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PERFECT PLASTICITY WITH DAMAGE AND HEALING AT SMALL STRAINS, ITS MODELING, ANALYSIS, AND COMPUTER IMPLEMENTATION*

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Abstract. The quasistatic, Prandtl–Reuss perfect plasticity at small strains is combined with a gradient, reversible (i.e., admitting healing) damage which influences both the elastic moduli and the yield stress. Existence of weak solutions of the resulting system of variational inequalities is proved by a suitable fractional-step discretization in time with guaranteed numerical stability and convergence. After finite-element approximation, this scheme is computationally implemented and illustrative two-dimensional simulations are performed. The model allows, e.g., for application in geophysical modeling of reoccurring rupture of lithospheric faults. Resulting incremental problems are solved in MATLAB by quasi-Newton method to resolve the elastoplasticity component of the solution, while the damage component is obtained by solving a quadratic programming problem.

Key words. Prandtl–Reuss perfect plasticity, bounded-deformation space, incomplete damage, fractional-step time discretization, finite-element method, quasi-Newton method, quadratic programming, nonsmooth continuum mechanics, geophysical applications

AMS subject classifications. 35K87, 49N10, 65K15, 74A30, 74C05, 74R20, 86A17, 90C53

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1. Introduction.

There is a vast amount of literature about plasticity and about damage separately, both in mathematics and in civil or mechanical engineering. Much less literature addresses various combinations of plasticity and damage; cf., e.g., [3, 4, 11, 12, 29, 31, 58]. In engineering, this is usually called ductile damage; cf., e.g., [21, 33, 34, 35, 40]. Also a lot of geophysical models combine reversible damage (called rather aging) with some sort of plasticity (often modeled as not entirely independent of damage, however); cf., e.g., [37].

The goal of this article is to devise a model that would allow for

- modeling of thin plastic shear bands surrounded by wider damage zones (as typically occurs in geophysical modeling of lithospheric faults with very narrow core) with possible healing of damage (as considered in geophysical modeling to allow reoccurring damaging), and simultaneously
- rigorous proof of existence of weak solutions of the resulting system of variational inequalities proved by a suitable fractional-step discretization in time with guaranteed numerical stability and convergence, and
- efficient numerical implementation of the time-discrete model.

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We depart from the standard linearized, associative, rate-independent plasticity at small strain as presented, e.g., in [28]. Simultaneously, we use a rather standard scalar (i.e., isotropic) damage as introduced by L. M. Kachanov in the late 1960s and presented, e.g., in [18], considered here however as rate dependent and reversible in the sense that a possible healing is allowed. To avoid serious mathematical and computation difficulties, we have in mind primarily an incomplete damage through a higher-order damage-independent term, although the standard elastic tensor can allow for a complete damage; cf. $\mathbb{H}$ and $\mathbb{C} = \mathbb{C}(\zeta)$ below. An important aspect of the model is that not only the conservative part but also the dissipative part are subjected to damage, i.e., not only the elastic moduli but also the yield stress will be considered as damageable. This relatively simple and lucid mechanism will, however, lead to a possibly very complex response of the model.

To make the model accessible to analysis, we work within the setting of small strains, and we also take into account surface-energy effects by including in the free energy a term dependent on the gradient of the total strain. This is also known as a concept of so-called second-grade nonsimple materials (cf., e.g., [19, 46, 57]), alternatively referred to as the concept of hyperstresses or couple stresses [48, 62]; for reasons we use it here cf. Remark 2.5 below.

In view of applications we have in mind, we suppress any hardening effects and thus we consider the *Prandtl–Reuss* elastic/*perfectly plastic* model; in fact, considering kinematic or isotropic hardening would make a lot of aspects much easier. A plastic yield stress dependent on damage is in some variants used in the Cam–Clay model (cf., e.g., [14, 36, 64]) or in the Perzyna model with damage (cf. [58]) and also in [3, 4, 11, 12]. Let us also point out that damage with healing without plasticity (as sometimes considered in the mathematical literature) would have only very limited application because damaged material typically can undergo substantial deformation and the healing should not be performed toward the original configuration.

We focus on the isothermal variant of the model. In contrast to [54], we consider rate-independent plasticity without any gradient, so that concentration of plastic and total strains and development of sharp shear bands are possible. Also, related to this concentration, both plastification and damage are driven by the elastic stress (which is still well controlled) rather than the total strain (which may concentrate); for plasticity itself, see also [53].

The presented model has potential applications in geophysical modeling of reoccurring rupture of lithospheric faults or of nucleation of new faults. A narrow so-called core of the fault can be modeled by the perfect plasticity and a relatively wide damage zone around it can arise by the gradient-damage model. After a combination with inertial effects (and possibly a viscoelastic rheology, e.g., of Jeffreys type), this model involves seismic waves and can serve for earthquake simulations where these waves are emitted during fast rupture; cf. Remarks 2.3 and 2.4 below for some modifications of the presented model toward these applications. Another possible modification, although going beyond the scope of this paper, might use the structure of the stored energy similar to what is used in a phenomenological models for polycrystalline shape-memory alloys where our damage variable is in a position of temperature and plastic strain is a transformation strain subjected to some additional constraints; see, e.g., [23, Example 5.15].

The plan of the paper is as follows. In section 2 we formulate the model and cast a suitable definition of the weak solution, and we state a basic existence result which is proved later in section 3 by a constructive time discretization method. A further finite-

$d=2,3$ dimension of the problem, $\mathbb{R}^{d\times d}_{\rm dev}:=\{A\in\mathbb{R};\ {\rm tr}\,A=0\}$, $u:Q\to\mathbb{R}^d$ displacement, $\pi:Q\to\mathbb{R}^{d\times d}_{\rm dev}$ plastic strain, $\zeta:Q\to[0,1]$ damage variable, $a:\mathbb{R}\to\mathbb{R}^+$ damage-dissipation potential, $b:[0,1]\to\mathbb{R}$ stored energy of damage, $e_{\rm el}$ elastic strain, $e_{\rm el}=e(u)-\pi$, $e=e(u)=\frac12\nabla u^\top+\frac12\nabla u$ total small-strain tensor, $\mathbb{C}:[0,1]\to\mathbb{R}^{3^4}$ elasticity tensor dependent on ζ,	$\mathfrak{h}$ hyperstress (3rd-order) tensor, $\mathbb{H}$ a (small) hyperelasticity tensor, $S=\sigma_{\rm Y}(\cdot)B_1:[0,1]\rightrightarrows\mathbb{R}^{d\times d}_{\rm dev}$, with B_1 the unit ball in $\mathbb{R}^{d\times d}_{\rm dev}$, $\sigma_{\rm Y}:[0,1]\to\mathbb{R}^+$ plastic yield stress dependent on ζ, $g:Q\to\mathbb{R}^d$ applied bulk force, $w_{\rm D}:\Sigma_{\rm D}\to\mathbb{R}^d$ prescribed time-dependent boundary displacement, $f:\Sigma_{\rm N}\to\mathbb{R}^d$ applied traction force, $\kappa>0$ scale coefficient of the gradient of damage

FIG. 1. *Summary of the basic notation used thorough the paper.*

element discretization is then outlined. This allows for computer implementation of the model presented in section 4, whose efficiency and some physical aspects are demonstrated in section 5 on an illustrative example with geophysical motivation.

2. The model, its weak formulation, and existence result. Hereafter, we suppose that the damageable elastoplastic body occupies a bounded smooth domain $\Omega\subset\mathbb{R}^d$, $d=2$ or 3. We denote by $\vec{n}$ the outward unit normal to $\partial\Omega$. We further suppose that the boundary of Ω splits as

$$\partial\Omega:=\Gamma=\Gamma_{\rm D}\cup\Gamma_{\rm N}$$

with $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$ open subsets in the relative topology of $\partial\Omega$, disjoint from one an other, each of them with a smooth ($(d{-}1)$-dimensional) boundary, and covering $\partial\Omega$ up to $(d{-}1)$-dimensional zero measure. Considering $T>0$ a fixed time horizon, we set

$$Q:=(0,T)\times\Omega,\qquad \Sigma:=(0,T)\times\Gamma,\qquad \Sigma_{\rm D}:=(0,T)\times\Gamma_{\rm D},\qquad \Sigma_{\rm N}:=(0,T)\times\Gamma_{\rm N}.$$

Further, $\mathbb{R}^{d\times d}_{\rm sym}$ and $\mathbb{R}^{d\times d}_{\rm dev}$ will denote the set of symmetric or symmetric trace-free (= deviatoric) $(d{\times}d)$-matrices, respectively. For the reader's convenience, Figure 1 summarizes the basic notation used in what follows.

The *state* is formed by the triple $q:=(u,\pi,\zeta)$. Considering still a (small but fixed) regularizing parameter $\varepsilon>0$, the governing equation/inclusions read as

$$\text{(2.1a)}\qquad {\rm div}\big(\mathbb{C}(\zeta)e_{\rm el}-{\rm div}\,\mathfrak{h}\big)+g=0 \qquad \text{(momentum equilibrium)},$$
$$\text{with}\quad \mathfrak{h}=\mathbb{H}\nabla e_{\rm el}\ \text{ and }\ e_{\rm el}=e(u){-}\pi$$

$$\text{(2.1b)}\qquad \partial\delta^*_{S(\zeta)}(\dot\pi)\ni{\rm dev}\big(\mathbb{C}(\zeta)e_{\rm el}-{\rm div}\,\mathfrak{h}\big)\qquad \text{(plastic flow rule)},$$

$$\text{(2.1c)}\qquad \partial a(\dot\zeta)+\frac12\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el}$$
$$-\kappa\,{\rm div}\big((1{+}\varepsilon|\nabla\zeta|^{r-2})\nabla\zeta\big)+N_{[0,1]}(\zeta)\ni b'(\zeta)\qquad \text{(damage flow rule)}$$

with δ_S the indicator function to S and δ^*_S its convex conjugate, while $N_{[0,1]}(\cdot)$ is the normal cone to the indicated convex set, here the interval $[0,1]$. Moreover, $[\mathbb{C}(\zeta)e]_{ij}$ and $[\mathbb{H}\nabla e]_{ijk}$ mean $\sum_{k,l=1}^d\mathbb{C}_{ijkl}(\zeta)e_{kl}$ and $\sum_{m,n=1}^d\mathbb{H}_{ijmn}\frac{\partial}{\partial x_k}e_{mn}$, respectively.

We employed two regularizing terms with a regularizing tensor $\mathbb{H}$ and a regularizing parameter $\varepsilon>0$ with an exponent to be assumed suitably big, namely, $r>d$. This regularization facilitates analytical well-posedness of the problem and, because the gradient-damage term degenerates at $\nabla\zeta = 0$, its influence is presumably small if ε is small and $\nabla\zeta$ not too large. Moreover, $\mathbb{H}$ in (2.1a) prevents a complete damage at least when we assume $\mathbb{C}(\zeta)$ positive semidefinite. Actually, (2.1b) represents the thermodynamical-force balance governing damage evolution, while the corresponding flow rule is written rather in the (equivalent) form

$$\dot{\pi} \in N_{S(\zeta)}\Big(\mathrm{dev}\big(\mathbb{C}(\zeta)e_{\mathrm{el}} - \mathrm{div}\,\mathfrak{h}\big)\Big)$$

with N_S the set-valued normal-cone mapping; recall that by standard convex-analysis calculus, it holds that $[\partial\delta_S^*]^{-1} = \partial\delta_S^{**} = \partial\delta_S = N_S$. An analogous remark applies to (2.1c).

A remarkable attribute of this model is a damage-dependent yield-stress domain $S = S(\zeta)$. Typically, developing damage makes S smaller and vice versa, i.e., $S(\cdot) : [0,1] \rightrightarrows \mathbb{R}^{d\times d}_{\mathrm{dev}}$ is nondecreasing with respect to the ordering of subsets by inclusion. Likewise, typically also $b(\cdot)$ and $\mathbb{C}(\cdot)$ are nondecreasing, the later with respect to the Löwner's ordering, i.e., $\mathbb{C}(z_1) - \mathbb{C}(z_2)$ is positive semidefinite for $z_1 \geq z_2$. Rate-dependency of damage evolution prevents nonphysically too-early damaging/plastification and, due to the driving force $b'(\zeta)$, also allows simply for reverse damage evolution (a so-called healing) by using a convex function $a : \mathbb{R} \to \mathbb{R}^+$ in (2.1c) having naturally its minimum at 0. The microstructural interpretation of b is a stored energy related with microcracks/microvoids arising by damage, reflecting the fact that any surface in the bulk bears some extra energy. Minimization of this energy naturally leads to a tendency for healing of these material defects. Of course, (2.1) is to be completed by appropriate boundary conditions for (2.1a), (2.1c), e.g.,

$$u = w_{\mathrm{D}} \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{2.2a}$$

$$\big(\mathbb{C}(\zeta)e_{\mathrm{el}} - \mathrm{div}\,\mathfrak{h}\big)\cdot\vec{n} - \mathrm{div}_{\mathrm{S}}(\mathfrak{h}\vec{n}) = f \qquad \text{on } \Gamma_{\mathrm{N}}, \tag{2.2b}$$

$$\nabla\zeta\cdot\vec{n} = 0 \quad \text{and} \quad \mathfrak{h}{:}(\vec{n}\otimes\vec{n}) = 0 \qquad \text{on } \Gamma \tag{2.2c}$$

with $\vec{n}$ denoting the unit outward normal to Ω. Moreover, $\mathrm{div}_{\mathrm{S}}$ is the surface-divergence operator, which may be introduced as follows [26]: given a vector field $v : \Gamma \to \mathbb{R}^d$, we extend it to a neighborhood of Γ, and we let its surface gradient (valued in $\mathbb{R}^{d\times d}$) be defined as $\nabla_{\mathrm{S}} v = \mathbb{P}_{\mathrm{S}}\nabla v$, where $\mathbb{P}_{\mathrm{S}} = \mathbb{I} - \vec{n}\otimes\vec{n}$ is the projector on the tangent space of Γ; we then let the surface divergence of v be the scalar field $\mathrm{div}_{\mathrm{S}} v = \mathbb{P}_{\mathrm{S}} : \nabla_{\mathrm{S}} v = \mathrm{tr}(\mathbb{P}_{\mathrm{S}}\nabla v\mathbb{P}_{\mathrm{S}})$. Given a tensor field $\mathbb{A} : \Gamma \to \mathbb{R}^{d\times d}$, we let $\mathrm{div}_{\mathrm{S}}\mathbb{A} : \Gamma \to \mathbb{R}^d$ be the unique vector field such that $\mathrm{div}_{\mathrm{S}}(\mathbb{A}^T a) = a\cdot\mathrm{div}_{\mathrm{S}}\mathbb{A}$ for all constant vector fields $a : \Gamma \to \mathbb{R}^d$. Furthermore, the symbols "$\cdot$" and "$:$" denote a contraction between the one or two indices, respectively. Later, we will use "$\vdots$" for a contraction between three indices. Thus, componentwise, the second condition in (2.2b) reads as $\sum_{j,k=1}^d \mathfrak{h}_{ijk}n_j n_k = 0$.

Of course, an inhomogeneous variant of (2.2b) or some mixed Dirichlet/Neumann conditions in the normal/tangent conditions could be considered with straightforward modifications of the following text. We will consider an initial-value problem for (2.1)–(2.2) by asking for

$$u(0) = u_0, \qquad \pi(0) = \pi_0, \quad \text{and} \quad \zeta(0) = \zeta_0. \tag{2.3}$$

In fact, as $\dot{u}$ does not occur in (2.1), u_0 is rather formal and will essentially be determined by π_0 and ζ_0 via (2.14h) below.

The system (2.1) with the boundary conditions (2.2) has, in its weak formulation, the structure of an abstract Biot equation (or here rather inclusion):

$$\partial_{\dot q}\mathscr{R}(q;\dot q) + \partial\mathscr{E}(t,q) \ni 0 \tag{2.4}$$

with suitable time-dependent stored-energy functional $\mathscr{E}$ and the state-dependent (pseudo)potential of dissipative forces $\mathscr{R}$. Equally, one can write (2.4) as a generalized gradient flow

$$\dot q \in \partial_\xi\mathscr{R}^*\big(q;-\partial\mathscr{E}(t,q)\big), \tag{2.5}$$

where $\xi \mapsto \mathscr{R}^*(q;\xi)$ denotes the conjugate functional to $v \mapsto \mathscr{R}(q;v)$.

The perfect-plasticity model itself received considerable attention a long time ago; see, e.g., [7, 13, 16, 30, 40, 50]. The peculiarity is that the displacement no longer lives in the conventional Sobolev H^1-space but rather in the space of *functions with bounded deformations* introduced by Suquet [61], defined as

$$\mathrm{BD}(\bar\Omega;\mathbb{R}^d) := \big\{u\in L^1(\Omega;\mathbb{R}^d);\ e(u)\in\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\big\}, \tag{2.6}$$

where $\mathrm{Meas}(\bar\Omega) \cong C(\bar\Omega)^*$ denotes the space of Borel measures on the closure of Ω. The other notation we will use is rather standard: beside the standard notation for the Lebesgue L^p-space we already used in (2.6) for $p = 1$, we further use $W^{k,p}$ for Sobolev space whose kth derivatives are in L^p-spaces, the abbreviation $H^k = W^{k,2}$, and $L^p(0,T;X)$ for Bochner spaces of Bochner-measurable mappings $(0,T) \to X$ with X a Banach space. Also, $W^{k,p}(0,T;X)$ denotes the Banach space of mappings from $L^p(0,T;X)$ whose kth distributional derivative in time is also in $L^p(0,T;X)$. Further, $C([0,T];X)$ and $C_{\mathrm{weak}}([0,T];X)$ will denote the Banach space of continuous and weakly continuous mappings $[0,T] \to X$, respectively. Moreover, we denote by $\mathrm{BV}([0,T];X)$ the Banach space of the mappings $[0,T] \to X$ that have a bounded variation on $[0,T]$ and by $\mathrm{B}([0,T];X)$ the space of Bochner-measurable, everywhere defined, and bounded mappings $[0,T] \to X$.

After considering smooth time-dependent Dirichlet boundary conditions w_{D} on Σ_{D} which allows for an extension onto Q, let us denote it by u_{D}, such that

$$\big(\mathbb{C}(\zeta)e(u_{\mathrm{D}}) - \mathrm{div}\,\mathfrak{h}_{\mathrm{D}}\big)\cdot\vec n - \mathrm{div}_{\mathrm{S}}(\mathfrak{h}_{\mathrm{D}}\vec n) = f \qquad \text{on } \Gamma_{\mathrm{N}}, \tag{2.7a}$$

$$\mathfrak{h}_{\mathrm{D}}{:}(\vec n\otimes\vec n) = 0 \quad \text{with} \quad \mathfrak{h}_{\mathrm{D}} = \mathbb{H}\nabla e(u_{\mathrm{D}}) \qquad \text{on } \Gamma \tag{2.7b}$$

for any admissible ζ, and making a substitution of $u+u_{\mathrm{D}}$ instead of u into (2.1)–(2.2), we arrive to the problem with time-constant (even homogeneous) Dirichlet boundary conditions. More specifically,

$$e_{\mathrm{el}} \ \text{ in (2.1b) is replaced by } \ e_{\mathrm{el}} = e(u{+}u_{\mathrm{D}}){-}\pi \ \text{ and} \tag{2.8a}$$

$$w_{\mathrm{D}} \ \text{ in (2.2a) replaces by 0.} \tag{2.8b}$$

The state space is then the Banach space

$$\begin{aligned} U := \big\{&(u,\pi,\zeta)\in\mathrm{BD}(\bar\Omega;\mathbb{R}^d)\times\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})\times W^{1,r}(\Omega);\\ &e(u){-}\pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\quad u\odot\vec n\mathrm{d}S{+}\,\pi = 0 \ \text{ on } \ \Gamma_{\mathrm{D}}\big\}, \end{aligned} \tag{2.9a}$$

where $a \odot b$ means the symmetrized tensorial product $\frac{1}{2}(a \otimes b + b \otimes a)$, and the functionals governing the problem (2.4) leading to (2.1)–(2.2) with the substitution (2.8) are

$$
\mathscr{E}(t,u,\pi,\zeta) := \begin{cases} \displaystyle\int_\Omega \frac{1}{2}\mathbb{C}(\zeta)\big(e(u{+}u_{\rm D}(t)){-}\pi\big) : \big(e(u{+}u_{\rm D}(t)){-}\pi\big) \\ \displaystyle\quad + \frac{1}{2}\mathbb{H}\nabla(e(u{+}u_{\rm D}(t)){-}\pi) \vdots \nabla(e(u{+}u_{\rm D}(t)){-}\pi) \\ \displaystyle\quad - b(\zeta) - g(t){\cdot}u + \kappa\left(\frac{1}{2}|\nabla\zeta|^2{+}\frac{\varepsilon}{r}|\nabla\zeta|^r\right) {\rm d}x \\ \displaystyle\quad - \int_{\Gamma_{\rm N}} f(t){\cdot}u \,{\rm d}S \qquad\qquad \text{if } \zeta{\in}[0,1] \text{ a.e. on } \Omega, \\ \quad\ \infty \qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{2.9b}
$$

$$
\mathscr{R}(\zeta;\dot{\pi},\dot{\zeta}) := \int_{\bar{\Omega}} \big[\boldsymbol{\delta}^*_{S(\zeta)}(\dot{\pi})\big]({\rm d}x) + \int_\Omega a(\dot{\zeta})\,{\rm d}x, \tag{2.9c}
$$

where $\boldsymbol{\delta}^*_{S(\zeta)}$ denotes the conjugate to the indicator function $\boldsymbol{\delta}_{S(\zeta)}$ to the convex set $S(\zeta)$ and where the first integral in (2.9c) is an integral of a Borel measure; counting the assumption (2.14f) below, this measure is $\sigma_{\rm Y}(\zeta)|\dot{\pi}|$ with $|\dot{\pi}|$ the total variation of π with respect to time. The norm on U is

$$
\big\|(u,\pi,\zeta)\big\|_U := \|u\|_{L^1(\Omega;\mathbb{R}^d)} + \|e(u)\|_{{\rm Meas}(\bar{\Omega};\mathbb{R}^{d\times d}_{\rm sym})} \\ + \|\pi\|_{{\rm Meas}(\Omega;\mathbb{R}^{d\times d}_{\rm dev})} + \|e(u){-}\pi\|_{H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} + \|\zeta\|_{W^{1,r}(\Omega)}.
$$

The boundary condition $u \odot \vec{n}{\rm d}S + \pi = 0$ on $\Gamma_{\rm D}$ occurring in (2.9a) is a proper variational modification of the homogeneous Dirichlet condition on the displacement in the situation that the plastic strain may concentrate (and thus form a shear band) just on $\Gamma_{\rm D}$; cf. [13, section 2.3] for deep technical details about stress-strain duality in perfect plasticity. Note in particular that if π does not concentrate (i.e., vanishes) on $\Gamma_{\rm D}$, then $u \odot \vec{n} = 0$ yields the expected condition $u = 0$ on $\Gamma_{\rm D}$ because of the algebraic inequality $|a \odot b| \geq |a|\,|b|/\sqrt{2}$; cf. again [13].

We can now state the weak formulation of the initial-boundary-value problem (2.1)–(2.3). As for the plastic part, we use the concept of the so-called energetic solution devised by Mielke and Theil [44] (cf. also [41, 42]), based on the energy (in)equality and the so-called stability and further employed in the viscous context in [51] with the stability condition modified to a semistability; cf. (2.11a) below. Another feature of the following definition is that we rely on a regularity of the damage ζ so that ${\rm div}((1{+}\varepsilon|\nabla\zeta|^{r-2})\nabla\zeta)$ is in duality with $\dot{\zeta}$ and thus, in fact, the damage flow rule (2.1c) holds even a.e. Q. Actually, we do not need such regularity for the definition itself because the usual weak formulation of (2.1c), which would involve (not well controlled) $\nabla\dot{\zeta}$ resulting from usage of Green's formula, could still be treated by applying a by-part integration in time to get rid of the term $((1{+}\varepsilon|\nabla\zeta|^{r-2})\nabla\zeta){\cdot}\nabla\dot{\zeta}$. Rather, this regularity is essential for the energy conservation.

Definition 2.1 (weak solution). *The triple* (u,π,ζ) *with*

$$
u \in {\rm B}([0,T];{\rm BD}(\bar{\Omega};\mathbb{R}^d)), \tag{2.10a}
$$

$$
\pi \in {\rm B}([0,T];{\rm Meas}(\bar{\Omega};\mathbb{R}^{d\times d}_{\rm dev})) \cap {\rm BV}([0,T];{\rm Meas}(\bar{\Omega};\mathbb{R}^{d\times d}_{\rm dev})), \tag{2.10b}
$$

$$
\zeta \in {\rm B}([0,T];W^{1,r}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap C([0,T]{\times}\bar{\Omega}) \tag{2.10c}
$$

such that also

$$e_{\rm el} = e(u{+}u_{\rm D}) - \pi \in {\rm B}([0,T]; H^1(\Omega;\mathbb{R}^{d\times d})) \quad \textit{and} \tag{2.10d}$$

$$\mathrm{div}\big((1{+}\varepsilon|\nabla\zeta|^{r-2})\nabla\zeta\big) \in L^2(Q) \tag{2.10e}$$

is called a weak solution to the initial-boundary-value problem (2.1)–(2.3) *with the substitution* (2.8) *if*

(i) *the semistability*

$$\mathscr{E}(t,u(t),\pi(t),\zeta(t)) \le \mathscr{E}(t,\widetilde{u},\widetilde{\pi},\zeta(t)) + \mathscr{R}(\zeta(t);\widetilde{\pi}-\pi(t),0) \tag{2.11a}$$

holds for all $t\in[0,T]$ *and for all* $(\widetilde{u},\widetilde{\pi}) \in {\rm BD}(\bar{\Omega};\mathbb{R}^d){\times}{\rm Meas}(\bar{\Omega};\mathbb{R}^{d\times d}_{\rm dev})$ *with* $u\odot\vec{n}{\rm d}S+\pi=0$ *on* $\Gamma_{\rm D}$ *and with* $e(u){-}\pi{\in}H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$*,*

(ii) *the variational inequality*

$$\int_Q a(v) + \Big(\frac12\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el} - \kappa\,\mathrm{div}\big((1{+}\varepsilon|\nabla\zeta|^{r-2})\nabla\zeta\big) - b'(\zeta)+\xi\Big)(v-\dot\zeta)\,{\rm d}x\,{\rm d}t \ge \int_Q a(\dot\zeta)\,{\rm d}x\,{\rm d}t \tag{2.11b}$$

holds for all $v{\in}L^2(Q)$ *and some* $\xi{\in}L^2(Q)$ *such that* $\xi{\in}N_{[0,1]}(\zeta)$ *a.e. on* Q*,*

(iii) *the energy equality*

$$\mathscr{E}(T,u(T),\pi(T),\zeta(T)) + \int_{[0,T]\times\bar{\Omega}} \big[\boldsymbol{\delta}^*_{S(\zeta)}(\dot\pi)\big]({\rm d}x{\rm d}t) + \int_Q \widehat{a}(\dot\zeta)\,{\rm d}x{\rm d}t = \mathscr{E}(0,u_0,\pi_0,\zeta_0) + \int_0^T \partial_t\mathscr{E}(t,u(t),\pi(t),\zeta(t))\,{\rm d}t \tag{2.11c}$$

holds with $\widehat{a}:\mathbb{R}\to\mathbb{R}$ *being the single-valued, continuous function defined by* $\widehat{a}(z):=z\partial a(z)$.

(iv) *and also the initial conditions* (2.3) *hold.*

Let us note that, counting cancellation of some terms in $\mathscr{E}(t,u(t),\pi(t),\zeta(t)) - \mathscr{E}(t,\widetilde{u},\widetilde{\pi},\zeta(t))$, the semistability (2.11a) means that

$$\begin{aligned}\int_\Omega &\frac12\mathbb{C}(\zeta(t))\big(e(u(t){+}2u_{\rm D}(t)){-}\pi(t)(t)\big):\big(e(u(t)){-}\pi(t)\big)\\ &+\frac12\mathbb{H}\nabla\big(e(u(t){+}2u_{\rm D}(t)){-}\pi(t)(t)\big)\,\vdots\,\nabla\big(e(u(t)){-}\pi(t)\big)\,{\rm d}x\\ \le \int_\Omega &\frac12\mathbb{C}(\zeta(t))\big(e(\widetilde{u}{+}2u_{\rm D}(t)){-}\widetilde{\pi}\big):\big(e(\widetilde{u}){-}\widetilde{\pi}\big)\\ &+\frac12\mathbb{H}\nabla\big(e(\widetilde{u}{+}2u_{\rm D}(t)){-}\widetilde{\pi}\big)\,\vdots\,\nabla\big(e(\widetilde{u}){-}\widetilde{\pi}\big)\,{\rm d}x + \int_{\bar{\Omega}}\big[\boldsymbol{\delta}^*_{S(\zeta(t))}(\widetilde{\pi}{-}\pi(t))\big]({\rm d}x).\end{aligned} \tag{2.12}$$

The last integral (2.12) is not a Lebesgue integral but an integral according the measure $\boldsymbol{\delta}^*_{B_1}(\widetilde{\pi}{-}\pi(t))$. Due to the special ansatz (2.14f) below, this integral will be the total variation $|\widetilde{\pi}{-}\pi(t)|$, namely, $\int_{\bar{\Omega}}\sigma_{\rm Y}(\zeta(t))|\widetilde{\pi}{-}\pi(t)|({\rm d}x)$. Similarly, the integral on the left-hand side of (2.11c) equals $\int_{[0,T]\times\bar{\Omega}}\sigma_{\rm Y}(\zeta)|\dot\pi|({\rm d}x{\rm d}t)$. Further note that although traces of functions from ${\rm BD}(\bar{\Omega};\mathbb{R}^d)$ are in $L^1(\Gamma;\mathbb{R}^d)$, one has to be aware of jumps that can occur at the boundary, i.e., the measure $e(u)$ may concentrate on the boundary Γ. Thus, the classical boundary condition $u = 0$ on $\Gamma_{\rm D}$ arising by the additive shift (2.8b) is replaced by the more involved relation $u \odot \vec{n} {\rm d}S + \pi = 0$ on $\Gamma_{\rm D}$ in (2.9a). This relation has to be understood as an equality of measures on $\Gamma_{\rm D}$:

$$\forall \, \text{measurable } A \subset \Gamma_{\rm D}: \qquad \int_A u \odot \vec{n} {\rm d}S = \int_A {\rm d}\pi = \pi(A).$$

The relation simply means that any jump of u on the boundary has to be due to a localized plastic deformation. See. [13] for analytical details. Finally, let us comment on the last term in (2.11c) which, in view of (2.9b), involves the expression

$$\begin{aligned}\partial_t \mathscr{E}(t,u,\pi,\zeta) = \int_\Omega & \mathbb{C}(\zeta)\big(e(u{+}u_{\rm D}(t)){-}\pi\big) : e(\dot{u}_{\rm D}(t)) \\ & + \mathbb{H}\nabla(e(u{+}u_{\rm D}(t)){-}\pi) \vdots \nabla e(\dot{u}_{\rm D}(t)) - \dot{g}(t){\cdot}u \, {\rm d}x - \int_{\Gamma_{\rm N}} \dot{f}(t){\cdot}u \, {\rm d}S.\end{aligned} \tag{2.13}$$

Let us collect the assumptions on the data and on the loading we will rely on, some of them already mentioned above:

(2.14a) $\Omega \subset \mathbb{R}^d$ bounded C^2-domain, $\Gamma_{\rm D}$ has a $(d{-}2)$-dimensional C^2-boundary,

(2.14b) $a : \mathbb{R} \to \mathbb{R}$ convex, smooth on $\mathbb{R}\backslash\{0\}$, $a(0) = 0$, and
$\exists \, \epsilon > 0 \ \forall z{\in}\mathbb{R}: \ \ \epsilon|z|^2 \le a(z) \le (1{+}|z|^2)/\epsilon,$

(2.14c) $b : [0,1] \to \mathbb{R}$ continuously differentiable, nondecreasing, concave,

(2.14d) $\mathbb{C} : [0,1] \to \mathbb{R}^{d\times d\times d\times d}$ continuously differentiable, positive-semidefinite,
$\forall \, i,j,k,l = 1,\dots,d: \ \ \mathbb{C}_{ijkl}(\cdot) = \mathbb{C}_{jikl}(\cdot) = \mathbb{C}_{klij}(\cdot),$
$\forall \, e{\in}\mathbb{R}^{d\times d}_{\rm sym}: \ \mathbb{C}(\cdot)e{:}e : [0,1] \to \mathbb{R}$ nondecreasing, convex,
$\exists \, \mathbb{C}_{\rm D}(\zeta), \, c_{\rm S}(\zeta): \quad \mathbb{C}(\zeta)e : e = \mathbb{C}_{\rm D}(\zeta) {\rm dev}\, e : {\rm dev}\, e + c_{\rm S}(\zeta)({\rm tr}\, e)^2,$

(2.14e) $\mathbb{H}$ positive definite, $\mathbb{H}_{ijkl} = \mathbb{H}_{jikl} = \mathbb{H}_{klij}$,
$\exists \, \mathbb{H}_{\rm D}, \, H_{\rm S}: \qquad \mathbb{H}\nabla e \vdots \nabla e = \mathbb{H}_{\rm D} \nabla {\rm dev}\, e \vdots \nabla {\rm dev}\, e + H_{\rm S} \nabla {\rm tr}\, e \cdot \nabla {\rm tr}\, e,$

(2.14f) $S(\zeta) = \sigma_{\rm Y}(\zeta) B_1, \quad \sigma_{\rm Y} : [0,1] \to (0,\infty)$ continuous nondecreasing,
with $B_1 \subset \mathbb{R}^{d\times d}_{\rm dev}$ a unit ball,

(2.14g) $w_{\rm D} {\in} W^{1,1}(0,T;H^{3/2}(\Gamma_{\rm D};\mathbb{R}^d))$ and $\exists \, u_{\rm D} {\in} W^{1,1}(0,T;H^2(\Omega;\mathbb{R}^d))$
satisfying (2.7) and $u_{\rm D}|_{\Gamma_{\rm D}} = w_{\rm D}$,
$g \in W^{1,1}(0,T;L^1(\Omega;\mathbb{R}^d)), \quad f \in W^{1,1}(0,T;L^1(\Gamma_{\rm N};\mathbb{R}^d)),$
$\exists \, \sigma_{\rm SL} : [0,T] \to L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \ \exists \, \alpha > 0: \quad \sigma_{\rm SL}\vec{n} = f$ on $[0,T]{\times}\Gamma_{\rm N}$ and
${\rm div}\, \sigma_{\rm SL} + g = 0$ and $|{\rm dev}\, \sigma_{\rm SL}| \le \sigma_{\rm Y}(0) - \alpha$ on $[0,T]{\times}\Omega$,

(2.14h) $(u_0,\pi_0,\zeta_0) \in {\rm BD}(\bar\Omega;\mathbb{R}^d){\times}{\rm Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev}){\times}W^{1,r}(\Omega),$
$0 \le \zeta_0 \le 1$ a.e. on Ω, and
$\forall (\widetilde{u},\widetilde{\pi}) \in {\rm BD}(\bar\Omega;\mathbb{R}^d){\times}{\rm Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\rm dev}),$
$e(\widetilde{u}){-}\widetilde{\pi} {\in} H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym}), \quad \widetilde{u} \odot \vec{n} \, {\rm d}S {+} \widetilde{\pi} = 0$ on $\Gamma_{\rm D}$:
$\mathscr{E}(0,u_0,\pi_0,\zeta_0) \le \mathscr{E}(0,\widetilde{u},\widetilde{\pi},\zeta_0) + \mathscr{R}(\zeta_0;0,\widetilde{\pi}{-}\pi_0),$

(2.14i) $\kappa > 0, \quad \varepsilon > 0, \quad r > d.$

The smoothness assumption (2.14a) and the "elastic" invariance of the orthogonal subspaces of deviatoric and volumetric components (2.14d), (2.14e) copy the assumptions used in [13] for perfect plasticity in simple materials without damage in a variant with spatially varying yield stress as in [14, 17, 59]. The stress $\sigma_{\rm SL}$ in the condition (2.14g) qualifies the loading f and g in such a way that the infinite sliding of some parts of the body is excluded; this is a usual requirement called a safe-load condition, connected to perfect plasticity, here adopted to the situation that the yield stress $\sigma_{\rm Y}$ may vary with damage similarly as in [17, Remark 2.9]. It should also be noted that this safe-load condition works similarly for nonsimple materials. Further note that (2.14h) represents in particular the semistability of the initial condition and, with the other assumption, makes the energy conservation (2.11c) possible. Note also that (2.14b) ensures that $\widehat{a}$ used in (2.11c) is single-valued although a itself may be set-valued at 0. In (2.14f), one can easily consider a bit more general situation when B_1 would be convex, closed, and $0 \in \operatorname{int} B_1$.

The main analytical result justifying rigorously the model (2.1)–(2.3) is the following.

THEOREM 2.2. *Under the assumptions* (2.14)*, at least one weak solution to the initial-boundary-value problem* (2.1)*–*(2.3) *according to Definition* 2.1 *does exist.*

We will prove this existence result in section 3 by a constructive time discretization method (cf. Lemma 3.1 with Proposition 3.3), which later in sections 4 and 5 allows for efficient computer implementation of the model. The uniqueness of the solution, however, can hardly be expected.

Remark 2.3 (the dynamical model). During fast rupture, inertial effects may be not negligible and even sometimes an important aspect of the model. Then, (2.1a) augments by the inertial force $\varrho\ddot{u}$ with $\varrho > 0$ denoting the mass density as

$$\varrho\ddot{u} - \operatorname{div}\sigma = g \quad \text{ with } \quad \sigma = \mathbb{C}(\zeta)e_{\rm el} - \operatorname{div}\mathfrak{h}. \tag{2.15}$$

Relying on that the inertial term $\varrho\ddot{u}$ is controlled in the space $L^2(0,T;H^2(\Omega;\mathbb{R}^3)^*)$ $\cap\ C_{\rm weak}([0,T];L^2(\Omega;\mathbb{R}^3))$ or actually even in a slightly better space counting that $\operatorname{dev}\sigma \in L^\infty(Q;\mathbb{R}^{d\times d}_{\rm sym})$, the weak formulation of (2.15) arising by double by-part integration in time should accompany (2.11) with $\mathscr{E}$ augmented by the inertial energy $\int_\Omega \frac{\varrho}{2}|\dot{u}|^2{\rm d}x$ but with (2.11a) holding only a.e. on $[0,T]$ and (2.11c) only as an inequality. The functional in (3.5a) then augments by $\varrho\tau^{-2}|u-2u_\tau^{k-1}+u_\tau^{k-2}|^2/2$. Actually, it seems a matter of a physically explainable fact that some difficulties with the energy conservation occurs probably due to integration of elastic waves with nonlinearly responding shear bands even if a Kelvin–Voigt–type viscoelastic rheology would be involved; cf. also [53, Remark 6]. In this dynamical case, the fast damage phases and subsequent fast plastic slips, called (tectonic) *earthquakes*, typically emit elastic (seismic) waves. However, although with some justification on a theoretical level, the computational modeling requires fine special techniques to suppress, e.g., parasitic numerical attenuation and the direct combination of elastic waves with the inelastic processes is difficult.

Remark 2.4 (a non-Hookean model). The concept of nonsimple materials allows an important generalization that $\mathscr{E}(t,\cdot,\zeta,\cdot)$ is not quadratic and even nonconvex. More specifically, instead of the coercive term $(e_{\rm el},\zeta)\mapsto\mathbb{C}(\zeta)e_{\rm el}{:}e_{\rm el} = \frac{1}{2}\lambda(\zeta)I_1^2+\mu(\zeta)I_2$ as used also here in (4.1) below, [38] proposed

$$(e_{\rm el},\zeta)\mapsto\frac{1}{2}\lambda(\zeta)I_1^2+\mu(\zeta)I_2-\gamma(\zeta)I_1\sqrt{I_2} \quad \text{ with } \ I_1=\operatorname{tr}e_{\rm el}, \quad I_2=|e_{\rm el}|^2. \tag{2.16}$$

The elastic stress is then $(\lambda(\zeta)-\gamma(\zeta)\sqrt{I_2})\operatorname{tr}e_{\rm el} + (2\mu(\zeta)-\gamma(\zeta)I_1/\sqrt{I_2})e_{\rm el}$, while the driving stress for damage is $\sigma_{\rm dam} = \frac{1}{2}\lambda'(\zeta)I_1^2+\mu'(\zeta)I_2-\gamma'(\zeta)I_1/\sqrt{I_2}$ and can now be positive even without the contribution of the b-term. Such a model is widely used in geophysics, where it is believed to be responsible for instability of heavily damaged rocks and leads to healing even without the b-term used in our model, but where it is used without the $\mathbb{H}$-term and thus without any rigorous justification of such models; cf., e.g., [27, 39] and references there. To preserve coercivity of the model due to boundary conditions and the $\mathbb{H}$-term, one can think about a certain softening under very large strain by replacing 2-homogeneous form (2.16) by an energy with only a linear growth

$$(e_{\rm el},\zeta)\mapsto\frac{\lambda(\zeta)I_1^2+2\mu(\zeta)I_2-2\gamma(\zeta)I_1\sqrt{I_2}}{\sqrt{4+\epsilon I_2}} \tag{2.17}$$

with $\epsilon>0$ presumably small. A certain conceptual inconsistency remains in damage-dependence of $\mathbb{C}$ but not of $\mathbb{H}$, although $\mathbb{H}$ is assumed to be only small in applications. Note that (3.5a) then represents a coercive but nonconvex minimization problem and one should seek a global minimizer to ensure (3.9a). The nonsimple-material concept allows for a simple modification of the convergence proof in semistability and in the damage flow by compactness: more specifically, the binomial trick in (3.17) is applied only to the dissipation and the $\mathbb{H}$-terms, while (3.18) is even simpler because $\mathbb{C}'(\zeta)e_{\rm el}$ is now bounded in $L^\infty(\Omega;\mathbb{R}^{d\times d})$.

Remark 2.5 (a simple-material model). Considering $\mathbb{H}=0$ would bring various difficulties. In particular, the $L^2(Q)$-estimate of the driving force $\frac12\mathbb{C}'(\zeta)e_{\rm el}{:}e_{\rm el}$, which would need a regularity of $e_{\rm el}$ that, however, does not seem available for plasticity models without hardening, would become problematic. Note that the higher integrability of $e_{\rm el}\otimes e_{\rm el}$ will be used, e.g., in (3.18) and in (3.21) too. One should note that the alternative idea to consider a nonlinear damage independent contribution to the stress of the type $+\varepsilon|e_{\rm el}|^2e_{\rm el}$ would not allow use of the binomial trick in Step 3 in the proof of Proposition 3.3 below, while the strong convergence of $e_{\rm el}$ seems also not obvious to prove. A certain possibility might be in considering a visco–elastic Kelvin–Voigt model with the stress $\mathbb{D}(\zeta)\dot e_{\rm el}+\mathbb{C}(\zeta,e_{\rm el})$ with a nonlinear, monotone $\mathbb{C}(\zeta,\cdot)$ having at most the growth $|\mathbb{C}(\zeta,e_{\rm el})|\le C(1+|e_{\rm el}|^{1/2})$ so that $\int_0^1\partial_\zeta\mathbb{C}(\zeta,te_{\rm el})\,{\rm d}t$ can still be estimated in $L^2(Q)$ due to the $\mathbb{D}$-term which can even depend on ζ as in [43].

3. The discretization, its stability, and convergence. To implement the initial-boundary-value problem (2.1)–(2.3) computationally, we need to make a time and space discretization.

Let us first make only a time discretization with, for notational simplicity, a constant time step $\tau>0$. As the inertial effects are not considered and thus the system is only first-order in time, the dependence of $\tau>0$ on the time levels is easy to consider for numerical analysis and to implement (as actually used in section 4 below).

As $\mathscr{E}$ is convex in terms of (u,π) and separately in ζ too, and also as $\mathscr{R}$ additively splits $(\dot u,\dot\pi)$ from $\dot\zeta$, the natural *fractional-step strategy* leading to an efficient and numerically stable semi-implicit formula follows this splitting (u,π) from ζ. More specifically, it reads as

$$\operatorname{div}\Big(\mathbb{C}(\zeta_\tau^{k-1})e_{{\rm el},\tau}^k-\operatorname{div}\mathfrak{h}_\tau^k\Big)+g_\tau^k=0 \tag{3.1a}$$
$$\text{with}\quad e_{{\rm el},\tau}^k=e(u_\tau^k{+}u_{\rm D}(k\tau)){-}\pi_\tau^k,\qquad \mathfrak{h}_\tau^k=\mathbb{H}\nabla e_{{\rm el},\tau}^k,\qquad g_\tau^k:=g(k\tau),$$

$$N_{S(\zeta_\tau^{k-1})}\Big(\frac{\pi_\tau^k{-}\pi_\tau^{k-1}}{\tau}\Big)\ni\operatorname{dev}\Big(\mathbb{C}(\zeta_\tau^{k-1})e_{{\rm el},\tau}^k-\operatorname{div}\mathfrak{h}_\tau^k\Big), \tag{3.1b}$$

$$\partial a\Big(\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)+\frac{1}{2}\mathbb{C}'(\zeta_\tau^k)e_{\mathrm{el},\tau}^k:e_{\mathrm{el},\tau}^k - \kappa\,\mathrm{div}\big((1+\varepsilon|\nabla\zeta_\tau^k|^{r-2})\nabla\zeta_\tau^k\big)+N_{[0,1]}(\zeta_\tau^k)\ni b'(\zeta_\tau^k), \tag{3.1c}$$

together with the corresponding boundary conditions

$$u_\tau^k=0 \qquad \text{on } \Gamma_{\rm D}, \tag{3.2a}$$

$$\big(\mathbb{C}(\zeta_\tau^{k-1})e_{\mathrm{el},\tau}^k-\mathrm{div}\,\mathfrak{h}_\tau^k\big)\cdot\vec{n}-\mathrm{div}_{\rm S}(\mathfrak{h}_\tau^k\vec{n})=f_\tau^k \qquad \text{on } \Gamma_{\rm N} \ \text{ with } \ f_\tau^k:=f(k\tau), \tag{3.2b}$$

$$\nabla\zeta_\tau^k\cdot\vec{n}=0 \quad \text{and} \quad \mathfrak{h}_\tau^k{:}(\vec{n}\otimes\vec{n})=0 \qquad \text{on } \Gamma, \tag{3.2c}$$

to be solved first for (u_τ^k,π_τ^k) from (3.1a), (3.1b) with (3.2a), (3.2b) and then for ζ_τ^k from (3.1c)–(3.2c) recursively for $k=1,\dots,T/\tau$. Both of these boundary-value problems have potentials and thus lead to minimization problems. Moreover, as $\mathbb{C}'$ and $-b'$ are nondecreasing (again with respect to the Löwner's ordering) and a is convex as assumed in (2.14), both of these boundary-value problems lead to convex variational problems; cf. (3.5) below.

Let us define the piecewise affine interpolant u_τ by

$$u_\tau(t):=\frac{t-(k{-}1)\tau}{\tau}u_\tau^k+\frac{k\tau-t}{\tau}u_\tau^{k-1} \qquad \text{for } t\in[(k{-}1)\tau,k\tau] \tag{3.3a}$$

with $k=0,\dots,T/\tau$. Besides, we define also the left-continuous piecewise constant interpolant $\overline{u}_\tau$ and the right-continuous piecewise constant interpolant $\underline{u}_\tau$ by

$$\overline{u}_\tau(t):=u_\tau^k \qquad \text{for } t\in((k{-}1)\tau,k\tau]\ , \ \ k=1,\dots,T/\tau, \tag{3.3b}$$

$$\underline{u}_\tau(t):=u_\tau^{k-1} \qquad \text{for } t\in[(k{-}1)\tau,k\tau)\ , \ \ k=1,\dots,T/\tau. \tag{3.3c}$$

Similarly, we define also π_τ, $\overline{\pi}_\tau$, $\underline{\pi}_\tau$, $\overline{\zeta}_\tau$, ζ_τ, $\bar{g}_\tau$, etc.

LEMMA 3.1 (existence and stability of discrete solutions). *The recursive boundary-value problem* (3.1)–(3.2) *has a weak solution* $(u_\tau^k,\pi_\tau^k,\zeta_\tau^k)$ *with* $u_\tau^k\in\mathrm{BD}(\bar\Omega;\mathbb{R}^d)$, $\pi_\tau^k\in W^{1,r}(\Omega)$, *and* $\zeta_\tau^k\in\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ *with* $e_{\mathrm{el},\tau}^k=e(u_\tau^k)-\pi_\tau^k\in H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *satisfying the a priori estimates*

$$\big\|\overline{u}_\tau\big\|_{L^\infty(0,T;\mathrm{BD}(\bar\Omega;\mathbb{R}^d))}\le C, \tag{3.4a}$$

$$\big\|\overline{\pi}_\tau\big\|_{L^\infty(0,T;\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))\,\cap\,\mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))}\le C, \tag{3.4b}$$

$$\big\|e(u_\tau){-}\pi_\tau\big\|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))}\le C, \tag{3.4c}$$

$$\big\|\zeta_\tau\big\|_{L^\infty(0,T;W^{1,r}(\Omega))\,\cap\,H^1(0,T;L^2(\Omega))}\le C, \tag{3.4d}$$

$$\big\|\mathrm{div}((1{+}\varepsilon|\nabla\overline{\zeta}_\tau|^{r-2})\nabla\overline{\zeta}_\tau)\big\|_{L^2(Q)}\le C. \tag{3.4e}$$

Proof. The existence of weak solutions to (3.1) can be justified by the direct method when realizing the variational structure: the boundary-value problem (3.1a), (3.1b) with (3.2a), (3.2b) represents a minimization problem

$$\left\{\begin{array}{ll}\text{Minimize} & (u,\pi)\mapsto\mathscr{E}(k\tau,u,\pi,\zeta_\tau^{k-1})+\mathscr{R}(\zeta_\tau^{k-1};\pi{-}\pi_\tau^{k-1},0)\\ \text{subject to} & u\in\mathrm{BD}(\bar\Omega;\mathbb{R}^d),\quad \pi\in\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}),\\ & e(u){-}\pi\in H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\quad u\odot\vec{n}\mathrm{d}S{+}\,\pi=0 \text{ on } \Gamma_{\rm D},\end{array}\right. \tag{3.5a}$$

while the boundary-value problem (3.1c)–(3.2c) represents a minimization problem

$$\text{(3.5b)}\qquad \begin{cases} \text{Minimize} & \zeta\mapsto \mathscr{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta)+\tau\mathscr{R}\Big(\zeta_\tau^{k-1};0,\dfrac{\zeta-\zeta_\tau^{k-1}}{\tau}\Big)\\ \text{subject to} & \zeta\in W^{1,r}(\Omega),\quad 0\le\zeta\le 1\ \text{ on }\Omega,\end{cases}$$

whose solutions do exist by coercivity, convexity, and lower semicontinuity arguments. Here the safe-load qualification (2.14g) of f and g is to be used.

Further, we test (3.1), respectively, by $u_\tau^k-u_\tau^{k-1}$, $\pi_\tau^k-\pi_\tau^{k-1}$, and $\zeta_\tau^k-\zeta_\tau^{k-1}$. Relying on the convexity of $\mathscr{E}(k\tau,\cdot,\cdot,\zeta_\tau^{k-1})$ and of $\mathscr{E}(k\tau,u_\tau^k,\pi_\tau^k,\cdot)$, we obtain the estimates

$$\text{(3.6a)}\quad \mathscr{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1})+\int_{\bar\Omega}\sigma_{\rm Y}(\zeta_\tau^{k-1})|\pi_\tau^k-\pi_\tau^{k-1}|({\rm d}x)\le \mathscr{E}(k\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1}),$$

$$\text{(3.6b)}\quad \mathscr{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^k)+\tau\int_\Omega \widehat a\Big(\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)\,{\rm d}x\le \mathscr{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1})$$

with $\widehat a$ from (2.11c). By summing these estimates, we can enjoy the cancellation of the terms $\mathscr{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^{k-1})$ in (3.6a) and (3.6b), and we thus obtain

$$\begin{aligned}\text{(3.7)}\quad &\mathscr{E}(k\tau,u_\tau^k,\pi_\tau^k,\zeta_\tau^k)+\tau\widehat{\mathscr{R}}\Big(\zeta_\tau^{k-1};\frac{\pi_\tau^k-\pi_\tau^{k-1}}{\tau},\frac{\zeta_\tau^k-\zeta_\tau^{k-1}}{\tau}\Big)\\ &\le \mathscr{E}(k\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})\\ &=\mathscr{E}((k-1)\tau,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})+\int_{(k-1)\tau}^{k\tau}\partial_t\mathscr{E}(t,u_\tau^{k-1},\pi_\tau^{k-1},\zeta_\tau^{k-1})\,{\rm d}t\end{aligned}$$

with the dissipation rate $\widehat{\mathscr{R}}$ defined as

$$\text{(3.8)}\qquad \widehat{\mathscr{R}}(\zeta;\dot\pi,\dot\zeta):=\int_{\bar\Omega}\sigma_{\rm Y}(\zeta)|\dot\pi|({\rm d}x)+\int_\Omega\widehat a(\dot\zeta)\,{\rm d}x\qquad\text{with }\ \widehat a(\dot\zeta)=\dot\zeta\partial a(\dot\zeta).$$

By summing (3.7) over k we enjoy a "telescopic" cancellation effect. Realizing (2.13) and (2.14g), by the discrete Gronwall inequality, we obtain (3.4a)–(3.4d).

Having estimated $\partial a(\dot\zeta_\tau)+\frac12\mathbb{C}'(\overline\zeta)\overline e_{{\rm el},\tau}:\overline e_{{\rm el},\tau}-b'(\overline\zeta_\tau)$ as a bounded set in $L^2(Q)$ uniformly with respect to $\tau>0$, we can estimate also ${\rm div}((1{+}\varepsilon|\nabla\zeta_\tau^k|^{r-2})\nabla\zeta_\tau^k)$ in the same space. For this, we test (3.1c) by $-{\rm div}((1{+}\varepsilon|\nabla\zeta_\tau^k|^{r-2})\nabla\zeta_\tau^k)$. Here, the important ingredient is, written rather formally, the estimate

$$\begin{aligned}&\int_\Omega N_{[0,1]}(\zeta_\tau^k)\big(-{\rm div}((1{+}\varepsilon|\nabla\zeta_\tau^k|^{r-2})\nabla\zeta_\tau^k)\big)\,{\rm d}x\\ &\qquad=-\int_\Omega\partial\boldsymbol{\delta}_{[0,1]}(\zeta_\tau^k)\big({\rm div}((1{+}\varepsilon|\nabla\zeta_\tau^k|^{r-2})\nabla\zeta_\tau^k)\big)\,{\rm d}x\\ &\qquad=\int_\Omega\nabla\big(\partial\boldsymbol{\delta}_{[0,1]}(\zeta_\tau^k)\big)\cdot(1{+}\varepsilon|\nabla\zeta_\tau^k|^{r-2})\nabla\zeta_\tau^k\,{\rm d}x\\ &\qquad=\int_\Omega\partial^2\boldsymbol{\delta}_{[0,1]}(\zeta_\tau^k)\cdot\nabla\zeta_\tau^k\cdot(1{+}\varepsilon|\nabla\zeta_\tau^k|^{r-2})\nabla\zeta_\tau^k\,{\rm d}x\ge 0,\end{aligned}$$

which is due to the positive-semidefiniteness of the (generalized) Jacobian $\partial^2\boldsymbol{\delta}_{[0,1]}$ of the convex function $\boldsymbol{\delta}_{[0,1]}$ and which is to be proved rigorously by a mollification of $\boldsymbol{\delta}_{[0,1]}$; cf. [55, Lemma 1] for technical details. Thus we obtain (3.4e). ∎

LEMMA 3.2 (discrete analogue of (2.11)). *With the notation* (3.3) *and* $\overline e_{{\rm el},\tau}=e(\overline u_\tau{+}\overline u_{{\rm D},\tau})-\overline\pi_\tau$, *the discrete solution obtained by the recursive scheme* (3.1)–(3.2) *satisfies*

$$\mathscr{E}(t,\overline{u}_\tau(t),\overline{\pi}_\tau(t),\underline{\zeta}_\tau(t)) \le \mathscr{E}(t,\widetilde{u},\widetilde{\pi},\underline{\zeta}_\tau(t)) + \mathscr{R}(\underline{\zeta}_\tau(t);\widetilde{\pi}-\overline{\pi}_\tau(t),0) \tag{3.9a}$$

for all $t\in[0,T]$ *and all* $(\widetilde{u},\widetilde{\pi})$ *as in* (2.11a)*, and*

$$\int_Q a(v) + \Big(\frac{1}{2}\mathbb{C}'(\underline{\zeta}_\tau)\overline{e}_{\mathrm{el},\tau} : \overline{e}_{\mathrm{el},\tau} - \kappa\,\mathrm{div}\big((1{+}\varepsilon|\nabla\overline{\zeta}_\tau|^{r-2})\nabla\overline{\zeta}_\tau\big) - b'(\overline{\zeta}_\tau) + \overline{\xi}_\tau\Big)(v-\dot{\zeta}_\tau)\,\mathrm{d}x\,\mathrm{d}t \ge \int_Q a(\dot{\zeta}_\tau)\,\mathrm{d}x\,\mathrm{d}t \tag{3.9b}$$

holds for all $v\in L^2(Q)$ *and for some* $\overline{\xi}_\tau\in L^2(Q)$ *such that* $\overline{\xi}_\tau\in N_{[0,1]}(\overline{\zeta}_\tau)$ *a.e. on* Q*, and eventually the energy (im)balance holds:*

$$\mathscr{E}(T,u_\tau(T),\pi_\tau(T),\zeta_\tau(T)) + \int_0^T \widehat{\mathscr{R}}(\underline{\zeta}_\tau;\dot{\pi}_\tau,\dot{\zeta}_\tau)\,\mathrm{d}t \le \mathscr{E}(0,u_0,\pi_0,\zeta_0) + \int_0^T \partial_t\mathscr{E}(t,\underline{u}_\tau(t),\underline{\pi}_\tau(t),\underline{\zeta}_\tau(t))\,\mathrm{d}t \tag{3.9c}$$

with the overall dissipation rate $\widehat{\mathscr{R}}$ *from* (3.8)*. Moreover, the a priori estimate holds:*

$$\big\|\overline{\xi}_\tau\big\|_{L^2(Q)} \le C. \tag{3.10}$$

Proof. The boundary-value problem (3.1a,b)–(3.2a,b) represents a minimization problem (3.5a) at the level k. As (u_τ^k,π_τ^k) is its minimizer, any other state, in particular $(u_\tau^{k-1},\pi_\tau^{k-1})$, must yield equal of greater values. Then, by using a triangle inequality facilitated by the 1-homogeneity of $\mathscr{R}(\zeta;\cdot,\dot{\zeta})$, we obtain (3.9a); actually, this is a standard argument in the theory of rate-independent processes [41, 42, 44].

In the case of the boundary-value problem (3.1c)–(3.2c), the variational inequality (3.9b) represents just the conventional weak formulation of the minimization problem (3.5b) summed for all time levels. Then, (3.9c) follows by summing (3.7) for $k=1,\dots,T/\tau$.

Eventually, the estimate (3.10) follows by comparison from the inclusion $\overline{\xi}_\tau\in b'(\overline{\zeta}_\tau)-\frac{1}{2}\mathbb{C}'(\underline{\zeta}_\tau)\overline{e}_{\mathrm{el},\tau}:\overline{e}_{\mathrm{el},\tau}+\kappa\,\mathrm{div}((1{+}\varepsilon|\nabla\overline{\zeta}_\tau|^{r-2})\nabla\overline{\zeta}_\tau)-\partial a(\dot{\zeta}_\tau)$ and by the already obtained estimates. ∎

PROPOSITION 3.3 (convergence). *Let the assumptions* (2.14) *be satisfied and the approximate solution* $(\overline{u}_\tau,\overline{\pi}_\tau,\overline{\zeta}_\tau,\overline{\xi}_\tau)$ *be obtained by the recursive scheme* (3.1)–(3.2)*. Then there is a subsequence and* (u,π,ζ,ξ) *such that*

$$\overline{u}_\tau(t)\to u(t) \quad \text{weakly* in } \mathrm{BD}(\bar{\Omega};\mathbb{R}^d), \tag{3.11a}$$

$$\overline{\pi}_\tau(t)\to \pi(t) \quad \text{weakly* in } \mathrm{Meas}(\bar{\Omega};\mathbb{R}^{d\times d}_{\mathrm{dev}}), \tag{3.11b}$$

$$\overline{e}_{\mathrm{el},\tau}(t)=e(\overline{u}_\tau(t))-\overline{\pi}_\tau(t)\to e(u(t))-\pi(t)=e_{\mathrm{el}}(t) \quad \text{weakly in } H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{3.11c}$$

$$\overline{\zeta}_\tau(t)\to\zeta(t) \quad \text{and} \quad \underline{\zeta}_\tau(t)\to\zeta(t) \quad \text{weakly in } W^{1,r}(\Omega) \tag{3.11d}$$

holding for any $t\in[0,T]$*, and further also*

$$\underline{\zeta}_\tau\to\zeta \quad \text{strongly in } L^\infty(Q) \text{ and} \tag{3.11e}$$

$$\overline{\xi}_\tau\to\xi \quad \text{weakly in } L^2(Q) \tag{3.11f}$$

with $\overline{\xi}_\tau$ *from Lemma* 3.2*. Moreover, any* (u, π, ζ) *obtained by such a way is a weak solution according Definition* 2.1 *with* ξ *in* (2.11b) *taken from* (3.11f)*.*

Proof. For clarity of exposition, we divide the proof into five particular steps.

Step 1: *Selection of a converging subsequence.* By Banach's selection principle, we select a weakly* converging subsequence with respect to the norms from the estimates (3.4) and (3.10); namely, for some u, π, ζ, and ξ we have

(3.12a) $\overline{u}_\tau \to u$ weakly* in $L^\infty(0,T;\mathrm{BD}(\bar\Omega;\mathbb{R}^d))$,

(3.12b) $\overline{\pi}_\tau \to \pi$ weakly* in $L^\infty(0,T;\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})) \cap \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}))$,

(3.12c) $\overline{e}_{\mathrm{el},\tau} = e(\overline{u}_\tau) - \overline{\pi}_\tau \to e_{\mathrm{el}} = e(u) - \pi$ weakly* in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$,

(3.12d) $\zeta_\tau \to \zeta$ weakly* in $L^\infty(0,T;W^{1,r}(\Omega)) \cap H^1(0,T;L^2(\Omega))$,

(3.12e) $\mathrm{div}((1+\varepsilon|\nabla\overline{\zeta}_\tau|^{r-2})\nabla\overline{\zeta}_\tau) \to \mathrm{div}((1+\varepsilon|\nabla\zeta|^{r-2})\nabla\zeta)$ weakly in $L^2(Q)$,

(3.12f) $\overline{\xi}_\tau \to \xi$ weakly in $L^2(Q)$;

actually, (3.12e) uses also the maximal monotonicity of the involved nonlinear operator. Moreover, by the BV-estimates and the Helly's selection principle, we can also count with (3.11b) and $\overline{\zeta}_\tau(t) \to \zeta(t)$ weakly in $L^2(\Omega)$, and then by the a priori $W^{1,r}$-estimate (3.4d) also both the first and the second convergence in (3.11d); both limits in (3.11d) are actually the same because the limit ζ is continuous in time into $L^2(\Omega)$ due to the a priori H^1-estimate (3.4d).

By the compact embedding $W^{1,r}(\Omega) \Subset C(\bar\Omega)$ and by the Arzelà–Ascoli modification of the Aubin–Lions theorem (cf. [52, Lemma 7.10]), we have the compact embedding $C_{\mathrm{weak}}([0,T];W^{1,r}(\Omega)) \cap H^1(0,T;L^2(\Omega)) \Subset C([0,T];C(\bar\Omega)) = C(\bar Q)$. Thus, from the estimate (3.4d), we obtain $\zeta_\tau \to \zeta$ in $C(\bar Q)$. Further, we have

$$
\begin{aligned}
(3.13)\qquad \big\|\underline{\zeta}_\tau - \zeta_\tau\big\|^2_{L^\infty(0,T;L^2(\Omega))} &= \sup_{0\le t\le T}\int_\Omega \big|\underline{\zeta}_\tau(t,x) - \zeta_\tau(t,x)\big|\,\mathrm{d}x \\
&\le \int_\Omega \Big(\sup_{0\le t\le T}|\underline{\zeta}_\tau(t,x) - \zeta_\tau(t,x)|^2\Big)\mathrm{d}x \\
&= \int_\Omega \max_{k=1,\dots,T/\tau}\big|\zeta_\tau^k - \zeta_\tau^{k-1}\big|^2\mathrm{d}x \le \int_\Omega \sum_{i=1}^{T/\tau}\big|\zeta_\tau^k - \zeta_\tau^{k-1}\big|^2\mathrm{d}x \\
&= \int_\Omega \tau\sum_{i=1}^{T/\tau}\tau\Big|\frac{\zeta_\tau^k - \zeta_\tau^{k-1}}{\tau}\Big|^2\mathrm{d}x = \tau\int_Q\big|\dot\zeta_\tau\big|^2\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Then, using the Gagliardo-Nirenberg inequality $\|z\|_{L^\infty(\Omega)} \le C_\varepsilon\|z\|^\varepsilon_{L^2(\Omega)}\|z\|^{1-\varepsilon}_{W^{1,r}(\Omega)}$ for some small $0<\varepsilon<1$ depending on $r>d$, we can interpolate (3.13), i.e., $\|\underline{\zeta}_\tau - \zeta_\tau\|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{\tau}\|\dot\zeta_\tau\|_{L^2(Q)}$, with $\|\underline{\zeta}_\tau - \zeta_\tau\|_{L^\infty(0,T;W^{1,r}(\Omega))} \le C$ to obtain $\|\underline{\zeta}_\tau - \zeta_\tau\|_{L^\infty(Q)} \to 0$. Thus (3.11e) is proved.

Step 2: *Energy inequality.* The convergence (3.12) allows already for passage in the limit in the inequality (3.9c) by lower semicontinuity in the left-hand side and by continuity in the right-hand side of (3.9c).

The limit passage in $\mathscr{E}(T, u_\tau(T), \pi_\tau(T), \zeta_\tau(T))$ is by the convexity of $\mathscr{E}(T,\cdot,\cdot,\zeta)$ and the compactness in ζ, while for $\int_0^T \partial_t\mathscr{E}(t, \underline{u}_\tau(t), \underline{\pi}_\tau(t), \underline{\zeta}_\tau(t))\,\mathrm{d}t$ we use the continuity of $\partial_t\mathscr{E}(t,\cdot,\cdot,\cdot)$ from (2.13) and the Lebesgue theorem; in more detail, we use the assumptions (2.14g) and the weak convergence (3.11c).

The only remaining (and nontrivial) term is the dissipation $\widehat{\mathscr{R}}$-term. Let us note that, as the discrete flow rule $N_{S(\underline{\zeta}_\tau)}(\dot{\pi}_\tau) \ni \mathrm{dev}(\mathbb{C}(\underline{\zeta}_\tau)\overline{e}_{\mathrm{el},\tau} - \mathrm{div}\,\mathfrak{h}_\tau^k)$ as well as the dissipation rate $\sigma_{\mathrm{Y}}(\underline{\zeta}_\tau)|\dot{\pi}_\tau|$ uses $\underline{\zeta}_\tau$ and not just ζ_τ, we needed to prove (3.11e) in Step 1. Therefore, we have at our disposal the estimate

$$\big\|(\sigma_{\mathrm{Y}}(\underline{\zeta}_\tau) - \sigma_{\mathrm{Y}}(\zeta))|\dot{\pi}_\tau|\big\|_{\mathrm{Meas}(\bar{Q})} \le \ell_{\sigma_{\mathrm{Y}}}\big\|\underline{\zeta}_\tau - \zeta\big\|_{L^\infty(Q)}\big\|\dot{\pi}_\tau\big\|_{\mathrm{Meas}(\bar{Q})} \to 0 \tag{3.14}$$

with $\ell_{\sigma_{\mathrm{Y}}}$ the modulus of Lipschitz continuity of σ_{Y} on $[0,1]$; cf. the assumption (2.14f). Then, using also $\zeta_\tau \to \zeta$ in $C(\overline{Q})$ already proved, we obtain

$$\begin{aligned}\liminf_{\tau\to 0}\int_0^T \widehat{\mathscr{R}}(\underline{\zeta}_\tau;\dot{\pi}_\tau,\dot{\zeta}_\tau)\,\mathrm{d}t &= \liminf_{\tau\to 0}\int_{\bar{Q}}\sigma_{\mathrm{Y}}(\underline{\zeta}_\tau)\big|\dot{\pi}_\tau\big|(\mathrm{d}x\mathrm{d}t)\\ &= \lim_{\tau\to 0}\int_{\bar{Q}}\big(\sigma_{\mathrm{Y}}(\underline{\zeta}_\tau)-\sigma_{\mathrm{Y}}(\zeta)\big)\big|\dot{\pi}_\tau\big|(\mathrm{d}x\mathrm{d}t) + \liminf_{\tau\to 0}\int_{\bar{Q}}\sigma_{\mathrm{Y}}(\zeta)\big|\dot{\pi}_\tau\big|(\mathrm{d}x\mathrm{d}t)\\ &\ge\ \ 0\ +\int_{\bar{Q}}\sigma_{\mathrm{Y}}(\zeta)\big|\dot{\pi}\big|(\mathrm{d}x\mathrm{d}t);\end{aligned} \tag{3.15}$$

for the used weak* lower semicontinuity of $\dot{\pi} \mapsto \int_{\bar{Q}}\sigma_{\mathrm{Y}}(\zeta)|\dot{\pi}|(\mathrm{d}x\mathrm{d}t)$ we refer, e.g., to [6, 20].

Step 3: *Limit passage in the semistability* (3.9a) *toward* (2.11a). For any $(\widetilde{u},\widetilde{\pi})$ used in (3.9a), we have to find at least one so-called mutual recovery sequence $\{(\widehat{u}_\tau,\widehat{\pi}_\tau)\}_{\tau>0}$ in the sense that

$$\begin{aligned}\limsup_{\tau\to 0}\mathscr{E}(t,\widetilde{u}_\tau,\widetilde{\pi}_\tau,\underline{\zeta}_\tau(t)) + \mathscr{R}(\underline{\zeta}_\tau(t);\widetilde{\pi}_\tau-\overline{\pi}_\tau(t),0) - \mathscr{E}(t,\overline{u}_\tau(t),\overline{\pi}_\tau(t),\underline{\zeta}_\tau(t))\\ \le \mathscr{E}(t,\widetilde{u},\widetilde{\pi},\zeta(t)) + \mathscr{R}(\underline{\zeta}_\tau(t);\widetilde{\pi}-\pi(t),0) - \mathscr{E}(t,u(t),\pi(t),\zeta(t)).\end{aligned}$$

We choose

$$\widetilde{u}_\tau = \overline{u}_\tau(t) + \widetilde{u} - u(t) \qquad\text{and}\qquad \widetilde{\pi}_\tau = \overline{\pi}_\tau(t) + \widetilde{\pi} - \pi(t). \tag{3.16}$$

Then, by using the cancellation and the binomial formula of the type $a^2 - b^2 = (a{+}b)(a{-}b)$ here in the form $\mathbb{C}\widetilde{e}{:}\widetilde{e} - \mathbb{C}e{:}e = \mathbb{C}(\widetilde{e}{+}e){:}(\widetilde{e}{-}e)$ and $\mathbb{H}\nabla\widetilde{e}\,\vdots\,\nabla\widetilde{e} - \mathbb{H}\nabla e\,\vdots\,\nabla e = \mathbb{H}\nabla(\widetilde{e}{+}e)\,\vdots\,\nabla(\widetilde{e}{-}e)$ (cf. (2.12)), and by making the substitution (3.16), we have

$$\begin{aligned}&\lim_{\tau\to 0}\mathscr{E}(t,\widetilde{u}_\tau,\widetilde{\pi}_\tau,\underline{\zeta}_\tau(t)) + \mathscr{R}(\underline{\zeta}_\tau(t);\widetilde{\pi}_\tau-\overline{\pi}_\tau(t),0) - \mathscr{E}(t,\overline{u}_\tau(t),\overline{\pi}_\tau(t),\underline{\zeta}_\tau(t))\\ &\quad= \lim_{\tau\to 0}\bigg(\int_\Omega \frac{1}{2}\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\widetilde{u}_\tau{+}\overline{u}_\tau(t){+}2u_{\mathrm{D}}(t)){-}\widetilde{\pi}_\tau{-}\overline{\pi}_\tau(t)\big)\\ &\qquad\qquad : \big(e(\widetilde{u}_\tau{-}\overline{u}_\tau(t)){-}\widetilde{\pi}_\tau{+}\overline{\pi}_\tau(t)\big) - g(t)\cdot(\widetilde{u}_\tau{-}\overline{u}_\tau(t))\\ &\qquad + \frac{1}{2}\mathbb{H}\nabla\big(e(\widetilde{u}_\tau{+}\overline{u}_\tau(t){+}2u_{\mathrm{D}}(t)){-}\widetilde{\pi}_\tau{-}\overline{\pi}_\tau(t)\big)\\ &\qquad\qquad \vdots\,\nabla\big(e(\widetilde{u}_\tau{-}\overline{u}_\tau(t)){-}\widetilde{\pi}_\tau{+}\overline{\pi}_\tau(t)\big)\,\mathrm{d}x\\ &\qquad + \int_{\bar{\Omega}}\big[\sigma_{\mathrm{Y}}(\underline{\zeta}_\tau(t))\big|\widetilde{\pi}_\tau{-}\overline{\pi}_\tau(t)\big|(\mathrm{d}x) - \int_{\Gamma_{\mathrm{N}}} f(t)\cdot(\widetilde{u}_\tau{-}\overline{u}_\tau(t))\,\mathrm{d}S\bigg)\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&= \lim_{\tau\to 0}\bigg(\int_\Omega \frac12\mathbb{C}(\underline{\zeta}_\tau(t))\big(e(\widetilde u_\tau+\overline u_\tau(t)+2u_{\rm D}(t))-\widetilde\pi_\tau-\overline\pi_\tau(t)\big)\\
&\qquad\qquad\qquad\qquad : \big(e(\widetilde u-u(t))-\widetilde\pi+\pi(t)\big)-g(t)\cdot(\widetilde u-u(t))\\
&\quad+\frac12\mathbb{H}\nabla\big(e(\widetilde u_\tau+\overline u_\tau(t)+2u_{\rm D}(t))-\widetilde\pi_\tau-\overline\pi_\tau(t)\big)\,\vdots\,\nabla\big(e(\widetilde u-u(t))-\widetilde\pi+\pi(t)\big)\,{\rm d}x\\
&\quad+\int_{\bar\Omega}\sigma_{\rm Y}(\underline{\zeta}_\tau(t))\big|\widetilde\pi-\pi(t)\big|({\rm d}x)\bigg)-\int_{\Gamma_{\rm N}}f(t)\cdot(\widetilde u-u(t))\,{\rm d}S\\
&=\int_\Omega \frac12\mathbb{C}(\underline{\zeta}(t))\big(e(\widetilde u+\overline u(t)+2u_{\rm D}(t))-\widetilde\pi-\overline\pi(t)\big):\big(e(\widetilde u-u(t))-\widetilde\pi+\pi(t)\big)\\
&\quad+\frac12\mathbb{H}\nabla\big(e(\widetilde u+\overline u(t)+2u_{\rm D}(t))-\widetilde\pi-\overline\pi(t)\big)\,\vdots\,\nabla\big(e(\widetilde u-u(t))-\widetilde\pi+\pi(t)\big)\,{\rm d}x\\
&\quad+\int_{\bar\Omega}\sigma_{\rm Y}(\zeta)\big|\widetilde\pi-\pi(t)\big|({\rm d}x)-\int_\Omega g(t)\cdot(\widetilde u-u(t))\,{\rm d}x-\int_{\Gamma_{\rm N}}f(t)\cdot(\widetilde u-u(t))\,{\rm d}S\\
&=\mathscr{E}(t,\widetilde u,\widetilde\pi,\zeta(t))+\mathscr{R}(\zeta(t);\widetilde\pi-\pi(t),0)-\mathscr{E}(t,u(t),\pi(t),\zeta(t)).
\end{aligned}
$$

Note that we used also $\sigma_{\rm Y}(\underline{\zeta}_\tau(t))|\widetilde\pi-\pi(t)|\to\sigma_{\rm Y}(\zeta)|\widetilde\pi-\pi(t)|$ in ${\rm Meas}(\bar\Omega)$ due to the continuity assumption (2.14f) on $\sigma_{\rm Y}$ and due to the convergence $\underline{\zeta}_\tau(t)\to\zeta(t)$ in $C(\bar\Omega)$ which follows from the second estimates in (3.11d) and the compact embedding $W^{1,r}(\Omega)\subset C(\bar\Omega)$.

Step 4: *Limit passage in the damage flow rule* (3.9b) *toward* (2.11b). We need to prove that $\overline e_{{\rm el},\tau}\to e_{\rm el}$ strongly in $L^2(Q;\mathbb{R}^{d\times d}_{\rm sym})$. Toward this goal, we first realize that $\nabla\overline e_{{\rm el},\tau}(t)\to\nabla e_{\rm el}(t)$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d\times d})$ as pronounced in (3.11c); here we use the uniqueness of stresses (counting the already selected subsequence (3.12) and its limit); cf. the arguments in [13, Theorem 5.9] and in [40, section 4.2.3] for simple materials without damage. Here, using also absolute continuity valid due to viscosity in damage flow rule we obtain

$$
\begin{aligned}
\frac12\frac{\rm d}{{\rm d}t}\Big(\big\langle\mathbb{H}\nabla(e_{\rm el}^{(1)}-e_{\rm el}^{(2)}),\nabla(e_{\rm el}^{(1)}-e_{\rm el}^{(2)})\big\rangle+\big\langle\mathbb{C}(\zeta)(e_{\rm el}^{(1)}-e_{\rm el}^{(2)}),e_{\rm el}^{(1)}-e_{\rm el}^{(2)}\big\rangle\Big)\\
=-\frac12\big\langle\mathbb{C}'(\zeta)\dot\zeta(e_{\rm el}^{(1)}-e_{\rm el}^{(2)}),e_{\rm el}^{(1)}-e_{\rm el}^{(2)}\big\rangle\\
\le\max_{0\le z\le 1}|\mathbb{C}'(z)|\big\|\dot\zeta\big\|_{L^2(\Omega)}\big\|e_{\rm el}^{(1)}-e_{\rm el}^{(2)}\big\|^2_{L^4(\Omega;\mathbb{R}^{d\times d})}.
\end{aligned}
\tag{3.18}
$$

Note that for $\mathbb{H}=0$ and $\mathbb{C}'=0$, it reduces to the simple inequality $\langle\sigma_{\rm el}^{(1)}-\sigma_{\rm el}^{(2)}),\dot e_{\rm el}^{(1)}-\dot e_{\rm el}^{(2)}\rangle\le 0$ used in [13, 40]. Here, we should integrate (3.18) over $[0,t]$, use positive-definiteness of $\mathbb{H}$ and $\mathbb{C}(\cdot)$, and eventually use Gronwall's inequality, which works here certainly even for $d\le 4$ for which the embedding $H^2(\Omega)\subset W^{1,4}(\Omega)$ holds. In this way, we obtain $e_{\rm el}^{(1)}=e_{\rm el}^{(2)}$. Thus, using the compact embedding, we also know that $\overline e_{{\rm el},\tau}(t)\to e_{\rm el}(t)$ strongly in $L^{6-\epsilon}(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ if $d\le 3$. Then, by the uniform bounds in time and by Lebesgue's theorem used, e.g., to $t\mapsto\|\overline e_{{\rm el},\tau}(t)-e_{\rm el}(t)\|_{L^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}$, we can see that $\overline e_{{\rm el},\tau}\to e_{\rm el}$ strongly even in $L^{1/\epsilon}(0,T;L^{6-\epsilon}(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))$ with each small $\epsilon>0$.

Then the only difficult remaining terms are $\kappa\int_Q{\rm div}((1+\varepsilon|\nabla\overline\zeta_\tau|^{r-2})\nabla\overline\zeta_\tau)\dot\zeta_\tau\,{\rm d}x{\rm d}t$ and $\int_Q\overline\xi_\tau(-\dot\zeta_\tau)\,{\rm d}x{\rm d}t$ because so far we know only the weak convergence of $\dot\zeta_\tau$, of ${\rm div}((1+\varepsilon|\nabla\overline\zeta_\tau|^{r-2})\nabla\overline\zeta_\tau)$, and of $\overline\xi_\tau$ in $L^2(Q)$. We indeed cannot expect the limit, but we can proceed the following estimate:

$$
\begin{aligned}
(3.19)\qquad \limsup_{\tau\to 0}\int_Q &\mathrm{div}((1{+}\varepsilon|\nabla\overline{\zeta}_\tau|^{r-2})\nabla\overline{\zeta}_\tau)\dot{\zeta}_\tau\,\mathrm{d}x\mathrm{d}t\\
&= -\liminf_{\tau\to 0}\int_Q (1{+}\varepsilon|\nabla\overline{\zeta}_\tau|^{r-2})\nabla\overline{\zeta}_\tau{\cdot}\nabla\dot{\zeta}_\tau\,\mathrm{d}x\mathrm{d}t\\
&\le \limsup_{\tau\to 0}\int_\Omega \frac{1}{2}\big|\nabla\zeta_0|^2+\frac{\varepsilon}{r}\big|\nabla\zeta_0\big|^r-\frac{1}{2}\big|\nabla\zeta_\tau(T)\big|^2-\frac{\varepsilon}{r}\big|\nabla\zeta_\tau(T)\big|^r\,\mathrm{d}x\\
&\le \int_\Omega \frac{1}{2}\big|\nabla\zeta_0|^2+\frac{\varepsilon}{r}\big|\nabla\zeta_0\big|^r-\frac{1}{2}\big|\nabla\zeta(T)\big|^2-\frac{\varepsilon}{r}\big|\nabla\zeta(T)\big|^r\,\mathrm{d}x\\
&= \int_Q \mathrm{div}((1{+}\varepsilon|\nabla\zeta|^{r-2})\nabla\zeta)\dot{\zeta}\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where we used (3.11d) at $t=T$ and where the last equality relies on the regularity property $\mathrm{div}((1{+}\varepsilon|\nabla\zeta|^{r-2})\nabla\zeta)\in L^2(Q)$ and can be proved either by a mollification in space [47, formula (3.69)] and or in time by a time-difference technique [25, formula (2.15)].

The convergence in the inclusion $\overline{\xi}_\tau\in N_{[0,1]}(\overline{\zeta}_\tau)$ is easy due to the maximal monotonicity of $N_{[0,1]}(\cdot)$ and the convergences (3.11f) and $\overline{\zeta}_\tau\to\zeta$ strongly in $L^2(Q)$, which can be proved by a generalized version of the Aubin–Lions theorem (cf. [52, Corollary 7.9]), or here even in $L^\infty(Q)$ was proved as in Step 1. Having proved $\xi\in N_{[0,1]}(\zeta)$, we can also see that

$$
\begin{aligned}
(3.20)\qquad \limsup_{\tau\to 0}\int_Q \overline{\xi}_\tau(-\dot{\zeta}_\tau)\,\mathrm{d}x\mathrm{d}t &= \limsup_{\tau\to 0}\bigg(\int_\Omega \boldsymbol{\delta}_{[0,1]}(\zeta_0)\,\mathrm{d}x-\int_\Omega \boldsymbol{\delta}_{[0,1]}(\zeta_\tau(T))\,\mathrm{d}x\bigg)\\
&\le \int_\Omega \boldsymbol{\delta}_{[0,1]}(\zeta_0)\,\mathrm{d}x-\int_\Omega \boldsymbol{\delta}_{[0,1]}(\zeta(T))\,\mathrm{d}x=\int_Q \xi(-\dot{\zeta})\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

which is needed for the limit passage in (3.9b); in fact, even the limit and the equality hold in (3.20).

Step 5: *Energy equality.* We test (2.1c), which holds a.e. on Q by $\dot{\zeta}$. This test is legal as all terms in (2.1c) as well as $\dot{\zeta}$ are in $L^2(Q)$. We again use the last equality in (3.19). Moreover, as $\xi\in\partial\boldsymbol{\delta}_{[0,1]}(\zeta)$, we have $\int_Q \xi\dot{\zeta}\,\mathrm{d}x\mathrm{d}t=\int_\Omega \boldsymbol{\delta}_{[0,1]}(\zeta(T))-\boldsymbol{\delta}_{[0,1]}(\zeta(0))\mathrm{d}x=0-0=0$. We thus obtain

$$
\begin{aligned}
(3.21)\qquad \int_\Omega &\frac{\kappa}{2}\big|\nabla\zeta(T)\big|^2+\frac{\varepsilon\kappa}{r}\big|\nabla\zeta(T)\big|^r-b(\zeta(T))\,\mathrm{d}x\\
&+\int_Q \frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}:e_{\rm el}+\widehat{a}(\dot{\zeta})\,\mathrm{d}x\mathrm{d}t=\int_\Omega \frac{\kappa}{2}\big|\nabla\zeta_0|^2+\frac{\varepsilon\kappa}{r}\big|\nabla\zeta_0\big|^r-b(\zeta_0)\,\mathrm{d}x.
\end{aligned}
$$

Furthermore, we test formally (2.1a) by $\dot{u}$ and (2.1b) by $\dot{\pi}$. The rigorous calculations use the approximation of the Stieltjes-type integral by Riemann sums and semistability; cf. [53, formulas (76)–(82)], which adapts a technique developed in the theory of rate-independent processes [15, 41]. Here, as $\mathbb{C}$ is not constant, we will still see the term $(\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}{:}e_{\rm el})\dot{\zeta}$ which results by the formal substitution $\mathbb{C}(\zeta)e_{\rm el}{:}\dot{e}_{\rm el}=\frac{\partial}{\partial t}\frac{1}{2}\mathbb{C}(\zeta)e_{\rm el}{:}e_{\rm el}-(\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}{:}e_{\rm el})\dot{\zeta}$; note that $\mathbb{C}(\zeta)e_{\rm el}{:}\dot{e}_{\rm el}$ is not well defined since $\dot{e}_{\rm el}$ is not well controlled. Thus we obtain

$$\begin{aligned}(3.22)\qquad &\int_\Omega \frac{1}{2}\mathbb{C}(\zeta(T))e_{\rm el}(T){:}e_{\rm el}(T)+\frac{1}{2}\mathbb{H}\nabla e_{\rm el}(T)\,\vdots\,\nabla e_{\rm el}(T)\,{\rm d}x\\ &+\int_{[0,T]\times\bar\Omega}\sigma_{\rm Y}(\zeta)\big|\dot\pi\big|({\rm d}x{\rm d}t)=\int_Q\Big(\frac{1}{2}\mathbb{C}'(\zeta)e_{\rm el}{:}e_{\rm el}\Big)\dot\zeta\,{\rm d}x{\rm d}t\\ &+\int_\Omega\frac{1}{2}\mathbb{C}(\zeta_0)e_{\rm el}(0){:}e_{\rm el}(0)+\frac{1}{2}\mathbb{H}\nabla e_{\rm el}(0)\,\vdots\,\nabla e_{\rm el}(0)\,{\rm d}x.\end{aligned}$$

Summing (3.21) and (3.22) then gives the energy balance (2.11c). ∎

Further, to implement the model computationally, we need to make a spatial discretization of the time-discrete scheme (3.1)–(3.2). Toward this goal, we use the lowest-order conformal finite-element method (FEM). In view of the used regularity (3.4e), the straightforward discretization therefore employs P2-elements for u and ζ and P1-elements for π. Rigorously speaking, due to the assumed smoothness (2.14a), one should consider FEM on a nonpolyhedral, curved domain. The minimization problems (3.5) are then to be restricted on the corresponding finite-dimensional subspaces, and the solution thus obtained is denoted by $u_{\tau h}^k$, $\pi_{\tau h}^k$, and $\zeta_{\tau h}^k$, with $h>0$ denoting the mesh size. By this way, we obtain also the piecewise constant and affine interpolants in time, denoted by $\overline{u}_{\tau h}$ and $u_{\tau h}$, $\overline{\pi}_{\tau h}$, and $\pi_{\tau h}$, and eventually $\overline{\zeta}_{\tau h}$ and $\zeta_{\tau h}$. Also, $\overline{\xi}_{\tau h}$ can be obtained analogously as before in Lemma 3.2.

PROPOSITION 3.4 (convergence of the FEM discretization). *Let* (2.14) *be satisfied, and the* P2-*FEM for u and ζ and* P1-*FEM for π are applied with $h>0$ the mesh size. Then*

(i) *the a priori estimates* (3.4) *and* (3.10) *hold when modified for $u_{\tau h}$, $\pi_{\tau h}$, $\zeta_{\tau h}$, and $\overline{\xi}_{\tau h}$ with C independent of $\tau>0$ and now of $h>0$, too;*

(ii) *moreover, these fully discrete solutions converge (in terms of subsequences) in the mode as* (3.11) *toward weak solutions in accordance with Definition* 2.1 *when simultaneously $\tau\to 0$ and $h\to 0$.*

The modification of the proof of this joint convergence of time-and-space discretization is rather routine, the explicit construction of the mutual recovery sequence (3.16) taking additionally a finite-element approximation like in [7], namely, $\widetilde{u}_{\tau h}=\overline{u}_{\tau h}(t)+\Pi_h^{(2)}(\widetilde{u}-u(t))$ and $\widetilde{\pi}_{\tau h}=\bar{\pi}_{\tau h}(t)+\Pi_h^{(1)}(\widetilde{\pi}-\pi(t))$ with $\Pi_h^{(1)}$ and $\Pi_h^{(2)}$ denoting a projector onto the P1- and P2-finite-elements spaces, respectively; we omit details about this modification.

Remark 3.5 (damage discretized by P1-elements). The damage flow rule (2.1c) itself suggests use of only P1-elements for ζ which are, naturally, more easy to implement than the P2-elements used in Proposition 3.4. Then, however, (3.4e) cannot be expected for the FEM approximation of ζ and also a direct P1-FEM analogue of (3.9b) cannot hold. Instead of (3.9b), we have

$$\begin{aligned}(3.23)\qquad &\int_Q\bigg(a(v)+\Big(\frac{1}{2}\mathbb{C}'(\underline{\zeta}_{\tau h})\overline{e}_{{\rm el},\tau}:\overline{e}_{{\rm el},\tau}-b'(\overline{\zeta}_{\tau h})+\overline{\xi}_{\tau h}\Big)(v-\dot\zeta_{\tau h})\\ &+\kappa\left((1{+}\varepsilon|\nabla\overline{\zeta}_{\tau h}|^{r-2})\nabla\overline{\zeta}_{\tau h}\right)\cdot\nabla(v{-}\dot\zeta_{\tau h})\bigg)\,{\rm d}x\,{\rm d}t\geq\int_Q a(\dot\zeta_{\tau h})\,{\rm d}x\,{\rm d}t\end{aligned}$$

for any v valued in the finite-dimensional P1-finite-element subspace. Yet, the sequence $\{\nabla\dot\zeta_{\tau h}\}_{\tau>0,h>0}$ cannot be expected to stay bounded. Thus, for the limit passage, instead of (3.23) one should rather use the discrete by-part integration (summation) in time like we did in (3.19), leading to

$$\begin{aligned}(3.24)\qquad &\int_Q \bigg(a(v) + \Big(\frac{1}{2}\mathbb{C}'(\underline{\zeta}_{\tau h})\overline{e}_{\mathrm{el},\tau} : \overline{e}_{\mathrm{el},\tau} - b'(\overline{\zeta}_{\tau h}) + \overline{\xi}_{\tau h}\Big)(v - \dot{\zeta}_{\tau h}) \\ &\quad + \kappa\left((1{+}\varepsilon|\nabla\overline{\zeta}_{\tau h}|^{r-2})\nabla\overline{\zeta}_{\tau h}\right)\cdot\nabla v\bigg)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \frac{\kappa}{2}|\nabla\zeta_0|^2 + \frac{\varepsilon\kappa}{r}|\nabla\zeta_0|^r\,\mathrm{d}x \\ &\quad \geq \int_Q a(\dot{\zeta}_{\tau h})\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \frac{\kappa}{2}|\nabla\zeta_{\tau h}(T)|^2 + \frac{\varepsilon\kappa}{r}|\nabla\zeta_{\tau h}(T)|^r\,\mathrm{d}x,\end{aligned}$$

which holds for any v with values in the P1-finite-element space. Now, however, we do not have the estimates (3.4e) and (3.10). Anyhow, the limit passage seems possible by using the strategy proposed by Colli and Visintin [10] (cf. also [52, section 11.1.2]), allowing for the stored energy $\mathscr{E}$ taking values $+\infty$ but relying on boundedness of $\mathscr{R}$, as it is indeed our situation. The convergence is, of course, in a weaker mode than (3.11). Only after this limit passage can we prove the regularity (2.10e) and go back to the weak formulation (2.11b) by using also the arguments which we have used for the last equality in (3.19).

Remark 3.6 (plasticity discretized by P1-/P0-elements). One can consider e_{el} as an independent internal variable and then understand $e(u){-}e_{\mathrm{el}}{-}\pi = 0$ involved in (2.1a) as a linear holonomic contraint. This suggest a discretization combined with a penalization of this constrant, leading to

$$(3.25\mathrm{a})\qquad U := \big\{(u,\pi,\zeta,e_{\mathrm{el}})\in \mathrm{BD}(\bar\Omega;\mathbb{R}^d)\times\mathrm{Meas}(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})\times W^{1,r}(\Omega)\times H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}});\ u\odot\vec{n}\mathrm{d}S + \pi = 0 \ \text{ on }\ \Gamma_{\mathrm{D}}\big\},$$

$$(3.25\mathrm{b})\qquad \mathscr{E}_\delta(t,u,\pi,\zeta) := \begin{cases} \displaystyle\int_\Omega \frac{1}{2}\mathbb{C}(\zeta)e_{\mathrm{el}}:e_{\mathrm{el}} + \frac{1}{2}\mathbb{H}\nabla e_{\mathrm{el}} \vdots \nabla e_{\mathrm{el}} - b(\zeta) - g(t)\cdot u \\ \displaystyle\quad +\kappa\Big(\frac{1}{2}|\nabla\zeta|^2 + \frac{\varepsilon}{r}|\nabla\zeta|^r\Big)\,\mathrm{d}x - \int_{\Gamma_{\mathrm{N}}} f(t)\cdot u\,\mathrm{d}S \\ \displaystyle\quad + \int_\Omega \frac{1}{2\delta}\big|e(u{+}u_{\mathrm{D}}(t)){-}e_{\mathrm{el}}{-}\pi\big|^2\,\mathrm{d}x & \text{if } \zeta\in[0,1] \text{ on } \Omega, \\ \infty & \text{otherwise,}\end{cases}$$

instead of (2.9a,b). Then, for $\delta > 0$, it suffices to use P1-elements for u, e_{el}, and (if counting also with Remark 3.5) for ζ, too, and P0-elements for π. Of course, one has to pay the price for using a lower-order FEM consisting in a higher number of variables caused by involving the additional internal variable e_{el}, and theoretical convergence like in Proposition 3.4 must be substantially weakened to get only a successive limit for $h \to 0$, and only then $\delta \to 0$, and eventually $\tau \to 0$. Note also that the choice of the P1-elements for e_{el} enforces the continuity of e_{el} over the element boundaries (i.e., edges in the two-dimentional or faces in the three-dimensional case), but the fields $e(u)$ and π will be approximated by P0-elements that are discontinuous over element boundaries. The additive decomposition $e(u) = e_{\mathrm{el}} + \pi$ can thus be satisfied only in the limit.

4. Implementation of the fully discrete model. The implementation of the model addressed in Proposition 3.4 is rather cumbersome because of the high-order FEM involved while the lower-order FEM outlined in Remark 3.6 allows only for a successive mode of convergence being of rather theoretical interest only. For classical elastoplasticity models with or without hardening (thus including perfect plasticity), implementations of higher polynomial basis functions (known as *hp*-FEM; cf. [56]) in elastoplasticity are explained in details in [22, 32, 45] but for simple materials only, i.e., for $\mathbb{H} = 0$, which allows for exploiting the local (valid in each continuum points) dependency of π on u and elliminate the variables π from minimization routines. For $\mathbb{H} > 0$, such dependency becomes nonlocal and one should rather make minimization for the full couple (u, π), as suggested in (3.1a), (3.1b). After a Mosco-type transformation, (cf., e.g., [1, Remark 4.7] or [42, section 3.6.3]), it may lead to a recursive second-order cone programming for which efficient codes do exist; cf. [5, 60].

Therefore we dare make few shortcuts: P1-elements are used for damage ζ according to Remark 3.5 and for u too, while P0-elements for π like in Remark 3.6, but we neglect the (anyhow usually small and even not reliably known) hyperelasticity moduli, i.e., $\mathbb{H} = 0$, believing that it does not substantially influence the particular computational experiments in section 5. Consequently, only P1-elements can be used for displacement u and P0-elements for the plastic strain π. Only the case $d = 2$ is treated, so the previous analytical part have required $r > 2$ and we dare make another (indeed negligible) shortcut by considering $r = 2$ (and therefore by putting $\epsilon = 0$ the damage-gradient term in (2.9b) become quadratic). The last shortcut is that we consider a nonsmooth, Lipschitz domain; see Figure 2. It should, however, be emphasized that the smoothness (2.14a) of Ω was needed only because it makes the technically demanding perfect plasticity in [13] work and it is not needed for any H^2-regularity. Let us point out that such H^2-regularity would hardly be satisfied because of a combination of the Dirichlet and the Neumann boundary conditions on an adjacent part of the boundary with a 180° angle, which is known to create singularities in solutions but which is admitted in [13]. In the particular geometry in Figure 2, it is important that the nonsmooth corners are far away from the regions where plastification evolves so that their influence on the plasticity can well be neglected in the particular computational experiments in section 5.

The material is assumed isotropic with properties linearly dependent on damage. The isotropic elasticity tensor is assumed as

$$\mathbb{C}_{ijkl}(\zeta) := [(\lambda_1 - \lambda_0)\zeta + \lambda_0]\delta_{ij}\delta_{kl} + [(\mu_1 - \mu_0)\zeta + \mu_0](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{4.1}$$

where λ_1, μ_1 and λ_0, μ_0 are two sets of Lamé parameters satisfying

$$\lambda_1 \geq \lambda_0 \geq 0, \qquad \mu_1 \geq \mu_0 > 0.$$

Here, δ denotes the Kronecker symbol. This choice implies that the elastic-moduli tensor satisfies (2.14d) and it is even positive-definite-valued (and therefore invertible). Values of $\mathbb{C}_{\rm D}(\zeta)$ and $c_{\rm S}(\zeta)$ in (2.14d) follow from a decomposition of the elastic strain energy $\frac{1}{2}\mathbb{C}(\cdot)e{:}e$ into the deviatoric and the volumetric parts of the strain tensor e. The stored energy of damage compliant with (2.14c) is assumed in the form

$$b(\zeta) := b_1\,\zeta, \tag{4.2}$$

where $b_1 > 0$ means the specific energy stored in the microcracks/microvoids created by damaging the material. By healing, this energy can be recovered. The plastic yield stress compliant with (2.14f) is assumed in the form

$$\sigma_{\rm Y}(\zeta) = \big(\sigma_{{\rm Y},1} - \sigma_{{\rm Y},0}\big)\zeta + \sigma_{{\rm Y},0}, \tag{4.3}$$

where $\sigma_{{\rm Y},1} \geq \sigma_{{\rm Y},0} > 0$. The damage-dissipation potential is assumed in the piecewise quadratic form

$$a(\dot\zeta) := \frac{1}{2}a_1(\dot\zeta^+)^2 + \frac{1}{2}a_2(\dot\zeta^-)^2 + a_3(\dot\zeta^-), \tag{4.4}$$

where $\dot\zeta^+ = \max\{0, \dot\zeta\}$ and $\dot\zeta^- = \max\{-\dot\zeta, 0\}$ and a_1, a_2, a_3 are given (material) nonnegative parameters. Values of a_1 and a_2 determine rate-dependent parts of healing and damage model components and the value of a_3 a rate-independent damage activation. The form of $a(\cdot)$ satisfies (2.14b).

With respect to the fractional-step strategy of section 3, we solve first for $(u^k_{\tau h}, \pi^k_{\tau h})$ from the elastoplastic minimization problems (3.5a) and then ζ^k_τ from the damage minimization problem (3.5b) recursively for $k = 1, \ldots, T/\tau$. In view of the above shortcuts and simplifications, the minimization problems (3.5a) and (3.5b) rewrite as

$$(4.5)\quad (u^k_{\tau h}, \pi^k_{\tau h}) = \underset{u,\pi}{\operatorname{argmin}} \int_\Omega \bigg(\frac{1}{2}\mathbb{C}(\zeta^{k-1}_{\tau h})\big(e(u{+}u^k_{\mathrm{D},\tau h}){-}\pi\big) : \big(e(u{+}u^k_{\mathrm{D},\tau h}){-}\pi\big) - g^k_{\tau h}{\cdot}u + \sigma_{\mathrm{Y}}(\zeta^{k-1}_{\tau h})|\pi{-}\pi^{k-1}_{\tau h}|\bigg)\,\mathrm{d}x - \int_{\Gamma_\mathrm{N}} f^k_{\tau h}{\cdot}u\,\mathrm{d}S,$$

$$(4.6)\quad \zeta^k_{\tau h} = \underset{\zeta}{\operatorname{argmin}} \int_\Omega \bigg(\frac{1}{2}\mathbb{C}(\zeta)\big(e(u^k_{\tau h}{+}u^k_{\mathrm{D},\tau h}){-}\pi^k_{\tau h}\big) : \big(e(u^k_{\tau h}{+}u^k_{\mathrm{D},\tau h}{-}\pi^k_{\tau h}\big){-}b_1\zeta + \frac{1}{2}\kappa|\nabla\zeta|^2 + \frac{1}{2\tau}a_1(\zeta{-}\zeta^{k-1}_{\tau h})^+ + \frac{1}{2\tau}a_2(\zeta{-}\zeta^{k-1}_{\tau h})^- + a_3(\zeta{-}\zeta^{k-1}_{\tau h})^-\bigg)\,\mathrm{d}x,$$

where u is searched over P1-elements satisfying Dirichlet boundary conditions, π over P0-elements satisfying elementwise trace-free condition $\mathrm{tr}\,\pi = 0$, and ζ over P1-elements satisfying the nodal box constraint $\zeta \in [0, 1]$. The form of (4.5) corresponds to the minimization problem of perfect plasticity with the elasticity tensor and the plastic yield stress depending on the damage variable in the previous time level. The energy in (4.5) is transformed to an energy in the variable u only by substituting the elementwise dependency of π on u; see [2, 9] for more details. Then, the quasi-Newton iterative method is applied to solve $u^k_{\tau h}$ while $\pi^k_{\tau h}$ is reconstructed from it. More details on this specific elastoplasticity solver can be found, e.g., in [9, 23, 24].

The damage minimization problem (4.6) represents a minimization of a nonsmooth but strictly convex functional. It can be reformulated to a modified problem

$$(4.7\mathrm{a})\qquad \underset{\zeta,z_+,z_-}{\operatorname{argmin}} \int_\Omega \bigg(\frac{1}{2}\mathbb{C}(\zeta)\big(e(u^k_{\tau h}{+}u^k_{\mathrm{D},\tau h}){-}\pi^k_{\tau h}\big) : \big(e(u^k_{\tau h}{+}u^k_{\mathrm{D},\tau h}{-}\pi^k_{\tau h}\big) - b_1\zeta + \frac{1}{2}\kappa|\nabla\zeta|^2 + \frac{1}{2\tau}a_1(z_+)^2 + \frac{1}{2\tau}a_2(z_-)^2 + a_3 z_-\bigg)\,\mathrm{d}x,$$

$$(4.7\mathrm{b})\qquad \text{where}\quad z_+ = (\zeta{-}\zeta^{k-1}_{\tau h})^+, z_- = (\zeta{-}\zeta^{k-1}_{\tau h})^-,$$

are additional "update" variables. It should be noted that ζ and $\zeta^{k-1}_{\tau h}$ are P1-functions and therefore z_+ and z_- are not P1-functions in general on elements where nodal values of $\zeta{-}\zeta^{k-1}_{\tau h}$ alternate signs. However, if we restrict z_+, z_- to P1-functions while (4.7b) is required at nodal points, then (4.7a) actually represents a conventional quadratic-programming (QP) problem, in which we require a linear and box constraints

$$(4.8)\qquad \zeta = \zeta^{k-1}_{\tau h} + z_+ - z_-, \qquad z_+ \in [0, 1 - \zeta^{k-1}_{\tau h}], \qquad z_- \in [0, \zeta^{k-1}_{\tau h}].$$

A quadratic cost functional of this QP problem has a positive-semidefinite Jacobian, since there are no Dirichlet boundary conditions on the damage variable ζ. Note that the optimal pair (z_+, z_-) must satisfy $z_+z_-{=}0$ in all nodes, i.e., both variables cannot be positive. This can be easily seen by contradiction: If $z_+z_-{>}0$ in some node, then a different pair $(z_+ - \min\{z_+, z_-\}, z_- - \min\{z_+, z_-\})$ would again satisfy the constraints (4.8) but would provide a smaller energy value in (4.7a).

Our MATLAB implementation is available for download at MATLAB Central [63]. It is based on an original elastoplasticity code related to multisurface models [8]. The code is simplified to work with one surface variable only (which corresponds to the classical model of kinematic hardening) and sets the hardening parameter to zero to enforce perfect plasticity. It partially utilizes vectorization techniques of [49] and works reasonably fast also for finer triangular meshes.

5. Illustrative computational simulations. We consider a time-simulation of a two-dimensional continuum visualized in Figure 2 describing two "plates" moving horizontally in opposite directions with the constant velocity $\pm 10^{-8}$m/s$\doteq$30 cm/yr. The model has applications in geophysics, specifically in modeling of tectonic and seismic processes in crustal parts of the earth lithosphere in the relatively short or very short time scales (meaning substantially less than a million years) where the small-strain concept and solid mechanics are well relevant. The hardening is naturally considered zero. The damage variable is in the position of a so-called aging. The healing together with the damage-dependent plastic yield stress allow for periodically alternating fast damage and slow healing under external loading with constant velocity, which is a typical stick/slip-type events of flat partly damaged subdomains (so-called lithospheric faults) manifested by reoccurring earthquakes.

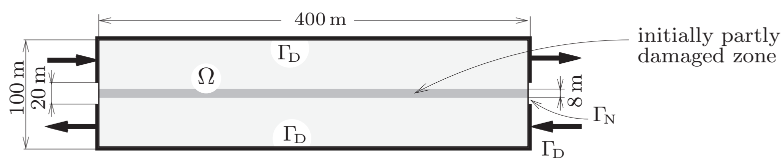

FIG. 2. *Geometry used for the computational experiment, imitating the fault between two plates moving horizontally in opposite directions. The time-dependent Dirichlet conditions are prescribed on* Γ_D*, using the constant velocity* $\pm 10^{-8}$m/s $\doteq$ 30 cm/year.

The domain Ω is assumed to be occupied by an elastic continuum specified by an isotropic homogeneous elasticity tensor in the form (4.1) with $\lambda_1 = 7.5$ GPa and $\mu_1 = 11.25$ GPa (which corresponds to Young's modulus $E_{\text{Young}} = 27$ GPa and the Poisson ratio $\nu = 0.2$ in the nondamage state), while the damaged material uses ten-times-smaller moduli, i.e., $\lambda_0 = 0.75$ GPa and $\mu_0 = 1.125$ GPa in (4.1). The yield stress σ_y in (4.3) ranges between the values $\sigma_{Y,1} = 2$ MPa and $\sigma_{Y,0} = \sigma_{Y,1} \times 10^{-12}$. The damage-dissipation potential (4.4) is specified by constants $a_1 = 100$ GPa s and $a_3 = 10$ Pa, while the damage viscosity a_2 will vary. The stored energy of damage is $b_1 = 0.001$ J/m^3 with the damage length-scale coefficient $\kappa = 0.001$ J/m. The initial conditions ensure that $\pi_0 = 0$, $\zeta_0 = 1$ (or $\zeta_0 = 1/2$ in a narrow central horizontal strip).

The first numerical test is run for discrete times in the interval $0 \le t \le 400$ ks with the equidistant time partition using the time step $\tau = 1$ ks. The spatial discretization of the domain Ω used a uniform triangular mesh with 4608 elements and 2373 nodes; this mesh is available by setting "refinement = 2" in the code [63], while finer uniform meshes can be generated by putting higher values of the "refinement" parameter. Thus, 400 time steps are computed and Figure 3 displays space-distributions of the shifted damage $1 - \zeta$, of the Frobenius norm of the plastic strain π, and of the von Mises stress $|\operatorname{dev}(\sigma)|$ at selected instants depicted on a domain from Figure 2 but deformed and intentionally visualized as magnified by the factor 12,500.

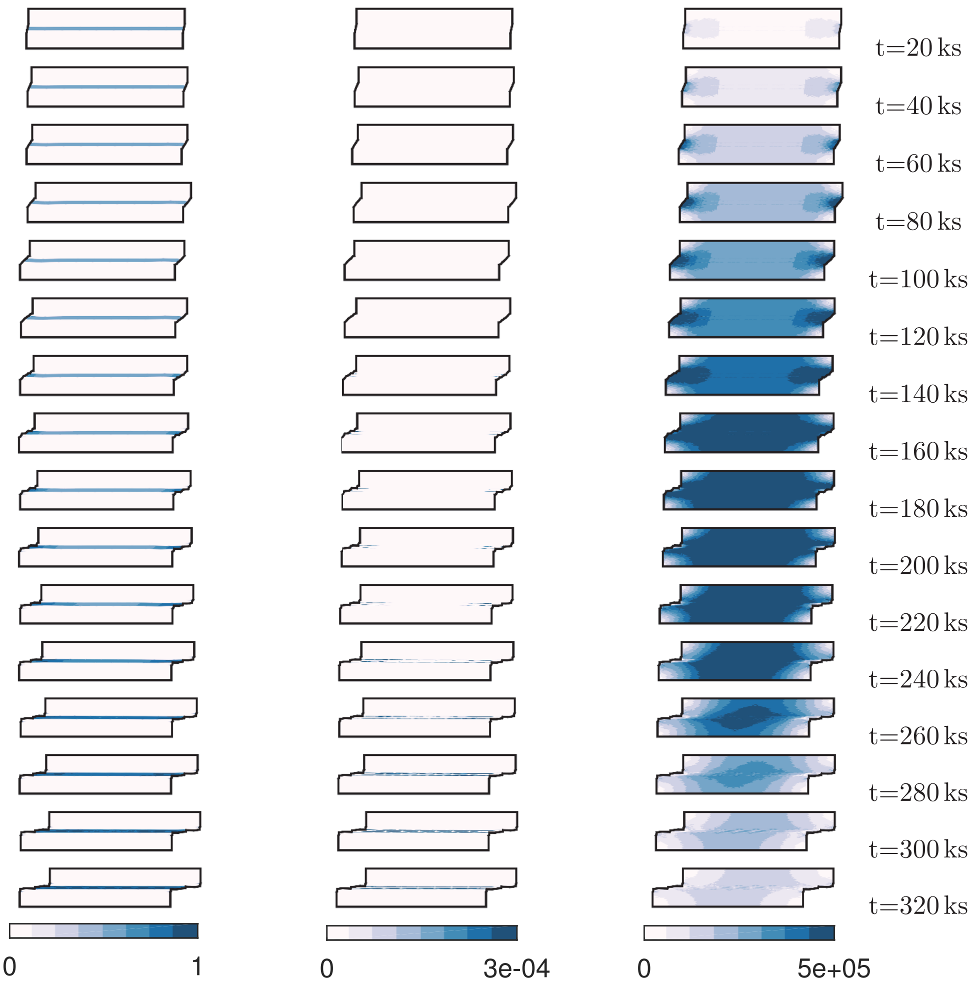

FIG. 3. *Evolution of space-distributions of damage (the left column, displaying* $1-\zeta$*) of the plastic strain (the middle column, displaying the Frobenius norm* $|\pi|$*) and of the von Mises stress (the right column, displaying* $|\operatorname{dev}(\sigma)|$*). The displacement of the deformed domain is displayed magnified by the factor* $12,500$*. Distributions were computed for damage viscosity* $a_2 = 10\,\mathrm{MPa\,s}$.

In order to see how the quality of discrete solutions depends on the time step τ, similar numerical tests are run for two additional time steps $\tau = 5\,\mathrm{ks}$ and $\tau = 10\,\mathrm{ks}$. The resulting energy balance (3.9c) is displayed in Figure 4. Naturally, it is best fulfilled for the smallest considered time-step $\tau = 1\,\mathrm{ks}$. Figure 5 shows the (horizontal component of the) reaction force which is here evaluated (very roughly) as an average from element values of von Mises stresses in the narrow central horizontal strip (i.e., the fault zone) shown in Figure 2. A comparison of Figures 4 and 5 indicates that the energy balance (3.9c) is better satisfie in the purely elastoplastic regime than within the undergoing damage. This becomes even more apparent if the damage process is speeded up by setting a smaller value $a_2 = 0.1\,\mathrm{MPa\,s}$; cf. the left-hand parts of Figures 4 and 5 versus the right-hand parts.

Dependence of the reaction-force evolution for varying viscosity of damage is shown in Figure 6 for a_2 as in Figures 4–5 compared also with a smaller viscos-

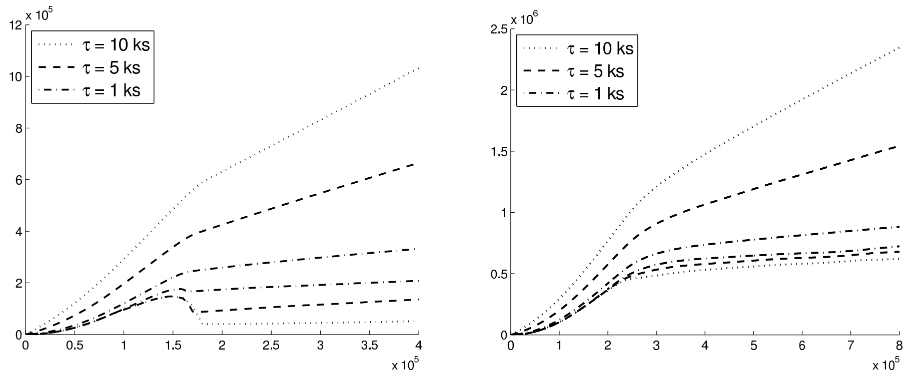

FIG. 4. *Evolution of the stored and dissipated energy (= the left-hand side of* (3.9c) *for T varying) and the work of external loading (= the right-hand side of* (3.9c) *for T as a current time t) calculated for three different values of the time steps $\tau = 10,\ 5,\ 1\,\mathrm{ks}$, documenting the convergence of* (3.9c) *toward the energy equality* (2.11c) *proved in Proposition* 3.3*. For less viscous damage this convergence is naturally slower than for a more viscous damage; cf. the left figure for $a_2 = 0.1\,\mathrm{MPa\,s}$ versus. the right one for $a_2 = 10\,\mathrm{MPa\,s}$.*

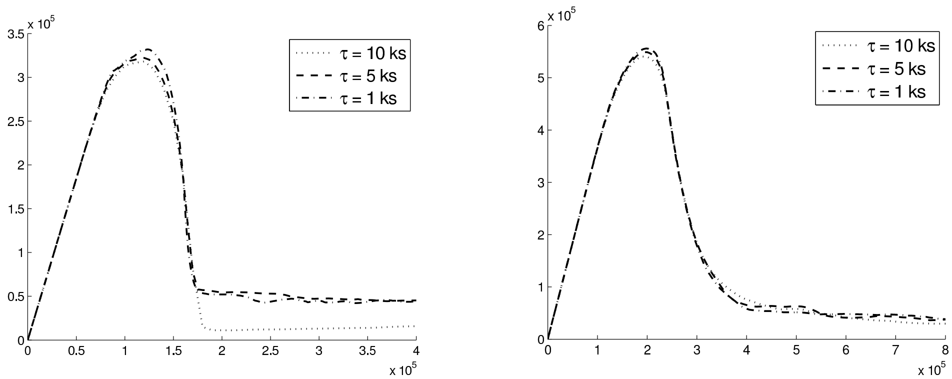

FIG. 5. *Evolution of the reaction force corresponding to Figure* 4*; the time scales in the left and the right figures are different. Notably, the force response is well converged even in situations (here for $\tau = 5\,\mathrm{ks}$) when the energetics on Figure* 4 *still exhibit big gaps.*

ity $a_2 = 1\,\mathrm{kPa\,s}$ which already provides a response essentially identical to the even smaller viscosity $a_2 = 0.01\,\mathrm{kPa\,s}$ (not displayed in Figure 6) where conservation of energy is numerically still more difficult to achieve. This indicates a certain tendency for convergence toward the model using rate-independent damage combined with rate-dependent healing (as in [42, section 5.2.7]) and with perfect plasticity, which is theoretically not justified, however.

Let us remark that the a posteriori information obtained from the residuum in the discrete energy balance (3.9c) written at a current time t (as also used in Figure 4) can be used to control adaptively the time step in a way to keep the numerical error in the energy under an a priori prescribed tolerance and, on the other hand, not to waste computational time by making too-small time steps in periods of slow evolution. We intentionally presented our numerical simulation on equidistant time partitions, but for actual geophysical simulations with very big difference in time scale between fast damage (earthquakes) and very slow healing, such an adaptivity is necessary.

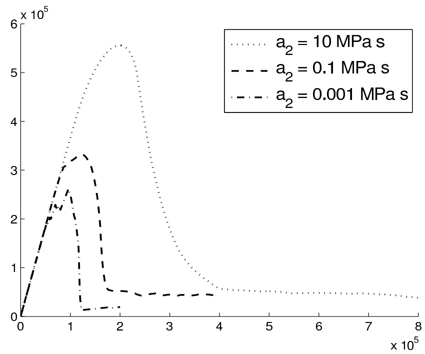

Fig. 6. *Dependence of the repulsive-force response on the viscosity of damage, the cases* $a_2 = 10$ *and* $0.1\,\mathrm{MPa\,s}$ *are (parts of) Figure* 5 *and are here compared also with even less viscous damage for* $a_2 = 1\,\mathrm{kPa\,s}$ *which gives essentially the same response as for the nearly inviscid case* $a_2 = 0.01\,\mathrm{kPa\,s}$ *(not displayed, however); the time step* $\tau = 1\,\mathrm{ks}$*. For decreasing viscosity, the rupture occurs earlier and propagates faster, showing a tendency to converge to an inviscid rate-independent (and theoretically not justified) damage model.*

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