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# Analytical-Algebraic Approach to Solving Chaotic System

Zdeněk Beran* and Sergej Čelikovský†
*Institute of Information Theory and Automation of the Czech Academy of Sciences, Pod Vodárenskou Věží 4, 182 08 Prague 8, Czech Republic*
**beran@utia.cas.cz*
*†celikovs@utia.cas.cz*

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The aim of this paper is to present the application of the analytical series technique to study properties of the nonlinear chaotic dynamical systems. More specifically, Laplace–Adomian decomposition method is applied to Rössler system and the so-called generalized Lorenz system. Some advantages and possible applications of this approach are discussed. Results are illustrated by numerical computations.

*Keywords*: Laplace transform; Laplace–Adomian decomposition; Adomian polynomials; nonlinear systems; chaos.

## 1. Introduction

A common approach to solve a particular problem for a given dynamical system is to transform this system from its original space of definition into an alternative space, where the corresponding solution is more easily achieved. Such a transformation has to be a homeomorphism in order to get the inverse transformation to be able to reinterpret the solution in the original space.

The well-known example of that approach is Laplace transform. Laplace transform is the useful tool to solve both ordinary and partial differential equations and it has enjoyed much success in this realm. This transform belongs to the class of integral transforms. Integral transforms date back to the work of Léonard Euler (1763 and 1769), who considered them essentially in the form of the inverse Laplace transform in solving the second-order class of linear ordinary differential equations. Even Laplace in his great work "Théorie analytique des probabilités" (1812) credits Euler with introducing integral transforms. Actually, it was Spitzer (1878) who attached the name of Laplace to this famous transform. In the late 19th century, the Laplace transform was extended to its complex form by Poincaré and Pincherle, rediscovered by Petzval and extended to two variables by Picard with further investigations conducted by Abel and many others. The first application of the modern Laplace transform occurs in the work of Bateman (1910), who transforms equations arising from Rutherfords work on radioactive decay. The modern approach was given particular impetus by Doetsch in the 1920s and 1930s; he applied the Laplace transform to differential, integral, and integro-differential equations. This body of work culminated in his fundamental 1937 text "Theorie und Anwendungen der Laplace Transformation". No account of the Laplace transformation would be complete without mentioning the work of Oliver Heaviside, who produced (mainly in the context of electrical engineering) a vast body of what is now termed as operational calculus. This material is scattered throughout his three volumes "Electromagnetic Theory" (1894, 1899, 1912) and bears many similarities to the Laplace transform method. Although

Heaviside's calculus was not entirely rigorous, it did find favor with electrical engineers as a useful technique for solving their problems. Considerable research went into trying to make the Heaviside calculus rigorous and to connect it with the Laplace transform. One such effort was that of Thomas John I'Anson Bromwich (1875–1929), who, among others, discovered and used the inverse transform.

Perhaps the most useful purpose of Laplace transform is to replace the problem of solving linear differential equations by the problem of solving a system of algebraic equations which belongs, indeed, to a substantially more explored area. The most utilized area of that approach is control theory. On the other hand, the main disadvantage here is that Laplace transform based approach addresses linear problems only. To proceed also with nonlinear problems, several new approaches were elaborated during the last decades.

Among them, a prominent role is played by the approach developed by Adomian [1994] who discovered new approximation polynomials, named Adomian polynomials later, that allow together with Laplace transform to obtain a kind of decomposition of the nonlinear system according to its nonlinearities. Such an approach results in the iterative scheme that embodies the fast convergence. Adomian's discovery started a very fruitful research in a broad area of applications. The equations modeling these applications range over algebraic polynomial equations, transcendental equations, ordinary differential equations, partial differential equations, difference equations and delay differential equations, among others. The coefficients of the corresponding differential equations may be time and space dependent, moreover, they may represent random processes as well. Finally, the equations may be nonlinear. Summarizing, the combination of the Adomian approach with Laplace transform enabled to engage the latter to investigate the nonlinear problems, too. The amount of applications grew very soon, the interested reader is referred to the following literature and references within there:

biology: [Abbaoui, 1995; Cherruault, 1994; Sen, 1998],

economy: [Wazwaz, 2004],

physics: [Alabdullatif *et al.*, 2007; Ismail *et al.*, 2004a; Ismail *et al.*, 2004b; Kaya, 2004; Khuri, 1998],

engineering: [Biazar *et al.*, 2006; Bokhari *et al.*, 2009; Olawanle & Ade, 2008; Serdal, 2005; Wazwaz, 2004],

integral equations: [Wazwaz, 2011].

The purpose of this paper is to develop Adomian method to treat the nonlinear part of the system to be used to study properties of chaotic systems, namely, the well-known Rössler system, [Gaspard, 2005], and the so-called generalized Lorenz system introduced in [Čelikovský & Vaněček, 1994; Čelikovský & Chen, 2002, 2005]. Using these examples, one can see that even a relatively low number of Adomian terms provide a good precision of approximation. Moreover, computations are performed time-pointwise and therefore they need not respect the time flow as in the case of classical numerical difference schemes, like the Runge–Kutta one. Last, but not least, differential equation can be effectively replaced by an algebraic expression which in our opinion opens new possibilities, e.g. in using chaotic continuous-time systems in encryption algorithms [Čelikovský & Lynnyk, 2012], avoiding usual problems caused by potential numerical instabilities in difference numerical schemes. Note, that these computations also show yet another advantage of the so-called generalized Lorenz canonical form [Čelikovský & Chen, 2002, 2005], namely, the diagonality of the linear approximation of this canonical form enables efficient use of the Adomian decomposition method combined with Laplace transform.

The rest of the paper is organized as follows. The next section compiles the preliminaries and necessary definitions including some useful results needed later on. Section 3 presents the main results of the paper — the analytical expression of the solution of famous Rössler system and the above mentioned generalized Lorenz system. Computational applications of these expressions are presented here as well. Some conclusions and outlooks are drawn in the final section.

## 2. Definitions and Preliminary Results

The well-known concepts from Laplace transform theory and Adomian decomposition method will be repeated here. Furthermore, three algorithms to implement them will be suggested.

Let $\mathcal{R}, \mathcal{N}$ be the space of real and positive integers, correspondingly. Consider the autonomous dynamical system

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T$, $f : \mathcal{R}^n \to \mathcal{R}^n$, $f(x(t)) = [f_1(x(t)), \ldots, f_n(x(t))]^T$, and presume that initial conditions $x(0) = [x_1(0), x_2(0), \ldots, x_n(0)]^T$ are given. Decompose the right-hand side of (1) as follows:

$$\begin{aligned} f_i(x(t)) &= g_i(x_1(t), \ldots, x_n(t)) \\ &\quad + F_i(x_1(t), \ldots, x_n(t)), \quad i = 1, \ldots, n, \end{aligned} \tag{2}$$

where $g_i(x_1(t), \ldots, x_n(t))$ and $F_i(x_1(t), \ldots, x_n(t))$ are the linear and the nonlinear parts of $f_i(x_1(t), \ldots, x_n(t))$, respectively. The possible decomposition (2) is not unique and its selection may be used to facilitate the analysis described later on. Further, let $\mathcal{L}\{\phi(t)\}$ stand for the image of a smooth function $\phi(t)$ under Laplace transform and recall, that $\mathcal{L}\{\dot{\phi}(t)\} = s\mathcal{L}\{\phi(t)\} - \phi(0)$. Therefore, for all $i = 1, \ldots, n$,

$$\mathcal{L}\{\dot{x}_i(t)\} = sX_i(s) - x_i(0), \quad X_i(s) := \mathcal{L}\{x_i\}.$$

Applying Laplace transform to both sides of (1) and (2) gives by linearity of $g_i$'s for all $i = 1, \ldots, n$ the following equality

$$\begin{aligned} sX_i(s) - x_i(0) &= g_i(X_1(s), \ldots, X_n(s)) \\ &\quad + \mathcal{L}\{F_i(x_1, \ldots, x_n)\}, \end{aligned}$$

which, in turn, leads by straightforward computations to

$$\begin{aligned} X_i(s) &= \frac{x_i(0)}{s} + \frac{1}{s} g_i(X_1(s), \ldots, X_n(s)) \\ &\quad + \frac{1}{s}\mathcal{L}\{F_i(x_1, \ldots, x_n)\}, \\ X_i(s) &:= \mathcal{L}\{x_i\}. \end{aligned} \tag{3}$$

Now, the Adomian decomposition method to handle the nonlinear terms in (1) and (2) can be described as follows. First, represent each $x_i(t)$, $i = 1, \ldots, n$, as the following expansion

$$x_i(t) = \sum_{k=0}^{\infty} x_{ik}(t). \tag{4}$$

Further, introduce a parameter $\lambda$ into expansion (4) and write

$$x_i(t) = \sum_{k=0}^{\infty} \lambda^k x_{ik}(t), \quad i = 1, \ldots, n. \tag{5}$$

Parameter $\lambda \in \mathcal{R}$ is the so-called "formal grouping parameter" [Cherruault & Adomian, 1993] helping to obtain the expansion of the nonlinear terms $F_i(x)$, $i = 1, \ldots, n$, having favorable homogeneity properties. More specifically, the aim is to define the so-called *Adomian polynomials* $A_{ik}$, $i = 1, \ldots, n$, $k = 0, 1, \ldots$, such that

$$\begin{aligned} &F_i\left(\sum_{k=0}^{\infty} \lambda^k x_{1k}, \sum_{k=0}^{\infty} \lambda^k x_{2k}, \ldots, \sum_{k=0}^{\infty} \lambda^k x_{nk}\right) \\ &\quad = \sum_{k=0}^{\infty} \lambda^k A_{ik}. \end{aligned} \tag{6}$$

Each polynomial $A_{ik}$, $i = 1, \ldots, n$, is to be defined in such a way that it depends on $x_{i0}, x_{i1}, \ldots, x_{ik}$, $i = 1, \ldots, n$, only:

$$\begin{aligned} A_{i0} &= A_{i0}(x_{10}, \ldots, x_{n0}), \\ A_{i1} &= A_{i1}(x_{10}, \ldots, x_{n0}, x_{11}, \ldots, x_{n1}), \\ A_{i2} &= A_{i2}(x_{10}, \ldots, x_{n0}, x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}). \end{aligned} \tag{7}$$

In the resulting series for the solution $x_i = \sum_{k=0}^{\infty} \times \lambda^k x_{ik}$, each $x_{ik}$ is to be calculated recursively, namely $A_{ik}$, $k \geq 0$, $i = 1, \ldots, n$, are defined as follows

$$A_{i0}(x_{10}, x_{20}, \ldots, x_{n0}) = F_i(x_{10}, x_{20}, \ldots, x_{n0}), \quad i = 1, 2, \ldots, n,$$

$$A_{ik} = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[F_i\left(\sum_{j=0}^{k} \lambda^j x_{1j}, \sum_{j=0}^{k} \lambda^j x_{2j}, \ldots, \sum_{j=0}^{k} \lambda^j x_{nj}\right)\right]_{\lambda=0}, \quad k = 1, 2, \ldots, \quad i = 1, 2, \ldots, n.$$

Note, that Adomian polynomials $A_{ik}$ can be evaluated for all forms of smooth nonlinearity $F_i(x)$, $i = 1, \ldots, n$. In particular, setting $\lambda = 1$ in (6) gives the desired expansion of $F_i$, $i = 1, \ldots, n$, namely

$$F_i(x_1, \ldots, x_n) = \sum_{k=0}^{\infty} A_{ik}(x_{10}, \ldots, x_{n0}, \ldots, x_{1k}, \ldots, x_{nk}), \quad i = 1, \ldots, n. \tag{8}$$

**Example 2.1.** Consider system (1) and (2) with $n = 3$ and $F_1 \equiv 0$, $F_2 \equiv 0$, $F_3 \equiv F(x_1, x_3)$, i.e. the following nonlinear dynamical system (recall, that $g_i$'s stand for the linear part of the right-hand side)

$$\begin{aligned} \dot{x}_1 = g_1(x_1, x_2, x_3), \quad \dot{x}_2 = g_2(x_1, x_2, x_3) \\ \dot{x}_3 = g_3(x_1, x_2, x_3) + F(x_1, x_3). \end{aligned} \tag{9}$$

Note, that famous Rössler system is a particular case of the above system and therefore the current example would serve well to analyze it later on, cf. Sec. 3. As the nonlinearity is present in the last row only, $A_{ik} \equiv 0$, $i = 1, 2$; $k = 0, 1, \ldots$ and denote $A_{3k} := A_k$ for all $k \in \mathcal{N}$. Straightforward though laborious computations show that

$$\begin{aligned}
A_0 &= F(x_{10}, x_{30}), \\
A_1 &= x_{11}\frac{\partial F}{\partial x_1}(x_{10}, x_{30}) + x_{31}\frac{\partial F}{\partial x_3}(x_{10}, x_{30}), \\
A_2 &= x_{12}\frac{\partial F}{\partial x_1}(x_{10}, x_{30}) + x_{32}\frac{\partial F}{\partial x_3}(x_{10}, x_{30}) + \frac{1}{2!}x_{11}^2\frac{\partial^2 F}{\partial x_1^2}(x_{10}, x_{30}) \\
&\quad + \frac{1}{2!}x_{31}^2\frac{\partial^2 F}{\partial x_3^2}(x_{10}, x_{30}) + x_{11}x_{31}\frac{\partial^2 F}{\partial x_1 \partial x_3}(x_{10}, x_{30}) \\
A_3 &= x_{13}\frac{\partial F}{\partial x_1}(x_{10}, x_{30}) + x_{33}\frac{\partial F}{\partial x_3}(x_{10}, x_{30}) + x_{11}x_{12}\frac{\partial^2 F}{\partial x_1^2}(x_{10}, x_{30}) \\
&\quad + (x_{11}x_{32} + x_{12}x_{31})\frac{\partial^2 F}{\partial x_1 \partial x_3}(x_{10}, x_{30}) + x_{31}x_{32}\frac{\partial^2 F}{\partial x_3^2}(x_{10}, x_{30}) + \frac{1}{3!}x_{11}^3\frac{\partial^3 F}{\partial x_1^3}(x_{10}, x_{30}) \\
&\quad + \frac{1}{3!}x_{31}^3\frac{\partial^3 F}{\partial x_3^3}(x_{10}, x_{30}) + \frac{1}{2!}x_{11}^2 x_{31}\frac{\partial^3 F}{\partial x_1^2 \partial x_3}(x_{10}, x_{30}) + \frac{1}{2!}x_{11}x_{31}^2\frac{\partial^3 F}{\partial x_1 \partial x_3^2}(x_{10}, x_{30}),
\end{aligned} \tag{10}$$

while Adomian polynomials $A_4, A_5, \ldots$ can be computed analogously. This concludes the example.

Now, let us combine the Adomian technique with Laplace transform of linear part of (2). Using (3), (4) and (8)

$$\mathcal{L}\left\{\sum_{k=0}^{\infty} x_{ik}\right\} = \frac{x_i(0)}{s} + \frac{1}{s}\mathcal{L}\left\{g_i\left(\sum_{k=0}^{\infty} x_{1k}, \ldots, \sum_{k=0}^{\infty} x_{nk}\right)\right\} + \frac{1}{s}\mathcal{L}\left\{\sum_{k=0}^{\infty} A_{ik}\right\}, \quad i = 1, 2, \ldots, n. \tag{11}$$

Introducing $X_{ik}(s) := \mathcal{L}\{x_{ik}(t)\}$ and using the linearity of $g_i$'s, the equality (11) implies

$$\sum_{k=0}^{\infty} X_{ik}(s) = \frac{x_i(0)}{s} + \sum_{k=0}^{\infty} g_i\left(\frac{1}{s}X_{1k}(s), \ldots, \frac{1}{s}X_{nk}(s)\right) + \sum_{k=0}^{\infty}\frac{1}{s}\mathcal{L}\{A_{ik}\}, \quad i = 1, 2, \ldots, n. \tag{12}$$

Matching both sides of (12) can be done in several ways. Let us mention three of them. They will be referred to as the Algorithms 1–3 in the sequel. Algorithms 1–3 solve (12) formally algebraically only and convergence issues will be addressed when appropriate later on.

**Algorithm 1.** *This algorithm formally and explicitly solves* (*12*) *in the following way*:

$$X_{i0}(s) = \frac{x_i(0)}{s}, \quad i = 1, \ldots, n,$$

$$\begin{aligned} X_{i,k+1}(s) &= g_i\left(\frac{X_{1k}(s)}{s}, \ldots, \frac{X_{nk}(s)}{s}\right) \\ &\quad + \frac{1}{s}\mathcal{L}\{A_{ik}(x_{10}(t), \ldots, x_{n0}(t), \ldots, \\ &\quad x_{1k}(t), \ldots, x_{nk}(t)\}, \quad i = 1, \ldots, n, \end{aligned} \tag{13}$$

*where $k = 0, 1, \ldots$ is the Adomian expansion term number. Indeed, each recursive step explicitly computes $(k+1)$th Adomian expansion term via those* *already computed before. Note, that the recursive steps in (13) can be repeated analytically again and again in the following way. One can easily see using recursive computations that any $x_{ik}(t)$ has the form $x_{ik}(t) = c_{ik}t^k$, $c_{ik} \in \mathcal{R}$. Therefore by properties of Adomian polynomials, namely, by their homogeneity one has*

$$A_{ik}(x_{10}(t), \ldots, x_{n0}(t)) = \overline{A_{ik}}(c_{10}, \ldots, c_{n0})t^k,$$
$$\mathcal{L}\{A_{ik}\} = \overline{A_{ik}}\mathcal{L}\{t^k\} = \overline{A}_{ik}\frac{k!}{s^{k+1}}.$$

*In other words, one can see that $X_{i,k+1}(s) = \frac{c_{i,k+1}}{s^{k+2}}$, where $c_{i,k+1} \in \mathcal{R}$ are some real constants that can be determined recursively. Rather than giving the general expression with abusive notation, the reader is referred to examples presented later on, namely, to Rössler system and to the so-called generalized Lorenz system.*

**Algorithm 2.** *This algorithm formally and semi-explicitly solves (12) in the following way:*

$$X_{i0} = \frac{x_i(0)}{s} + \frac{1}{s}g_i(X_{10}(s), \ldots, X_{n0}(s)), \quad i = 1, \ldots, n,$$
$$X_{i,k+1}(s) = g_i\left(\frac{X_{1k}(s)}{s}, \ldots, \frac{X_{nk}(s)}{s}\right) + \frac{1}{s}\mathcal{L}\{A_{ik}(x_{10}(t), \ldots, x_{n0}(t), \ldots, x_{1k}(t), \ldots, x_{nk}(t)\}, \quad i = 1, \ldots, n, \tag{14}$$

*where $k = 0, 1, \ldots$ is the Adomian expansion term number. Indeed, only the initial step is implicit here and involves straightforward solving of the linear algebraic equation to get the first Adomian expansion term. All other recursive steps just determine explicitly $(k + 1)$th Adomian expansion terms via those already computed before. These recursive steps are the same as in Algorithm 1 and therefore similar comments apply to their computability as before.*

**Algorithm 3.** *This algorithm formally and implicitly solves (12) in the following way:*

$$X_{i0} = \frac{x_i(0)}{s} + \frac{1}{s}g_i(X_{10}(s), \ldots, X_{n0}(s)), \quad i = 1, \ldots, n,$$
$$X_{i,k+1}(s) - g_i\left(\frac{X_{1,k+1}(s)}{s}, \ldots, \frac{X_{n,k+1}(s)}{s}\right) = \frac{1}{s}\mathcal{L}\{A_{ik}(x_{10}(t), \ldots, x_{n0}(t), \ldots, x_{1k}(t), \ldots, x_{nk}(t)\}, \quad i = 1, \ldots, n, \tag{15}$$

*where $k = 0, 1, \ldots$ is the Adomian expansion term number. The above recursive relation can be regarded as the implicit one as its left-hand side is a linear function of $X_{1,k+1}, \ldots, X_{n,k+1}$, which is invertible except for isolated values of $s$, while the nonlinearity on the right-hand side depends on $X_{1,j}, \ldots, X_{n,j}$, $j = 0, 1, \ldots, k$, only. These recursive steps can be realized analytically (symbolically) as well. Indeed, each Adomian polynomial gives a generalized polynomial of time, so its Laplace transform can be computed analytically.*

Note, that Algorithm 2 is more computationally complex than Algorithm 1 while Algorithm 3 is more complex than Algorithm 2. Advantage of Algorithm 3 is, roughly saying, that it does not approximate any exponential by its expansion. Finally, note that all three algorithms obviously give the same solutions, they just filter differently the infinite sums into partial finite sums in order to match all terms in (12).

The convergence of the Adomian method is of great importance, nevertheless, its general analysis is not available, some partial results are given in [Cherruault & Adomian, 1993].

**Example 2.2.** Algorithm 1 was applied in [Hashim *et al.*, 2006] to the well-known Lorenz system

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= \beta z + xy, \end{aligned} \tag{16}$$

where $\sigma = 10.0$, $\rho = 28.0$, $\beta = 8/3$ is the well-known collection of parameters providing its chaotic behavior. The following explicit solution of Lorenz system (16) was obtained in [Hashim *et al.*, 2006]

$$\begin{aligned} x(t) &= \sum_{m=0}^{\infty} a_m \frac{t^m}{m!}, \\ y(t) &= \sum_{m=0}^{\infty} b_m \frac{t^m}{m!}, \\ z(t) &= \sum_{m=0}^{\infty} c_m \frac{t^m}{m!}, \end{aligned} \tag{17}$$

where the coefficients $a_m, b_m, c_m \in \mathcal{R}$ are given by the following recursive relations

$$
\begin{aligned}
&a_0 = x(0), \quad b_0 = y(0), \quad c_0 = z(0),\\
&a_m = -\sigma a_{m-1} + \sigma b_{m-1}, \quad m \geq 1,\\
&b_m = \rho a_{m-1} - b_{m-1}\\
&\qquad - (m-1)! \sum_{k=0}^{m-1} \frac{a_k c_{m-k-1}}{k!(m-k-1)!}, \quad m \geq 1,\\
&c_m = \beta c_{m-1} + (m-1)! \sum_{k=0}^{m-1} \frac{a_k b_{m-k-1}}{k!(m-k-1)!},\\
&\qquad\qquad m \geq 1.
\end{aligned}
\tag{18}
$$

Moreover, it was demonstrated in [Hashim *et al.*, 2006] that the 10-term decomposition solution is comparable to the fourth-order Runge–Kutta numerical solution even in the case of chaotic behavior of Lorenz system. This result will be extended in this paper both to the case of the well-known Rössler system [Gaspard, 2005] and to the so-called generalized Lorenz system [Čelikovský & Vaněček, 1994; Čelikovský & Chen, 2002, 2005].

## 3. Main Results

In this section, the main paper results are presented, namely, the computation of the Adomian representations for the well-known Rössler system and for the generalized Lorenz system [Čelikovský & Vaněček, 1994; Čelikovský & Chen, 2002, 2005]. Moreover, interesting analysis of the generalized Lorenz system with respect to values of certain parameters in its nonlinear part is provided using numerical computations stemming from the Adomian method. To start with, compute Adomian method based recurrent relations for the case of Rössler system [Gaspard, 2005]

$$
\begin{aligned}
\dot{x} = -y - z, \quad \dot{y} = x + ay,\\
\dot{z} = b - cz + xz.
\end{aligned}
\tag{19}
$$

Let $\mathcal{L}[\phi(t)]$ stand for Laplace transform of the function $\phi(t)$. Applying Laplace transform to all parts of Eqs. (19) gives

$$
\begin{aligned}
\mathcal{L}[\dot{x}] = -\mathcal{L}[y] - \mathcal{L}[z], \quad \mathcal{L}[\dot{y}] = \mathcal{L}[x] + a\mathcal{L}[y],\\
\mathcal{L}[\dot{z}] = \mathcal{L}[b] - c\mathcal{L}[z] + \mathcal{L}[xz].
\end{aligned}
\tag{20}
$$

Denote $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$, $Z(s) = \mathcal{L}\{z(t)\}$, then (20) leads to

$$
\begin{aligned}
sX(s) - x(0) &= -Y(s) - Z(s),\\
sY(s) - y(0) &= X(s) + aY(s),\\
sZ(s) - z(0) &= \frac{b}{s} - cZ(s) + \mathcal{L}[xz]
\end{aligned}
\tag{21}
$$

and one has by straightforward computations that

$$
\begin{aligned}
X(s) &= \frac{x(0)}{s} - \frac{1}{s}Y(s) - \frac{1}{s}Z(s),\\
Y(s) &= \frac{y(0)}{s} + \frac{1}{s}X(s) + \frac{a}{s}Y(s),\\
Z(s) &= \frac{z(0)}{s} + \frac{b}{s^2} - \frac{c}{s}Z(s) + \frac{1}{s}\mathcal{L}[xz].
\end{aligned}
\tag{22}
$$

To derive the recurrent relations solving (22), let us represent the solution in the time domain as follows:

$$
\begin{aligned}
x(t) = \sum_{k=0}^{\infty} \bar{x}_k(t), \quad y(t) = \sum_{k=0}^{\infty} \bar{y}_k(t),\\
z(t) = \sum_{k=0}^{\infty} \bar{z}_k(t).
\end{aligned}
\tag{23}
$$

The overbar introduced in (23) is meant to distinguish the expansion functions from the expansion coefficients introduced later on. To satisfy initial conditions $x(0)$, $y(0)$, $z(0)$, the terms $\bar{x}_k(t)$, $\bar{y}_k(t)$, $\bar{z}_k(t)$, $k \in \mathcal{R}$ in (23) are to be computed in such a way that

$$\bar{x}_0(0) = x(0), \quad \bar{y}_0(0) = y(0), \quad \bar{z}_0(0) = z(0), \tag{24}$$

$$\bar{x}_k(0) = \bar{y}_k(0) = \bar{z}_k(0) = 0, \quad k = 1, 2, \ldots. \tag{25}$$

Next, to handle the nonlinear term $\mathcal{L}[xz]$ note, that (19) has the same structure as Example 2.1 with $x_1 := x$, $x_2 := y$, $x_3 := z$ and $F := xz$. Therefore, using (10) with $x_1 := x$, $x_2 := y$, $x_3 := z$, one gets straightforwardly

$$A_k = \sum_{i=0}^{k} \bar{x}_i \bar{z}_{k-i}, \quad k \geq 0, \quad \mathcal{L}[xz] = \sum_{k=0}^{\infty} \mathcal{L}[A_k].$$

The above evaluations allow to present the recurrent procedure solving the Rössler system. For simplicity and shortness, the expansion solving (22) will be obtained using Algorithm 1, but Algorithms 2 and 3 may be applied with some modifications as well. Denote again $X_k(s) := \mathcal{L}[\bar{x}_k(t)]$,

$Y_k(s) := \mathcal{L}[\bar{y}_k(t)]$, $Z_k(s) := \mathcal{L}[\bar{z}_k(t)]$, $k = 0, 1, \ldots$. Based on Algorithm 1, the relation (22) is decomposed into the following series of relations

$$\begin{aligned}
X_0(s) &= \frac{x(0)}{s}, \quad Y_0(s) = \frac{y(0)}{s}, \quad Z_0(s) = \frac{z(0)}{s},\\
X_1(s) &= -\frac{1}{s}(X_0(s) + Z_0(s)),\\
Y_1(s) &= \frac{1}{s}(X_0(s) + aY_0(s)),\\
Z_1(s) &= \frac{b}{s^2} - \frac{1}{s}(cZ_0(s) - \mathcal{L}[A_0]),\\
X_{k+1}(s) &= -\frac{1}{s}(X_k(s) + Z_k(s)),\\
Y_{k+1}(s) &= \frac{1}{s}(X_k(s) + aY_k(s)),\\
Z_{k+1}(s) &= -\frac{1}{s}(cZ_k(s) - \mathcal{L}[A_k]), \quad k = 1, 2, \ldots.
\end{aligned} \tag{26}$$

To solve the above series of relations, one can proceed as follows. First, determine $\bar{x}_0(t)$, $\bar{y}_0(t)$, $\bar{z}_0(t)$ and define $x_0, y_0, z_0 \in \mathcal{R}$ in such a way that:

$$\bar{x}_0(t) \equiv x(0) := x_0, \quad \bar{y}_0(t) \equiv y(0) := y_0,$$
$$\bar{z}_0(t) \equiv z(0) := z_0.$$

Next, to determine $\bar{x}_1(t)$, $\bar{y}_1(t)$, $\bar{z}_1(t)$, note, that $\mathcal{L}[A_0] = \mathcal{L}[\bar{x}_0(t)\bar{z}_0(t)] = \frac{1}{s}x_0 z_0$, and therefore

$$\begin{aligned}
X_1(s) &= -\frac{1}{s^2}(x_0 + y_0),\\
Y_1(s) &= \frac{1}{s^2}(x_0 + ay_0),\\
Z_1(s) &= \frac{b}{s^2} - \frac{1}{s^2}(cz_0 - x_0 z_0).
\end{aligned}$$

This allows to determine $\bar{x}_1(t)$, $\bar{y}_1(t)$, $\bar{z}_1(t)$ and to define $x_1, y_1, z_1$:

$$\bar{x}_1(t) = x_1 t, \quad \bar{y}_1(t) = y_1 t, \quad \bar{z}_1(t) = z_1 t,$$
$$x_1 := x_0 + y_0, \quad y_1 := x_0 + ay_0,$$
$$z_1 := b - cz_0 + x_0 z_0.$$

Note, that (25) with $k = 1$ was used here, i.e. $\bar{x}_1(0) = \bar{y}_1(0) = \bar{z}_1(0) = 0$. Furthermore, $\bar{x}_2(t)$, $\bar{y}_2(t)$, $\bar{z}_2(t)$ and $x_2, y_2, z_2 \in \mathcal{R}$ are determined using

$$\begin{aligned}
X_2(s) &= -\frac{1}{s}(X_1(s) + Z_1(s)) = -\frac{1}{s^3}(x_1 + z_1),\\
Y_2(s) &= \frac{1}{s}(X_1(s) + aZ_1(s)) = \frac{1}{s^3}(x_1 + az_1),\\
Z_2(s) &= -\frac{1}{s}(cZ_1(s) - \mathcal{L}[A_1])\\
&= -\frac{1}{s^3}(cz_1 - \mathcal{L}[(x_0 z_1 + x_1 z_0)t])\\
&= \frac{1}{s^3}(x_0 z_1 + x_1 z_0) - \frac{cz_1}{s^3}.
\end{aligned} \tag{27}$$

In other words

$$\begin{aligned}
X_2(s) &= \frac{x_2}{s^3}, \quad Y_2(s) = \frac{y_2}{s^3}, \quad Z_2(s) = \frac{z_2}{s^3},\\
&x_2 := -x_1 - z_1, \quad y_2 := x_1 + ay_1,\\
&z_2 := -cz_1 + x_0 z_1 + x_1 z_0,\\
\bar{x}_2(t) &= x_2\frac{t^2}{2}, \quad \bar{y}_2(t) = y_2\frac{t^2}{2}, \quad \bar{z}_2(t) = z_2\frac{t^2}{2}.
\end{aligned} \tag{28}$$

The last relation uses (25) with $k = 2$. To obtain the general terms $X_{k+1}(s)$, $Y_{k+1}(s)$, $Z_{k+1}(s)$, $\bar{x}_{k+1}(t)$, $\bar{y}_{k+1}(t)$, $\bar{z}_{k+1}(t)$ and $x_{k+1}$, $y_{k+1}$, $z_{k+1}$, $k = 2, 3, \ldots$, consider

$$\begin{aligned}
\mathcal{L}[A_k] &= \mathcal{L}\left[\sum_{i=0}^{k} \bar{x}_i(t)\bar{z}_{k-i}(t)\right]\\
&= \mathcal{L}\left[\sum_{i=0}^{k} x_i z_{k-i}\frac{t^i}{i!}\frac{t^{k-i}}{(k-i)!}\right]\\
&= \sum_{i=0}^{k} \frac{x_i z_{k-i}}{i!(k-i)!}\mathcal{L}[t^k]\\
&= \frac{k!}{s^{k+1}}\sum_{i=0}^{k} x_i z_{k-i}\frac{1}{i!(k-i)!}.
\end{aligned}$$

Therefore one has for $k = 2, 3, \ldots$

$$\begin{aligned}
X_{k+1}(s) &= \frac{x_{k+1}}{s^{k+2}}, \quad Y_{k+1} = \frac{y_{k+1}}{s^{k+2}},\\
Z_{k+1}(s) &= \frac{z_{k+1}}{s^{k+2}},\\
&x_{k+1} := -x_k - z_k, \quad y_{k+1} := x_k + ay_k,\\
&z_{k+1} := -cz_k + \sum_{i=0}^{k} x_i z_{k-i}\frac{k!}{i!(k-i)!},\\
\bar{x}_{k+1}(t) &= x_{k+1}\frac{t^{k+1}}{(k+1)!},
\end{aligned}$$

$$\bar{y}_{k+1}(t) = y_{k+1}\frac{t^{k+1}}{(k+1)!},$$

$$\bar{z}_{k+1}(t) = z_{k+1}\frac{t^{k+1}}{(k+1)!}. \tag{29}$$

Summarizing, the following theorem has been just proved.

**Theorem 1.** *Consider Rössler system*

$$\dot{x} = -y - z,$$
$$\dot{y} = x + ay, \tag{30}$$
$$\dot{z} = b - cz + xz.$$

*Then the formal solution of (30) starting from the initial conditions* $x(0), y(0), z(0)$ *is given by the following series*

$$x(t) = \sum_{k=0}^{\infty} x_k \frac{t^k}{k!}, \tag{31}$$

$$y(t) = \sum_{k=0}^{\infty} y_k \frac{t^k}{k!}, \tag{32}$$

$$z(t) = \sum_{k=0}^{\infty} z_k \frac{t^k}{k!}, \tag{33}$$

*where coefficients* $x_k, y_k, z_k$ *are given by the following recurrent relations*

$$x_0 = x(0), \quad y_0 = y(0), \quad z_0 = z(0), \tag{34}$$

$$x_1 = -y_0 - z_0, \quad y_1 = x_0 + ay_0,$$
$$z_1 = b - cz_0 + x_0 z_0, \tag{35}$$

$$x_{k+1} = -y_k - z_k, \quad y_{k+1} = x_k + ay_k, \quad k \geq 1, \tag{36}$$

$$z_{k+1} = -cz_k + \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} x_i z_{k-i}, \quad k \geq 1. \tag{37}$$

To illustrate Theorem 1, recurrent relations (34)–(37), were evaluated numerically for the case $k = 0, 1, \ldots, 30$ and discretized by $t = 0, 1, \ldots, 800$. The results are shown in Fig. 1. The graphical output shown there is comparable to the similar results in the literature, [Gaspard, 2005; Wikipedia, 2015; Shinn, 2010].

Another object of our investigation is the so-called generalized Lorenz system introduced in the series of papers [Čelikovský & Vaněček, 1994; Čelikovský & Chen, 2002, 2005] and it is defined as follows.

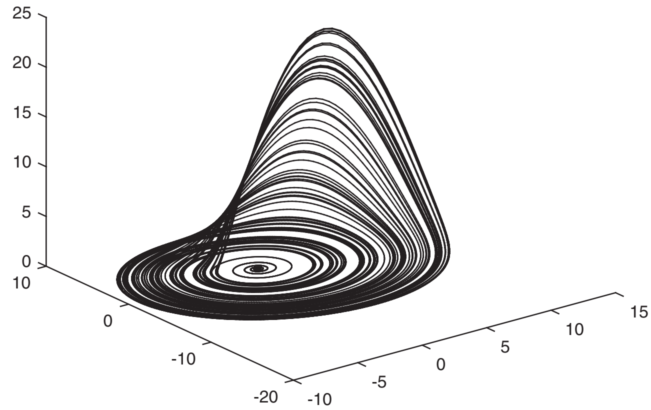


Fig. 1. Rössler system.

**Definition 3.1.** The nonlinear system of ordinary differential equations in $\mathcal{R}^3$ of the following form is called the *generalized Lorenz system*:

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & \lambda_3 \end{bmatrix} x + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x, \tag{38}$$

where $x = [x_1, x_2, x_3]^T$, $\lambda_3 \in \mathcal{R}$, and $A$ is a $(2 \times 2)$ real matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{39}$$

with eigenvalues $\lambda_1, \lambda_2 \in \mathcal{R}$ such that

$$-\lambda_2 > \lambda_1 > -\lambda_3 > 0. \tag{40}$$

Moreover, the generalized Lorenz system is said to be *nontrivial* if it has at least one solution that goes neither to zero nor to infinity nor to a limit cycle.

For the nontrivial generalized Lorenz system, the following theorem was proved [Čelikovský & Chen, 2002].

**Theorem 2.** *For the nontrivial generalized Lorenz system (38)–(40), there exists a nonsingular linear change of coordinates,* $z = Tx$, *which takes (38) into the following generalized Lorenz canonical form*

$$\dot{z} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} z + (1, -1, 0)z \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & \tau & 0 \end{bmatrix} z, \tag{41}$$

*where* $z = [z_1, z_2, z_3]^T$ *and parameter* $\tau \in (-1, \infty)$.

The form (41) is usually referred to as the *generalized Lorenz canonical form* [Čelikovský & Chen, 2002] and it can be rewritten as follows

$$\begin{aligned}\dot{z}_1 &= \lambda_1 z_1 - (z_1 - z_2)z_3\\ \dot{z}_2 &= \lambda_2 z_2 - (z_1 - z_2)z_3\\ \dot{z}_3 &= \lambda_3 z_3 + (z_1 - z_2)z_1 + \tau(z_1 - z_2)z_2.\end{aligned} \tag{42}$$

It is worthwhile to mention the dependance of the system nonlinearity only on one scalar parameter $\tau$ that drives the whole nonlinear dynamics of the system. The system (42) will be used to study the generalized Lorenz system by the Adomian approach. Another significant advantage here is that the linear part of (42) is easily solvable due to its diagonal matrix. Analogously to Rössler system, the solution of (42) is sought in the form

$$z_1(t) = \sum_{k=0}^{\infty} \overline{z}_{1k}(t) = \sum_{k=0}^{\infty} z_{1k}\frac{t^k}{k!}, \tag{43}$$

$$z_2(t) = \sum_{k=0}^{\infty} \overline{z}_{2k}(t) = \sum_{k=0}^{\infty} z_{2k}\frac{t^k}{k!}, \tag{44}$$

$$z_3(t) = \sum_{k=0}^{\infty} \overline{z}_{3k}(t) = \sum_{k=0}^{\infty} z_{3k}\frac{t^k}{k!}. \tag{45}$$

Again, variables with overbars stand for the functions while those without bars are real numbers, called as the expansion coefficients. First, set $z_{10} = z_1(0)$, $z_{20} = z_2(0)$, $z_{30} = z_3(0)$, where $z_1(0)$, $z_2(0)$, $z_3(0)$ are the initial conditions of the problem (42). Next, note that the Adomian polynomials decomposition for the nonlinear part of (42) is as follows:

$$\begin{aligned}&(z_1 - z_2)z_1\\ &\quad= \sum_{k=0}^{\infty} A_{jk}\\ &\quad= \sum_{k=0}^{\infty}\sum_{i=0}^{k} (\overline{z}_{1i} - \overline{z}_{2i})\overline{z}_{1,k-i}, \quad j = 1, 2,\\ &(z_1 - z_2)z_1 + \tau(z_1 - z_2)z_2\\ &\quad= \sum_{k=0}^{\infty} A_{3k}\\ &\quad= \sum_{k=0}^{\infty}\sum_{i=0}^{k} (\overline{z}_{1i} - \overline{z}_{2i})\overline{z}_{1,k-i}\\ &\qquad+ \tau\sum_{k=0}^{\infty}\sum_{i=0}^{k} (\overline{z}_{1i} - \overline{z}_{2i})\overline{z}_{2,k-i}.\end{aligned} \tag{46}$$

The expansion coefficients $z_{1k}$, $z_{2k}$, $z_{3k}$ in (43) are therefore as follows

$$z_{10} = z_1(0), \quad z_{20} = z_2(0), \quad z_{30} = z_3(0), \tag{47}$$

$$\begin{aligned}z_{11} &= \lambda_1 z_{10} - A_{10},\\ z_{21} &= \lambda_2 z_{20} - A_{20},\\ z_{31} &= \lambda_3 z_{30} + A_{30},\end{aligned} \tag{48}$$

$$\begin{aligned}z_{1,k+1} &= \lambda_1 z_{1k} - \sum_{i=0}^{k} \frac{k!}{i!(k-i)!}\\ &\quad\times (z_{1i} - z_{2i})z_{3,k-i}, \quad k \geq 1,\end{aligned} \tag{49}$$

$$\begin{aligned}z_{2,k+1} &= \lambda_2 z_{2k} - \sum_{i=0}^{k} \frac{k!}{i!(k-i)!}\\ &\quad\times (z_{1i} - z_{2i})z_{3,k-i}, \quad k \geq 1,\end{aligned} \tag{50}$$

$$\begin{aligned}z_{3,k+1} &= \lambda_3 z_{3k} + \sum_{i=0}^{k} \frac{k!}{i!(k-i)!}(z_{1i} - z_{2i})z_{1,k-i}\\ &\quad+ \tau\sum_{i=0}^{k} \frac{k!}{i!(k-i)!}(z_{1i} - z_{2i})z_{2,k-i}, \quad k \geq 1.\end{aligned} \tag{51}$$

Detailed derivation of the above recursive formulas mimics that of Rössler system earlier in this paper and therefore they are skipped for shortness. The above evaluation of the generalized Lorenz canonical form (42) was clearly based on Algorithm 1 described in Sec. 2. Nevertheless, the major advantage of the generalized Lorenz canonical form is the diagonality of its linear approximation, see [Čelikovský & Chen, 2002] for discussion. Yet another advantage of that diagonality is that Algorithm 3 described in Sec. 2 of the current paper is easily implementable. Indeed, one has obviously by (42) and the mentioned Algorithm 3 described in Sec. 2 that

$$Z_i^0(s) = \frac{z_i(0)}{s - \lambda_i}, \quad i = 1, 2, 3, \ldots.$$

Next, one has by (42) and (46) such that

$$Z_1^1(s) = \frac{z_2^0 z_3^0}{\lambda_2+\lambda_3-\lambda_1}\left(\frac{1}{s-\lambda_2-\lambda_3}-\frac{1}{s-\lambda_1}\right) - \frac{z_1^0 z_3^0}{\lambda_1+\lambda_3-\lambda_1}\left(\frac{1}{s-\lambda_1-\lambda_3}-\frac{1}{s-\lambda_1}\right),$$

$$Z_2^1(s) = \frac{z_2^0 z_3^0}{\lambda_2+\lambda_3-\lambda_2}\left(\frac{1}{s-\lambda_2-\lambda_3}-\frac{1}{s-\lambda_2}\right) - \frac{z_1^0 z_3^0}{\lambda_1+\lambda_3-\lambda_2}\left(\frac{1}{s-\lambda_1-\lambda_3}-\frac{1}{s-\lambda_2}\right),$$

$$Z_3^1(s) = \frac{(z_1^0)^2}{2\lambda_1-\lambda_3}\left(\frac{1}{s-2\lambda_1}-\frac{1}{s-\lambda_3}\right) - \frac{\tau(z_2^0)^2}{2\lambda_2-\lambda_3}\left(\frac{1}{s-2\lambda_2}-\frac{1}{s-\lambda_3}\right) + \frac{(\tau-1)z_1^0 z_2^0}{\lambda_1+\lambda_2-\lambda_3}\left(\frac{1}{s-\lambda_1-\lambda_2}-\frac{1}{s-\lambda_3}\right).$$

Finally, assume recursively, that it holds for some $k \geq 1$ and $i = 1, 2, 3$

$$Z_i^k(s) = \sum_{l=1}^{N_k} \frac{L_l^{i,k}}{(s-\alpha_l^{i,k})^{p_l^{i,k}}}, \quad L_l^{i,k} \in \mathcal{R},\ \ \alpha_l^{i,k} \in \mathcal{R},\ \ p_l^{i,k} \in \mathcal{N},\ N_k \in \mathcal{N},\ l = 1, \ldots, N_k.$$

Indeed, both $Z_i^0(s)$ and $Z_i^1(s)$, $i = 1, 2, 3$, have the above form. Now, one has by (42) and (46) such that

$$Z_i^{k+1}(s) = \frac{1}{s-\lambda_i}\mathcal{L}(A_k^i), \quad i = 1, 2, 3.$$

Note, that (here $\mathcal{L}^{-1}$ stands for the inverse Laplace transform)

$$A_k^i = A_k^i(\mathcal{L}^{-1}(Z_1^0), \mathcal{L}^{-1}(Z_2^0), \mathcal{L}^{-1}(Z_3^0), \ldots, \mathcal{L}^{-1}(Z_1^k), \mathcal{L}^{-1}(Z_2^k), \mathcal{L}^{-1}(Z_3^k)) = \sum_{j=0}^{k}(\mathcal{L}^{-1}(Z_1^j) - \mathcal{L}^{-1}(Z_2^j))\mathcal{L}^{-1}(Z_i^{k-j}), \quad i = 1, 2, 3.$$

Moreover, $\mathcal{L}^{-1}(Z_i^j)$, $i = 1, 2, 3$, $j = 0, 1, \ldots, k$, is by the above recursive assumption a generalized polynomial (sum of monomials containing powers of $t$ and exponents of $t$) and therefore $\mathcal{L}(A_k^i)$, $i = 1, 2, 3$, is again of the form

$$\mathcal{L}(A_k^i) = \mathcal{L}\left(\sum_{j=0}^{k}(\mathcal{L}^{-1}(Z_1^j) - \mathcal{L}^{-1}(Z_2^j))\mathcal{L}^{-1}(Z_i^{k-j})\right) = \sum_{l=1}^{\overline{N}_k}\frac{\overline{L}_l^{i,k}}{(s-\overline{\alpha}_l^{i,k})^{\overline{p}_l^{i,k}}}, \quad \overline{L}_l^{i,k} \in \mathcal{R},\ \overline{\alpha}_l^{i,k} \in \mathcal{R},\ \overline{p}_l^{i,k} \in \mathcal{N},\ \overline{N}_k \in \mathcal{N},\ l = 1, \ldots, \overline{N}_k.$$

Finally, the constants $\overline{L}_l^{i,k}$, $\overline{\alpha}_l^{i,k}$, $\overline{p}_l^{i,k}$, $\overline{N}_k$, are the explicit functions of $L_l^{i,k}$, $\alpha_l^{i,k}$, $p_l^{i,k}$, $N_k$. Therefore,

$$Z_i^{k+1}(s) = \frac{1}{s-\lambda_i}\sum_{l=1}^{\overline{N}_k}\frac{\overline{L}_l^{i,k}}{(s-\overline{\alpha}_l^{i,k})^{\overline{p}_l^{i,k}}} := \sum_{l=1}^{N_{k+1}}\frac{L_l^{i,k+1}}{(s-\alpha_l^{i,k+1})^{p_l^{i,k+1}}},$$

where $L_l^{i,k+1} \in \mathcal{R}$, $\alpha_l^{i,k+1} \in \mathcal{R}$, $N_{k+1} \in \mathcal{N}$, $p_l^{i,k+1} \in \mathcal{N}$, $l = 1, \ldots, N_{k+1}$, are some constants depending on $L_l^{i,k}$, $\alpha_l^{i,k}$, $p_l^{i,k}$, $N_k$, $l = 1, \ldots, N_k$. The corresponding explicit recursive expressions can be easily obtained, but they are skipped for shortness. In such a way, taking inverse Laplace transform one has that the solution of (42) is approximated by generalized polynomials uniquely defined by $N_k \in \mathcal{N}$, $L_l^{i,k}, \alpha_l^{i,k}, p_l^{i,k}, l = 1, \ldots, N_k, k = 1, 2, \ldots, i = 1, 2, 3$.

Graphical results of the simulations performed by Algorithm 1 of Sec. 2 are presented in Figs. 2–6.

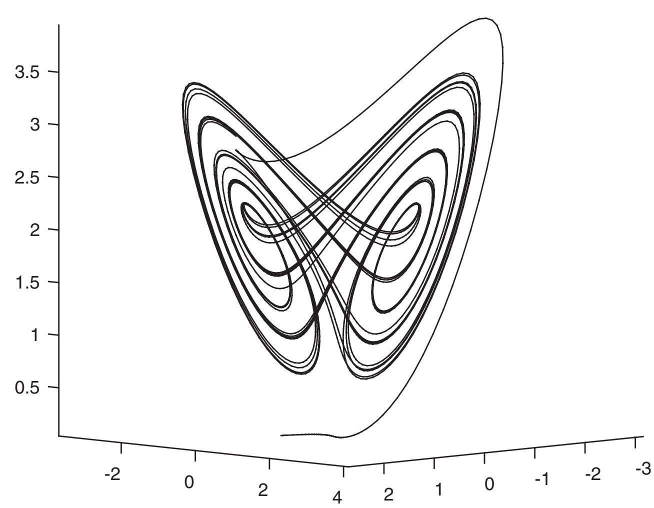


Fig. 2. Generalized Lorenz system $\lambda_1 = -3$, $\lambda_2 = 5$, $\lambda_3 = 1$, $\tau = 0$.

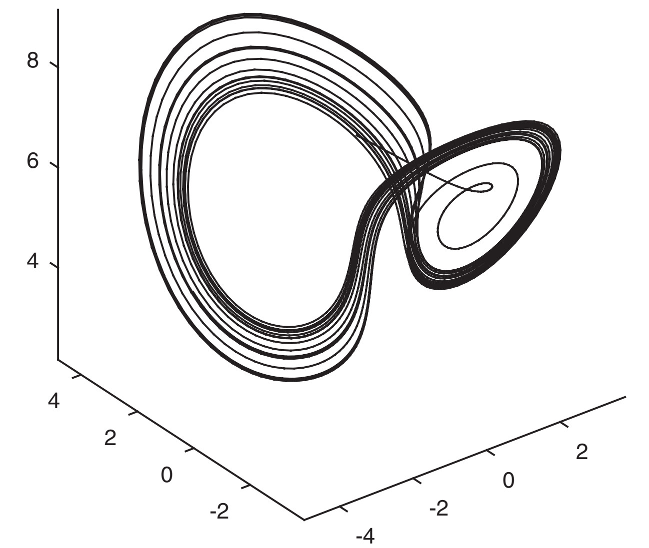


Fig. 3. Generalized Lorenz system $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$, $\tau = 0$.

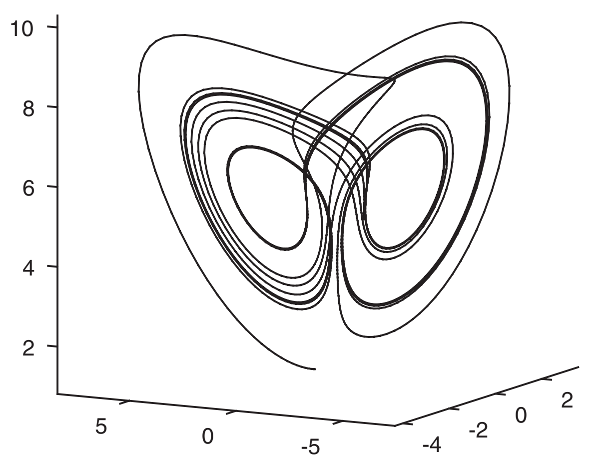


Fig. 4. Generalized Lorenz system $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$, $\tau = 0.168$.

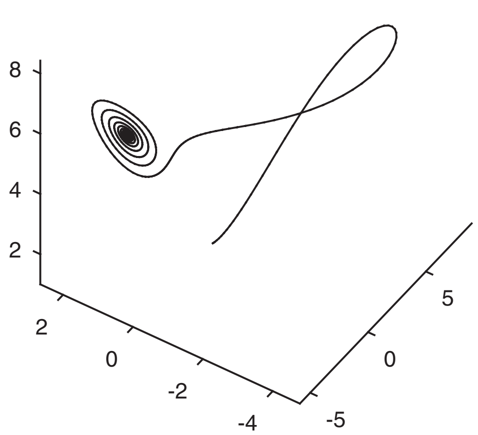


Fig. 5. Generalized Lorenz system $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$, $\tau = 0.9$.

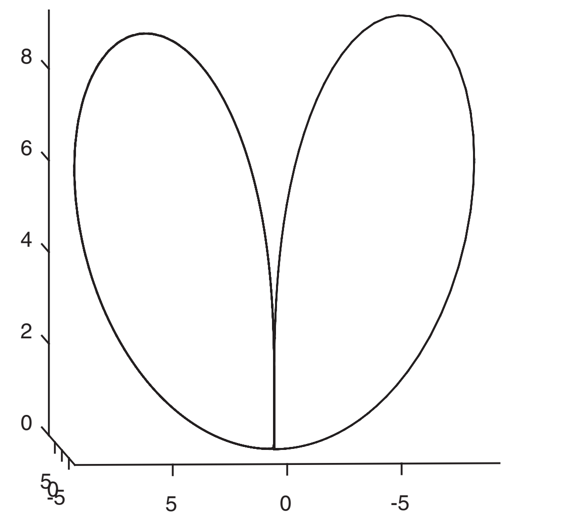


Fig. 6. Generalized Lorenz system $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -7$, $\tau = 0.017889$.

Note, that these results are astonishingly similar to the simulations made in [Čelikovský & Chen, 2002] using Runge–Kutta numerical schemes.

Let us shortly comment on these results. Figures 2 and 3 show the so-called "simplest chaotical systems" introduced already in [Čelikovský & Chen, 2002]. Indeed, these systems have their right-hand sides with integer parameters only. Moreover, the attractor in Fig. 3 has completely different topology than the classical Lorenz system. As a matter of fact, as already noted in [Čelikovský & Chen, 2002, 2005], these cases correspond to the well-known Chen system [Chen & Ueta, 1999] in different coordinates.

Figure 6 then illustrates typical chaos near the homoclinicity, as predicted by Shilnikov [Shilnikov, 1969; Shilnikov *et al.*, 2004, 2009], see e.g. [Wiggins, 1988, 2003] for nice and detailed presentation of that theory. Indeed, zooming in, Fig. 6 shows chaotic-like behavior, though the attractor is very narrow and approximates the homoclinic orbit as well. Note, that the unpredictability of behavior is related exclusively to a small neighborhood of the origin, where trajectory turns to the left or right, depending on a very small state of differences. The remaining parts of the trajectories behave in a well-predictable way.

## 4. Conclusions and Outlooks

This paper has demonstrated that even in the case of nonlinear systems there are analytical-algebraic approaches of getting results in a fashion resembling

Laplace transform technique for linear systems. In particular, Rössler system and generalized Lorenz system were represented and simulated in their chaotic regimes using Laplace–Adomian decomposition method. The major advantage here is that one can compute highly precise approximation on the pre-selected time interval and enhance this precision inside selected time subintervals only. In other words, precision at some time moment does not affect the precision at another time moment. Other interesting future applications may be in the area of chaos-based cryptography replacing the usual chaotic differential equation by its approximation by properly truncated Adomian based iterative expansion.

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