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Nash equilibrium in a pay-as-bid electricity market: Part 1 – existence and characterization*

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ABSTRACT
We consider a model of a pay-as-bid electricity market based on a multi-leader-common-follower approach where the producers as leaders are at the upper level and the regulator as a common follower is at the lower level. We fully characterize Nash equilibria for this model by describing necessary and sufficient conditions for their existence as well as providing explicit formulas of such equilibria in the market.

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1. Introduction

In many countries, electricity markets were deregulated and privatized since the 1990s. Recently, new models appeared in the literature as this topic abounds with open questions from a broad spectrum of disciplines. The immanent complexity of these markets is due to a central authority operating the electrical power system in which it is established. In particular, it is required to balance electricity flows on the network at all times, by balancing supply and demand. This authority also guarantees all users of the power grid fair and transparent treatment, and seeks to promote fluid exchanges. There are different denominations for this authority depending on the country and market design. In this article, it is referred as an Independent System Operator (ISO). Since electricity cannot be stored, some ISOs use a balancing mechanism to balance electricity supply and demand close to real time, e.g. the balancing mechanism set up in France in 2013 as a pay-as-bid market.[1] When an ISO predicts that there will be a discrepancy, for a given period, between the planned production of electricity and the demand, the ISO calls for bids from producers to adjust the electricity production. Such a discrepancy usually comes from changes either in electricity demand due to weather forecast updates, or in dispatch due to technical incident or due to network congestion. Here, we do not consider possible adjustments of the demand, like e.g. erasement process.

The resulting balancing market has several specific characteristics. First, it is organized as a multi-leader-common-follower problem, where the interaction between each market participant is modelled as a bilevel problem with the ISO's problem at the lower level in the role of a follower, and all producers and consumers at the upper level in the role of leaders. Second, it is a pay-as-bid market in which each producer (or consumer) provides the ISO with a bid (function) used to derive directly its revenues (or expenses). Note that in our model and for the sake of simplicity we aggregate the demand of consumers. This aggregated demand, thereafter referred as demand, is assumed to be given, whereas it is not precisely known in real-world electricity markets. Therefore, it is a common practice for a producer to do some 'demand sampling' around a reference value to elaborate his bid.

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*Dedicated to Professor Franco Giannessi on the occasion of his 80th birthday and to Professor Diethard Pallaschke on the occasion of his 75th birthday.

Our aim in this couple of articles (Part 1 and Part 2) is to determine the Nash equilibria of a pay-as-bid market. In Part 1, we characterize the equilibria depending on different values of the demand. In Part 2, we characterize the best response of a producer that is the optimal bid(s) maximizing his profit, see [2]. For each given producer it is assumed that the demand is known as well as an estimate of the bids of the other producers. Such a characterization of the best response is also the cornerstone in the proof of the existence result for the Nash equilibria. This describes how both Part 1 and Part 2 are interconnected.

In the approaches proposed in [3–8], the bids are assumed to be convex quadratic functions of the production quantity. Actually, in most electricity markets, only piecewise linear bids or block orders may be allowed. However, quadratic function with non-negative coefficients capture well the typical behaviour of aggregated block offers and, at the same time, it is amenable to further analysis. Note that a classical way to ensure uniqueness of the solution to the ISO's problem, see e.g. the above quoted references, is to assume that all producers are bidding true quadratic functions. In this work, we allow producers to bid linearly or quadratically and uniqueness of the solution to the ISO's problem is obtained, thanks to the 'equity property' assumption explicitly claiming that the ISO treats all producers in the same manner. It is also important to enlighten that only few existence results of equilibria in electricity markets appeared in the literature. Actually, the models are usually based on true quadratic bids only. And existence results are obtained under quite restrictive assumption; see e.g. [5, Corollary 1] in the particular case of a 'two-island market'. But allowing linear bids plays a fundamental role to obtain Nash equilibria since, at equilibrium, the bids of the producers with a positive production quantity are linear. This follows from the forthcoming existence result stated in Theorem 3.1.

Part 1 of this couple of articles is organized as follows. In Section 2, we define the model of a pay-as-bid electricity market that will be considered as well as the needed notation. In Section 3, the existence and characterization of Nash equilibrium is considered while in Section 4 we illustrate through an example the sensitivity of the Nash equilibrium with respect to the demand.

2. Problem setting and best response of a producer

2.1. Notations and problem setting

The basic notations along with some assumptions follow: $D > 0$ is the demand, $\mathcal{N} = \{1, \dots, N\}$ is the set of producers ($N > 1$) and for $i \in \mathcal{N}$ we use $a_i, b_i \geq 0$ to denote the coefficients of the ith producer's bid $a_i q_i + b_i q_i^2$. And $A_i \geq 0, B_i > 0$ stand for the coefficients of the true production cost function $A_i q_i + B_i q_i^2$. The increasing behaviour of the production cost function is well known and reflects that a producer typically ranks up its production units in merit order. Transportation thermal losses as well as possible congestions of transmission lines are not taken into account, thus the transmission network is not considered in this model.

Furthermore, $q_i \geq 0$ represents the non-negative production quantity of the ith producer. Considering $q \in \mathbb{R}^N_+$, we use $q_{-i} \in \mathbb{R}^{N-1}_+$ to denote the vector $(q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_N)$, and the same convention is used for other vectors hereinafter. A producer is said to be *active in the market* if his bid has been accepted by the ISO, i.e. the corresponding production quantity is positive. Finally, $\mathbb{R}_{++} = \mathbb{R}_+ \setminus \{0\}$, then $\mathbb{R}^N_{++} = (\mathbb{R}_{++})^N$, and for $x, y \in \mathbb{R}^2$ we write $x \succ y$ if $x_i \geq y_i$ for $i = 1, 2$ where at least one inequality is strict.

Thus each producer provides the ISO with a quadratic bid $a_i q_i + b_i q_i^2$. Then, knowing the bid vectors $a = (a_1, \dots, a_N) \in \mathbb{R}^N_+$ and $b = (b_1, \dots, b_N) \in \mathbb{R}^N_+$, the ISO computes the production quantity to be dispatched to the producers $q = (q_1, \dots, q_N) \in \mathbb{R}^N_+$ to minimize the total expenses while satisfying the demand, see e.g. [3,5,6]. For the sake of uniqueness of the solution to the ISO's problem, we use the *equity property* assumption, see [2], which reads:

$$\text{(E)} \qquad (a_i, b_i) = (a_j, b_j) \Longrightarrow q_i = q_j, \quad \forall i, j \in \mathcal{N},$$

thus formalizing the 'fairness' of the ISO. Hence the optimization problem ISO(a, b, D) is as follows

$$\text{ISO(a,b,D)} \qquad \min_q \sum_{i \in \mathcal{N}} (a_i q_i + b_i q_i^2)$$
$$\text{s.t.} \begin{cases} q_i \geq 0, \quad \forall i \in \mathcal{N} \\ \left[(a_i, b_i) = (a_j, b_j) \Rightarrow q_i = q_j\right], \quad \forall i, j \in \mathcal{N} \\ \sum_{i \in \mathcal{N}} q_i = D. \end{cases}$$

Then producer $i \in \mathcal{N}$ aims at maximizing his profit $\pi_i(a, b, D)$ given by

$$\pi_i(a, b, D) = (a_i - A_i)\, q_i(a, b, D) + (b_i - B_i)\, q_i(a, b, D)^2$$

manipulating his own strategic variables $a_i, b_i \geq 0$ with the rest of variables $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$ kept fixed. Note that $q_i(a, b, D)$ stands for the unique solution of the ISO's problem ISO(a, b, D), as shown in [2, Theorem 2.1] where the equity property plays a fundamental role for uniqueness. In other words, the ith producer's problem $P_i(a_{-i}, b_{-i}, D)$ reads

$$P_i(a_{-i}, b_{-i}, D) \qquad \tilde{\pi}_i = \sup_{a_i, b_i \geq 0} \pi_i(a_i, a_{-i}, b_i, b_{-i}, D).$$

Note that for the sake of simplicity we omit coefficients (A_i, B_i) in the notation for $\pi_i(a, b, D)$ and $P_i(a_{-i}, b_{-i}, D)$, and similarly we omit (a_{-i}, b_{-i}, D) when writing $\tilde{\pi}_i$. Then for a fixed set of producers $\mathcal{N}$ and any $(A, B, D) \in \mathbb{R}_+^N \times \mathbb{R}_{++}^N \times \mathbb{R}_{++}$, the set of all Nash equilibria for the corresponding market is denoted by

$$\mathcal{E}(A, B, D) = \left\{ (\tilde{a}, \tilde{b}) \in \mathbb{R}_+^{2N} : \pi_i(\tilde{a}, \tilde{b}, D) = \sup_{a_i, b_i \geq 0} \pi_i(a_i, \tilde{a}_{-i}, b_i, \tilde{b}_{-i}, D), \forall i \in \mathcal{N} \right\} \tag{1}$$

For a bid $(\tilde{a}, \tilde{b}) \in \mathcal{E}(A, B, D)$ at equilibrium, $(\tilde{a}_i, \tilde{b}_i)$ solves problem $P_i(a_{-i}, b_{-i}, D)$ for any producer $i \in \mathcal{N}$. The aim of this work is to evaluate the set $\mathcal{E}(A, B, D)$ for any value of the true cost coefficients $(A, B) \in \mathbb{R}_+^N \times \mathbb{R}_{++}^N$ and of the demand $D > 0$, and to come up with basic economic interpretation of this result.

2.2. Best response of a producer

To find all Nash equilibria of the modelled electricity market, we need first to analyse the best responses of a given producer, that is the bid, if exists, maximizing his profit. We assume that the demand is known as well as an estimate of the bids of the other producers. This analysis is mainly the aim of Part 2 – [2] and we thus only recall in this subsection some notations and a key result which will be useful to characterize the Nash equilibria of the market. We define

$$\lambda^m(a) = \min_{i \in \mathcal{N}} a_i,$$

and then several critical parameters of ISO(a, b, D), namely a critical marginal price $\lambda^c(a, b)$, a critical value of the demand $D^c(a, b)$, and a set of producers bidding critical (linear) bids $\mathcal{N}^c(a, b) \subset \mathcal{N}$

$$\begin{cases} \lambda^c(a,b) = \min_{i \in \mathcal{N}: b_i = 0} a_i, \\ \mathcal{N}^c(a,b) = \left\{ i \in \mathcal{N} \ : \ a_i = \lambda^c(a,b) \text{ and } b_i = 0 \right\}, \\ D^c(a,b) = \sum_{i \in \mathcal{N}: a_i < \lambda^c(a,b)} \dfrac{\lambda^c(a,b) - a_i}{2b_i}. \end{cases} \tag{2}$$

We note that $a_i < \lambda^c(a, b)$ implies $b_i > 0$ and so $D^c(a, b)$ is well-defined. If there is no $i \in \mathcal{N}$ such that $a_i < \lambda^c(a, b)$, we set $D^c(a, b) = 0$. Looking to [2, Theorem 2.1], all active producers bid linearly in this case. If there is no producer bidding linearly, i.e. we have $b_i > 0$ for all $i \in \mathcal{N}$, we set $\lambda^c(a, b) = D^c(a, b) = +\infty$. For the cardinality of $\mathcal{N}^c(a, b)$ it is denoted by $N^c(a, b)$. The above defined critical parameters have clear economic meanings discussed in detail in [2, Remark 1]. Next we define a set $\Gamma = \left\{(a, b, \lambda) \in \mathbb{R}_+^{2N+1} \ : \ 0 \le \lambda \le \lambda^c(a, b)\right\}$ (considering strict right inequality for the case of $\lambda^c(a, b) = +\infty$) and the function $F : \Gamma \to \mathbb{R}_+$ by

$$F(a,b,\lambda) = \sum_{i \in \mathcal{N}: a_i < \lambda} \frac{\lambda - a_i}{2b_i}. \tag{3}$$

Note that for $\lambda \le \lambda^m(a)$ the set $\{i \in \mathcal{N} : a_i < \lambda\}$ is empty and so $F(a, b, \lambda) = 0$. Note also that for $\lambda > \lambda^c(a, b)$ formula (3) is ill-posed as there exists $i \in \mathcal{N}$ such that $a_i < \lambda$ and $b_i = 0$. For any $(a, b) \in \mathbb{R}_+^{2N}$, function $\lambda \to F(a, b, \lambda)$ is continuous, piecewise linear and strictly increasing on $[\lambda^m(a), \lambda^c(a, b)[$, see [2, p.4 and Lemma A.1]. Thus, we may define a function $\lambda(a, b, D) : \mathbb{R}_+^{2N+1} \to \mathbb{R}_+$ as follows

$$\lambda(a,b,D) = \begin{cases} \lambda \in [\lambda^m(a), \lambda^c(a,b)[, \text{ s.t. } F(a,b,\lambda) = D & \text{if } D \in [0, D^c(a,b)[\\ \lambda^c(a,b), & \text{if } D \ge D^c(a,b). \end{cases} \tag{4}$$

For any $(a, b) \in \mathbb{R}_+^{2N}$ function $\lambda(a, b, D)$ is continuous and piecewise linear in D, and for any $D > 0$ satisfies $\lambda^m(a) \le \lambda(a, b, D) \le \lambda^c(a, b)$. The value of $\lambda(a, b, D)$ is identified in [2, Proposition 2.4] with the marginal price in the market.

Next for any $(a, b) \in \mathbb{R}_+^{2N}$, we define the following sets

$$\Gamma^- = \left\{(a,b,\tilde{\lambda}) \in \mathbb{R}_+^{2N+1} : \tilde{\lambda} \in]\lambda^m(a), \lambda^c(a,b)]\right\},$$
$$\Gamma^+ = \left\{(a,b,\tilde{\lambda}) \in \mathbb{R}_+^{2N+1} : \tilde{\lambda} \in [\lambda^m(a), \lambda^c(a,b)]\right\}.$$

Denoting a directional partial derivatives with respect to variable x by $\partial_x^{\pm}$, see [2, p.9], we may define functions $m^{\pm} : \Gamma^{\pm} \to \mathbb{R}_+$ as follows

$$\begin{cases} m^-(a,b,\tilde{\lambda}) = \partial_D^- \lambda(a,b,F(a,b,\tilde{\lambda})) & \text{for } \tilde{\lambda} \in]\lambda^m(a), \lambda^c(a,b)], \\ m^+(a,b,\tilde{\lambda}) = \partial_D^+ \lambda(a,b,F(a,b,\tilde{\lambda})) & \text{for } \tilde{\lambda} \in [\lambda^m(a), \lambda^c(a,b)[, \\ m^+(a,b,\tilde{\lambda}) = 0 & \text{for } \tilde{\lambda} = \lambda^c(a,b), \end{cases}$$

where $F(a, b, \tilde{\lambda})$ corresponds to the production quantity given the marginal price $\tilde{\lambda}$, see (4). For computation and properties of $m^{\pm}(a, b, \tilde{\lambda})$ we refer to [2, Lemma 2.3].

To analyse the minimization problem $P_i(a_{-i}, b_{-i}, D)$ of producer $i \in \mathcal{N}$ in detail, we need to extend the previous notation to describe the market without producer $i \in \mathcal{N}$, i.e. a market consisting only of producers in $\mathcal{N} \setminus \{i\}$. We define

$$\lambda^c(a_{-i}, b_{-i}) = \min_{j \in \mathcal{N} \setminus \{i\}, b_j = 0} a_j,$$

and similarly to (2) also the other critical parameters $\mathcal{N}^c(a_{-i}, b_{-i})$, $D^c(a_{-i}, b_{-i})$ of $\mathrm{ISO}(a_{-i}, b_{-i}, D)$. Similarly, we define functions $F(a_{-i}, b_{-i}, \lambda)$ and $\lambda(a_{-i}, b_{-i}, D)$ following (3) and (4). Using this notation, we may introduce several production quantities being important for finding the best response of a producer:

$$q_i^\star(a_{-i}, b_{-i}) = \frac{\lambda^c(a_{-i}, b_{-i}) - A_i}{2B_i}, \tag{5}$$

$$q_i^0(a_{-i}, b_{-i}) = F(a_{-i}, b_{-i}, A_i) \text{ provided } A_i \le \lambda^c(a_{-i}, b_{-i}), \tag{6}$$

$$q_i^m(a_{-i}, b_{-i}) = \frac{\lambda^m(a_{-i}) - A_i}{2B_i + m^+(a_{-i}, b_{-i}, \lambda^m(a_{-i}))}, \tag{7}$$

$$q_i^c(a_{-i}, b_{-i}) = \begin{cases} \dfrac{\lambda^c(a_{-i}, b_{-i}) - A_i}{2B_i + m^-(a_{-i}, b_{-i}, \lambda^c(a_{-i}, b_{-i}))} & \text{for } \lambda^m(a_{-i}) < \lambda^c(a_{-i}, b_{-i}), \\ 0 & \text{for } \lambda^m(a_{-i}) = \lambda^c(a_{-i}, b_{-i}). \end{cases} \tag{8}$$

As shown in [2, Lemma 2.5], $q_i^\star(a_{-i}, b_{-i})$ sometimes corresponds to an ideal production quantity that is, given the production cost coefficients A_i and B_i, the production quantity providing the maximum of the profit function for producer i.

Now, let us recall the main result from [2] describing the best response of producer $i \in \mathcal{N}$ with respect to various values of the demand $D > 0$.

Corollary 2.1 (of [2, Theorem 3.1]): *Let $D > 0$, $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$ for some $i \in \mathcal{N}$ and consider the problem*

$$P_i(a_{-i}, b_{-i}, D) \qquad \tilde{\pi}_i = \sup_{a_i, b_i \ge 0} \pi_i(a_i, a_{-i}, b_i, b_{-i}, D).$$

Then one of the following alternatives holds:

(a) *if either $A_i \ge \lambda^c(a_{-i}, b_{-i})$ or $D \in \,]0, q_i^0(a_{-i}, b_{-i})]$ then $\tilde{\pi}_i = 0$,*
(b) *if $D \in \,]q_i^0(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i})[$ then $\tilde{\pi}_i > 0$ and there is a unique best response $(\tilde{a}_i, \tilde{b}_i)$ given by $\tilde{b}_i = 0$ and $\tilde{a}_i \in [\lambda^m(a_{-i}), \lambda^c(a_{-i}, b_{-i})[$ satisfying*

$$\begin{cases} \tilde{a}_i = \lambda^m(a_{-i}) & \text{if } D \le q_i^m(a_{-i}, b_{-i}), \\ \dfrac{\tilde{a}_i - A_i}{2B_i + m^-(a_{-i}, b_{-i}, \tilde{a}_i)} \le D - F(a_{-i}, b_{-i}, \tilde{a}_i) \le \dfrac{\tilde{a}_i - A_i}{2B_i + m^+(a_{-i}, b_{-i}, \tilde{a}_i)} & \text{if } D > q_i^m(a_{-i}, b_{-i}), \end{cases} \tag{9}$$

(c) *if $D \ge D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i})$ and $D \ne D^c(a_{-i}, b_{-i}) + (N^c(a_{-i}, b_{-i}) + 1)\, q_i^\star(a_{-i}, b_{-i})$ then $\tilde{\pi}_i > 0$ and the best response does not exist,*
(d) *if $D = D^c(a_{-i}, b_{-i}) + (N^c(a_{-i}, b_{-i}) + 1)\, q_i^\star(a_{-i}, b_{-i})$ then $\tilde{\pi}_i > 0$ and there is a unique best response $(\tilde{a}_i, \tilde{b}_i) = (\lambda^c(a_{-i}, b_{-i}), 0)$.*

Fixing $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$, we see that for some values of D there is no maximizer in problem $P_i(a_{-i}, b_{-i}, D)$ and so the best response of producer $i \in \mathcal{N}$ does not exist. This situation is fully analysed in [2, Theorem 3.1], where a sequence of bids yielding a supremum of the profit is provided and the concept of limiting best response is used.

3. Main results

In this section, we fully characterize the set $\mathcal{E}(A,B,D)$ of Nash equilibria in the market (see definition (1)). To this end, we order all producers according to the linear coefficients of their true cost of production, thus we can assume that $A_1 \le A_2 \le \cdots \le A_N$. Then, we define accordingly

$$\mathcal{N}_{act}(A,B,D) = \{i \in \mathcal{N} \ : \ A_i < \lambda(A,B,D)\} = \{1,\dots,N_{act}(A,B,D)\}.$$

Note that $N_{act}(A,B,D) \ge 1$ since we always have $\lambda(A,B,D) > A_1$ using monotonicity of F stated in [2, Lemma A.1] and observing $F(A,B,A_1) = 0$. By similar arguments $\lambda(A,B,D) < A_2$ implies $A_1 < A_2$. Now, the main result of this article is as follows.

Theorem 3.1: *Let $(A,B,D) \in \mathbb{R}^N_+ \times \mathbb{R}^N_{++} \times \mathbb{R}_{++}$ be given. Then, Nash equilibrium exists if and only if*

(a) *either $\lambda(A,B,D) < A_2$,*
(b) *or $\lambda(A,B,D) > A_2$ and the following condition is satisfied*

$$N_{act}(A,B,D)(A_i - A_1) = 2D(B_1 - B_i), \quad \forall i \le N_{act}(A,B,D). \tag{10}$$

In equilibrium, the set of active producers is $\mathcal{N}_{act}(A,B,D)$. Moreover, in case (a) it holds $N_{act}(A,B,D) = 1$ and the set of Nash equilibria reads

$$\mathcal{E}(A,B,D) = \bigcup_{\tilde{\lambda} \in]A_1,A_2]} \left\{ (a,b) \in \mathbb{R}^{2N}_+ : \begin{array}{l} D \le \dfrac{\tilde{\lambda} - A_1}{2B_1} \dfrac{\sum_{j>1;a_j=\tilde{\lambda}} \frac{1}{b_j}}{\frac{1}{B_1} + \sum_{j>1;a_j=\tilde{\lambda}} \frac{1}{b_j}}, \\ (a_1,b_1) = (\tilde{\lambda},0), \lambda^m(a_{-1}) = \tilde{\lambda}, \\ (a_i,b_i) \succ (\tilde{\lambda},0), \ \forall i > 1 \end{array} \right\}, \tag{11}$$

and in case (b) we have

$$\mathcal{E}(A,B,D) = \left\{ (a,b) \in \mathbb{R}^{2N}_+ : \begin{array}{l} (a_i,b_i) = (\lambda(A,B,D),0), \ \forall i \le N_{act}(A,B,D) \\ (a_i,b_i) \succ (\lambda(A,B,D),0), \ \forall i > N_{act}(A,B,D) \end{array} \right\}. \tag{12}$$

Note that the set defined by formula (11), respectively (12), is not empty in case (a), respectively (b), of the above theorem. Indeed, in the forthcoming Corollary 4.1, some particular elements of these sets will be described.

To prove Theorem 3.1 we first provide several auxiliary lemmas describing properties of the following subsets of producers:

$$\begin{aligned} \mathcal{N}_0(a,b\,D) &= \{i \in \mathcal{N} \ : \ \pi_i(a,b,D) = 0, \text{ and } A_i \ge \lambda(a_{-i},b_{-i},D)\}, \\ \mathcal{N}_1(a,b,D) &= \{i \in \mathcal{N} \ : \ D \in]q_i^0(a_{-i},b_{-i}), D^c(a_{-i},b_{-i}) + q_i^c(a_{-i},b_{-i})[, \\ &\qquad a_i \in [\lambda^m(a_{-i}), \lambda^c(a_{-i},b_{-i})[\text{ satisfies (9)}, b_i = 0\}, \\ \mathcal{N}_2(a,b,D) &= \left\{ i \in \mathcal{N} \ : \ \begin{array}{l} D = D^c(a_{-i},b_{-i}) + (N^c(a_{-i},b_{-i}) + 1)\, q_i^\star(a_{-i},b_{-i}), \\ a_i = \lambda^c(a_{-i},b_{-i}), b_i = 0 \end{array} \right\}, \end{aligned}$$

omitting the true cost coefficients $(A,B) \in \mathbb{R}^N_+ \times \mathbb{R}^N_{++}$ in the notation for the sake of simplicity again. In the lemma below, we show that in Nash equilibrium these groups of producers form a partition of $\mathcal{N}$.

Lemma 3.2: *For any $(A,B,D) \in \mathbb{R}^N_+ \times \mathbb{R}^N_{++} \times \mathbb{R}_{++}$ it holds $(a,b) \in \mathcal{E}(A,B,D)$ if and only if $\mathcal{N} = \mathcal{N}_0(a,b,D) \cup \mathcal{N}_1(a,b,D) \cup \mathcal{N}_2(a,b,D)$.*

Proof: By definition of a Nash equilibrium, any producer $i \in \mathcal{N}$ provides the ISO with the best response bid, i.e. one of the bids described by statements (a), (b), (d) of Corollary 2.1. Similarly, once $\mathcal{N} = \mathcal{N}_0(a,b,D) \cup \mathcal{N}_1(a,b,D) \cup \mathcal{N}_2(a,b,D)$ we are necessarily at equilibrium $(a,b) \in \mathcal{E}(A,B,D)$. To finish the proof we observe $\mathcal{N}_0(a,b,D) = \{i \in \mathcal{N} | \pi_i(a,b,D) = 0 \text{ and } [A_i \geq \lambda^c(a_{-i},b_{-i}) \text{ or } D \leq q_i^0(a_{-i},b_{-i})]\}$ due to Corollary 2.1, definition of $q_i^0(a_{-i},b_{-i})$ and $\lambda(a_{-i},b_{-i},D)$. □

To further describe the producers in $\mathcal{N}_0(a,b,D)$ the following lemma is useful.

Lemma 3.3: *At equilibrium, the producers of subset $\mathcal{N}_0(a,b,D)$ are non active, that is, for any $(a,b) \in \mathcal{E}(A,B,D)$ and any $i \in \mathcal{N}_0(a,b,D)$ it holds $q_i(a,b,D) = 0$.*

Proof: For brevity we use λ as a shortcut to denote $\lambda(a,b,D)$ and assume $q_i(a,b,D) > 0$ for a contradiction. Using [2, Theorem 2.1] we know that either $a_i < \lambda$ and $q_i(a,b,D) = \frac{\lambda - a_i}{2b_i}$, or $a_i = \lambda$ and $q_i(a,b,D) = \frac{D - D^c(a,b)}{N^c(a,b)}$. Moreover, we have $\pi_i(a,b,D) = 0$ by definition of $\mathcal{N}_0(a,b,D)$ and since $q_i(a,b,D) > 0$ we necessarily have $a_i - A_i + (b_i - B_i)q_i(a,b,D) = 0$. Altogether, we obtain either $\lambda + a_i - 2A_i = B_i q_i(a,b,D)$ or $\lambda - A_i = B_i q_i(a,b,D)$, respectively. Finally, observing $\lambda \leq \lambda(a_{-i},b_{-i},D) \leq A_i$ due to [2, Lemma A.2] and $i \in \mathcal{N}_0(a,b,D)$, we have $B_i q_i(a,b,D) < 0$, a contradiction with $q_i(a,b,D) > 0$. □

Then, from Corollary 2.1 we observe that if $(a,b) \in \mathcal{E}(A,B,D)$ then for any $i \in \mathcal{N}_1(a,b,D) \cup \mathcal{N}_2(a,b,D)$ we have $\pi_i(a,b,D) > 0$. Thus, recalling Lemmas 3.2 and 3.3, we may state the following remark.

Remark 1: Note that once there is Nash Equilibrium in the market, $(a,b) \in \mathcal{E}(A,B,D)$, we have $q_i(a,b,D) = 0 \Leftrightarrow \pi_i(a,b,D) = 0 \Leftrightarrow i \in \mathcal{N}_0(a,b,D)$, and similarly $q_i(a,b,D) > 0 \Leftrightarrow \pi_i(a,b,D) > 0 \Leftrightarrow i \in \mathcal{N}_1(a,b,D) \cup \mathcal{N}_2(a,b,D)$.

The following properties of Nash equilibria will be of future use.

Lemma 3.4: *Let $(a,b) \in \mathcal{E}(A,B,D)$, then*

(i) $D > D^c(a,b) = 0$ *and* $\lambda(a,b,D) = \lambda^c(a,b)$,
(ii) *for any* $i \in \mathcal{N}_0(a,b,D)$ *it holds* $(a_i,b_i) \succ (\lambda(a,b,D),0)$,
(iii) $\mathcal{N}^c(a,b) = \mathcal{N}_1(a,b,D) \cup \mathcal{N}_2(a,b,D)$.

Proof: Using $D > 0$ and Lemma 3.3 there has to be some $j \in \mathcal{N}_1(a,b,D) \cup \mathcal{N}_2(a,b,D)$ such that $q_j(a,b,D) > 0$. For such j we have $b_j = 0$, thus also $a_j = \lambda(a,b,D)$ and $D > D^c(a,b)$ due to [2, Theorem 2.1]. Then, $\lambda(a,b,D) = \lambda^c(a,b)$ results directly from the definition of $\lambda(a,b,D)$. Thus, the proof of (i) is done except for statement $D^c(a,b) = 0$ which will be shown later.

Now for any $i \in \mathcal{N}_0(a,b,D)$ and according to Lemma 3.3 we have $q_i(a,b,D) = 0$, and by [2, Theorem 2.1] there are three variants of bids leading to $q_i(a,b,D) = 0$. However, variant $D = D^c(a,b)$ with $a_i = \lambda(a,b,D)$ and $b_i = 0$ is avoided due to the statement (i). Thus, we showed (ii).

By combining the statements (ii) and $\lambda(a,b,D) = \lambda^c(a,b)$ we see $\mathcal{N}_0(a,b,D) \cap \mathcal{N}^c(a,b) = \emptyset$. Now, for any $i \in \mathcal{N}_1(a,b,D) \cup \mathcal{N}_2(a,b,D)$ we have $b_i = 0$ and $a_i \leq \lambda^c(a_{-i},b_{-i})$, thus $a_i = \lambda^c(a,b)$, $i \in \mathcal{N}^c(a,b)$, and so (iii) is proved.

Finally, we can deduce $D^c(a,b) = 0$ from the definition using statements (ii), (iii) and $\lambda(a,b,D) = \lambda^c(a,b)$ to finish the proof of (i). □

Let us now describe splitting of the active producers between the sets $\mathcal{N}_1(a,b,D)$ and $\mathcal{N}_2(a,b,D)$ for (a,b) being a Nash equilibrium. We will see that there are only two possibilities.

Proposition 3.5: *Let $(a,b) \in \mathcal{E}(A,B,D)$, then one of these alternatives is satisfied:*

(a) $|\mathcal{N}_1(a,b,D)| = 1$ *and* $\mathcal{N}_2(a,b,D) = \emptyset$.
(b) $\mathcal{N}_1(a,b,D) = \emptyset$ *and* $|\mathcal{N}_2(a,b,D)| \geq 2$,

Note that both alternative configurations of the market in equilibrium described by the above proposition have clear economic meaning:

- case (a) corresponds to a 'monopolistic configuration' where only one producer is active in the market.
- case (b) corresponds to 'competitive configuration' where several producers are active in the market.

Proof: We prove this proposition in three steps. First we show $|\mathcal{N}_1(a,b,D)| \leq 1$ by contradiction. To this end, observe that if $k,l \in \mathcal{N}_1(a,b,D)$, $k \neq l$ then $a_k < \lambda^c(a_{-k},b_{-k}) \leq a_l < \lambda^c(a_{-l},b_{-l}) \leq a_k$ where both definitions of $\mathcal{N}_1(a,b,D)$ and λ^c are used twice.

Next, we show that $|\mathcal{N}_1(a,b,D)| = 1$ implies $\mathcal{N}_2(a,b,D) = \emptyset$. Assuming $k \in \mathcal{N}_1(a,b,D)$ and $l \in \mathcal{N}_2(a,b,D)$ for a contradiction, we may write $\lambda^c(a,b) = a_k < \lambda^c(a_{-k},b_{-k}) \leq a_l = \lambda^c(a_{-l},b_{-l}) = \lambda^c(a,b)$ where we use the definitions of $\lambda^c(a_{-i},b_{-i},D)$ and $\mathcal{N}_1(a,b,D)$, the fact that $k \neq l$, and the definition of $\mathcal{N}_2(a,b,D)$.

Finally, for $\mathcal{N}_1(a,b,D) = \emptyset$ we show that $|\mathcal{N}_2(a,b,D)| \geq 2$. First, if $\mathcal{N}_2(a,b,D) = \emptyset$ then we have $\mathcal{N} = \mathcal{N}_0(a,b,D)$, and by using Lemma 3.3 we observe $D = 0$, a contradiction with $D > 0$. Second, assuming $\mathcal{N}_2(a,b,D) = \{k\}$ and using Lemma 3.4(iii) we have $\mathcal{N}^c(a,b) = \{k\}$, thus $a_k < \lambda^c(a_{-k},b_{-k})$, a contradiction with $k \in \mathcal{N}_2(a,b,D)$. □

4. A numerical example

The set of Nash equilibria $\mathcal{E}(A,B,D)$ as given by Theorem 3.1 is too rich to be well illustrated by a figure. This is caused by the fact that in Corollary 2.1 we do not prescribe any particular bid to non-profiting producers, i.e. producers $i \in \mathcal{N}$ such that $\tilde{\pi}_i = 0$, see also the definition of $\mathcal{N}_0(a,b,D)$. We have already shown that these producers do not produce, see Lemma 3.3, but still their actual bids have influence on the marginal price in equilibrium, see Theorem 3.1(b). To simplify the problem, we will further consider just equilibria $(a,b) \in \mathcal{E}(A,B,D)$ such that condition

$$\text{(H)} \qquad \tilde{\pi}_i = 0 \Rightarrow (a_i,b_i) = (A_i,B_i), \quad \forall i \in \mathcal{N}$$

is satisfied. Denoting a set of such equilibria $\tilde{\mathcal{E}}(A,B,D)$, we show that it contains at maximum one element, and may be fully described in terms of the data (A,B,D).

Corollary 4.1 (of Theorem 3.1): *Let $(A,B,D) \in \mathbb{R}^N_+ \times \mathbb{R}^N_{++} \times \mathbb{R}_{++}$ be given. Then, Nash equilibrium such that (H) is satisfied exists if and only if*

(a) *either*

$$D \leq \frac{A_2 - A_1}{2B_1} \frac{\sum_{j;A_j=A_2} \frac{1}{B_j}}{\frac{1}{B_1} + \sum_{j;A_j=A_2} \frac{1}{B_j}}, \tag{13}$$

(b) *or $D > \dfrac{A_2 - A_1}{2B_1}$ and the following condition is satisfied*

$$N_{act}(A,B,D)(A_i - A_1) = 2D(B_1 - B_i), \quad \forall i \leq N_{act}(A,B,D). \tag{14}$$

Moreover, in both cases such an equilibrium is unique, $\tilde{\mathcal{E}}(A,B,D) = \{(\tilde{a},\tilde{b})\}$, in case (a) given by

$$(\tilde{a}_1,\tilde{b}_1) = (A_2,0), \quad (\tilde{a}_i,\tilde{b}_i) = (A_i,B_i), \quad \forall i > 1,$$

and in case (b) by

$$(a_i,b_i) = (\lambda(A,B,D),0), \quad \forall i \leq N_{act}(A,B,D),$$
$$(a_i,b_i) = (A_i,B_i), \quad \forall i > N_{act}(A,B,D).$$

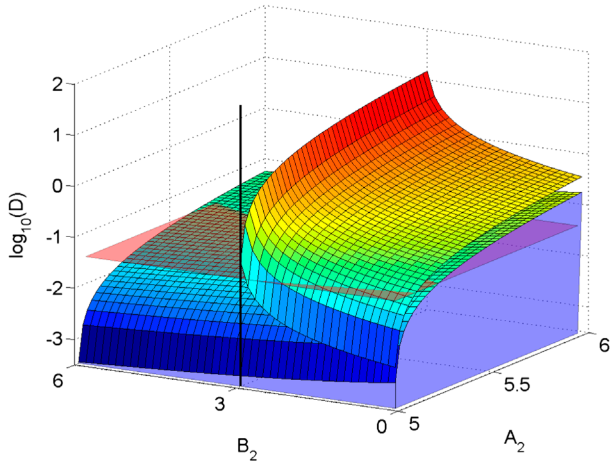

Figure 1. This figure shows the set $\Omega = \Omega_a \cup \Omega_{b1} \cup \Omega_{b2}$ of all points (A_2, B_2, D) such that Nash equilibrium satisfying (H) exists provided $(A_1, B_1) = (5, 3)$, as given in Example 4.2. Component Ω_a corresponds to the blue volume, component Ω_{b1} is given by the above curved surface, and Ω_{b2} is the black line. The red plane given by $D = 0.04$ denotes the cutting plane used to plot Figure 2.

Proof of Corollary 4.1: We find all $(a, b) \in \mathcal{E}(A, B, D)$ such that $(a_i, b_i) = (A_i, B_i)$ for all $i > N_{act}(A, B, D)$. First, assuming $\lambda(A, B, D) < A_2$ and using Theorem 3.1(a), we necessarily obtain $\tilde{\lambda} = \lambda^m(a_{-1}) = \lambda^m(A_{-1}) = A_2$, thus arriving at condition (13). Moreover, this condition implies $D < \frac{A_2 - A_1}{2B_1}$, or equivalently $\lambda(A, B, D) < A_2$ since $(A_2 - A_1)/(2B_1) = F(A, B, A_2)$ and using [2, Lemma A.1], and so we may drop the latter inequality from our assumptions. To show statement (b), we use (12), set $(a_i, b_i) = (A_i, B_i)$ for all $i > N_{act}(A, B, D)$, and then reformulate assumption $\lambda(A, B, D) > A_2$ in terms of D in the same way as in the case (a), the proof is done. □

Example 4.2: To illustrate the properties of a set of Nash equilibria satisfying (H), consider a simple market with two producers $\mathcal{N} = \{1, 2\}$, such that $(A_1, B_1) = (5, 3)$. Note that we still assume $A_1 \leq A_2$ as discussed in the beginning of Section 3. Then we may define a set

$$\Omega = \left\{(A_2, B_2, D) \in \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+ \;:\; \tilde{\mathcal{E}}(A_1, A_2, B_1, B_2, D) \neq \emptyset\right\}.$$

From Corollary 4.1 we can deduce that Ω consists of a union of two subsets Ω_a and Ω_b. Moreover, for Ω_b we see that $\Omega_b = \Omega_{b1} \cup \Omega_{b2}$, with all these sets given as follows:

$$\begin{aligned}
\Omega_a &= \left\{(A_2, B_2, D) \in]A_1, +\infty[\times \mathbb{R}_{++} \times \mathbb{R}_+ \;:\; D \leq \frac{1}{2}\frac{A_2 - A_1}{B_1 + B_2}\right\},\\
\Omega_{b1} &= \left\{(A_2, B_2, D) \in]A_1, +\infty[\times]0, B_1[\times \mathbb{R}_+ \;:\; D = \frac{A_2 - A_1}{B_1 - B_2}\right\},\\
\Omega_{b2} &= \{(A_2, B_2, D) \in \{A_1\} \times \{B_1\} \times \mathbb{R}_+\}.
\end{aligned}$$

These sets of Nash equilibria are depicted in Figures 1 and 2.

5. Proof of Theorem 3.1

To prove Theorem 3.1, we denote the right-hand side of (11) and (12) by $\mathcal{E}_1(A, B, D)$ and $\mathcal{E}_2(A, B, D)$, respectively, and state several auxiliary lemmas.

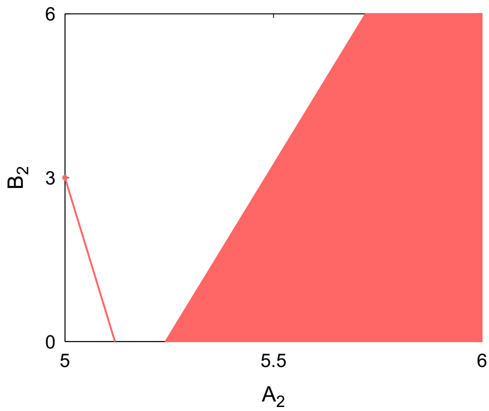

Figure 2. In this figure we depict all points (A_2, B_2) such that Nash equilibrium satisfying (H) exists provided $(A_1, B_1) = (5, 3)$, as given in Example 4.2, and $D = 0.04$. This value of demand corresponds to the red cutting plane plotted in Figure 1.

Lemma 5.1: *Let $(a, b) \in \mathbb{R}^{2N}_+$ be such that $a_1 = \lambda^m(a_{-1})$. Then $D \le q_1^m(a_{-1}, b_{-1})$ if and only if*

$$D \le \frac{a_1 - A_1}{2B_1} \frac{\sum_{j>1; a_j=a_1} \frac{1}{b_j}}{\frac{1}{B_1} + \sum_{j>1; a_j=a_1} \frac{1}{b_j}}. \tag{15}$$

Proof: From the definition of $q_i^m(a_{-i}, b_{-i})$, see (7), we observe that condition $D \le q_1^m(a_{-1}, b_{-1})$ in our setting actually reads

$$D \le \frac{a_1 - A_1}{2B_1 + m^+(a_{-1}, b_{-1}, a_1)}.$$

This is, after a short calculation, equivalent to (15) due to [2, Lemma 2.3]. □

We will now demonstrate that any equilibrium of the market corresponding to a monopolistic configuration has to be an element of $\mathcal{E}_1(A, B, D)$.

Lemma 5.2: *Let $(a, b) \in \mathcal{E}(A, B, D)$ be such that $|\mathcal{N}_1(a, b, D)| = 1$ and $\mathcal{N}_2(a, b, D) = \emptyset$, then $(a, b) \in \mathcal{E}_1(A, B, D)$.*

Proof: Let $(a, b) \in \mathcal{E}(A, B, D)$ and denote $\mathcal{N}_1(a, b, D) = \{k\}$ and $\lambda_{eq} = \lambda(a, b, D)$. Then we have $(a_i, b_i) \succ (\lambda_{eq}, 0)$ for all $i \in \mathcal{N}_0(a, b, D)$ due to Lemma 3.4(ii), thus according to Lemma 3.2 also $\lambda^m(a_{-k}) \ge \lambda_{eq}$. Now, we observe $\lambda_{eq} = \lambda^c(a, b) = a_k \ge \lambda^m(a_{-k})$ using Lemma 3.4(i) and $k \in \mathcal{N}_1(a, b, D)$. Thus, necessarily $a_k = \lambda_{eq} = \lambda^m(a_{-k})$. Now, we prove that $a_k = \lambda^m(a_{-k})$ and $k \in \mathcal{N}_1(a, b, D)$ implies

$$D \le q_k^m(a_{-k}, b_{-k}) = \frac{a_k - A_k}{2B_k + m^+(a_{-k}, b_{-k}, a_k)}. \tag{16}$$

Indeed, if $D > q_k^m(a_{-k}, b_{-k})$ then one obtains $D - F(a_{-k}, b_{-k}, a_k) \le q_k^m(a_{-k}, b_{-k})$ in (9) due to $k \in \mathcal{N}_1(a, b, D)$, a contradiction since $F(a_{-k}, b_{-k}, a_k) = 0$ because $a_k = \lambda^m(a_{-k})$.

For any $j \in \mathcal{N}_0(a, b, D)$ we have $A_j \ge \lambda(a_{-j}, b_{-j}, D) \ge \lambda_{eq}$ using [2, Lemma A.2], thus also $\lambda^m(A_{-k}) \ge \lambda_{eq} = a_k$. Moreover, $a_k > A_k$ due to (16) together with $D > 0$. Thus, we may identify $k = 1$ with respect to the ordering of producers, and the proof is complete by setting $\tilde{\lambda} = \lambda_{eq}$, rewriting (16) according to Lemma 5.1, and realizing that $\lambda^m(A_{-k}) = \lambda^m(A_{-1}) = A_2$. □

Symmetrically to the previous lemma, we will now show that any equilibrium of the market corresponding to a non-monopolistic configuration must be an element of $\mathcal{E}_2(A, B, D)$.

Lemma 5.3: *Let $(a, b) \in \mathcal{E}(A, B, D)$ be such that $\mathcal{N}_1(a, b, D) = \emptyset$ and $|\mathcal{N}_2(a, b, D)| \ge 2$, then $(a, b) \in \mathcal{E}_2(A, B, D)$.*

Proof: Let us denote $\lambda_{eq} = \lambda(a, b, D)$ and show first that $\mathcal{N}_{act}(A, B, D) = \mathcal{N}_2(a, b, D)$. Using Lemma 3.4 and the definition of $\mathcal{N}_2(a, b, D)$ we observe $D^c(a, b) = 0$, $\mathcal{N}^c(a, b) = \mathcal{N}_2(a, b, D)$ and $\lambda_{eq} = \lambda^c(a, b) = \lambda^c(a_{-i}, b_{-i}) = a_i$ for any $i \in \mathcal{N}_2(a, b, D)$. Then we have $0 \leq D^c(a_{-i}, b_{-i}) \leq D^c(a, b) = 0$ and so

$$q_i^\star(a_{-i}, b_{-i}) = \frac{D}{|\mathcal{N}_2(a, b, D)|} = \frac{\lambda_{eq} - A_i}{2B_i} > 0 \tag{17}$$

still for any $i \in \mathcal{N}_2(a, b, D)$.

Now, for any $j \in \mathcal{N}_0(a, b, D)$ we observe

$$\lambda_{eq} \leq \lambda(a_{-j}, b_{-j}, D) \leq A_j \tag{18}$$

due to [2, Lemma A.2] and the definition of $\mathcal{N}_0(a, b, D)$. Thus, $\{j : A_j < \lambda_{eq}\} \cap \mathcal{N}_0(a, b, D) = \emptyset$ and, according to Lemma 3.2, $\{j : A_j < \lambda_{eq}\} \subset \mathcal{N}_2(a, b, D)$. Using the inequality stated in (17), we observe that the previous inclusion is, actually, an equality . Since $B_i > 0$ for any $i \in \mathcal{N}$, it holds $\lambda^c(A, B) = +\infty$. Thus recalling the definition of F and using (17), we have $F(A, B, \lambda_{eq}) = \sum_{i \in \mathcal{N}_2(a,b,D)} \frac{\lambda_{eq} - A_i}{2B_i} = D$, or equivalently $\lambda_{eq} = \lambda(A, B, D)$. This prove that $\mathcal{N}_{act}(A, B, D) = \{j : A_j < \lambda_{eq}\} = \mathcal{N}_2(a, b, D)$.

Now, assumption $|\mathcal{N}_2(a, b, D)| \geq 2$ implies $\lambda(A, B, D) > A_2$. To verify (10) we may use (17). Indeed, for any $i \leq N_{act}(A, B, D)$ we have $N_{act}(A, B, D)(\lambda_{eq} - A_i) = 2DB_i$, thus also $N_{act}(A, B, D)(\lambda_{eq} - A_1) = 2DB_1$, and finally (10). To finish the proof, we observe that $N_{act}(A, B, D) < j \leq N$ is now equivalent to $j \in \mathcal{N}_0(a, b, D)$ using (18), and so $(a_j, b_j) \succ (\lambda_{eq}, 0)$ due to Lemma 3.4(ii). □

Lemma 5.4: *Let $(A, B, D) \in \mathbb{R}_+^N \times \mathbb{R}_{++}^N \times \mathbb{R}_{++}$, then $\mathcal{E}_1(A, B, D) \subset \mathcal{E}(A, B, D)$.*

Proof: To show $\mathcal{E}_1(A, B, D) \subset \mathcal{E}(A, B, D)$ we may equivalently prove that $\mathcal{N} = \mathcal{N}_0(a, b, D) \cup \mathcal{N}_1(a, b, D) \cup \mathcal{N}_2(a, b, D)$ for any $(a, b) \in \mathcal{E}_1(A, B, D)$, see Lemma 3.2. To this end we will prove that $1 \in \mathcal{N}_1(a, b, D)$ and $\mathcal{N} \setminus \{1\} \subset \mathcal{N}_0(a, b, D)$ for any $(a, b) \in \mathcal{E}_1(A, B, D)$, since $\mathcal{N}_0(a, b, D) \cup \mathcal{N}_1(a, b, D) \cup \mathcal{N}_2(a, b, D) \subset \mathcal{N}$ always. Take any $(a, b) \in \mathcal{E}_1(A, B, D)$ and denote the respective value $\tilde{\lambda}$ in (11) by $\lambda_{eq} \in]A_1, A_2]$. From the definition of $\mathcal{E}_1(A, B, D)$ we have $D \leq q_1^m(a_{-1}, b_{-1})$ due to $a_1 = \lambda^m(a_{-1})$ using Lemma 5.1, thus also (9) for $i = 1$. Additionally, the last equality together with $(a_i, b_i) \succ (\lambda_{eq}, 0)$ for all $i > 1$ implies $\lambda^m(a_{-1}) < \lambda^c(a_{-1}, b_{-1})$. This means that $q_1^c(a_{-1}, b_{-1})$ is given by the first part of (8), and using formulas in [2, Lemma 2.3] also $m^+(a_{-1}, b_{-1}, \lambda^m(a_{-1})) \geq m^-(a_{-1}, b_{-1}, \lambda^c(a_{-1}, b_{-1}))$. Thus $q_1^m(a_{-1}, b_{-1}) < q_1^c(a_{-1}, b_{-1})$ and so $D \leq q_1^m(a_{-1}, b_{-1}) < D^c(a_{-1}, b_{-1}) + q_1^c(a_{-1}, b_{-1})$. Moreover, we have $D > q_1^0(a_{-1}, b_{-1}) = F(a_{-1}, b_{-1}, A_1) = 0$ since $A_1 < a_1 = \lambda^m(a_{-1})$ and so $1 \in \mathcal{N}_1(a, b, D)$. Next, we take any $k \in \mathcal{N}$ such that $k > 1$, and show $k \in \mathcal{N}_0(a, b, D)$. From the definition of $\mathcal{E}_1(A, B, D)$ and $D^c(a, b)$ we observe $D^c(a, b) = F(a, b, \lambda^c(a, b)) = 0$ since $\lambda^c(a, b) = a_1 = \lambda^m(a_{-1})$, then $q_k(a, b, D) = 0$ as given by [2, Theorem 2.1] due to $(a_k, b_k) \succ (\lambda_{eq}, 0)$, and so $\pi_k(a, b, D) = 0$. Then we show $D^c(a_{-k}, b_{-k}) = 0$ again from the definition since $\lambda^c(a_{-k}, b_{-k}) = a_1$. Finally, $\lambda(a_{-k}, b_{-k}, D) = \lambda^c(a_{-k}, b_{-k}) = a_1 \leq A_k$ using $a_1 \leq \lambda^m(A_{-1})$, and so $k \in \mathcal{N}_0(a, b, D)$. Thus we showed $\mathcal{E}_1(A, B, D) \subset \mathcal{E}(A, B, D)$. □

Lemma 5.5: *Let $(A, B, D) \in \mathbb{R}_+^N \times \mathbb{R}_{++}^N \times \mathbb{R}_{++}$ be such that $\lambda(A, B, D) > A_2$ and condition (10) is satisfied, then $\mathcal{E}_2(A, B, D) \subset \mathcal{E}(A, B, D)$.*

Proof: By similar arguments as in the proof of Lemma 5.4 it suffices to show $\{1, \ldots, N_{act}(A, B, D)\} \subset \mathcal{N}_2(a, b, D)$ and $\{N_{act}(A, B, D)+1, \ldots, N\} \subset \mathcal{N}_0(a, b, D)$ for any $(a, b) \in \mathcal{E}_2(A, B, D)$. From definition we see that $\lambda(A, B, D) > A_2$ is equivalent to $N_{act}(A, B, D) \geq 2$. Then for any $(a, b) \in \mathcal{E}_2(A, B, D)$ and for any $i \in \mathcal{N}$ it holds $a_i \geq \lambda^c(a, b) = \lambda^c(a_{-i}, b_{-i}) = \lambda(A, B, D)$, and so also $D^c(a, b) = D^c(a_{-i}, b_{-i}) = 0$ and $N^c(a, b) = N_{act}(A, B, D)$ due to the definition of $\mathcal{E}_2(A, B, D)$.

Further, consider $i \leq N_{act}(A, B, D)$, then $N^c(a_{-i}, b_{-i}) + 1 = N^c(a, b) = N_{act}(A, B, D)$, thus if we show also $D = N_{act}(A, B, D)\, q_i^\star(a_{-i}, b_{-i})$ we will prove $i \in \mathcal{N}_2(a, b, D)$. To this end we show that the last equality is implied by (10). First observe that $\bar{\lambda} = A_i + 2B_i(D/N_{act}(A, B, D))$ is well defined for any $i \leq N_{act}(A,B,D)$ due to (10). Thus, we have also

$$\frac{\bar{\lambda} - A_i}{2B_i} = \frac{D}{N_{act}(A,B,D)},$$

and by summing over all $i \in \mathcal{N}_{act}(A,B,D)$ we obtain $F(A,B,\bar{\lambda}) = D$. In other words $\bar{\lambda} = \lambda(A,B,D)$ and so $\frac{\bar{\lambda}-A_i}{2B_i} = q_i^\star(a_{-i},b_{-i})$ using $\lambda(A,B,D) = \lambda^c(a_{-i},b_{-i})$ from the previous paragraph.

On the other hand, if $i > N_{act}(A,B,D)$ than from the definition of $\mathcal{E}_2(A,B,D)$ we have $q_i(a,b,D) = 0$ using [2, Theorem 2.1], thus also $\pi_i(a,b,D) = 0$. Moreover, we may write $A_i \geq \lambda(A,B,D) = \lambda^c(a_{-i},b_{-i}) = \lambda(a_{-i},b_{-i},D)$ using the definition of $N_{act}(A,B,D)$, the previous considerations and $D > D^c(a_{-i},b_{-i}) = 0$. In other words $i \in \mathcal{N}_0(a,b,D)$, and so $\mathcal{E}_2(A,B,D) \subset \mathcal{E}(A,B,D)$. □

Lemma 5.6: *Let $(A,B,D) \in \mathbb{R}^N_+ \times \mathbb{R}^N_{++} \times \mathbb{R}_{++}$ be such that $\mathcal{E}_1(A,B,D) \neq \emptyset$, then $\lambda(A,B,D) < A_2$.*

Proof: For any $(a,b) \in \mathcal{E}_1(A,B,D)$ we have $D \leq q_1^m(a_{-1},b_{-1})$, $\lambda^m(a_{-1}) = a_1 \leq A_2$, and after a short calculation also $\lambda^m(a_{-1}) < \lambda^c(a_{-1},b_{-1})$. Then, moreover, $m^+(a_{-1},b_{-1},\lambda^m(a_{-1})) > 0$ due to [2, Lemma 2.3], and so

$$D \leq q_1^m(a_{-1},b_{-1}) \leq \frac{A_2 - A_1}{2B_1 + m^+(a_{-1},b_{-1},\lambda^m(a_{-1}))} < \frac{A_2 - A_1}{2B_1} = F(A,B,A_2).$$

Now we may conclude using monotonicity of F, see [2, Lemma A.1]. □

Lemma 5.7: *Let $(A,B,D) \in \mathbb{R}^N_+ \times \mathbb{R}^N_{++} \times \mathbb{R}_{++}$ and $(a,b) \in \mathcal{E}(A,B,D)$. Then*

$$\lambda(A,B,D) > A_2 \text{ and } (10) \quad \Longleftrightarrow \quad \mathcal{N}_1(a,b,D) = \emptyset,\ |\mathcal{N}_2(a,b,D)| \geq 2.$$

Proof: One implication is due to Lemma 5.3, to show the other take $(a,b) \in \mathcal{E}(A,B,D)$ and assume that $\lambda(A,B,D) > A_2$ and (10) hold. Then we observe that $(a,b) \notin \mathcal{E}_1(A,B,D)$ due to Lemma 5.6, and so $|\mathcal{N}_1(a,b,D)| = 1$ and $\mathcal{N}_2(a,b,D) = \emptyset$ is avoided using Lemma 5.2. Thus, we necessarily have $\mathcal{N}_1(a,b,D) = \emptyset$ and $|\mathcal{N}_2(a,b,D)| \geq 2$ as given by Proposition 3.5. □

Proof of Theorem 3.1: The proof will be composed of three cases.

Case (i): first, assume that $\lambda(A,B,D) > A_2$ and (10) are satisfied. Then $\mathcal{E}_2(A,B,D) \subset \mathcal{E}(A,B,D)$ due to Lemma 5.5. To show the opposite inclusion, consider any $(a,b) \in \mathcal{E}(A,B,D)$. Using Lemma 5.7 we obtain $\mathcal{N}_1(a,b,D) = \emptyset$ and $|\mathcal{N}_2(a,b,D)| \geq 2$, and so $(a,b) \in \mathcal{E}_2(A,B,D)$ due to Lemma 5.3. Thus, if $\lambda(A,B,D) > A_2$ and (10) holds then $\mathcal{E}_2(A,B,D) = \mathcal{E}(A,B,D)$.

Case (ii): now, consider (A,B,D) such that $\lambda(A,B,D) < A_2$. It immediately implies that $\mathcal{N}_{act}(A,B,D) = \{1\}$. Moreover, we have $\mathcal{E}_1(A,B,D) \subset \mathcal{E}(A,B,D)$ due to Lemma 5.4. Consider now any $(a,b) \in \mathcal{E}(A,B,D)$. Using Lemma 5.7 and Proposition 3.5 we obtain $|\mathcal{N}_1(a,b,D)| = 1$ and $\mathcal{N}_2(a,b,D) = \emptyset$. Thus, $(a,b) \in \mathcal{E}_1(A,B,D)$ due to Lemma 5.2 and thus $\mathcal{E}_1(A,B,D) = \mathcal{E}(A,B,D)$ providing $\lambda(A,B,D) < A_2$.

Case (iii): to end the proof, let us assume that (A,B,D) are such that none of the previous case (i) and (ii) occurs. It can be easily seen that it is equivalent to assume that (A,B,D) satisfies either $\lambda(A,B,D) = A_2$, or $\left[\lambda(A,B,D) > A_2 \text{ and (10) is not satisfied}\right]$. It implies that, in both cases, assertion $\left[\lambda(A,B,D) > A_2 \text{ and (10)}\right]$ of Lemma 5.7 do not hold true. Therefore, combining with Proposition 3.5, we immediately have, for any $(a,b) \in \mathcal{E}(A,B,D)$, $|\mathcal{N}_1(a,b,D)| = 1$ and $\mathcal{N}_2(a,b,D) = \emptyset$. According to Lemma 5.2, (a,b) is an element of $\mathcal{E}_1(A,B,D)$ and a contradiction is obtained since, by Lemma 5.6, $\lambda(A,B,D) < A_2$. It shows that $\mathcal{E}(A,B,D) = \emptyset$, that is no Nash equilibrium exist in this case.

Let us observe that in both cases (i) and (ii) above, at equilibrium, the active producers are the ones of $\mathcal{N}_{act}(A,B,D)$. Indeed, first following both formulas (11) and (12), and according to [2, Theorem 2.1], the producers that are not in $\mathcal{N}_{act}(A,B,D)$ are not active. Based on the same theorem, producers in $\mathcal{N}_{act}(A, B, D)$ have to produce $D/\mathcal{N}^c(a, b) > 0$ since for any $(a, b) \in \mathcal{E}(A, B, D)$, $\lambda^m(a) = \lambda^c(a, b)$, and so $D^c(a, b) = 0$. □

6. Conclusion

We considered a pay-as-bid balancing market that we modelled as a multi-leader-common-follower problem. This is a bilevel problem which is from a producer's viewpoint to determine the optimal bids for a given demand, and from the ISO's viewpoint to determine the optimal dispatch of the demand among the producers.

The main result is that we proved the existence and came up with explicit formulas for the Nash equilibria. Such a result is noteworthy in more than one respect. First, we considered a larger class of bid functions compared to that commonly used in the literature. Allowing linear bids instead of pure quadratic bids proved to be fruitful, as the bids of active producers have shown to be linear at equilibrium. Second, few comparable results are reported in the literature, and the considered models are with quite restrictive assumptions.

A noticeable feature follows from the equity property. Since at the equilibrium the ISO shares equally the demand among the active producers, it may appear that the producers could not follow their best response. The mismatch could be interpreted as a 'price of fairness'.

As a perspective work, we could think of considering explicit production bounds. First, it would alleviate the main limitation of our model, even though we showed that implicit bounds are in fact enforced. Second, such bounds could be used in the bidding process, and eventually would have an impact on the share of the production quantity assigned to each active producer.

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