Journal of Economic Dynamics & Control 85 (2017) 21–45

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Estimation of financial agent-based models with simulated maximum likelihood

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ARTICLE INFO

Article history:
Received 3 June 2016
Revised 2 June 2017
Accepted 22 September 2017
Available online 29 September 2017

JEL classification:
C14
C51
C63
D84
G02
G12

Keywords:
Heterogeneous agent model
Simulated maximum likelihood
Estimation
Intensity of choice
Switching

ABSTRACT

This paper proposes a general computational framework for empirical estimation of financial agent-based models, for which criterion functions have unknown analytical form. For this purpose, we adapt a recently developed nonparametric simulated maximum likelihood estimation based on kernel methods. In combination with the model developed by Brock and Hommes (1998), which is one of the most widely analysed heterogeneous agent models in the literature, we extensively test the properties and behaviour of the estimation framework, as well as its ability to recover parameters consistently and efficiently using simulations. Key empirical findings indicate the statistical insignificance of the switching coefficient but markedly significant belief parameters that define heterogeneous trading regimes with a predominance of trend following over contrarian strategies. In addition, we document a slight proportional dominance of fundamentalists over trend-following chartists in major world markets.

1. Introduction

After the failure of traditional financial models in the major financial crisis of 2007–2008, agent-based approaches attracted greater attention from both academicians and practitioners and hence have gradually replaced traditional financial models in the recent literature. The financial agent-based models (FABMs)[1] in particular reflects the well-documented and systematic human departure from the representative agent's complete rationality towards reasonably realistically bounded rationality (Simon, 1957). An essential achievement of this field is the ability to replicate the stylised facts of financial data and account for the emergence of asset market bubbles followed by sudden crashes. Recently, numerous projects have proposed a courageous attempt to complement or even alternate current mainstream policy-making models with agent-based approaches, which is possible only if one can estimate these models with empirical data.

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[1] For a general overview of financial agent-based modelling and its development, Chen et al. (2012); Hommes (2006), and LeBaron (2006) provide excellent surveys.

https://doi.org/10.1016/j.jedc.2017.09.006

Although empirical estimation is an important part of the modelling cycle and is crucial for model validation, empirical estimation of FABMs is still in its nascence. Looking at the last fifteen years of financial literature, we observe neither a general consensus on the estimation methodology nor conclusive results. Fagiolo et al. (2007, pg. 202) even emphasise that there is “no consensus at all about how (and if) agent-based models should be empirically validated.” Generally, there are two critical challenges in the estimation. First, the highly nonlinear and complex nature of these systems prevents researchers from using classical estimation methods, because the objective function often has no analytical expression. Second, possible over-parametrisation, the high number of degrees of freedom, and optional model settings, as well as stochastic dynamics, further escalate the complexity of the problem. The emerging properties of these models cannot be analytically deduced, and a possible estimation via method of moments, “while fine in theory, might be too computationally costly to undertake” (LeBaron and Tesfatsion, 2008, pg. 249). Thus, a considerable simulation capacity for the numerical analysis is required.

Literature focusing on the estimation of FABMs attempts to use several direct and indirect estimation methods. In terms of direct estimation, the nonlinear least squares and quasi maximum likelihood are applied in most cases rather than the classical ordinary least squares or maximum likelihood due to the complexity of the models. In these applications, the key structural features of agent-based models are sometimes restrained to obtain a simplified approach that can be estimated directly. However, for many models, the aggregation equation, which would contain all parameters of interest, cannot be derived analytically and therefore the application of direct estimation techniques is not feasible. Hence, indirect estimation based on the simulation of artificial data by the model through which the aggregation concepts such as moments are derived is used instead. Simulation-based methods “are very applicable and may dramatically open the empirical accessibility of agent-based models in the future,” as noted by Chen et al. (2012, pg. 204). However, the use of these methods for validation of agent-based models in economics is relatively rare thus far. Simulation-based methods already used for estimation include the method of simulated moments, or generally the simulated minimum distance (Grazzini and Richiardi, 2015), which are based on minimising the weighted distance between two sets of simulated and observed moments. However, as noted by Fernández-Villaverde and Rubio-Ramírez (2010, pg. 23), the main difficulty remains in the selection of proper moments characterising the parameters, because the selection of different moments may lead to considerably different point estimates. Grazzini and Richiardi (2015, pg. 151) further state that although simulation-based maximum likelihood estimation is more complex, it could be used instead. This option has not yet been explored in the literature. Alternatively, Grazzini et al. (2017) also propose simulated likelihood methods for agent-based models (ABMs), but from a Bayesian inference perspective. They use non-parametric kernel density estimation for approximating the likelihood function and apply their methodology to a simple stock market model with one parameter and to a behavioural macroeconomic model with nine parameters.

This paper takes a step forward and proposes a more general computational framework for empirical validation of full-fledged FABMs utilising a non-parametric simulated maximum likelihood estimator (NPSMLE) recently developed by Kristensen and Shin (2012). The main advantage of this framework is that under general conditions met by FABMs, it can approximate the conditional density of the data-generating process from numerically simulated observations. Thus, the unknown likelihood function can be replaced by the simulated likelihood in the estimation and parameters can be recovered in a traditional manner. We extensively test the capability of this method for FABMs estimation purposes using a large Monte Carlo study. We marry the customised estimation methodology with the most widely analysed model of Brock and Hommes (1998). The key feature of the model is the evolutionary switching of agents between simple trading strategies based on previously realised profits, so called adaptive belief system, governed by the switching parameter of the intensity of choice. This parameter is responsible for the high nonlinearity of the system and possibly even chaotic price motion. We show that the NPSMLE successfully estimates the switching parameter in this generally challenging framework. We thus presuppose that it is likely to be more generalisable and useful for estimation of other agent-based models in the future.

2. The route to empirical estimation of FABMs: a short review of existing approaches

The design of FABMs is to a large extent motivated by empirical evidence accumulated in the late 1980s and early 1990s about the behaviour of real financial agents (Allen and Taylor, 1990; Frankel and Froot, 1990). These studies conclude that interactions between two main types of expectations control the dynamics of financial markets. Fundamental traders, who believe that possible mispricing is likely to be corrected over short periods by arbitrageurs and thus the market price tends to revert to its fundamental value, constitute a stabilising market force. Technical analysts, also called ‘noise traders’ or chartists, believe that a currently observable trend will continue in the short-run; these analysts constitute a destabilising market force that is responsible for the development of speculative bubbles. These two types of traders could be understood as available trading strategies, because an intelligent market agent is likely to adapt his or her strategy over time based on its relative historical performance. The time-varying evolution of market fractions between these two types of trading strategies is thus the essence of many artificial markets. In the seminal (Brock and Hommes, 1998) discrete-choice FABM, this concept is embodied via the switching parameter of the intensity of choice, which defines the overall willingness of market agents to switch between potential trading strategies. ‘N-type models’ comprise fundamentalists and several types of chartists. These models, in which the autonomy of agents is constrained by a predetermined class of beliefs, have been found to successfully mimic many financial stylised facts (Chen et al., 2012).

By virtue of their relatively simple design, highly stylised 2-type and 3-type models that account for the most robust heterogeneous features of real markets have been the subjects of empirical estimation in the literature. Therefore, such models are the focus of this paper. Attempts to statistically estimate the parameters of various FABMs are summarized in Tables 6, 7, and 8 in Appendix A. From a bird's-eye view, we observe a strong dominance of models derived from the adaptive belief system in the tradition of the Brock and Hommes (1998) original framework. Eight models out of 47 are based on interactive agent hypothesis (Lux, 1995), and only three are based on the ant type of system (Kirman, 1993). The prevalence of the adaptive belief system in the recent literature is the principal reason why we analyse the original (Brock and Hommes, 1998) model later in this study.

Regarding estimation frameworks, three main frameworks prevail over the others: nonlinear least squares, quasi maximum likelihood, and the method of simulated moments. Further, we observe a general tendency to estimate parameters that determine the heterogeneous behavioural rules of agents, such as belief coefficients that define individual trading strategies and the intensity of choice or corresponding concepts (e.g., mutation, herding tendency, and switching thresholds). These parameters are clearly important for the economic interpretation of given models, and authors strive to limit the set of estimated coefficients to those that are the most relevant to model dynamics. However, surprisingly, no considerable curse of dimensionality is directly observable; rather, studies with a relatively large number of estimated parameters also reveal favourable results in terms of estimation performance and the ability to estimate switching parameters.

It is important to note that almost one-half of these studies employ daily datasets comprising thousands of observations. The availability of long historical daily-frequency datasets is one of the most important features that distinguishes financial agent-based models from macroeconomic agent-based models. However, other studies use low-frequency data, ranging from weekly to annual observations. The main reason for the use of low-frequency data is that dividends, earnings, and other variables related to the fundamental value of stocks are only available at low frequencies, for example, quarterly or annually. Among studies in which a fundamental value must be approximated, stock market and foreign exchange (FX) studies largely prevail, followed by housing market data, commodities, and gold.

Although literature estimating the intensity of choice is relatively scarce, the hypothetical existence of behavioural switching receives significant attention. As aptly summarised by Chen et al. (2012, pg. 202), "supposing that we are given the significance of the intensity of choice in generating some stylized facts, then the next legitimate question will be: can this intensity be empirically determined, and if so, how big or how small is it?". However, it is important to emphasise here that the magnitude of this unit-free variable cannot be directly rigorously compared across various models, assets, or time periods, because it is conditional on the specific model design and the specific dataset. Nonetheless, the intensity of choice is a crucial and very robust driver of the data-generating process behind switching FABMs and to a large extent determines the behaviour of the system in a very consistent manner: zero intensity of choice fixes market fractions and does not allow for any evolutionary switching, whereas high values imply a wild switching for the vast majority of model specifications, assets, or periods. In the majority of relevant studies, the estimated values are primarily single digits but often close to zero and statistically insignificant, which reflects an economic intuition of a potentially detectable but realistically low switching frequency between major types of trading strategies.

Looking at goodness-of-fit results, the vast majority of models exhibit almost suspiciously good fit, explaining between 70–97% of the variation in the data. However, it is important to add that these results are most often based on low-frequency datasets and therefore relatively straightforward estimation methods might relatively easily find a well-fitting model for at most several hundred observations. With respect to ordinary least squares and quasi maximum likelihood, these estimation techniques provide generally biased estimates due to possible model misspecifications. For various methods of moments, roughly one-half of models are rejected based on the specification test. Indeed, some of the most recent contributions (Franke and Westerhoff, 2011; 2012; 2016) show acceptance of a particular model but only for selected datasets from several options (Chen and Lux, 2016; Ghonghadze and Lux, 2016). Although application of the method of simulated moments offers a tool for direct mutual comparison of models, it struggles with practical technical issues that necessitate further development of the method. The most recent contribution of Chen and Lux (2016, pg. 17) explains the main problem preventing proper identification in all of these studies: "We have to cope with multiple local minima as well as with relatively flat surfaces in certain regions of the parameter space. Any standard optimization algorithm could, thus, not be expected to converge to a unique solution from different initial conditions". Another closely related problem is a roughness of the surface of the objective function, which further embarrasses standard methods of optimisation searches. Application of methods based on the maximum likelihood principle exhibits a problem relatively similar to that of methods of moments: the objective function is often very flat, typically in the direction of the intensity of choice. Problematic identification is then manifested in large standard deviations that prevent contributive interpretation of the results. Recently, Bolt et al. (2014) report that the likelihood is not very informative and the model accuracy is not sensitive for the switching parameter. The shape of the objective function has not yet been sufficiently studied, with an exception being Hommes and Veld (2015), who emphasise a very flat shape for the intensity of choice selection that hampers validity of the test to reject the null hypothesis of switching, especially for small samples. In contrast, smoothness of the objective function does not seem to be an issue for maximum likelihood methods. This important finding is further confirmed later in this study.

To summarise, although simulation-based methods are generally applicable and not constrained by strict theoretical assumptions, they are not yet developed enough to generate unambiguous conclusions. Future progress, especially via solving related, rather technical issues, is likely to be largely encouraged and fostered by the recent rapid development of high-speed computational facilities. In this regard, our paper is an important step forward.

3. Simulation-based estimation of FABMs via NPSMLE

This section proposes a general computational framework for empirical estimation of full-fledged FABMs. As discussed in the previous section, few authors apply simulation-based methods of moments to overcome the problem of unavailability of the criterion functions. We take this a step further by adapting the simulated maximum likelihood estimator (MLE) based on nonparametric kernel methods recently suggested by Kristensen and Shin (2012). This methodology was developed for dynamic models where no closed-form representation of the likelihood function exists and thus we cannot derive the usual MLE. As we see later, NPSMLE constitutes an opportune estimation method for the general class of FABMs.

3.1. Agent-based models

The studied framework follows a common structure of (ABMs), naturally including FABMs. Grazzini and Richiardi (2015) draw attention to a representation of (ABMs) in which an agent $h \in \{1, \ldots, H\}$ at time $t \in \{1, \ldots, T\}$ is characterised completely by individual-specific state variables $z_{h,t} \in \mathbb{R}^j$. The evolution of $z_{h,t}$ follows:

$$z_{h,t} = \mathfrak{f}_h(z_{h,t-1}, Z_{\neg h,t-1}, \xi_{h,t}; \theta_h), \tag{1}$$

where $\mathfrak{f}_h, h \in \{1, \ldots, H\}$ define individual behavioural rules, $Z_{\neg h,t-1} = (z_{1,t-1}, \ldots z_{h-1,t-1}, z_{h+1,t-1}, \ldots z_{n,t-1})$ represent state variables of agents other than h, $\xi_{h,t}$ is an independent and identically distributed (i.i.d.) stochastic term, and θ_h, $h \in \{1, \ldots, H\}$, are vectors of unknown parameters. The data generating process is then specified by a system of H structural equations defined according to Eq. (1). This system can be compactly described in terms of a state equation:

$$Z_t = F(Z_{t-1}, \xi_t; \theta), \tag{2}$$

where $Z_{t-1} = (z_{1,t-1}, \ldots z_{H,t-1})$, vector of shocks $\xi_t = (\xi_{1,t}, \ldots \xi_{H,t})$, and vector of unknown parameters $\theta = (\theta_1, \ldots, \theta_H)$.

When collecting empirical data on an economy, one unfortunately observes only a vector of aggregate variables Y_t that can be defined via an observation equation projecting from Z to Y as a function of the underlying state variables:

$$Y_t = Q(Z_t, \kappa_t; \theta), \tag{3}$$

where κ_t represents additional stochastic shocks (e.g., measurement errors) to the observables.

Most importantly, in (ABMs), Eq. (2) does not have an explicit closed-form; rather, it is only implicitly defined by a system of Eq. (1). Moreover, Eq. (3) has no analytical formulation. Furthermore, a potential nonlinearity of $\mathfrak{f}_h$ and Q inhibits exploitation of expectations to eliminate the effect of κ_t. Therefore, "the only way to analyze the mapping of (Z_0, θ) into Y_t is by means of Monte Carlo analysis, by simulating the model for different initial states and values of the parameters, and repeating each simulation experiment many times to obtain a distribution of Y_t" (Grazzini and Richiardi, 2015, pg. 153).

3.2. Construction of the NPSMLE

Having defined a general family of FABMs, we advance with the construction of an estimation technique based on the simulated MLE proposed by Kristensen and Shin (2012). Let us assume processes (y_t, x_t), $y_t : t \mapsto \mathbb{R}^k$, $x_t : t \mapsto \mathcal{X}_t$, $t = 1, \ldots, \infty$. In general, the processes (y_t, x_t) can be non-stationary and x_t contains exogenous explanatory variables, including lagged dependent variables y_t. Suppose that we have T realisations $\{(y_t, x_t)\}_{t=1}^T$. Let us further assume that the time series $\{y_t\}_{t=1}^T$ is generated by a fully parametric model:

$$y_t = q_t(x_t, \varepsilon_t; \theta), \quad t = 1, \ldots, T, \tag{4}$$

where a function $q_t : \{x_t, \varepsilon_t; \theta\} \mapsto \mathbb{R}^k$, $\theta \in \Theta \subseteq \mathbb{R}^u$ is an unknown parameter vector (identical to θ in Eqs. (2) and (3) and ε_t is an i.i.d. sequence with known distribution $\mathcal{F}_\varepsilon$, which is (without a loss of generality) assumed not to depend on t or θ.

Regarding the general ABM description in the previous section, realisations y_t can either directly coincide with observables Y_t or be latent variables, as we discuss further within a specific application of the estimation method on the Brock and Hommes (1998) FABM in Section 3.3.

We further assume the model to have an associated conditional density $p_t(y|x;\ \theta)$, i.e.,

$$P(y_t \in A | x_t = x) = \int_A p_t(y|x; \theta) dy, \quad t = 1, \ldots, T, \tag{5}$$

for any Borel set $A \subseteq \mathbb{R}^k$.

Let us now suppose that $p_t(y|x;\ \theta)$ from Eq. (5) does not have a closed-form representation. In this situation, we are not able to derive the exact likelihood function of the model from Eq. (4). Thus, a natural estimator of θ, the maximiser of the conditional log-likelihood

$$\widetilde{\theta}_{ML} = \text{argmax}_{\theta \in \Theta} \mathcal{L}_T(\theta), \quad \mathcal{L}_T(\theta) = \sum_{t=1}^T \log p_t(y_t | x_t; \theta) \tag{6}$$

is not feasible.

However, we are still able to simulate observations from the model in Eq. (4) numerically.[2] The presented method allows us to compute a simulated conditional density, which we use to gain a simulated version of the MLE.

To obtain a simulated version of $p_t(y_t|x_t;\theta)$ $\forall\, t\in\{1,\dots,T\}, y_t\in\mathbb{R}^k, x_t\in\mathcal{X}_t$, and $\theta\in\Theta$, we first generate $N\in\mathbb{N}$ i.i.d. draws from $\mathcal{F}_\varepsilon$, $\{\varepsilon_i\}_{i=1}^N$, which are used to compute:

$$y_{t,i}^{\theta}=q_t(x_t,\varepsilon_i;\theta),\quad i=1,\dots,N. \tag{7}$$

These N simulated i.i.d. random variables, $\{y_{t,i}^{\theta}\}_{i=1}^N$, follow the target distribution by construction: $y_{t,i}^{\theta}\sim p_t(\cdot|x_t;\theta)$. Therefore, they can be used to estimate the conditional density $p_t(y|x;\,\theta)$ with kernel methods. We define:

$$\widehat{p}_t(y_t|x_t;\theta)=\frac{1}{N}\sum_{i=1}^{N}K_\eta(y_{t,i}^{\theta}-y_t), \tag{8}$$

where $K_\eta(\psi)=K(\psi/\eta)/\eta^k$, $K:\mathbb{R}^k\mapsto\mathbb{R}$ is a generic kernel, and $\eta>0$ is a bandwidth. Under regularity conditions on p_t and K (Kristensen and Shin, 2012, conditions A.2, A.4, K.1, and K.2, pg. 80–81), we get:

$$\widehat{p}_t(y_t|x_t;\theta)=p_t(y_t|x_t;\theta)+O_P(1/\sqrt{N\eta^k})+O_P(\eta^2),\quad N\longrightarrow\infty, \tag{9}$$

where the last two terms are $o_P(1)$ if $\eta\longrightarrow 0$ and $N\eta^k\longrightarrow\infty$.

Having obtained the simulated conditional density $\widehat{p}_t(y_t|x_t;\theta)$ from Eq. (8), we can now derive the simulated MLE of θ:

$$\widehat{\theta}_{NPSMLE}=\mathrm{argmax}_{\theta\in\Theta}\widehat{\mathcal{L}}_T(\theta),\quad \widehat{\mathcal{L}}_T(\theta)=\sum_{t=1}^{T}\log\widehat{p}_t(y_t|x_t;\theta). \tag{10}$$

The same draws are used for all values of θ, and we may also use the same set of draws from $\mathcal{F}_\varepsilon(\cdot)$, $\{\varepsilon_i\}_i^N$, across t. Numerical optimisation is facilitated if $\widehat{\mathcal{L}}_T(\theta)$ is continuous and differentiable in θ. Considering Eq. (8), if K and $\theta\mapsto q_t(x_t,\,\varepsilon_t;\,\theta)$ are $r\geq 0$ continuously differentiable, the same holds for $\widehat{\mathcal{L}}_T(\theta)$.

Under the regularity conditions, the fact that $\widehat{p}_t(y_t|x_t;\theta)\xrightarrow{P}p_t(y_t|x_t;\theta)$ implies that $\widehat{\mathcal{L}}_T(\theta)\xrightarrow{P}\mathcal{L}_T(\theta)$ as $N\longrightarrow\infty$ for a given $T\geq 1$. Thus, the simulated MLE, $\widehat{\theta}_{NPSMLE}$, retains the same properties as the infeasible MLE, $\widetilde{\theta}_{ML}$, as $T,N\longrightarrow\infty$, under suitable conditions.

3.2.1. Advantages and disadvantages

As noted by Kristensen and Shin (2012), one of the main advantages of the NPSML is its general applicability. In addition, the estimator works whether the observations y_t are i.i.d. or non-stationary because the density estimator based on i.i.d. draws is not affected by the dependence structures in the observed data. In addition, the estimator does not suffer from the curse of dimensionality, which is usually associated with kernel estimators. In general, high dimensional models (i.e., those with larger $k\equiv\dim(y_t)$, because we only smooth over y_t here) require a larger number of simulations to control the variance component of the resulting estimator. However, the summation in Eq. (10) reveals an additional smoothing effect, and the additional variance of $\widehat{\mathcal{L}}_T(\theta)$ caused by simulations retains the standard parametric rate $1/N$.

Conversely, the simulated log-likelihood function is a biased estimate of the actual log-likelihood function for fixed N and $\eta>0$. To obtain consistency, we need $N\longrightarrow\infty$ and $\eta\longrightarrow 0$. Thus, the parameter η must be chosen properly for a given sample and simulation size. In the stationary case, the standard identification assumption is:

$$\mathbb{E}[\log p(y_t|x_t;\theta)]<\mathbb{E}[\log p(y_t|x_t;\theta_0)]\ \forall\ \theta\neq\theta_0. \tag{11}$$

Under stronger identification assumptions, the choice of the parameter η might be less important, and one can prove the consistency of the estimator for any fixed $0<\eta<\bar{\eta}$ for some $\bar{\eta}>0$ as $N\longrightarrow\infty$ (Altissimo and Mele, 2009). In practice, this approach still requires us to know the threshold level $\bar{\eta}>0$, but from a theoretical viewpoint, it ensures that parameters can be well identified in large finite samples after a given $\bar{\eta}>0$ is set. Moreover, it suggests that the proposed methodology is fairly robust to the choice of η. Indeed, Kristensen and Shin (2012) show in their simulation study that NPSMLE performs well with a broad range of bandwidth choices.

3.2.2. Asymptotic properties

As the theoretical convergence of the simulated conditional density towards the true density is met, we would expect the $\widehat{\theta}_{NPSMLE}$ to have the same asymptotic properties as the infeasible $\widetilde{\theta}_{ML}$ for a properly chosen sequence $N=N(T)$ and $\eta=\eta(N)$. Kristensen and Shin (2012) show that $\widehat{\theta}_{NPSMLE}$ is first-order asymptotic equivalent to $\widetilde{\theta}_{ML}$ under a set of general conditions, even allowing for non-stationary and mixed discrete and continuous distribution of the response variable. Further, using additional assumptions, including stationarity, they provide results for the higher-order asymptotic properties of

[2] For cases in which the model in Eq. (4) is itself intractable and thus cannot be used to generate observations, Kristensen and Shin (2012) suggest a methodology for approximate simulations and define regularity conditions for the associated approximate NPSMLE $\widehat{\theta}_{APPROX}$ to have the same asymptotic properties as the simulated $\widehat{\theta}_{NPSMLE}$ defined in Eq. (10).

$\widehat{\theta}_{NPSMLE}$ and derive expressions of the bias and variance components of the $\widehat{\theta}_{NPSMLE}$ relative to the actual MLE due to kernel approximation and simulations.

A set of general conditions satisfied by most models must be verified so that $\widehat{p} \longrightarrow p$ is sufficiently fast to ensure asymptotic equivalence of $\widehat{\theta}_{NPSMLE}$ and $\widetilde{\theta}_{ML}$. Kristensen and Shin (2012) define a set of regularity conditions on the model and its associated conditional density that satisfy these general conditions for uniform rates of kernel estimators defined in Kristensen (2009). Moreover, kernel K from Eq. (8) must belong to a broad class of so-called bias high-order or bias reducing kernels. Among others, the Gaussian kernel, which we use in Section 4, satisfies this condition if $s \geq 2$, where s is the number of derivatives of p. Higher s increases the rate of convergence and determines the degree of $\widehat{p}$ bias reduction. Moreover, the general conditions typically required for consistency and well-defined asymptotic distribution (asymptotic normality) of MLEs in stationary and ergodic models are imposed on the actual log-likelihood function and the associated MLE to ensure that the actual $\widetilde{\theta}_{ML}$ in Eq. (6) is asymptotically well behaved.

3.3. The Brock and Hommes (1998) model

Since its introduction, the seminal (Brock and Hommes, 1998) model has remained one of the most widely analysed FABMs. Nevertheless, there is no consensus on how to estimate the model parameters. The model represents a stylised financial market application of the adaptive belief system —the endogenous evolutionary selection of heterogeneous expectation rules (Lucas, 1978). Essentially, it is an expectation feedback system in which variables depend partly on known values and partly on expectations about the future. The exact model specification we use to generate observations is a stylised simple version that is compactly described in Hommes (2006, pg. 1169)[3] and comprises three mutually dependent equations:

$$Ry_t = \sum_{h=1}^{H} n_{h,t-1} f_{h,t} + \varepsilon_t \equiv \sum_{h=1}^{H} n_{h,t-1}(g_h y_{t-1} + b_h) + \varepsilon_t, \tag{12}$$

$$n_{h,t-1} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^{H} \exp(\beta U_{h,t-1})}, \tag{13}$$

$$\begin{aligned} U_{h,t-1} &= (y_{t-1} - Ry_{t-2}) \frac{f_{h,t-2} - Ry_{t-2}}{a\sigma^2} \\ &\equiv (y_{t-1} - Ry_{t-2}) \frac{g_h y_{t-3} + b_h - Ry_{t-2}}{a\sigma^2}. \end{aligned} \tag{14}$$

Essentially, Eq. (12) is a heterogeneous market equilibrium risky asset pricing formula represented in deviations:

$$y_t = \mathfrak{p}_t - \mathfrak{p}_t^*, \tag{15}$$

where $\mathfrak{p}_t$ denotes observable price level and $\mathfrak{p}_t^*$ is the fundamental price. Next, a fixed gross rate $R = 1 + r$, i.e., r, represents the constant risk-free interest rate.

The fundamental price $\mathfrak{p}_t^*$ provides an important benchmark for asset valuation under rational expectations based on economic fundamentals. In a specific case of an i.i.d. stochastic risky asset dividend process d_t, $d_t : t \mapsto \mathbb{R}$, expectation $E_t\{d_{t+1}\} = \bar{d}$ is a constant and thus $\mathfrak{p}_t^*$ can be derived using the simple formula for perpetuity:

$$\mathfrak{p}_t^* = \sum_{i=1}^{\infty} \frac{\bar{d}}{(1+r)^i} = \frac{\bar{d}}{r}. \tag{16}$$

Following the model specification above, we treat the fundamental price $\mathfrak{p}_t^*$ as latent because Eq. (12) directly provides us with deviations y_t. Then, $n_{h,t-1}$ denote fractions of agents of classes $h \in \{1, \ldots, H\}$ at time $t-1$ that satisfy $\sum_{h=1}^{H} n_{h,t-1} = 1$; $f_{h,t}$ is a deterministic function that can differ across agent classes h and represents a simple linear 'h-type' trading strategy at time t; and g_h and b_h denote the trend and bias parameters, respectively, of the trading strategy f_h. Finally, ε_t is an i.i.d. sequence with a given distribution representing market uncertainty and unpredictable market events.

The original paper by Brock and Hommes (1998) analyses an artificial market comprising only a few simple trading strategies. The authors argue that only very simple forecasting rules can have a real impact on equilibrium prices because complicated strategies are unlikely to be understood and followed by a sufficient number of traders. The first agent class comprises fundamentalists who constitute the special case with $g_h = b_h = f_{h,t} = 0$. They believe that asset prices are determined solely by economic fundamentals according to the efficient market hypothesis (Fama, 1970) and thus always converge to their fundamental values. Another agent class includes the chartists, who believe that asset prices can be partially predicted based on various patterns observed in past data, e.g., by using simple technical trading rules and extrapolation techniques. If $b_h = 0$, trader h is called a pure trend follower if $0 < g_h \leq R$ and is deemed a strong trend follower if $g_h > R$. Next, trader h is called a contrarian if $-R \leq g_h < 0$ and a strong contrarian if $g_h < -R$. If $g_h = 0$, trader h is considered to be

[3] Note that we changed the notation slightly to emphasize the general timing of updating beliefs, as in Brock and Hommes (1998).

purely upward biased if $b_h > 0$ and purely downward biased if $b_h < 0$. Combined trading strategies with $g_h \neq 0$ and $b_h \neq 0$ are also certainly possible.

Eq. (13) defines market fractions $n_{h,t-1}$ of agent classes $h \in \{1, \dots, H\}$ that are derived under the discrete choice probability framework using the multinomial logit model. $U_{h,t-1}$ denote profitability measures for strategies $h \in \{1, \dots, H\}$, and $\beta \geq 0$ is the intensity of choice parameter that measures how fast agents are willing to switch between different trading strategies based on their past profitability.

Eq. (14) then derives the profitability measures $U_{h,t-1}$ based on past realised profits, risk-aversion coefficient $a > 0$, and beliefs about the conditional variance of excess returns σ^2.[4]

It is now useful to comment on interconnections between a special ABM case of the Brock and Hommes (1998) FABM and the general frameworks described above to specify the final estimation methodology and to clarify the applicability of the NPSMLE in a situation typical for (ABMs) when certain state variables Z_t can be latent or unobserved in practice, i.e., $Y_t \neq Z_t$.

First, we realistically set observables Y_t equal to price level $\mathfrak{p}_t$, $Y_t = \mathfrak{p}_t$ because as the fundamental price $\mathfrak{p}_t^*$ is considered latent. The definition of deviations y_t in Eq. (15) then relates Eqs. (3) and (12) and the respective stochastic shocks $\xi_{h,t}$ and κ_t with ε_t.

Individual behavioural rules $\mathfrak{f}_h, h \in \{1, \dots, H\}$ are reflected in trading strategies f_h via the trend and bias parameters, i.e., g_h and b_h, respectively. Eqs. (13) and (14) then specify the state Eq. (2) for state variables Z_t and the functional form of F. Finally, Eqs. (12), (15), and (16) specify the observation Eq. 3 for $Y_t = \mathfrak{p}_t$ and the functional form of Q.

Importantly, processes y_t, i.i.d. sequences ε_t, and unknown parameter vectors θ from Sections 3.2 and 3.3 fully coincide, making the simulation analysis in Section 4 particularly convenient. However, we must address an approximation of y_t and the estimation of the distributional parameters of $\mathcal{F}_\varepsilon$ within the empirical analysis in Section 5. Although x_t in Eq. (4) contains not only lagged y_t but also other (exogenous) variables specifying the dividend process, the fundamental price $\mathfrak{p}_t^*$, and risk-free and risk-aversion rates, etc., the coincidence of y_t, ε_t, and θ between the NPSMLE and the FABM completely specifies the estimation process, which is determined by a sample time series $\{y_t\}_{t=1}^T$ and distributional assumptions about $\mathcal{F}_\varepsilon$.

3.4. Overview of estimated parameters

Within the simulation study in Section 4, we first estimate the intensity of choice β (i.e., $\theta = \beta$) while keeping the other parameters fixed. In subsequent steps, we extend the unknown parameter vector by agents' belief coefficients g_2 and b_2 (i.e., $\theta = \{\beta, g_2, b_2\}$ for the 2-type model) and g_3 and b_3 (i.e., $\theta = \{\beta, g_2, b_2, g_3, b_3\}$ for the 3-type model). In the empirical Section 5, we jointly estimate the intensity of choice β; agents' belief coefficients g_2, b_2, and g_3; and the standard deviation of the market noise represented by stochastic term ε_t denoted as *noise intensity* (i.e., $\theta = \{\beta, g_2, b_2, \textit{noise intensity}\}$ for the 2-type model and $\theta = \{\beta, g_2, g_3, \textit{noise intensity}\}$ for the 3-type model). All other parameters are fixed or repeatedly randomly generated from fixed distributions, as described in Section 4.1.

4. Simulation study

Prior to estimating the parameters of such a complex model on real world data, we evaluate small sample performance of the proposed estimation strategy on the simulated data from Brock and Hommes (1998) model. Sections 4.1 and 4.2 describe a general setup for all following estimation exercises if not explicitly stated otherwise. An extensive simulation study will allow us to see the extent to which the estimation is able to recover the true values of parameters in the controlled environment.

Because of its conceptual importance, meticulous attention is devoted to the switching parameter of the intensity of choice β. Due to the frequent statistical insignificance of the switching coefficient found in the literature, we properly focus on a plausibility of zero β throughout the entire section. Hence, we always allow for the comparison of models estimated

[4] Additional memory can be introduced into the profitability measure, e.g., as a weighted average of past realised values:

$$U_{M,h,t-1} = U_{h,t-1} + \delta U_{M,h,t-2} - C_h, \tag{17}$$

where $0 \leq \delta \leq 1$ denotes the 'dilution parameter' of past memory in the profitability measure, $C_h \geq 0$ is the cost to obtain market information for fundamentalists, and $C_h = 0$ is the cost to obtain market information for chartists. However, following (Hommes, 2006) via Eq. (14), neither the 'dilution parameter' of past memory δ nor the potential information cost C for fundamentalists are implemented into the basic model setup to keep the dynamics of the model and the impact of the analysed modification as clear as possible. Indeed, for the majority of examples, Brock and Hommes (1998) also suppose the case with $\delta = C = 0$ to preserve the simplicity and analytical tractability of the analysis.
Another approach to agents' memories is suggested by Barunik et al. (2009), who propose that the profitability measure in Eq. (14) is extended via memory parameters m_h:

$$U_{h,t-1} = \frac{1}{m_h} \sum_{j=0}^{m_h-1} \left[(y_{t-1-j} - R y_{t-2-j}) \frac{f_{h,t-2-j} - R y_{t-2-j}}{a\sigma^2} \right]. \tag{18}$$

The memory for each individual trading strategy $h \in \{1, \dots, H\}$ is then a randomly generated integer from the uniform distribution $U(0, m_h)$.

with and without switching ($\beta > 0$ vs. $\beta = 0$). β is not only the crucial parameter influencing model dynamics; we also stress its conceptual importance because it represents the dominant approach to modelling the boundedly rational choices of agents in the current literature. Despite the focus on a single estimated parameter, capturing the effect of the switching coefficient β is generally challenging and requires robust performance of the optimisation algorithm.

4.1. Model simulation setup

We fix variables that are unimportant for model dynamics to support the estimation of key parameters. A daily constant gross interest rate is set to $R = 1 + r = 1.0001$, representing a circa 2.5% annual risk-free interest rate. We believe this is a reasonable approximation comparable to values commonly used in literature, and we also test for the robustness of our results to variations of this parameter. The linear term $1/a\sigma^2$ is fixed to 1. It is important to note that a and σ^2 are only scale factors for the profitability measure U not affecting relative proportions of $U_{h,t}$ and thus they do not influence the dynamics of model output; on the contrary, the dynamics of model output are usually characterised by time-varying variance. Next, we use a relatively small number of possible trading strategies, $H = 5$, for the general model setting. In Monte Carlo simulation, behavioural parameters are randomly generated from fixed distributions to obtain statistically valid inferences. Trend parameters g_h, $h \in \{2, 3, 4, 5\}$ are drawn from the normal distribution $N(0, 0.4^2)$, and bias parameters b_h, $h \in \{2, 3, 4, 5\}$ are drawn from the normal distribution $N(0, 0.3^2)$. A strict fundamental strategy in the sense of the original (Brock and Hommes, 1998, pg. 1245) article appears in the market by default, i.e., the first trading strategy is defined as $g_1 = b_1 = 0$ and thus some proportion of fundamentalists are always present in the market.

Our main focus is on the ability to recover the switching parameter of the intensity of choice β, which is typically found to be a single digit and is often close to zero. Hence, we set a meaningful range of $\beta = \{0, 0.1, 0.5, 1, 3, 5, 10\}$. These values resemble the economic intuition of a realistically low switching frequency between major types of trading strategies. Negative β does not make economic sense because it would cause inverse illogical switching towards less profitable strategies, whereas high values of β would cause unrealistically fast switching, which can hardly be observed among market agents in reality.[5]

Because the noise term significantly influences the model, its magnitude must be considered carefully. Noise represents market uncertainty and unpredictable events but must not overshadow the effect of the variables under scrutiny. Although theoretically the $\mathcal{F}_\varepsilon$ from which the $\{\varepsilon_i\}_{i=1}^N$ are drawn to simulate $\{y_{t,i}^\theta\}_{i=1}^N$ in Eq. (7) is a generic known distribution, assumptions about market noise can play a crucial role in the NPSMLE application to real-world data. Therefore, we test the model sensitivity and robustness of the proposed methodology using an extensive range of noise specifications based on normal distribution and considering 'miniscule' standard deviations $SD = 10^{-8}$ or 10^{-6} [used by Hommes (2013, pg. 170, 174, 177)], 'small' standard deviations such as $SD = 0.01$ [used by Hommes (2013, pg. 171)], standard normal $SD = 1$, and finally, a relatively large 'experimental' standard deviation $SD = 2$. A comprehensive list is provided in Table 9 in Appendix A. The normal distribution of market noise seems reasonably realistic, and a similar assumption has already been used in related studies, where "the non-linear models are fed with an exogenous stochastic process, but the noise process is 'nice', which in this case means that it is normally distributed", as noted by Amilon (2008, pg. 344). We also utilise the favourable theoretical properties of the Gaussian kernel (Kristensen and Shin, 2012, pg. 81) in Eq. (8).

4.2. NPSMLE simulation setup

We combine three different levels of the kernel estimation precision $N = \{100, 500, 1000\}$ with five lengths of the time series entering the algorithm $t = \{100, 500, 1000, 5000, 10000\}$, and we always simulate 100 extra observations to be discarded as an initial period in which the model dynamic is being established. To ensure the statistical validity of the results, 1000 random runs are conducted.[6] Following the general principle of a preliminary rough search with unconstrained parameter space followed by fine-tuning on a considerably restricted subset of the parameter space (Chen and Lux, 2016), we restrict the parameter space to $\langle -\beta, 3\beta \rangle$ for $\beta > 0$ and to $\langle -0.5, 0.5 \rangle$ for $\beta = 0$. We also allow for economically irrelevant negative values to avoid upward bias of the simulated estimator, especially for β close to 0. To estimate the conditional density $p_t(y|x; \theta)$, the Gaussian kernel and Silverman's 1986 rule of thumb for finding the optimal size of the bandwidth are used: $\eta = [4/(3N)]^{1/5}\widehat{\sigma}$, where $\widehat{\sigma}$ denotes the standard deviation of $\{y_{t,i}^\theta\}_{i=1}^N$.

4.3. Estimating intensity of choice β

We primarily verify how consistently and efficiently the estimator recovers β parameter in small samples. In Fig. 1, we depict a 'snapshot' of simulation results for three interesting values of the intensity of choice $\beta \in \{0, 0.5, 10\}$ combined with

[5] It is beyond the scope of this work to provide a deep analysis of the model behaviour, e.g., how the intensity of choice β influences the dynamics of the model that can under certain settings generate purely chaotic behaviour. Many studies have been devoted to this generally difficult issue in the past two decades. In this context, we refer the interested reader to the original paper of Brock and Hommes (1998) containing comprehensive model dynamics analysis; extensive studies by Hommes and Wagener (2009); Hommes (2006) and Chiarella et al. (2009); or a recent book by Hommes (2013) that summarises two decades of research on the heterogeneous expectations hypothesis.

[6] We also experimented with smaller numbers of random runs, and the results confirm that the use of 500 runs performs sufficiently well. These results are available upon request from the authors.

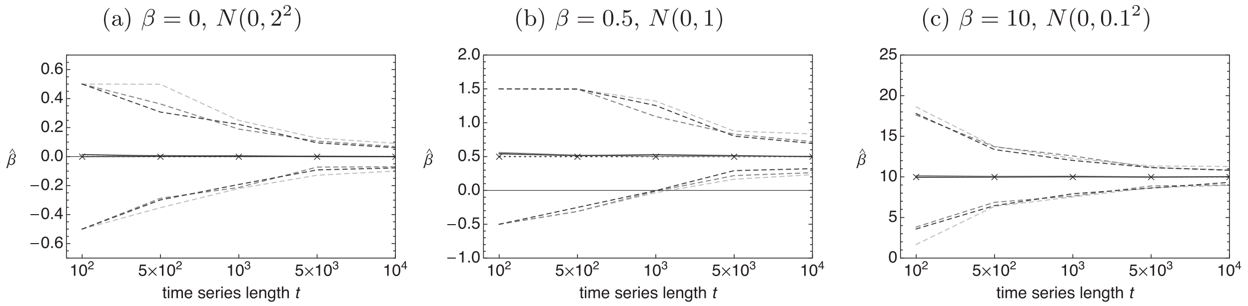

Fig. 1. Simulation results for estimation of $\beta \in \{0, 0.5, 10\}$. Results are based on 1000 random runs. Stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from given normal distribution. Black dotted lines with $\times$ depict the true β. Unbroken grey lines depict sample means of estimated β. Grey dashed lines depict 2.5% and 97.5% quantiles. The light grey colour represents results for $N = 100$, normal grey represents results for $N = 500$, and dark grey represents results for $N = 1000$.

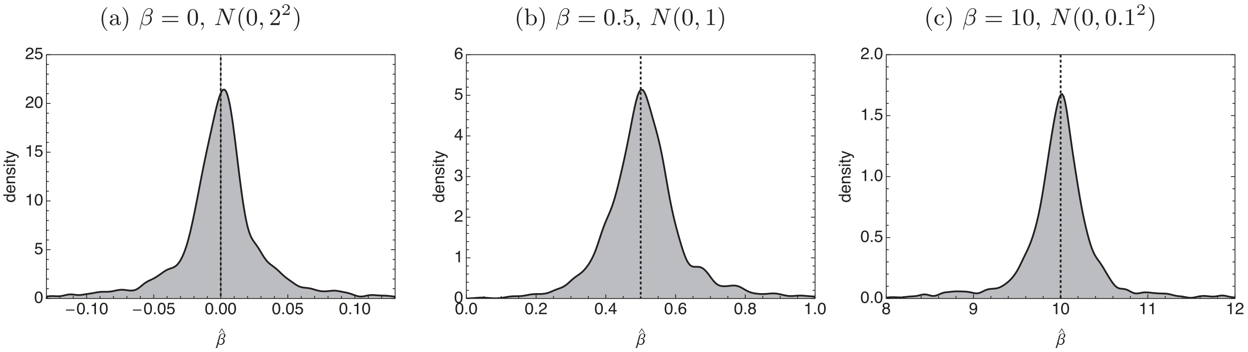

Fig. 2. Densities for selected $\widehat{\beta}$. Results are based on 1000 random runs, $t = 5000$, and $N = 1000$. Stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from given normal distribution. Black dotted lines depict the true β.

Table 1
Simulation results for β estimation with Gaussian noise.

β	$\widehat{\beta}$					
	Median	Mean	SD	LQ	HQ	μ
0	0.00	0.00	0.05	−0.09	0.09	2
.1	0.10	0.10	0.04	0.00	0.19	2
.5	0.50	0.51	0.13	0.29	0.80	1
1	0.99	1.00	0.19	0.69	1.33	1
3	3.00	3.02	0.40	2.33	3.89	1
5	5.00	5.00	0.57	3.99	6.06	0.1
10	10.00	9.97	0.73	8.62	11.15	0.1

Results are based on 1000 random runs, $t = 5000$, and $N = 1000$. Stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from $N(0, \mu^2)$. SD, LQ, and HQ stand for standard deviation, 2.5%, and 97.5% quantiles, respectively. Figures are rounded to 2 decimal digits.

three distinct specifications of the stochastic noise, as specified above. This process also allows for the comparison of model estimates with and without switching ($\beta > 0$ vs. $\beta = 0$). The method clearly reveals the true value of β with increasing precision as the number of observations increases and as kernel estimation precision N increases. Further, Fig. 1 reveals an important result from an economic perspective, which is that we should be able to detect not only very weak signs of behavioural switching in long-span daily financial data but also stronger signs of switching in macroeconomic data for which lower-frequency time series of shorter durations are typically available.

To limit the number of results reported in this paper, we report results for fixed length of generated time series $t = 5000$ and kernel estimation precision $N = 1000$; the rest of the results are available from the authors upon request. Fig. 2 and Table 1 confirm the ability of the estimation framework to recover the true value of β very precisely, with narrow confidence intervals. Hence, we are able to test the null hypothesis $H_0 : \beta = 0$ even for small values of β, which is crucial in empirical estimation.

Encouraged by these results, we study the robustness of the method with respect to various noises. These results, which are summarised in Table 9 in Appendix A, suggest the estimation is robust to changing noise. Generally, lower values of

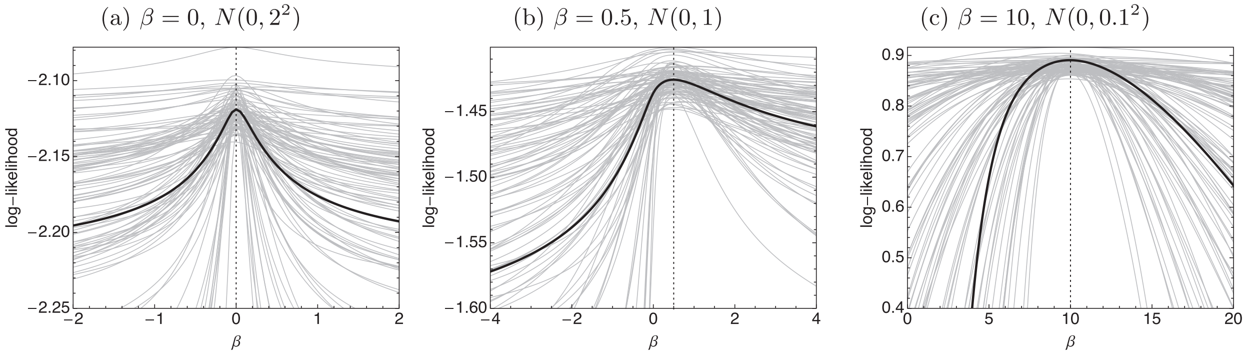

Fig. 3. Shape of the simulated log-likelihood function. Results are based on 100 random runs, $t = 5000$, and $N = 1000$. Stochastic noise ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from given normal distribution. Black dotted vertical lines depict the true β. Unbroken bold black lines depict sample averages.

the intensity of choice, $\beta = \{0.1, 0.5, 1\}$, are the most difficult to reveal and are most precisely estimated under extremely small or fairly large noises. However, these values essentially represent the extreme case in which there is no switching by agents among possible strategies, which restrains model dynamics because there is only a small difference between this model behaviour and the agents' absolute inertia case, where $\beta = 0$. Conversely, higher values of the intensity of choice, $\beta = \{5, 10\}$, require relatively small noises for the most precise detection of the switching effect [subparts (g) and (h) of Table 9]. These are important findings that highlight the necessity of a proper noise specification within the empirical estimation procedure.

4.3.1. Behaviour of the simulated log-likelihood function

Kristensen and Shin (2012, pg. 80-81) define a set of regularity conditions A.1-A.4 for the model and its associated conditional density that ensure sufficiently fast convergence of $\widehat{p} \longrightarrow p$ and thus the asymptotic equivalence of $\widehat{\theta}$ and $\widetilde{\theta}$. These conditions basically impose restrictions on data-generating functions and the conditional density that is being estimated. These assumptions are "quite weak and are satisfied by many models" (Kristensen and Shin, 2012, pg. 81). However, we are not able to verify these conditions analytically for the Brock and Hommes (1998) and thus must rely on computational tools. Therefore, we explore the smoothness condition, identification of parameters, and existence of a unique maximum by observing the simulated log-likelihood functions. For these purposes, Fig. 3 depicts the simulated log-likelihood function for the same three interesting values of the intensity of choice $\beta \in \{0, 0.5, 10\}$. We clearly observe the very smooth shape of the functions over the entire domain and a unique maximum generally shared by all 100 random runs. Bold black unbroken lines represent sample averages over these 100 runs. Based on the generally smooth shapes and unique optima of the simulated log-likelihood functions, we assume that the regularity conditions are met for the model and that the identification of parameters is assured.

4.3.2. Robustness checks

For the estimation of the β parameter, it is important that the NPSMLE recovers the true parameter under different beliefs that significantly impact model dynamics (Barunik et al., 2009; Kukacka and Barunik, 2013; Vacha et al., 2012). Table 10 in Appendix A reports results for different belief parameters g_h and b_h drawn randomly from the normal distribution centred at zero with variances $\{0.1^2, 0.2^2, 0.3^2, 0.4^2\}$. We observe a general ability of the method to accurately reveal the true value of the intensity of choice β for the vast majority of combinations of the simulation grid. Increasing the variance of the belief distribution generally obtains richer model dynamics that can be estimated more efficiently, as evidenced by the generally decreasing standard deviations of estimated β. We also experimented with different noise levels, but the results are essentially the same. These results are available upon request.

Another important issue is the extent to which the estimation methodology is robust to an assumption of inaccurate stochastic noise. This inaccuracy can be manifested either by a correct distribution with incorrect parameters or through a completely different distribution. This matter is especially important with respect to empirical application because we are rarely able to ascertain proper assumptions about noise in real-world data. To analyse this situation, we present the results of eight combinations of different distributions used for random generation of stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$. In the upper part of Table 2, we report cases where stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from the same distributions with different variances [subparts (a) and (b)], as well as from different distributions with same variances [subparts (c) and (d)]. Basically, we use combinations of normal and uniform distributions, and for different variances, we use specifications with centupled values. The uniform distribution is selected for its simplicity and because it is the maximum entropy probability distribution among its family of symmetric probability distributions. Conclusions for this robustness check are quite clear and can be summarised as follows.

When a distribution with lower variance is used [subparts (a) and (b)] to generate the stochastic noise ε_t, then for $\{\varepsilon_i\}_{i=1}^N$ to define the kernel approximation precision, the NPSMLE works but produces statistically insignificant estimates and very

Table 2
Results for β estimation with various combined noises.

β	(a) $\widehat{\beta}$, $\varepsilon_t \sim N(0, 0.1^2)$, $\{\varepsilon_i\}_{i=1}^N \sim N(0, 1)$					(b) $\widehat{\beta}$, $\varepsilon_t \sim U\left(-\frac{\sqrt{12}}{2} \times 10^{-1}, \frac{\sqrt{12}}{2} \times 10^{-1}\right)$, $\{\varepsilon_i\}_{i=1}^N \sim U\left(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2}\right)$				
	Median	Mean	SD	LQ	HQ	Med.	Mean	SD	LQ	HQ
0	0.02	0.01	0.36	–.50	0.50	0.08	0.03	0.43	-.50	0.50
.1	0.10	0.10	0.15	–.10	0.30	0.09	0.10	0.17	–.10	0.30
.5	0.51	0.50	0.66	–.50	1.50	0.46	0.48	0.85	–.50	1.50
1	0.98	1.01	1.14	–1.00	3.00	0.96	1.01	1.67	–1.00	3.00
3	2.99	3.17	2.61	–2.99	9.00	2.81	2.95	4.77	–3.00	9.00
5	4.97	5.40	4.02	–4.98	15.00	4.51	4.97	7.80	–5.00	15.00
10	10.09	11.35	7.07	–2.83	30.00	9.38	10.56	15.21	–10.00	30.00
	(c) $\widehat{\beta}$, $\varepsilon_t \sim N(0, 0.1^2)$, $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2} \times 10^{-1}, \frac{\sqrt{12}}{2} \times 10^{-1})$					(d) $\widehat{\beta}$, $\varepsilon_t \sim U(-\frac{\sqrt{12}}{2} \times 10^{-1}, \frac{\sqrt{12}}{2} \times 10^{-1})$, $\{\varepsilon_i\}_{i=1}^N \sim N(0, 0.1^2)$				
0	–.02	–.02	0.30	–.50	0.50	0.00	0.01	0.26	–.50	0.50
.1	0.10	0.10	0.16	–.10	0.30	0.10	0.10	0.14	–.10	0.30
.5	0.50	0.49	0.44	–.50	1.50	0.49	0.50	0.36	–.44	1.46
1	1.01	1.02	0.55	–.27	2.38	0.99	0.98	0.46	–.10	1.93
3	3.00	2.97	0.76	1.52	4.28	3.01	2.99	0.61	1.81	4.01
5	4.99	4.99	0.79	3.51	6.35	5.00	5.01	0.67	4.09	6.17
10	9.99	9.97	0.82	8.25	11.43	10.00	9.99	0.58	8.85	11.08
	(e) $\widehat{\beta}$, $\varepsilon_t = \zeta_t - 0.1\varepsilon_{t-1}$ $\{\varepsilon_i\}_{i=1}^N \sim N(0, 0.1^2)$					(f) $\widehat{\beta}$, $\varepsilon_t = \zeta_t - 0.1\varepsilon_{t-1}$ $\{\varepsilon_i\}_{i=1}^N \sim N(0, 1)$				
0	–.50	–.46	0.09	–.50	–.19	–.11	–.16	0.16	–.50	0.03
.1	–.10	–.10	0.01	–.10	–.10	–.01	–.01	0.08	–.10	0.14
.5	–.48	–.29	0.26	–.50	0.27	0.38	0.32	0.22	–.33	0.61
1	0.04	–.11	0.61	–1.00	0.75	0.81	0.76	0.29	0.03	1.21
3	1.93	1.52	1.29	–2.65	2.76	2.56	2.51	0.56	1.26	3.53
5	3.96	3.45	1.62	–1.91	4.85	4.33	4.28	1.02	2.24	6.30
10	8.93	8.43	1.78	3.87	9.99	8.94	8.91	1.95	4.85	12.74
	(g) $\widehat{\beta}$, $\varepsilon_t = \zeta_t + 0.1\varepsilon_{t-1}$ $\{\varepsilon_i\}_{i=1}^N \sim N(0, 0.1^2)$					(h) $\widehat{\beta}$, $\varepsilon_t = \zeta_t + 0.1\varepsilon_{t-1}$ $\{\varepsilon_i\}_{i=1}^N \sim N(0, 1)$				
0	0.50	0.48	0.07	0.26	0.50	0.13	0.18	0.19	–.09	0.50
.1	0.30	0.30	0.00	0.30	0.30	0.23	0.21	0.09	0.03	0.30
.5	1.50	1.33	0.24	0.78	1.50	0.66	0.74	0.29	0.39	1.50
1	2.14	2.23	0.59	1.30	3.00	1.23	1.35	0.44	0.78	2.65
3	4.21	4.68	1.44	3.26	9.00	3.54	3.71	0.86	2.49	6.02
5	6.21	6.84	1.89	5.13	12.51	5.76	5.96	1.23	4.16	9.36
10	11.12	11.82	2.46	9.97	18.63	10.98	11.47	2.52	7.96	17.45

Results are based on 1000 random runs, $t = 5000$, and $N = 1000$. Stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from the same distributions with different variances in (a) and (b) and from different distributions with the same variances in (c) and (d). In (e), (f), (g), and (h), ε_t follows an autocorrelated MA(1) process: $\varepsilon_t = \zeta_t \pm 0.1\varepsilon_{t-1}$, where $\zeta \sim N(0, \Sigma^2)$, so that ε_t has the same variance as $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distributions of given parameters. Sample medians, means, standard deviations (SD), and 2.5% (LQ) and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits.

uniformly distributed estimated values. Moreover, uniform distributions [subpart (b)] reveal inferior estimation precision, which can be attributed mainly to their different shape compared to normal distributions. Conversely, when different distributions with the same variances are used [subparts (c) and (d)], we obtain more precise estimates with markedly lower standard deviations, especially for higher β. These findings strongly confirm the need to properly specify the magnitude of stochastic noise but suggest laxer requirements for accurate specification of the noise distribution in the empirical application of NPSMLE.

Moreover, subparts (e)-(h) study properties of the estimation methodology when stochastic noises are characterised by negative and positive first-order autocorrelation. Negative autocorrelation is commonly assumed to be a feature of the microstructure noise that affects market prices (Bandi and Russell, 2006; 2008),[7] whereas positive autocorrelation serves as an opposite effect check. We implement ε_t as a weak MA(1) process with a first-order autocorrelated structure: $\varepsilon_t = \zeta_t \pm 0.1\varepsilon_{t-1}$, where $\zeta \sim N(0, \Sigma^2)$, so that ε_t has the same variance as $\{\varepsilon_i\}_{i=1}^N$ drawn from normal distributions of given parameters. It is important to highlight here that an autocorrelated noise does not follow the NPSMLE assumption that requires ε_t to be an i.i.d. sequence (see Eq. 4). Nonetheless, this imperfection provides us with another interesting robustness check with respect to application of the methodology to real data. Noises with negative autocorrelation cause a considerable

[7] We would like to thank an anonymous referee for this valuable comment.

Table 3
Results of 3-parameter estimation of a 2-type model.

β, g_2, b_2	(a) $\widehat{\beta}$			(b) $\widehat{g_2}$		(c) $\widehat{b_2}$	
	Median	Mean	SD	Mean	SD	Mean	SD
.0, 0.4, 0.3	0.01	0.01	0.33	0.40	0.03	0.30	0.01
.5, 0.4, 0.3	0.55	0.53	0.40	0.40	0.04	0.30	0.01
3, 0.4, 0.3	2.98	2.99	0.42	0.40	0.03	0.30	0.01
10, 0.4, 0.3	9.98	10.00	0.62	0.40	0.03	0.30	0.01
.0, -.4, -.3	0.03	0.01	0.41	-.40	0.03	-.30	0.01
.5, -.4, -.3	0.54	0.53	0.67	-.40	0.04	-.30	0.01
3, -.4, -.3	3.02	3.01	0.86	-.40	0.04	-.30	0.01
10, -.4, -.3	9.93	9.96	0.96	-.40	0.03	-.30	0.01
.0, 0.4, -.3	0.04	0.02	0.32	0.40	0.04	-.30	0.01
.5, 0.4, -.3	0.49	0.49	0.40	0.40	0.04	-.30	0.01
3, 0.4, -.3	3.01	3.01	0.45	0.40	0.04	-.30	0.01
10, 0.4, -.3	9.98	10.00	0.62	0.40	0.03	-.30	0.01
.0, -.4, 0.3	0.07	0.02	0.41	-.40	0.03	0.30	0.01
.5, -.4, 0.3	0.50	0.51	0.66	-.40	0.04	0.30	0.01
3, -.4, 0.3	3.00	2.99	0.86	-.40	0.04	0.30	0.01
10, -.4, 0.3	10.03	10.04	0.98	-.40	0.04	0.30	0.01

Results are based on 1000 random runs, $t = 5000$, and $N = 1000$. Stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from normal distribution $N(0, 0.1^2)$. Sample medians, means, and standard deviations (SD) are reported. Figures are rounded to 2 decimal digits.

downward bias of estimates [subparts (e) and (f)], whereas an upward bias effect is clearly observable for positive autocorrelation [subparts (g) and (h)]. Precision is naturally lower compared to Table 1 with assumptions met, and interestingly, the bias markedly decreases as stochastic noise variance increases [subparts (f) and (h)]. These results could have important consequences for empirical application—if the true market noise is autocorrelated, NPSMLE can be suspected to generate biased results.

4.4. 2-type and 3-type model estimation

An important advantage of FABMs is that their dynamics are driven primarily by a small number of crucial parameters. Hence, as a natural subsequent step, we might attempt to simultaneously estimate all essential coefficients in the simplest 2-type and 3-type models for which both theoretical and empirical rationales exist in the recent literature (Chen et al., 2012). With reference to Biondi et al. (2012, pg. 5534), "it has been advocated that the two broad categories of chartism and fundamentalism account for most possible investment strategies". Estimated parameters are selected consistently with the findings in Tables 6 and 7, i.e., we jointly estimate the intensity of choice β and agents' behavioural belief coefficients g_h and b_h.

The main differences compared with the previous setup are that the number of trading strategies is lower, $H = \{2, 3\}$, and that belief coefficients g_h and b_h are no longer drawn from distributions but rather kept fixed for all 1000 Monte Carlo runs. Other parameters follow the previous setup, and we define bounds for the parameter space of belief coefficients for the 2-type model as $\langle -3|g_2|, 3|g_2|\rangle$ and $\langle -3|b_2|, 3|b_2|\rangle$, respectively. For the 3-type model, it is also necessary to limit bounds for belief coefficients by zero from one side to avoid problems with insufficient specification of the model, for instance, $\langle 0, 3|g_2|\rangle$ and $\langle 0, 3|b_2|\rangle$ for a trend following an upward-biased strategy.

We first study a simple 2-type system consisting of two trading strategies and in which the fundamental strategy appears in the market again by default, i.e., $g_1 = b_1 = 0$. A discrete grid of the true intensity of choice β and chartistic beliefs g_2 and b_2 is defined to cover a purposeful set of values. To keep the number of combinations at a reasonable level, we opt for $\beta = \{0, 0.5, 3, 10\}$ and cover all combinations of trend-following ($g_2 > 0$), contrarian ($g_2 < 0$), upward-biased ($b_2 > 0$), and downward-biased ($b_2 < 0$) strategies. We refer the reader to the first column of Table 3 for a complete specification. We report only one specification of noise, namely, $\varepsilon_t \sim N(0, 0.1^2)$, because the results, e.g., for $\varepsilon_t \sim N(0, 1)$ essentially do not change.

In a nutshell, Table 3 confirms all main findings from the simulation analysis of the single parameter β estimation. The method is also generally able to accurately reveal the true values of estimated parameters in the 3-parameter joint estimation case. In addition, belief coefficients g_2 and b_2, which are of central importance in this section, are estimated as significant overall and with markedly high precision. Because of the altered setup, the estimation precision of the β parameter is not directly comparable to previous results; however, we still get generally conformable figures.

To verify smoothness conditions and the identification of parameters with multiple parameter estimation, Fig. 4 visualises 3D simulated sub-log-likelihood functions in a global shape when imaginarily combined in a 4D object. One can think of these functions as transversal cuts or profiles of the likelihood function in the planes of given parameters, keeping the remaining parameter fixed at $\beta = 0.5$, $g_2 = 0.4$, $b_2 = 0.3$. We clearly observe very smooth shapes of the surface, and regions

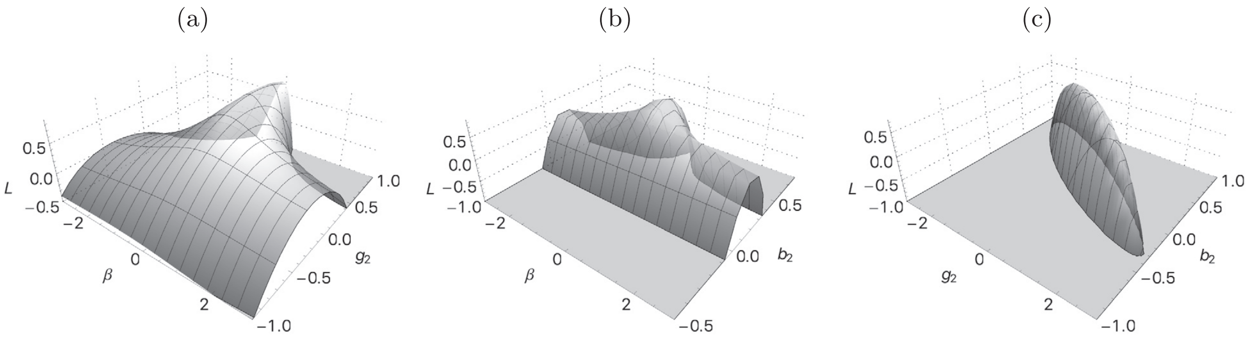

Fig. 4. Simulated sub-log-likelihood functions in 3D. Results are averaged over 30 random runs, $t = 5000$, and $N = 1000$. The complete set of true parameters: $\beta = 0.5$, $g_2 = 0.4$, $b_2 = 0.3$. Stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from normal distribution $N(0, 0.1^2)$. L denotes log-likelihood.

of possible maxima are easily detectable. Parameters g_2 and b_2 are well identified, which confirms the previous results. The most challenging part is the identification of the β coefficient in the direction of which the surface is very flat for a large interval of the domain. These findings are largely in accord with the conclusions of Bolt et al. (2014, pg. 15) and Hommes and Veld (2015), who claim that "the other parameters can to a large extent compensate for changes in β" and similarly report a very flat shape of the likelihood function. We can generally assume that the regularity conditions are met and that the identification of parameters is assured for the 2-type model estimation as well.

Finally, the results of a joint estimation of 5 parameters in the 3-type model are provided in Table 11 in Appendix A. We continue the strategy of defining a grid of chartistic beliefs and the conclusions are generally in accord with the results of the 2-type model estimation. The difference is mainly in the efficiency of estimates, which is by nature lower for the 3-type model than for the 2-type model.

5. Estimation on empirical data

Equipped with the performance study of the proposed methodology, we broaden the topic with an empirical application, estimating the Brock and Hommes (1998) model using a cross-section of world stock markets. We analyse S&P500 and NASDAQ for the U.S., DAX and FTSE for Europe, and NIKKEI 225 and HSI for Japan and Hong Kong, respectively.

5.1. The estimation setup

The setup of the empirical estimation algorithm follows the findings of the simulation study regarding the sufficient setting of the NPSMLE. We compute results for 1000 random runs and number of observations $t = 5000$, and the kernel approximation precision is set to $N = 500$. Because of a potential problem with the numerical stability of the model and the fairly rough shapes of the simulated log-likelihood functions when real data are introduced, we increase the number of randomly generated starting points to eight, which increases the computational burden considerably. The other setting remains the same as defined earlier. As we show in Section 4.3.2, a proper specification of stochastic market noise is crucial for estimation performance. Because noise intensity for various real markets is a rarely anticipated variable, we eschew the grid strategy here. Instead, we define noise intensity as a fraction of the standard deviations of the noise term and the data and add this new coefficient, denoted as *noise intensity*, to the list of estimated parameters. We thus apply a joint unconstrained multivariable function estimation of four interesting parameters: agents' belief coefficients g_h and b_h, intensity of choice β, and market *noise intensity*.

We start with the estimation of the 2-type model, which includes one chartistic strategy represented by g_2 and b_2, which are to be estimated. Then, we estimate the 3-type model, which includes two different chartistic strategies. Based on the results of the 2-type model estimation, we assume a zero bias for both strategies, i.e., $b_2 = b_3 = 0$. Moreover, to properly identify the two different strategies, we need to constrain the trend-following coefficient $g_2 > 0$ and the contrarian coefficient $g_3 < 0$.

5.2. Fundamental price approximation

Approximation of the fundamental price is inevitably the most challenging piece of the entire empirical estimation. In this process, we follow the existing literature, which approximates the fundamental price based on a moving average (ter Ellen and Zwinkels, 2010; Huisman et al., 2010; Winker et al., 2007). For example, Winker et al. (2007) use moving average (MA) over the last 200 observations of the DM/USD exchange rate time series for the period 1991/11/11 to 2000/11/9 as a proxy for the fundamental price. ter Ellen and Zwinkels (2010) use the MA of the Brent and WTI-Cushing monthly oil prices in USD over 24 months, i.e., from 1984/1 to 2009/8. Huisman et al. (2010) employ the MA of European forward electricity daily historical prices over three years for the base-load calendar year 2008 forward contracts and set the MA window to 3 as a calibration result of the optimal length.[8]

Long-term and short-term MA is also commonly used by practicing traders to extrapolate divergence from the fundamental value in technical analysis. Because the fundamental value of stocks is essentially unknown, market practitioners often tend to at least estimate whether the stock is over- or under-valued, whether possible mispricing is small or large, and whether the gap is going to increase or a rapid correction is more likely. Because the Brock and Hommes (1998) model is also formulated in deviations from the fundamental price, the MA approach to approximation seems reasonable. The MA filtering is a cornerstone of technical analysis and is therefore widely used by active traders: Allen and Taylor (1990, pg. 50) present empirical evidence on the perceived importance of technical analysis among London foreign exchange dealers and refer to prevalent mechanical indicators, including trend-following rules: "buy when a shorter MA cuts a longer MA from below". Taylor and Allen (1992) survey chief foreign exchange dealers operating in London and report that 64.3% of organisations use MA and/or other trend-following analytical techniques. Brock et al. (1992, pg. 1735) refer to MA technical rules as one of the two simplest and most widely used techniques: "when the short-period MA penetrates the long-period MA, a trend is considered to be initiated". Lui and Mole (1998, pg. 541, 535) repeat the largely analogical survey conducted by Taylor and Allen (1992) among Hong Kong foreign exchange dealers and report the same conclusion about the usefulness of MA at intraday, intramonth, and $>$ 1 month horizons. Goldbaum (1999, pg. 70, 71) describes how the MA trading rules translate into buy-sell indicators in practice, and Sullivan et al. (1999, pg. 1656) state that "MA cross-over rules…are among the most popular and common trading rules discussed in the technical analysis literature". Closely related to our work, Chiarella et al. (2006, pg. 1748) propose a model in which the demand of chartists is determined by the difference between a long-term MA and the current market price.

In our empirical estimation, we keep the strategy of using a wide range of possible settings to ensure the robustness of our findings. We present results for two specific MA window lengths, namely, 61 (MA61) and 241 (MA241) days. For the robustness check, we also tested other variants ranging from one month to two years, namely, 21, 121, and 481 days, and obtained comparable results. These results are available from the authors upon request. Instead of the usual 'historical' MA, which considers only past information for a given time, we use the 'centred' MA. Both MA versions were analysed and found to produce largely comparable results. The centred MA is therefore suggested to reduce the delay in information flow. Moreover, the centred MA incorporates a convenient property—that the price by definition converges to it—which is precisely the type of feature one would expect from the fundamental value. Although undoubtedly our fundamental price approximation differs from the true fundamental value, the MA filter produces a series of anticipated structures to be modelled. However, it is important to emphasise here that if the MA is a poor proxy for true fundamental value, bias may result in the subsequent empirical sections.

5.3. Data description

We use the daily close prices of six world stock market indices retrieved from Yahoo Finance as the base of our empirical dataset. For S&P500, the dataset covers the period from 1994/02/23 to 2013/12/31, i.e., 5000 observations in total. For the other indices, only the starting dates of the datasets vary due to different public holidays and calendar configurations around the world, i.e., 1994/04/22 for DAX, 1994/11/02 for FTSE, 1994/02/23 for NASDAQ, 1993/09/03 for NIKKEI 225, and 1994/06/13 for HSI. The fundamental price is simultaneously calculated as the centred MA and subtracted from the actual price; hence, for each index, we obtain a comparable number of 5000 deviations y_t with the same end date (2013/12/31) that are the subject of further estimation. Descriptive statistics of y_t series for all indices and two MA lengths for the fundamental value approximation are summarised in Table 12 and in Appendix A.

5.4. Full sample estimates of the 2-type model

We start with the full sample estimation summarised in Table 4. We observe broad similarities across all indices and notably statistically significant estimates of a positive belief parameter g_2, revealing the superiority of trend following over contrarian strategies in financial markets. In contrast, estimates of the intensity of choice β and the bias parameter b_2 are largely statistically insignificant. Although this is the expected result for the bias, because there is no obvious reason why trend-following strategies should be somehow biased in the long term, the insignificance of $\widehat{\beta}$ is an important and interesting result. We thus conflict with a subpart of the FABM estimation literature (see Section 2) but confirm the main results of, e.g., Bolt et al. (2014); Boswijk et al. (2007); ter Ellen et al. (2013); de Jong et al. (2009b); Westerhoff and Reitz (2005). Because the heterogeneity in trading regimes is confirmed by the significance of g_2, this outcome might not be cause for concern, as discussed in Boswijk et al. (2007, pg. 1995) and Hommes (2013, pg. 203), who emphasise that "this is a common result in non-linear switching regression models, where the parameter in the transition function is difficult to estimate

[8] In contrast, another class of FABMs of FX markets successfully utilises the purchasing power parity between two countries to approximate the fundamental value of the currency exchange rate [see e.g. Goldbaum and Zwinkels (2014); Manzan and Westerhoff (2007); Verschoor and Zwinkels (2013); Vigfusson (1997); Wan and Kao (2009); Westerhoff and Reitz (2003)]. Boswijk et al. (2007) and de Jong et al. (2009a) employ the static growth model proposed by Gordon (1962) for equity valuation, but this approach is infeasible for empirical validation of the original (Brock and Hommes, 1998) model. Several other papers simply use a random walk formula to drive the fundamental price (De Grauwe and Grimaldi, 2005; 2006; Franke, 2009; Winker et al., 2007).

Table 4
Empirical results of the 2-type switching model estimation.

Data, MA period	(a) $\widehat{\beta}$			(b) $\widehat{g}_2$			(c) $\widehat{b}_2$			(d) $\widehat{noise\ intensity}$			(e) L		
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD
SP500, 61	0.015	0.040	0.122	1.567	1.587	0.233	0.009	0.003	0.121	0.653	0.656	0.108	−1.486	−1.491	0.074
NASDAQ, 61	0.002	0.006	0.146	1.717	1.715	0.166	−.005	−.003	0.092	0.609	0.609	0.079	0.117	0.115	0.038
DAX, 61	0.001	0.018	0.112	1.640	1.646	0.215	0.008	0.001	0.117	0.590	0.601	0.099	−1.259	−1.264	0.081
FTSE, 61	−.000	0.008	0.113	1.671	1.668	0.201	0.001	0.004	0.117	0.597	0.602	0.092	−.988	−.995	0.059
HSI, 61	−.004	−.002	0.150	1.724	1.727	0.164	−.003	−.001	0.098	0.566	0.570	0.069	−.145	−.147	0.036
N225, 61	0.008	0.021	0.100	1.601	1.619	0.210	0.011	0.007	0.121	0.593	0.601	0.103	−1.544	−1.546	0.085
SP500, 241	−.021	−.009	0.195	1.878	1.875	0.127	0.007	0.001	0.111	0.388	0.397	0.064	0.029	0.019	0.074
NASDAQ, 241	0.038	0.047	0.189	1.882	1.869	0.152	−.009	−.003	0.105	0.423	0.419	0.078	0.422	0.424	0.111
DAX, 241	0.014	0.012	0.247	1.932	1.928	0.109	0.007	0.003	0.103	0.292	0.315	0.070	0.328	0.309	0.096
FTSE, 241	0.010	0.012	0.162	1.871	1.858	0.143	0.008	0.004	0.114	0.408	0.409	0.071	−.153	−.157	0.085
HSI, 241	0.008	0.006	0.204	1.914	1.908	0.130	0.004	0.002	0.103	0.351	0.366	0.080	0.369	0.363	0.118
N225, 241	0.003	0.012	0.147	1.865	1.850	0.165	−.000	−.000	0.125	0.394	0.392	0.101	−.793	−.793	0.160
Robustness check															
SP500 monthly, 13	−.016	−.031	0.272	0.886	0.891	0.238	0.006	0.002	0.121	0.863	0.869	0.104	−3.660	−3.666	0.054
SP500 weekly, 13	−.085	−.103	0.228	0.953	0.971	0.224	−.003	−.001	0.121	0.865	0.869	0.098	−2.336	−2.349	0.057
SP500 weekly, 49	0.044	0.089	0.146	1.119	1.168	0.241	0.005	0.002	0.118	0.724	0.720	0.103	−2.626	−2.615	0.073
SP500 R=1.001, 61	0.016	0.035	0.112	1.585	1.607	0.241	0.004	0.004	0.121	0.656	0.654	0.110	−1.480	−1.485	0.073
SP500 R=1.001, 241	0.001	−.001	0.196	1.889	1.880	0.134	0.005	0.002	0.111	0.393	0.397	0.062	0.029	0.019	0.073
SP500 m_h=40, 61	−.032	−.041	0.149	1.659	1.673	0.173	−.002	−.002	0.114	0.584	0.587	0.075	−1.393	−1.402	0.038
SP500 m_h=40, 241	−.028	−.020	0.298	1.908	1.905	0.107	−.004	−.003	0.099	0.338	0.355	0.056	0.100	0.086	0.054
SP500 m_h=80, 61	−.047	−.054	0.177	1.644	1.664	0.182	0.003	0.003	0.115	0.573	0.576	0.065	−1.385	−1.392	0.032
SP500 m_h=80, 241	−.036	−.014	0.309	1.908	1.906	0.105	−.002	−.004	0.097	0.335	0.352	0.054	0.106	0.091	0.056

Results are based on 1000 random runs, $t = 5000$, and $N = 500$ draws from normal distribution. Sample medians, means, and standard deviations (SD) are reported. L denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

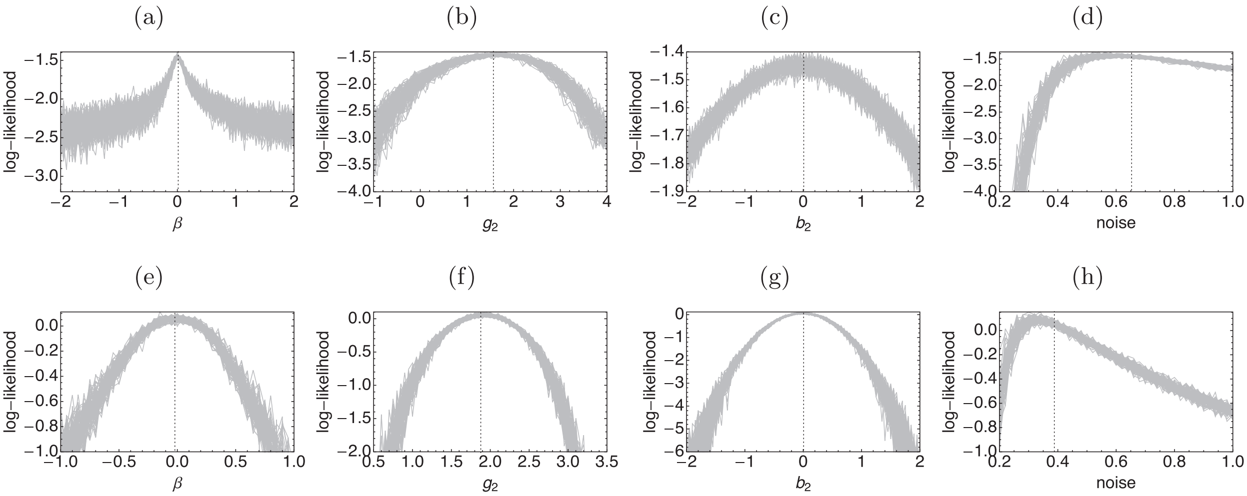

Fig. 5. Simulated sub-log-likelihood functions for single parameters. Results are based on 100 random runs, S&P500 data, and $N = 500$ draws from normal distribution. Upper part (a-d) depicts results based on MA61 fundamental price approximation, bottom part (e-h) based on MA241. Black dotted vertical lines depict estimated parameters (see Table 4).

and has a large standard deviation, because relatively large changes in β^* cause only small variation of the fraction n_t. Teräsvirta (1994) argues that this should not be worrisome as long as there is significant heterogeneity in the estimated regimes". Furthermore, as noted by Huisman et al. (2010, pg. 17, 20), "the significance of the intensity of choice is not a necessary condition for the switching to have added value to the fit of the model" and "the non-significant intensity of choice…indicates that the switching does not occur systematically". The magnitudes of trend parameter estimates $\widehat{g_2}$, which stay roughly between 1.6 and 1.9, might seem large, but it is important to note that they influence the price change only from the circa 50% implied by the insignificance of the intensity of choice β, keeping the population ratio of the two strategies relatively stable around 0.5/0.5. We also observe that the larger the MA window is, the larger the estimates $\widehat{g_2}$, but the smaller the standard deviations. Our explanation is that the longer period of the MA filter suggesting a longer fundamental trend, the more various short-term trends remain in the time series of price deviations y_t, that are likely to refine the estimate of the trend following coefficient g_2 and consequently reduce the standard error. However, we stress that the respective $\widehat{g_2}$ estimates based on MA61 and MA241 are generally statistically indistinguishable based on reported sample standard deviations considering usual significance levels.

Certain differences across markets can be seen in the (d) column of Table 4 between efficiently estimated values of the $\widehat{\textit{noise intensity}}$. It is worth mentioning that the highest stock market noise intensity is estimated for the U.S. indices, specifically, S&P500 in case of MA61-based fundamental value and NASDAQ in case of MA241. Conversely, the lowest values are estimated for DAX and the difference is circa 30% for MA241. The level difference in values between the upper part of Table 4, which depicts results for the MA61 fundamental price approximation, and the middle part, which shows results for the MA241, is perhaps again a technical consequence induced by different lengths of MA windows. Nevertheless, the main results also demonstrate considerable robustness with respect to the choice of the fundamental value specification. The lower values of the $\widehat{\textit{noise intensity}}$ might be explained by a better fundamental value approximation using a bigger MA window.

5.4.1. Behaviour of the simulated log-likelihood function

We also verify the smoothness conditions and unique maxima of the simulated log-likelihood functions within the empirical application to show how the parameters are identified. Fig. 5 draws partial 2D shapes of sub-log-likelihood functions in the direction of given parameters, assuming that the other parameters are fixed at estimated values from Table 4. Generally, we observe slightly rough shapes in the detail but a very consistent performance of the estimation method over all 100 random runs, leading to unique maxima consistent with the full sample estimates in all cases. Interestingly, based on visual inspections of subfigures (d) and (h), we suspect a small potential upward bias of the $\widehat{\textit{noise intensity}}$.

5.4.2. Robustness check of the 2-type model

For the robustness check, we consider two modifications of the estimation setup and different data frequencies. Equipped with the knowledge obtained from the previous empirical analysis, we compute results only for S&P500. In addition to weekly and monthly datasets, we estimate the model using an unrealistically high daily risk-free interest rate $R = 1.001$ and a nontrivial agent memory defined via parameters $m_h = \{40, 80\}\ \forall\ h \in \{1, \ldots, H\}$ in Eq. 18 for an average memory length resembling 20 or 40 days. Three new datasets cover the same span, i.e., we use weekly data from 1994/02/28 to 2013/12/30 (1035 observations) and monthly data from 1994/03/01 to 2013/12/02 (238 observations). The MA lengths are selected to resemble the 61 and 241 days as closely as possible, i.e., 13 and 49 weeks and 13 months. The assumed market risk-free rate is adjusted to reflect the modified data periodicity, that is, to $R = 1.0005$ and $R = 1.002$.

Results reported in the bottom section of Table 4 confirm that the important findings of the preceding empirical analysis remain unaffected under the robustness burden. Differences are mainly observable at the level of trend parameters $\widehat{g_2}$ and $\widehat{noise\ intensity}$, but the behaviour keeps the detected patterns within the original analysis. Results based on monthly and weekly data show considerably lower $\widehat{g_2}$—the values even fall below 1 for the monthly MA13 fundamental value specification—indicating a weak trend-following strategy. In other words, the lower the frequency of data, the lower the estimates $\widehat{g_2}$. This effect of a lower data frequency associated with more average aggregated dynamics of the price deviation series y_t is expected as monthly data are naturally smother than the corresponding weekly or daily time series. Memory increases the model fit only slightly; as the interconnected effect, it simultaneously decreases the $\widehat{noise\ intensity}$.

We also check that our estimation results are robust towards imposing under- or over-smoothing of the bandwidth size η. As another robustness check, we estimate the model on yearly rolling windows to uncover possible dynamics in time. The rolling approach particularly strongly supports the stability of model behaviour over time and thus confirms the validity of the full sample estimation results. Our empirical findings thus prove robustness to various data frequencies and subperiods and to modifications of interesting parameters in the model.

5.5. Full sample estimates of the 3-type model

Estimation of a more-flexible 3-type model reveals a strikingly similar big picture: the estimate of intensity of choice $\widehat{\beta}$ maintains its statistical insignificance, and the trend-following strategy coefficient $\widehat{g_2}$ retains its positive sign and high statistical significance. Estimated parameters for S&P500 are reported in Table 13 in Appendix A. The only new conclusion is a statistical insignificance of the contrarian strategy represented by coefficient $\widehat{g_3}$. Although its reported point estimates are negative, this is merely an effect of the enforced $g_3 < 0$ constraint. The real distribution mass of the estimates concentrates close to 0. The likelihood function is likely to be very flat in the dimension of the g_3 parameter because the effect of a very weak contrarian strategy is overshadowed if it is combined with a very strong trend-following strategy. The absolute value of $\widehat{g_2}$ is naturally higher because the trend-following strategy now impacts the price only via the 1/3 weight in the 3-type model (compared to 1/2 weight in the 2-type model), conditional on insignificant $\widehat{\beta}$. Taking those weights into account, we obtain very similar impacts of the trend-following strategy in both models.

5.6. Estimation of market fractions

A statistically insignificant intensity of choice parameter $\widehat{\beta}$ implies that any systematic evolutionary switching between trading strategies cannot be detected. It also implies a relatively stable population ratio of trading strategies $n_{1,\,t}/n_{2,\,t}\doteq 0.5/0.5$ in time, which means that the populations of different trading strategies are forced to maintain nearly the same magnitudes throughout the entire data span. Thus, the model almost boils down to a simple weighted AR(1) process and different types of trading strategies cannot be well identified because agents do not switch over time. We thus face a problem of nuisance parameters, i.e., $\widehat{g_2}$ and $\widehat{b_2}$ might to some extent lose their original model interpretations and we cannot fully trust their estimated magnitudes. Moreover, because the $\widehat{\beta}$ and $\widehat{g_3}$ coefficients are statistically insignificant, contrarians in the 3-type model behave as fundamentalists in terms of their price impact. These findings imply two important conclusions. First, the 3-type model does not really help us to capture additional features of the data-generating process; in contrast, it deviates the implied market fractions. Second, the 3-type model suggests that there might be more fundamentalists than chartists in real markets and therefore the nearly fixed population ratio of trading strategies $n_{1,\,t}/n_{2,\,t}\doteq 0.5/0.5$ in the 2-type model estimation does not likely capture real market population proportions.

Therefore, we trivialise the model by disabling evolutionary switching behaviour and fixing the population ratio of trading strategies at $n_{1,t}/n_{2,t} = const$. Eqs. (13) and (14) are replaced by Eq. (20), and the coefficient n_1 (which we also call the percentage *fraction* of fundamentalists) is to be estimated instead of the switching coefficient β:

$$Rx_t = \sum_{h=1}^{H} n_h f_{h,t} + \varepsilon_t \equiv \sum_{h=1}^{H} n_h (g_h x_{t-1} + b_h) + \varepsilon_t, \tag{19}$$

$$n_1 = 1 - n_2, \tag{20}$$

where $H = 2$ for the 2-type model. The modified setup does not distort the structure of the original model, but the population ratio of trading strategies n_1/n_2 and implied percentage *fraction* of fundamentalists ($g_1 = b_1 = 0$) in the market are now direct subjects of estimation. The other setup stays the same.

5.6.1. Full sample estimates of the 2-type fraction model and a robustness check

Outcomes of the full sample estimation of six stock market indices are reported in Table 5, and the main interest now lies in the behaviour of the new variable *fraction*. The behaviour of all other variables is on average very similar to their behaviour in the 2-type β model estimation; moreover, we no longer observe considerable distinctions caused by the MA window length for the fundamental value approximation. The $\widehat{fraction}$ coefficient is strongly statistically significant, with values closely around 0.56, leaving 44% of the market population to chartistic strategies. The *fraction* model therefore suggests

Table 5
Empirical results of the 2-type *fraction* model estimation.

Data, MA period	(a) $\widehat{fraction}$			(b) $\widehat{g_2}$			(c) $\widehat{b_2}$			(d) $\widehat{noise\ intensity}$			(e) L		
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD
SP500, 61	0.569	0.555	0.088	1.963	1.986	0.362	0.006	0.003	0.121	0.559	0.558	0.044	−4.022	−4.023	0.013
NASDAQ, 61	0.555	0.541	0.086	1.972	1.985	0.344	−.001	−.001	0.117	0.565	0.566	0.054	−5.118	−5.119	0.019
DAX, 61	0.559	0.547	0.090	2.006	2.021	0.387	0.008	0.005	0.123	0.489	0.490	0.035	−5.733	−5.733	0.011
FTSE, 61	0.556	0.545	0.085	1.980	1.997	0.357	−.009	−.007	0.121	0.519	0.520	0.037	−5.507	−5.507	0.011
HSI, 61	0.559	0.549	0.082	1.993	2.020	0.344	−.006	−.001	0.120	0.531	0.531	0.045	−6.999	−7.000	0.016
N225, 61	0.562	0.553	0.084	1.995	2.023	0.361	0.005	0.001	0.124	0.482	0.484	0.032	−6.700	−6.701	0.010
SP500, 241	0.562	0.556	0.084	2.161	2.228	0.425	0.002	−.001	0.133	0.331	0.348	0.053	−4.073	−4.088	0.049
NASDAQ, 241	0.562	0.556	0.086	2.191	2.244	0.434	−.002	−.001	0.133	0.311	0.345	0.077	−5.195	−5.231	0.089
DAX, 241	0.565	0.559	0.088	2.219	2.275	0.455	0.013	0.003	0.137	0.256	0.279	0.069	−5.782	−5.814	0.086
FTSE, 241	0.562	0.554	0.087	2.179	2.227	0.445	−.002	0.003	0.133	0.322	0.339	0.053	−5.547	−5.563	0.045
HSI, 241	0.562	0.554	0.084	2.204	2.253	0.446	−.009	−.006	0.137	0.276	0.304	0.070	−7.057	−7.095	0.091
N225, 241	0.570	0.562	0.087	2.232	2.281	0.460	−.012	−.005	0.138	0.255	0.272	0.058	−6.744	−6.767	0.069
Robustness check															
SP500 monthly, 13	0.606	0.592	0.114	1.429	1.451	0.306	−.000	0.000	0.116	0.806	0.801	0.064	−5.150	−5.153	0.014
SP500 weekly, 13	0.635	0.616	0.128	1.404	1.422	0.309	−.011	−.004	0.115	0.853	0.848	0.063	−4.598	−4.603	0.028
SP500 weekly, 49	0.485	0.483	0.093	1.591	1.624	0.292	0.012	0.006	0.117	0.606	0.606	0.057	−4.721	−4.722	0.017
SP500 R=1.001, 61	0.566	0.553	0.089	1.967	1.984	0.368	−.004	−.004	0.121	0.557	0.557	0.044	−4.022	−4.022	0.013
SP500 R=1.001, 241	0.570	0.557	0.085	2.220	2.239	0.430	0.003	−.001	0.132	0.332	0.349	0.054	−4.074	−4.089	0.047

Results are based on 1000 random runs, $t = 5000$, and $N = 500$ draws from normal distribution. Sample medians, means, and standard deviations (SD) are reported. L denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

overall proportional dominance of the fundamental strategy in the investigated world stock markets. Estimates of the trend-following coefficient g_2 are generally higher compared to the values from the 2-type switching model, but one must realise that here the trend-following strategy is relatively weaker in terms of market price impact because its proportion is lower than 0.5. If we consider estimates incorrectly implied by the 2-type switching model and compare it to $\widehat{fraction}$ and related $\widehat{g_2}$ estimated in this section according to Eq. 12, we deduce an almost similar impact. This result confirms our suspicion of an improper model specification with insignificant $\widehat{\beta}$, and we correct for this misspecification by introducing $\widehat{fraction}$ specification via Eq. 20. Evolutionary switching can be now captured through changes in the $\widehat{fraction}$ coefficient in a smooth form by using the rolling approach, as asserted by Teräsvirta (1994, pg. 217): "if one assumes that agents make only dichotomous decisions or change their behaviour discretely, it is unlikely that they do this simultaneously. Thus, if only an aggregated process is observed, then the regime changes in that process may be more accurately described as being smooth rather than discrete." Nonetheless, the rolling approach does not reveal any significant dynamics in the behaviour of $\widehat{fraction}$, which confirms the validity of the full sample estimation results. The rolling sample estimates are available upon request.

For the *fraction* model, we employ a similar (but now irrelevant effect of memory) robustness check and report the results in the bottom section of Table 5. Basic conclusions are identical to the previous findings: the $\widehat{fraction}$, $\widehat{g_2}$, and $\widehat{noise\ intensity}$ generally reveal strong statistical significance, and the bias parameter $\widehat{b_2}$ shows the opposite behaviour. Differences are again observable at the level of trend parameters $\widehat{g_2}$ and $\widehat{noise\ intensity}$ based on monthly and weekly data, whereas the results show lower $\widehat{g_2}$ and higher $\widehat{noise\ intensity}$ for daily data. The shapes of simulated log-likelihood functions are similar to those of previous models.

6. Conclusion

This paper proposes an innovative computational framework for empirical estimation of FABMs. Motivated by the lack of general consensus on estimation methodology, the paucity of examples of structural estimation of FABMs, and inconclusive results in recent FABM literature, we aim to customise and test a more general method for the estimation of FABMs that significantly reduces the importance of restrictive theoretical assumptions.

In a large simulation and estimation exercise, we show that the recently developed NPSMLE (Kristensen and Shin, 2012) married with the Brock and Hommes (1998) model generally estimates the parameters of various versions of the model precisely. We confirm that simulated MLE constitutes a flexible method to estimate complicated nonlinear models that are incompatible with traditional estimation approaches and have historically largely remained inestimable. Employing NPSMLE, we are generally able to estimate models for which the closed-form solution or theoretical approximation of the objective function does not exist. We also prove that by using simulation-based non-parametric methods, the parameters of such systems can be recovered reasonably well. Together with the rapidly increasing computational capabilities of personal computers, server clusters, and supercomputers, we anticipate a bright future and rapid development of simulation-based methods in the near future.

The crucial result of the empirical analysis is the statistical insignificance of the switching coefficient across major world market indices. Although this result is common in the existing literature, it conflicts with a segment of the estimation literature on FABMs that reports significant switching coefficients for various specific markets. Specifically, our estimation results of the 2-type model reveal markedly statistically significant belief parameters that define heterogeneous trading regimes with an absolute superiority of trend following over contrarian strategies. Our findings further indicate robustness with respect to the fundamental value specification and remain largely unaffected under the robustness burden of data frequencies other than daily, jumps in the market risk-free rate, and the introduction of agent memory. Graphical inspection of simulated log-likelihood functions reveals a slightly rough surface but a very consistent performance of the estimation method over all random runs leading to unique maxima. However, the adapted computational algorithm ably addresses the not-completely-smooth surface of the simulated log-likelihood function and thus the important identification feature is also verified for empirical application.

Acknowledgements

This paper benefited greatly from detailed discussions with E. Gerba, L. Vacha, and R.C.J. Zwinkels and from the comments of three anonymous referees, to whom we are especially grateful. We are also indebted to C. Hommes, B. LeBaron, and S. Barde and the participants of WEHIA 2014, CFE 2014, First Bordeaux-Milano Joint Workshop on Agent-Based Macroeconomics 2015, WEHIA 2015, CEF 2015, and a doctoral seminar at Charles University for their many useful comments and suggestions.

The research leading to these results received funding from the European Union's Seventh Framework Programme FP7/2007-2013 [grant agreement no. FP7-SSH- 612955 (FinMaP)]. Support from the Czech Science Foundation [grant project P402/12/G097 DYME - 'Dynamic Models in Economics'] is also gratefully acknowledged, and J. Kukacka gratefully acknowledges financial support from the Charles University Grant Agency (GAUK) http://www.cuni.cz/UK-33.html [grant number 192215].

Appendix A. Supplementary tables

Table 6
Estimation methods of FABMs I.

Model	Origin	Method	Parameters estimated	#	Data	Type	Fit	|IOC|
Alfarano et al. (2005)	IAH	ML	Herding tendency	2	d:5034–9761 o.	s,fx,g	-	-
Alfarano et al. (2006)	IAH	ML	Herding tendency	2	d:5495,6523 o.	s,fx	-	-
Alfarano et al. (2007)	IAH	ML	Herding tendency	2	d:1975–2001	s	-	-
Amilon (2008)	ABS	EMM/ML	Intensity of choice[a]	15	d:1980–2000	s	$p-v=0\%$	1.99(i),1.91(s)
Boswijk et al. (2007)	ABS	NLS	Belief coefficients/Intensity of choice	3	a:132 o.	s	$R^2=.82$	10.29(i),7.54(i)
de Jong et al. (2009b)	ABS	NLS	Belief coefficients/Intensity of choice	5	w:102 o.	fx	$adjR^2=.14$	1.52(i)
de Jong et al. (2010)	ABS	QML	Belief coefficients/Intensity of choice	7	m:238 o.	fx	-	0.0007(i)–6.29(s)
Diks and Weide (2005)	ABS	ML	(G)ARCH relations/Sign of MA(1) c.	3	d:3914 o.	fx	-	-
Ecemis et al. (2005)	AA	IEC	Market fractions/Behavioural rules	3	-	s	-	-
Gilli and Winker (2003)	ANT	MSM	Mutation/Conviction rate	3	d:1991–2000	fx	NA	-
Manzan and Westerhoff (2007)	ABS	OLS	Reaction c./Switching threshold	4	m:1/74–12/98	fx	NA	-
Reitz and Westerhoff (2007)	ABS	QML	Behavioural rules/Intensity of choice	6	m:365 o.	c	-	0.17(s)–.47(s)
Westerhoff and Reitz (2003)	ABS	QML	Behavioural rules/Intensity of choice	7	d:4431 o.	fx	-	0.02(s)–.17(s)
Winker and Gilli (2001)	ANT	MSM	Mutation/Conviction rate	2	d:1991–2000	fx	NA	-
Winker et al. (2007)	ANT	MSM	Mutation/Conviction rate	3	d:1991–2000	fx	$p-v<1\%$[b]	-

This table is adopted from Chen et al. (2012, pg. 203) and amended by the authors. Authors of the articles are alphabetised. The full meaning of the acronyms under “Origin”: AA stands for autonomous agents, ABS for adaptive belief system, ANT for the ant type of system, and IAH for interactive agent hypothesis. The full meaning of the acronyms under “Methods”: ML stands for maximum likelihood, EMM for efficient method of moments, NLS for nonlinear least squares, QML for quasi maximum likelihood, OLS for ordinary least squares, IEC for interactive evolutionary computation, and MSM (SMM) for method of simulated moments. “#” displays the total number of estimated parameters; “Data” describes data frequency (“d/w/m/q/a” for daily/weekly/monthly/quarterly/annual) and the number of observations (where no specific figure is provided, we report the starting and final years); “Type” shows the type of data(“s/fx/c/g/re” for stock markets/FX/commodity markets/gold/real estate); “Fit” reports the statistical fit of the estimation (R^2, its alternatives, p-value of the J-test of overidentifying restrictions to accept the model as a possible data generating process); and, where relevant, “|IOC|” displays the absolute estimated value of the switching parameter of the intensity of choice and “s”/“i” denotes its statistical significance/insignificance at the 5% level. Figures are rounded to 2 decimal digits.

[a] Chen et al. (2012) do not report other important parameters estimated, namely, belief coefficients, intensities of exogenous noises, risk aversion, information costs for fundamentalists, forgetting factors, and memory in the fitness measure.

[b] While p-val for GARCH(1,1) model $>$ 5%.

Table 7
Estimation methods of FABMs II. a).

Model	Origin	Method	Parameters estimated
Barunik and Vosvrda (2009)	Cusp	ML	Asymmetry and bifurcation factors/Location and scale coefficient
Barunik and Kukacka (2015)	Cusp	RV/ML	Asymmetry and bifurcation factors/Polynomial data approximation
Berardi et al. (2016)	ABS	calibration	Belief coefficient/Intensity of choice/Risk aversion/Fundamental value
Bolt et al. (2011)	ABS	NLS(?)	Expectations' bias/Discount factor/Belief coefficients/Intensity of choice
Bolt et al. (2014)	ABS	NLS	Belief coefficients/A-synchronous updating ratio/Intensity of choice
Cornea et al. (2013)	ABS	VAR/NLS	Fundamentalists' belief coefficient/Intensity of choice
Chen and Lux (2016)	IAH	MSM	Standard deviation of innovations/Herding tendency
Chiarella et al. (2014)	ABS	QML	Belief & market maker coefficients/Memory decay rate/Intensity of choice
Chiarella et al. (2015)	ABS	QML	Belief coefficients/Variance risk premium/Intensity of choice
de Jong et al. (2009a)	ABS	ML	Belief coefficients/Intensity of choice
Diks and Wang (2016)	Cusp	ML	Asymmetry and bifurcation factors/Scale coefficient
ter Ellen and Zwinkels (2010)	ABS	QML	Belief coefficients/Intensity of choice
ter Ellen et al. (2013)	ABS	OLS/NLS	Behavioural rules/Intensity of choice
Franke (2009)	ABS	MSM	Reaction coefficients/Switching threshold
Frijns et al. (2010)	ABS	EMS	Local volatility/Belief coefficients/Intensity of choice
Franke and Westerhoff (2011)	IAH	MSM	Behavioural rules/Flexibility/Predisposition coefficients
Franke and Westerhoff (2012)	ABS/IAH	MSM	Behavioural rules/Wealth/Predisposition/Misalignment coefficients
Franke and Westerhoff (2016)	IAH	MSM	Behavioural rules and noises/Predisposition/Herding/Misalignment coefficients
Ghonghadze and Lux (2016)	IAH	GMM	Standard deviation of innovations/Herding tendency
Grazzini et al. (2013)	Bass (1969)	ML, MSM	Probability of independent adoption/Peer pressure/Population size
Grazzini and Richiardi (2015)	-	SMD	-
Grazzini et al. (2017)	-	Bayesian inference	Behavioural parameter
Goldbaum and Zwinkels (2014)	ABS	OLS (iterative)	Belief coefficients
Hommes and Veld (2015)	ABS	NLS(?)	Belief coefficients/A-synchronous updating ratio(?)/Intensity of choice
Huisman et al. (2010)	ABS	QML	Belief coefficients/Intensity of choice
Kouwenberg and Zwinkels (2014)	ABS	QML	Belief coefficients/Intensity of choice
Kouwenberg and Zwinkels (2015)	ABS	QML	Price elasticity/Belief coefficients/Intensity of choice
Lof (2012)	ABS	NLS	Belief coefficients/Intensity of choice
Lof (2015)	ABS	VAR/NLS	Discount factors/Belief coefficients/Intensity of choice
Reitz and Slopek (2009)	ABS	QML	GARCH coefficients/Belief coefficients/Transition parameter
Recchioni et al. (2015)	ABS	calibration	Belief coefficient/Intensity of choice/Risk aversion/Fundamental value
Verschoor and Zwinkels (2013)	ABS	ML	Belief coefficients/Intensity of choice

This table follows the logic of Table 6 and summarises recent research not covered there. Authors of articles are alphabetised. The full meaning of the acronyms under "Origin": Cusp stands for the cusp catastrophe model, ABS for adaptive belief system, and IAH for interactive agent hypothesis. The full meaning of the acronyms under "Methods": ML stands for maximum likelihood, RV for realised volatility, NLS for nonlinear least squares, VAR for vector autoregression, MSM (SMM) for method of simulated moments, QML for quasi maximum likelihood, OLS for ordinary least squares, EMS for empirical martingale simulation by Duan and Simonato (1998), GMM for generalized method of moments, and SMD for simulated minimum distance. "?" means that the given information is unclear to the authors.

Table 8
Estimation methods of FABMs II. b).

Model	#	Data	Type	Fit	|IOC|
Barunik and Vosvrda (2009)	8 & 17	d:1987–1988, 2001–2002	s	$pseudo-R^2$ up to 0.8	-
Barunik and Kukacka (2015)	10	d:6739,409 o.	s	$pseudo-R^2 = .8, .86$	-
Berardi et al. (2016)	4	d:5579 o.	b	-	1.64(s),1.75(s),1.56(s)
Bolt et al. (2011)	5	q:164 o.	re	NA	2716(i),12420(i)
Bolt et al. (2014)	4	q:178 o.	re	NA	795(i)–26333(i)
Cornea et al. (2013)	2	q:204 o.	U.S. inflation	$R^2 = .78, .94$	4.78(s)
Chen and Lux (2016)	3	d:1/1980–12/2010	s/fx/g	$p-v \in \langle 4.6\%, 45.5\% \rangle$	-
Chiarella et al. (2014)	6	m:502,251 o.	s	-	0.44(s),.54(s),.69(s)
Chiarella et al. (2015)	5	w:2007–4/2013	CDS spreads	-	0.74(i)–6.84(s)
de Jong et al. (2009a)	10	q:112 o.	s	-	1.03(s),2.87(s)
Diks and Wang (2016)	3 & 5	q:1970-2013	re	-	-
ter Ellen and Zwinkels (2010)	7	m:295,319	c (crude oil)	-	1.19(s),1.36(s)
ter Ellen et al. (2013)	2–5	w:1/2003–2/2008	fx	$adjR^2$ up to 0.7	7.72(i)–454.4(i)
Franke (2009)	6	d:4115–6867 o.	s,fx	$p-v \in \langle 0\%, 2\% \rangle$	-
Frijns et al. (2010)	5	d:01–12/2000	s (index options)	-	107.34(i)
Franke and Westerhoff (2011)	6	d:6866,6861 o.	s,fx	$p-v = 12.8\%, 27.7\%$	-
Franke and Westerhoff (2012)	9	d:6866 o.	s	$p-v = 12.7\%–32.6\%$	-
Franke and Westerhoff (2016)	7	d:6866 o.	s	$p-v = 17.3\%$	-
Ghonghadze and Lux (2016)	3	d:1/1980–12/2009	s/fx/g	$p-v \in \langle 0.3\%, 67\% \rangle$	-
Grazzini et al. (2013)	3	-	-	-	-
Grazzini and Richiardi (2015)	1	d:400 o.	s	-	-
Grazzini et al. (2017)	1	d:1500 o.	s	-	-

(continued on next page)

Table 8 (*continued*)

Model	#	Data	Type	Fit	|IOC|
Goldbaum and Zwinkels (2014)	4	m:2825–2941 o.	fx (experts' forecasts)	$adjR^2 = .55–.79$	-
Hommes and Veld (2015)	4	q:252 o.	re	$R^2 = .95$	2.44(i)
Huisman et al. (2010)	4	d:694,753,1038 o.	c (electricity futures)	-	1.06(s),1.77(s),15.87(i)
Kouwenberg and Zwinkels (2014)	4	q:127,198 o.	re	-	2.98(s),1.36(s)
Kouwenberg and Zwinkels (2015)	5	q:204 o.	re	-	2.18(s)
Lof (2012)	7	q:208 o.	s	$R^2 = .97$	7.45(s), 4.74(s)
Lof (2015)	5	a:140 o.	s	$R^2 = .55$	type-specific: 0.8(i), 1.13(s),5.18(i)
Reitz and Slopek (2009)	6	m:252 o.	c (crude oil)	-	-
Recchioni et al. (2015)	4	d:245 o.	s	-	2.14(s),.59(i),.03(s),.36(i)
Verschoor and Zwinkels (2013)	5	m:107 o.	fx	-	2.64(i),14.51(i)

This table complements information in Table 7, following the logic of Table 6. Authors of articles are alphabetised. "#" describes the total number of estimated parameters; "Data" describes data frequency ("d/w/m/q/a" for daily/weekly/monthly/quarterly/annual) and number of observations (where no specific figure is provided, we report starting and final years); "Type" shows the type of data ("s/fx/c/g/re/b" for stock markets/FX/commodity markets/gold/real estate/banking indices); "Fit" reports the statistical fit of the estimation (R^2, its alternatives, p-value of the J-test of overidentifying restrictions to accept the model as a possible data generating process); and, where relevant, "|IOC|" displays the absolute estimated value of the switching parameter of the intensity of choice and "s"/"i" denotes its statistical significance/insignificance at the 5% level. Figures are rounded to 2 decimal digits.

Table 9
Simulation results for β estimation with Gaussian noise.

β	(a) $\widehat{\beta}$, $N(0, 10^{-16})$					(b) $\widehat{\beta}$, $N(0, 10^{-14})$				
	Median	Mean	SD	LQ	HQ	Median	Mean	SD	LQ	HQ
0	0.00	0.00	0.03	−.02	0.01	−.00	−.00	0.07	−.14	0.12
.1	0.10	0.10	0.02	0.08	0.12	0.10	0.10	0.05	−.05	0.21
.5	0.50	0.50	0.06	0.48	0.52	0.50	0.50	0.11	0.34	0.62
1	1.00	1.00	0.02	0.98	1.01	1.00	1.00	0.12	0.87	1.13
3	3.00	3.00	0.03	2.98	3.02	3.00	2.99	0.30	2.81	3.11
5	5.00	5.00	0.03	4.98	5.02	5.00	5.00	0.36	4.80	5.21
10	10.00	10.02	0.24	9.92	10.16	10.00	10.00	0.25	9.73	10.28
	(c) $\widehat{\beta}$, $N(0, 10^{-12})$					(d) $\widehat{\beta}$, $N(0, 10^{-10})$				
0	−.00	−.00	0.19	−.50	0.50	−.00	−.00	0.28	−.50	0.50
.1	0.10	0.10	0.11	−.10	0.30	0.11	0.10	0.15	−.10	0.30
.5	0.50	0.50	0.24	−.13	1.07	0.49	0.48	0.39	−.50	1.45
1	1.00	1.00	0.35	0.34	1.65	1.01	1.02	0.54	−.26	2.48
3	3.00	3.00	0.39	2.33	3.61	2.99	2.98	0.55	1.89	4.10
5	5.00	4.99	0.50	4.31	5.64	5.00	4.99	0.60	3.66	6.12
10	10.00	9.95	0.94	8.89	10.68	9.99	9.99	1.10	8.36	11.43
	(e) $\widehat{\beta}$, $N(0, 10^{-8})$					(f) $\widehat{\beta}$, $N(0, 10^{-6})$				
0	0.01	0.01	0.26	−.50	0.50	0.00	0.00	0.25	−.50	0.50
.1	0.10	0.10	0.15	−.10	0.30	0.10	0.10	0.15	−.10	0.30
.5	0.50	0.50	0.35	−.49	1.49	0.50	0.49	0.35	−.47	1.48
1	0.99	0.98	0.50	−.40	2.07	1.00	0.98	0.48	−.25	2.14
3	3.00	3.00	0.65	1.95	3.99	3.00	2.97	0.66	1.90	3.89
5	5.01	5.02	0.68	3.78	6.27	4.99	4.98	0.65	3.91	6.02
10	10.02	10.01	0.88	8.74	11.26	10.01	9.99	0.87	8.81	11.32
	(g) $\widehat{\beta}$, $N(0, 0.01^2)$					(h) $\widehat{\beta}$, $N(0, 0.1^2)$				
0	−.01	−.01	0.25	−.50	0.50	0.00	−.00	0.25	−.50	0.50
.1	0.11	0.10	0.14	−.10	0.30	0.10	0.10	0.15	−.10	0.30
.5	0.50	0.50	0.36	−.39	1.47	0.50	0.49	0.35	−.49	1.41
1	0.98	0.97	0.46	−.20	1.88	1.00	0.99	0.44	0.00	1.86
3	2.99	3.04	0.71	2.09	4.18	3.00	3.02	0.62	2.04	4.04
5	5.01	5.01	0.66	3.74	6.01	5.00	5.00	0.57	3.99	6.06
10	10.00	10.01	0.83	8.84	11.05	10.00	9.97	0.73	8.62	11.15
	(i) $\widehat{\beta}$, $N(0, 1)$					(j) $\widehat{\beta}$, $N(0, 2^2)$				
0	−.00	−.00	0.10	−.21	0.22	0.00	0.00	0.05	−.09	0.09
.1	0.10	0.11	0.08	−.10	0.30	0.10	0.10	0.04	0.00	0.19
.5	0.50	0.51	0.13	0.29	0.80	0.50	0.50	0.09	0.35	0.73
1	0.99	1.00	0.19	0.69	1.33	1.00	1.02	0.18	0.75	1.40
3	3.00	3.02	0.40	2.33	3.89	3.01	3.12	0.72	2.06	4.81
5	4.97	5.05	0.70	3.82	6.64	5.01	5.12	1.15	3.14	7.87
10	9.94	10.07	1.54	7.47	13.54	9.83	9.23	1.95	4.87	12.23

Results are based on 1000 random runs, $t = 5000$, and $N = 1000$. Stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from given normal distribution. Sample medians, means, standard deviations (SD), and 2.5% (LQ) and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits.

Table 10
Results for β estimation w.r.t. various distributions of g_h & b_h.

β	(a) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.1^2)$					(b) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.2^2)$				
	Median	Mean	SD	LQ	HQ	Med.	Mean	SD	LQ	HQ
0	−.02	−.01	0.44	−.50	0.50	0.02	0.02	0.36	−.50	0.50
.1	0.08	0.10	0.18	−.10	0.30	0.10	0.10	0.17	−.10	0.30
.5	0.58	0.53	0.82	−.50	1.50	0.51	0.50	0.54	−.50	1.50
1	1.00	0.98	1.39	−1.00	3.00	1.01	1.01	0.76	−.92	2.98
3	3.02	3.06	2.64	−2.99	8.98	2.99	2.96	1.08	0.70	5.08
5	5.06	5.22	3.51	−4.84	14.68	5.00	5.01	1.25	2.44	7.37
10	10.04	10.07	4.44	−.69	19.63	10.00	10.00	1.58	7.57	12.39
	(c) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.3^2)$					(d) $\widehat{\beta}$, g_h & $b_h \sim N(0, 0.4^2)$				
0	0.00	0.01	0.25	−.50	0.50	0.01	0.00	0.20	−.49	0.50
.1	0.09	0.09	0.14	−.10	0.30	0.10	0.10	0.12	−.10	0.30
.5	0.50	0.51	0.35	−.40	1.48	0.50	0.50	0.24	−.07	1.05
1	1.00	1.00	0.45	0.06	1.93	1.00	1.01	0.30	0.52	1.67
3	2.99	3.02	0.70	1.81	4.08	3.01	2.99	0.34	2.40	3.54
5	5.01	5.01	0.71	4.05	6.03	5.01	5.01	0.30	4.49	5.54
10	10.01	10.04	0.59	8.84	11.25	10.00	9.99	0.41	9.23	10.67

Results are based on 1000 random runs, $t = 5000$, and $N = 1000$. Belief parameters g_h and b_h are drawn from various normal distributions of the given parameter; stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from normal distribution $N(0, 0.1^2)$. Sample medians, means, standard deviations (SD), and 2.5% (LQ) and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits.

Table 11
Results of 5−parameter estimation of a 3−type model.

β, g_2, b_2, g_3, b_3	(a) $\widehat{\beta}$			(b) $\widehat{g_2}$			(c) $\widehat{b_2}$			(d) $\widehat{g_3}$			(e) $\widehat{b_3}$		
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD
.5, 0.8, 0.6, 0.4, 0.3	0.50	0.50	0.16	0.85	0.76	0.45	0.64	0.62	0.25	0.34	0.44	0.45	0.28	0.28	0.25
3, 0.8, 0.6, 0.4, 0.3	3.00	2.96	0.42	0.81	0.80	0.16	0.60	0.59	0.10	0.39	0.40	0.16	0.30	0.31	0.10
10, 0.8, 0.6, 0.4, 0.3	10.01	10.01	0.22	0.79	0.61	0.20	0.59	0.45	0.15	0.43	0.59	0.20	0.32	0.44	0.15
.5, -.8, −.6, 0.4, 0.3	0.33	0.65	0.58	−.79	−.87	0.39	−.72	−.74	0.29	0.40	0.46	0.40	0.42	0.44	0.29
3, −.8, −.6, 0.4, 0.3	3.00	3.44	2.52	−.80	−.92	0.33	−.60	−.70	0.27	0.41	0.52	0.33	0.30	0.40	0.26
10, −.8, −.6, 0.4, 0.3	10.00	10.00	0.90	−.80	−.80	0.03	−.60	−.60	0.02	0.40	0.40	0.05	0.30	0.30	0.02
.5, −.8, 0.6, 0.4, −.3	0.26	0.60	0.58	−.84	−.89	0.40	0.76	0.76	0.29	0.44	0.49	0.41	−.47	−.46	0.29
3, −.8, 0.6, 0.4, −.3	3.15	3.55	2.55	−.80	−.91	0.32	0.59	0.69	0.26	0.41	0.51	0.32	−.30	−.39	0.26
10, −.8, 0.6, 0.4, −.3	10.01	9.92	1.27	−.80	−.80	0.03	0.60	0.60	0.04	0.40	0.41	0.08	−.30	−.30	0.03

Results are based on 1000 random runs, $t = 5000$, and $N = 1000$. Stochastic noises ε_t and $\{\varepsilon_i\}_{i=1}^N$ are drawn from normal distribution $N(0, 0.1^2)$. Sample medians, means and standard deviations (SD) are reported. Figures are rounded to 2 decimal digits.

Table 12
Descriptive statistics of deviations from fundamental prices.

Data, MA period	Mean	Median	Min.	Max.	SD	Skew.	Kurt.	LQ	HQ	AC	AC x_t^2
SP500, 61	−.03	2.0	−145.2	113.0	26.5	−.65	5.5	−61.4	50.7	0.87	0.73
NASDAQ, 61	−.10	2.4	−753.5	639.0	81.8	0.03	13.6	−170.7	147.9	0.89	0.81
DAX, 61	−.28	10.1	−939.6	716.1	163.2	−.50	5.4	−361.8	311.6	0.89	0.80
FTSE, 61	−.12	7.5	−702.5	404.6	122.7	−.58	5.1	−275.4	235.7	0.88	0.73
HSI, 61	0.35	24.3	−4253.2	3463.5	562.2	−.29	6.7	−1186.7	1130.0	0.90	0.74
NIKKEI 225, 61	0.16	6.8	−2249.8	1900.2	434.7	−.30	4.2	−953.8	799.4	0.89	0.77
SP500, 241	−.60	2.8	−252.9	160.1	48.4	−.64	4.5	−112.7	85.7	0.96	0.90
NASDAQ, 241	−1.78	1.6	−756.4	1253.7	168.1	1.01	11.3	−350.9	322.0	0.97	0.96
DAX, 241	−3.35	1.4	−1531.3	1242.6	330.4	−.21	4.5	−728.0	669.1	0.97	0.94
FTSE, 241	−.73	11.9	−1072.4	721.0	210.6	−.56	4.4	−479.4	388.5	0.96	0.90
HSI, 241	2.26	−5.5	−6505.5	7099.5	1177.6	0.18	5.7	−2424.1	2282.0	0.98	0.94
NIKKEI 225, 241	−6.38	−22.6	−3497.8	2872.4	860.0	−.20	3.3	−1821.4	1581.3	0.97	0.92

Sample means, medians, minima, maxima, standard deviations (SD), skewnesses, kurtoses, 2.5% (LQ) and 97.5% (HQ) quantiles, and autocorrelations (AC) are reported. Figures are rounded to 1 or 2 decimal digits.

Table 13
Empirical results of the 3-type switching model estimation.

Data, MA p.	(a) $\widehat{\beta}$		(b) $\widehat{g_2}$		(c) $\widehat{g_3}$		(d) $\widehat{noise\ i}$.		(e) L	
	Med.	SD	Med.	SD	Med.	SD	Med.	SD	Med.	SD
SP500, 61	−.003	0.082	2.502	0.175	−.123	0.111	0.550	0.047	−.127	0.022
SP500, 241	0.007	0.050	2.674	0.217	−.032	0.142	0.403	0.045	−.289	0.094

Results are based on 500 random runs, $t = 5000$, and $N = 500$ draws from normal distribution. Sample medians and standard deviations (SD) are reported. L denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

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