $x_k\in\mathcal{M}^{-1}(\bar y)$ and therefore $\bar y\in\mathcal{M}(x_k)$, we see that $(x_k,\bar y)$ is a local minimizer for the problem

$$\min_{(x,y)\in\operatorname{gph}\mathcal{M}}\left(\kappa+\frac1k\right)\|x_k^*\|\|y-\bar y\|-\langle x_k^*,x-x_k\rangle+\frac2k\|x-x_k\|.$$

The respective optimality conditions [39, Theorem 8.15] imply the existence of some $y_k^* \in \mathcal{B}_{\mathbb{R}^d}$ and some $\eta_k^* \in \mathcal{B}_{\mathbb{R}^s}$ such that

$$0 \in \Big(- x_k^* + \frac{2}{k}\eta_k^*, \Big(\kappa + \frac{1}{k}\Big)\|x_k^*\| y_k^*\Big) + N_{\mathrm{gph}\,\mathcal{M}}(x_k, \bar{y}),$$

and by the definition of the limiting normal cone to $\mathrm{gph}\,\mathcal{M}$ at $(x_k, \bar{y})$ we can find elements $(\tilde{x}_k, \tilde{y}_k) \in \mathrm{gph}\,\mathcal{M}$ and $(\tilde{x}_k^*, \tilde{y}_k^*) \in \widehat{N}_{\mathrm{gph}\,\mathcal{M}}(\tilde{x}_k, \tilde{y}_k)$ such that

$$\|(\tilde{x}_k, \tilde{y}_k) - (x_k, \bar{y})\| \le \frac{t_k}{k} \quad \text{and} \quad \left\| \left(-x_k^* + \frac{2}{k}\eta_k^*, \left(\kappa + \frac{1}{k}\right)\|x_k^*\| y_k^*\right) + (\tilde{x}_k^*, \tilde{y}_k^*) \right\| \le \frac{1}{k}.$$

We infer $\|\tilde{y}_k^*\| \le (\kappa + \frac{1}{k})\|x_k^*\| + \frac{1}{k}$, and therefore by passing to a subsequence if necessary, we can assume that $\tilde{y}_k^*$ converges to some $y^* \in \mathbb{R}^d$. Since $\lim_{k\to\infty} \tilde{x}_k^* = \lim_{k\to\infty} x_k^* = x^*$ and

$$\lim_{k\to\infty} \frac{(\tilde{x}_k, \tilde{y}_k) - (\bar{x}, \bar{y})}{t_k} = \lim_{k\to\infty} \left(\frac{(\tilde{x}_k, \tilde{y}_k) - (x_k, \bar{y})}{t_k} + \lim_{k\to\infty} \frac{(x_k, \bar{y}) - (\bar{x}, \bar{y})}{t_k} \right) = (u, 0),$$

we conclude that $(x^*, y^*) \in N_{\mathrm{gph}\,\mathcal{M}}((\bar{x}, \bar{y}); (u, 0))$ and $\|y^*\| \le \kappa\|x^*\|$, and the first inclusion in (4.2) is shown. The second one is straightforward. $\square$

By combining (4.1) and Lemma 2.8(iii) we obtain the following corollary.

COROLLARY 4.2. *Assume that the multifunction $\mathcal{M} : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ is metrically subregular at $(\bar{x}, \bar{y}) \in \mathrm{gph}\,\mathcal{M}$. Then*

$$T_{\mathcal{M}^{-1}(\bar{y})}(\bar{x}) = \{u \,|\, 0 \in D\mathcal{M}(\bar{x}, \bar{y})(u)\}.$$

By combining the Mordukhovich criterion for the Aubin property of S with the definition of the directional limiting coderivative, and by invoking Proposition 4.1, and Corollary 4.2, we arrive at the next statement.

PROPOSITION 4.3. *Assume that the condition*

$$(q^*, 0) \in \{(q, u) \,|\, 0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\}^\circ \;\Rightarrow\; q^* = 0 \tag{4.3}$$

holds, assume that M is metrically subregular at $(\bar{p}, \bar{x}, 0)$, and assume that there do not exist vectors $(0, 0) \ne (q, u) \in \mathbb{R}^l \times \mathbb{R}^n$, $(q^*, v^*) \in \mathbb{R}^l \times \mathbb{R}^m$ with $q^* \ne 0$ such that

$$0 \in DM(\bar{p}, \bar{x}, 0)(q, u), \quad (q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*). \tag{4.4}$$

Then S has the Aubin property around $(\bar{p}, \bar{x})$.

Proof. The proof is obtained by contraposition. Assume on the contrary that S does not have the Aubin property around $(\bar{p}, \bar{x})$. Then we can infer from the Mordukhovich criterion [31], [39, Theorem 9.40] that $0 \ne q^* \in D^*S(\bar{p}, \bar{x})(0)$. By the definition of the limiting coderivative there are sequences $(p_k, x_k, q_k^*, u_k^*) \to (\bar{p}, \bar{x}, q^*, 0)$ such that $(q_k^*, -u_k^*) \in \widehat{N}_{\mathrm{gph}\,S}(p_k, x_k)$ for every k. Consider first the case that $(p_k, x_k) = (\bar{p}, \bar{x})$ holds for infinitely many k. Then, by passing to a subsequence and using the fact that the regular normal cone $\widehat{N}_{\mathrm{gph}\,S}(\bar{p}, \bar{x})$ is closed as a polar cone of $T_{\mathrm{gph}\,S}(\bar{p}, \bar{x})$, we obtain $(q^*, 0) \in \widehat{N}_{\mathrm{gph}\,S}(\bar{p}, \bar{x}) = (T_{\mathrm{gph}\,S}(\bar{p}, \bar{x}))^\circ$. Since M is metrically subregular at $(\bar{p}, \bar{x}, 0)$ and $\mathrm{gph}\,S = M^{-1}(0)$, by Corollary 4.2 we arrive at $(q^*, 0) \in \{(q, u) \,|\, 0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\}^\circ$, contradicting (4.3). Hence, $(p_k, x_k) = (\bar{p}, \bar{x})$ holds only for finitely many k, and so without loss of generality $(p_k, x_k) \ne (\bar{p}, \bar{x})$ for all k. By putting

$t_k := \|(p_k - \bar{p}, x_k - \bar{x})\|$ and by passing to a subsequence if necessary, we can assume that $(p_k - \bar{p}, x_k - \bar{x})/t_k$ converges to some (q, u) with $\|(q, u)\| = 1$, and we conclude that $(q^*, 0) \in N_{\mathrm{gph}\, S}((\bar{p}, \bar{x}); (q, u))$. Hence $(q, u) \in T_{\mathrm{gph}\, S}(\bar{p}, \bar{x})$, and by Proposition 4.1 we conclude $0 \in DM(\bar{p}, \bar{x}, 0)(q, u)$ and the existence of some v^* such that $(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*)$, contradicting (4.4). $\square$

We are now in the position to state the main result of this section.

THEOREM 4.4. *Assume that*

$$\{u \,|\, 0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\} \neq \emptyset \quad \forall\, q \in \mathbb{R}^l \tag{4.5}$$

holds, assume that M is metrically subregular at $(\bar{p}, \bar{x}, 0)$, and assume that for every $(0, 0) \neq (q, u) \in \mathbb{R}^l \times \mathbb{R}^n$ verifying $0 \in DM(\bar{p}, \bar{x}, 0)(q, u)$ the condition

$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*) \implies q^* = 0 \tag{4.6}$$

holds. Then S has the Aubin property around $(\bar{p}, \bar{x})$ and

$$DS(\bar{p}, \bar{x})(q) = \{u \,|\, 0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\},\ q \in \mathbb{R}^l. \tag{4.7}$$

Proof. In view of Proposition 4.3 and Corollary 4.2, we need only show that condition (4.5) implies condition (4.3). In fact, let $(q^*, 0) \in \{(q, u) \,|\, 0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\}^\circ$. Then, for every $q \in \mathbb{R}^l$ we can find some $u_q \in \{u \,|\, 0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\}$, and therefore $\langle q^*, q\rangle + \langle 0, u_q\rangle = \langle q^*, q\rangle \leq 0$ implying $q^* = 0$. $\square$

If we replace the requirement that M is metrically subregular at $(\bar{p}, \bar{x}, 0)$ by FOSCMS (2.5), we obtain the next corollary.

COROLLARY 4.5. *Assume that* (4.5) *holds, and assume that for every $(0, 0) \neq (q, u) \in \mathbb{R}^l \times \mathbb{R}^n$ verifying $0 \in DM(\bar{p}, \bar{x}, 0)(q, u)$ the condition*

$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*) \implies q^* = 0,\ v^* = 0 \tag{4.8}$$

is fulfilled. Then S has the Aubin property around $(\bar{p}, \bar{x})$ and (4.7) *holds.*

We now show that condition (4.5) is also necessary in order for the mapping S to have the Aubin property.

PROPOSITION 4.6. *If S has the Aubin property around $(\bar{p}, \bar{x})$, then* (4.5) *is fulfilled.*

Proof. If S has the Aubin property around $(\bar{p}, \bar{x})$, then $\{\bar{x}\} \subset S(p) + L\|p - \bar{p}\|\mathcal{B}_{\mathbb{R}^n}$ holds for all p in some neighborhood of $\bar{p}$. Consider now any direction $q \in \mathbb{R}^l$ and any sequence $t_k \searrow 0$. Then for every k sufficiently large we can find $x_k \in S(\bar{p} + t_k q)$ such that $\|x_k - \bar{x}\| \leq L t_k \|q\|$. By passing to a subsequence we can assume that the sequence $u_k := (x_k - \bar{x})/t_k$ converges to some u. Since $0 \in M(\bar{p} + t_k q_k, x_k) = M(\bar{p} + t_k q_k, \bar{x} + t_k u_k)$, the inclusion $0 \in DM(\bar{p}, \bar{x}, 0)(q, u)$ follows, and thus $\{u \,|\, 0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\} \neq \emptyset$. Because q was chosen arbitrarily, relation (4.5) follows.

Now let us compare the criteria of Corollary 4.5 with the criterion of [33, Corollary 4.60]. By taking $f \equiv 0$ and $Q = M$ in [33, Corollary 4.60] we obtain that the condition

$$(q^*, 0) \in D^*M(\bar{p}, \bar{x}, 0)(v^*) \implies q^* = 0,\ v^* = 0 \tag{4.9}$$

is sufficient for the Aubin property of S around $(\bar{p}, \bar{x})$. So, instead of the standard coderivative of M used in (4.9), we use in condition (4.8) the directional coderivative of M in certain directions, which is by definition not larger (typically smaller) than the standard coderivative. This indicates that in this way we arrive at substantially less restrictive sufficient conditions, ensuring the Aubin property of S. By Example 5, we will strikingly illustrate that the conditions of Corollary 4.5 are indeed weaker than (4.9).

Before we present this example, we work out the preceding theory for the case of a class of variational systems, where

$$M(p,x) = G(p,x) + Q(x), \tag{4.10}$$

with $G : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable and $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ being a closed-graph multifunction. It is well known that in this case (cf. [12, Proposition 4A.2]) at a fixed triple $(\bar{p}, \bar{x}, 0) \in \operatorname{gph} M$ one has

$$DM(\bar{p},\bar{x},0)(q,u) = \nabla_p G(\bar{p},\bar{x})q + \nabla_x G(\bar{p},\bar{x})u + DQ(\bar{x},\bar{y}^*)(u),$$

where $\bar{y}^* := -G(\bar{p},\bar{x})$.

Condition (4.5) thus amounts to the requirement that the *generalized equation* (GE)

$$0 \in \nabla_p G(\bar{p},\bar{x})q + \nabla_x G(\bar{p},\bar{x})u + DQ(\bar{x},\bar{y}^*)(u) \tag{4.11}$$

in variable u possesses a solution for all $q \in \mathbb{R}^l$. Further, by virtue of (2.4), condition (4.6) amounts to the implication

$$\left.\begin{array}{l} q^* = (\nabla_p G(\bar{p},\bar{x}))^T v^*, \\ 0 \in (\nabla_x G(\bar{p},\bar{x}))^T v^* + D^*Q((\bar{x},\bar{y}^*);(u, -\nabla_p G(\bar{p},\bar{x})q - \nabla_x G(\bar{p},\bar{x})u))(v^*) \end{array}\right\} \Longrightarrow q^* = 0, \tag{4.12}$$

and condition (4.8) amounts to a strengthened variant of (4.12), where on the right-hand side one has $q^* = 0$, $u^* = 0$. In contrast to the criterion from [33, Corollary 4.61], this means that instead of the solutions v^* to the standard adjoint GE

$$0 \in (\nabla_x G(\bar{p},\bar{x}))^T v^* + D^*Q(\bar{x},\bar{y}^*)(v^*) \tag{4.13}$$

with the standard limiting coderivative of $Q(\cdot)$, now we have to consider the respective *directional adjoint GE* for directions (q,u) solving (4.11). By the definition, the respective set of solutions is not larger (typically much smaller) than the set of solutions to (4.13).

In the following example we illustrate the efficiency of our technique for the special case when $Q(x) = N_\Gamma(x)$ with $\Gamma \subset \mathbb{R}^n$ being a convex polyhedron. Then, by virtue of (2.9), condition (4.11) amounts to

$$0 \in \nabla_p G(\bar{p},\bar{x})q + \nabla_x G(\bar{p},\bar{x})u + N_{K_\Gamma(\bar{x},\bar{y}^*)}(u). \tag{4.14}$$

Example 5. Consider the solution map $S : \mathbb{R} \rightrightarrows \mathbb{R}^2$ of the GE

$$0 \in M(p,x) = \begin{pmatrix} x_1 - p \\ -x_2 + x_2^2 \end{pmatrix} + N_\Gamma(x) \tag{4.15}$$

with $\Gamma = \{x \in \mathbb{R}^2 \,|\, \frac{1}{2}x_1 \le x_2 \le -\frac{1}{2}x_1\}$ at $(\bar{p},\bar{x}) = (0,(0,0)) \in \operatorname{gph} S$. We will now demonstrate that by means of Corollary 4.5 we can verify the Aubin property of S around $(\bar{p},\bar{x})$, whereas [33, Corollaries 4.60 and 4.61] are not applicable.

We have $\bar{y}^* = (0,0)$, $K_\Gamma(\bar{x},\bar{y}^*) = \{u \in \mathbb{R}^2 \,|\, \frac{1}{2}u_1 \le u_2 \le -\frac{1}{2}u_1\}$, and therefore, concerning the directions (q,u) verifying (4.14), one has to consider the following four situations:

(i) $q \le 0$, $u_1 = q$, $u_2 = 0$;
(ii) $q \le 0$, $u_1 = \frac{4}{3}q$, $u_2 = -\frac{2}{3}q$;
(iii) $q \le 0$, $u_1 = \frac{4}{3}q$, $u_2 = \frac{2}{3}q$;
(iv) $q \ge 0$, $u_1 = u_2 = 0$.

We observe that (4.5) is fulfilled. Since we interested only in nonzero directions (q,u), for our further analysis we can restrict to the case $q \ne 0$.

The faces of the critical cone are exactly the cones $\mathcal{F}_1 := \{(0,0)\}$, $\mathcal{F}_2 := \mathbb{R}^+(-1,\frac{1}{2})$, and $\mathcal{F}_3 := \mathbb{R}^+(-1,-\frac{1}{2})$ and the critical cone $\mathcal{F}_4 := K_\Gamma(\bar{x},\bar{y}^*)$ itself.

In case (i) we have

$$-\nabla_p G(\bar{p},\bar{x})q - \nabla_x G(\bar{p},\bar{x})u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and, by virtue of Theorem 2.12,

$$N_{\mathrm{gph}\, N_\Gamma}((\bar{x},\bar{y}^*);((q,0),(0,0))) = (\mathcal{F}_4 - \mathcal{F}_4)^\circ \times (\mathcal{F}_4 - \mathcal{F}_4) = \{(0,0)\} \times \mathbb{R}^2,$$

since the only face of $K_\Gamma(\bar{x},\bar{y}^*)$ containing $(q,0)$ with $q < 0$ is the critical cone itself. Thus,

$$D^* N_\Gamma((\bar{x},\bar{y}^*);((q,0),(0,0)))(v^*) = \{(0,0)\},$$

and the directional adjoint GE attains the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1^* \\ -v_2^* \end{pmatrix}. \tag{4.16}$$

In case (ii),

$$-\nabla_p G(\bar{p},\bar{x})q - \nabla_x G(\bar{p},\bar{x})u = \begin{pmatrix} -\frac{1}{3}q \\ \\ -\frac{2}{3}q \end{pmatrix},$$

$$N_{\mathrm{gph}\, N_\Gamma}\left((\bar{x},\bar{y}^*);\left(\left(\frac{4}{3}q,-\frac{2}{3}q\right),\left(-\frac{1}{3}q,-\frac{2}{3}q\right)\right)\right) = (\mathcal{F}_2 - \mathcal{F}_2)^\circ \times (\mathcal{F}_2 - \mathcal{F}_2),$$

$$D^* N_\Gamma\left((\bar{x},\bar{y}^*);\left(\left(\frac{4}{3}q,-\frac{2}{3}q\right),\left(-\frac{1}{3}q,-\frac{2}{3}q\right)\right)\right)(v^*) = \begin{cases} \mathcal{K}_1^\circ & \text{if } -v^* \in \mathcal{K}_1, \\ \emptyset & \text{otherwise,} \end{cases}$$

with $\mathcal{K}_1 := \mathcal{F}_2 - \mathcal{F}_2 = \mathbb{R}(-1,\frac{1}{2})$, and the directional adjoint GE attains the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} v_1^* \\ -v_2^* \end{pmatrix} + \mathbb{R}\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \quad -v^* \in \mathcal{K}_1, \tag{4.17}$$

which has the only solution $v^* = 0$.

Similarly to the second case, in case (iii) the directional adjoint GE attains the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} v_1^* \\ -v_2^* \end{pmatrix} + \mathbb{R}\begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix}, \quad -v^* \in \mathcal{K}_2 := \mathcal{F}_3 - \mathcal{F}_3 = \mathbb{R}\begin{pmatrix} -1 \\ -\frac{1}{2} \end{pmatrix}, \tag{4.18}$$

and again the unique solution is $v^* = 0$.

Finally, in case (iv),

$$-\nabla_p G(\bar{p},\bar{x})q - \nabla_x G(\bar{p},\bar{x})u = \begin{pmatrix} q \\ 0 \end{pmatrix},$$

$$N_{\operatorname{gph} N_\Gamma}((\bar{x},\bar{y}^*);((0,0),(q,0))) = (\mathcal{F}_1 - \mathcal{F}_1)^\circ \times (\mathcal{F}_1 - \mathcal{F}_1) = \mathbb{R}^2 \times \{(0,0)\},$$

$$D^* N_\Gamma((\bar{x},\bar{y}^*);((0,0),(q,0)))(v^*) = \begin{cases} \mathbb{R}^2 & \text{if } v^* = (0,0), \\ \emptyset & \text{otherwise,} \end{cases}$$

and the directional adjoint GE attains the form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} v_1^* \\ -v_2^* \end{pmatrix} + \mathbb{R}^2, \quad v^* = (0,0). \tag{4.19}$$

In this way we have analyzed all "suspicious" pairs of nonzero directions (q,u) given by (4.14) and concluded that all GEs (4.16)–(4.19) possess only the trivial solution $v^* = (0,0)$. Since $q^* = -v_1^*$, Corollary 4.5 implies that the solution map of GE (4.15) indeed has the Aubin property around $(\bar{p},\bar{x})$.

Now let us analyze the standard GE (4.13), which reads as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} v_1^* \\ -v_2^* \end{pmatrix} + D^* N_\Gamma(\bar{x},\bar{y}^*)(v^*) \tag{4.20}$$

for our example. Using the representation of the limiting normal cone $N_{\operatorname{gph} N_\Gamma}$ at $(\bar{x},\bar{y}^*)$ as stated in section 2, we obtain

$$N_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{x},\bar{y}^*) = \bigcup_{i=1}^{9} (\mathcal{K}_i^\circ \times \mathcal{K}_i)$$

with $\mathcal{K}_3 = \mathcal{F}_4 - \mathcal{F}_1 = K_\Gamma(\bar{x},\bar{y}^*)$, $\mathcal{K}_4 = \mathcal{F}_4 - \mathcal{F}_2 = \{v \in \mathbb{R}^2 \,|\, \frac{1}{2}v_1 + v_2 \leq 0\}$, $\mathcal{K}_5 = \mathcal{F}_4 - \mathcal{F}_3 = \{v \in \mathbb{R}^2 \,|\, \frac{1}{2}v_1 - v_2 \leq 0\}$, $\mathcal{K}_6 = \mathcal{F}_4 - \mathcal{F}_4 = \mathbb{R}^2$, $\mathcal{K}_7 = \mathcal{F}_2 - \mathcal{F}_1 = \mathbb{R}^+(-1,\frac{1}{2})$, $\mathcal{K}_8 = \mathcal{F}_3 - \mathcal{F}_1 = \mathbb{R}^+(-1,-\frac{1}{2})$, and $\mathcal{K}_9 = \mathcal{F}_1 - \mathcal{F}_1 = \{(0,0)\}$.

We see that for $v^* = (-1,2)$ we have $-v^* \in \mathcal{K}_4$ and $-(v_1^*,-v_2^*) = (1,2) \in \mathcal{K}_4^\circ \subset D^* N_\Gamma(\bar{x},\bar{y}^*)(v^*)$, verifying that v^* is a nontrivial solution of the GE (4.20). Another nontrivial solution of (4.20) is provided by $v^* = (-1,-2) \in -\mathcal{K}_5$. This implies that we cannot apply [33, Corollaries 4.60 and 4.61] to detect the Aubin property of the solution map S.

5. Conclusion. In both main sections of the paper (sections 3 and 4) we use as a basic tool the directional limiting coderivatives. The purpose for their usage, however, is different. Whereas in section 3 they are employed in verifying the calmness of $M_{\bar{p}}$—and in this role they could possibly be substituted by another calmness criterion—in section 4 they help us to capture the behavior of M along relevant directions, and in this role they cannot be substituted by any of the currently available generalized derivatives. This ability of directional limiting coderivatives could possibly be utilized also in analysis of other stability properties of S (than the calmness and the Aubin property). In particular, under the assumptions of Theorem 4.4 the mapping S has a single-valued Lipschitz localization around $(\bar{p},\bar{x})$ whenever we ensure the single-valuedness of S close to $(\bar{p},\bar{x})$. This may be done, e.g., by standard monotonicity assumptions imposed on $M_p(\cdot)$, but we believe that a suitable additional condition could be formulated directly in terms of graphical derivatives and directional limiting coderivatives of M at $(\bar{p}, \bar{x}, 0)$. This question we plan to tackle in our future research. The application potential of the directional limiting coderivative is further increased by the formula developed in Theorem 2.12, which enables us to perform an efficient computation of this object in the case of polyhedral constraints.

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