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On linearity of pan-integral and pan-integrable functions space

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ABSTRACT

This paper investigates the linearity and integrability of the $(+,\cdot)$-based pan-integrals on subadditive monotone measure spaces. It is shown that all nonnegative pan-integrable functions form a convex cone and the restriction of the pan-integral to the convex cone is a positive homogeneous linear functional. We extend the pan-integral to the general real-valued measurable functions. The generalized pan-integrals are shown to be symmetric and fully homogeneous, and to remain additive for all pan-integrable functions. Thus for a subadditive monotone measure the generalized pan-integral is linear functional defined on the linear space which consists of all pan-integrable functions. We define a p-norm on the linear space consisting of all p-th order pan-integrable functions, and when the monotone measure μ is continuous we obtain a complete normed linear space $L^p_{pan}(X,\mu)$ equipped with the p-norm, i.e., an analogue of classical Lebesgue space L^p.

1. Introduction

In nonlinear integral theory there are three types of important integrals, the Choquet integral [2], the pan-integral (based on the usual addition $+$ and multiplication $\cdot$ on reals) [37] and the concave integral [12,13]. These integrals coincide with the Lebesgue integral in all cases for σ-additive measures, i.e., these three nonlinear integrals (with respect to monotone measures) are particular generalizations of the Lebesgue integral. In general, they are significantly different from each other, the Choquet integral deals with finite chains of sets, the pan-integral is based on finite partitions (disjoint set systems, similar to the Lebesgue integral) while the concave integral is related to arbitrary finite set systems, see [21]. These integrals have been widely studied and many important results have been obtained, for more details see [3,4,8,15,27,28,33,35], etc., and the structure theory of integral has been enriched and developed in depth, see [7,10,11,17,18,21].

It is well-known that L^p space theory is a crucial aspect of classical measure theory [1,5,6]. Since the above mentioned three integrals are based on monotone measures and lack additivity, in general case the L^p space theory is not true for these integrals. Denneberg [3] showed a subadditivity theorem for the Choquet integral under the assumption that the monotone measure is submodular. The corresponding L^p space theory for the Choquet integral was developed. The similar results were presented by Shirali [30], and the related researches were presented in recent paper by Pap [29]. Note that the Choquet integral is a level set-based integral, which ensures its comonotonic additivity – a property weaker than additivity.

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In [19] the linearity and generalized linearity of fuzzy integral (including the Choquet integral and the Sugeno integral) were discussed (see also [9]). Unlike the Choquet integral, the pan-integral is a partition-based integral. The relationships between these two types of integrals were discussed in [16,25,26] (see also [35]). As a result, the pan-integral even lacks comonotonic additivity. From this point of view, it requires to examine whether or not the L^p space theory holds for the pan-integral. In this paper we will investigate this problem.

The concept of pan-integral introduced by Yang ([35,37]) involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers, i.e., it considers the commutative isotonic semiring $(\overline{R}_+, \oplus, \otimes)$ (see [14,27,32,35,36]). A related concept of generalized Lebesgue integral based on a generalized ring was proposed and discussed (see [39]). In this paper we only consider the pan-integrals based on the standard addition $+$ and multiplication $\cdot$.

This paper is structured as follows. After this introduction, in the next section we recall some basic facts about the monotone measures and the pan-integrals. In Sections 3 and 4 we consider the pan-integral for nonnegative measurable functions. We show that on a subadditive monotone measure space $(X, \mathcal{A}, \mu)$ the pan-integral with respect to μ is additive. Noting that the pan-integral based on the standard addition $+$ and multiplication $\cdot$ is positively homogeneous, then all nonnegative pan-integrable functions form a convex cone in the usual sense and thus the restriction of the pan-integral to the convex cone is a positive homogeneous linear functional. In Section 5, in the same way as the symmetric Choquet integral ([3,27]) we extend the pan-integral for nonnegative measurable functions to the class of general real-valued measurable functions (not necessarily nonnegative). We get a type of symmetric and fully homogeneous nonlinear integral. For a subadditive monotone measure μ the generalized pan-integral (with respect to μ) remains additive for all pan-integrable functions, and hence it is a linear functional defined on the linear space consisting of all pan-integrable functions. In Section 6, we show that all p-*th* order pan-integrable functions form a linear space under the conditions that the monotone measure is subadditive and continuous from below. On such the linear space, by using the Minkowski type inequality for pan-integral [38], we can define a p-norm in the usual manner as Lebesgue space L^p and then we obtain a complete normed linear space $L^p_{pan}(X, \mu)$ equipped with the p-norm, i.e., it is analogous to Lebesgue space L^p. Thus, in the framework of the generalized pan-integral the classical L^p space is generalized.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X. A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called a *monotone measure* [35] on $(X, \mathcal{A})$, if it satisfies the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;
(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

When μ is a monotone measure, the triple $(X, \mathcal{A}, \mu)$ is called a monotone measure space ([27,35]).

Note: In the literature, the monotone measure is also known as a monotone set function, a capacity, a fuzzy measure, or a nonadditive probability (constrained by $\mu(X) = 1$, sometimes also assuming the continuity of μ), etc. (see [2,3,15,22,27,31,34,36]).

In this paper we always assume that μ is a monotone measure on $(X, \mathcal{A})$. Recall that μ is said to be

(i) *subadditive*, if $\mu(A \cup B) \le \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$;
(ii) *submodular*, if $\mu(A \cup B) + \mu(A \cap B) \le \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$;
(iii) *supermodular*, if $\mu(A \cup B) + \mu(A \cap B) \ge \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$;
(iv) *null-additive* [35], if for any $A, B \in \mathcal{A}$, $\mu(B) = 0$ implies $\mu(A \cup B) = \mu(A)$;
(v) *continuous from below*, if for any $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$, $A_1 \subset A_2 \subset \ldots$ implies $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;
(vi) *continuous from above*, if for any $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$, $A_1 \supset A_2 \supset \ldots$ and $\mu(A_1) < \infty$ imply $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;
(vii) *continuous* if it is continuous both from below and from above.

Obviously, the submodularity of μ implies the subadditivity and both imply the null-additivity, but not vice versa.

In [37] (see also [23,35,39]) the concept of pan-integral was introduced, which involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see [20,27,32,35]). In this paper we only consider the pan-integrals based on the standard addition $+$ and multiplication $\cdot$.

A real-valued function $f : X \to (-\infty, +\infty)$ is said to be $\mathcal{A}$-measurable on $(X, \mathcal{A})$ (or simply "measurable" when there is no confusion) if $f^{-}(B) \in \mathcal{A}$ for every Borel set B of real numbers. Let $\mathcal{F}^+$ be the collection of all nonnegative real-valued measurable functions on $(X, \mathcal{A})$. We recall the following definition.

Definition 2.1. Let $(X, \mathcal{A}, \mu)$ be a monotone measure space, $A \in \mathcal{A}$ and $f \in \mathcal{F}^+$. The pan-integral of f on A with respect to μ, is defined by

$$\int_A^{pan} f d\mu = \sup_{\mathcal{E} \in \hat{\mathcal{P}}} \left\{ \sum_{E \in \mathcal{E}} \left[(\inf_{x \in A \cap E} f(x)) \cdot \mu(A \cap E) \right] \right\}, \tag{2.1}$$

where $\hat{\mathcal{P}}$ is the set of all finite measurable partitions of X. When $A = X$, $\int_X^{pan} f d\mu$ is written as $\int^{pan} f d\mu$. If $\int_A^{pan} f d\mu < \infty$, then we say that f is pan-integrable on A. When f is pan-integrable on X, we simply say that f is pan-integrable.

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Note: By a finite measurable partition of X we mean that it is a finite disjoint system of sets $\{A_i\}_{i=1}^n \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^n A_i = X$.

The above pan-integral of f can be expressed in the following form:

$$\int^{pan} f d\mu = \sup\left\{\sum_{i=1}^n \lambda_i \mu(A_i) : \sum_{i=1}^n \lambda_i \chi_{A_i} \leq f, \{A_i\}_{i=1}^n \subset \mathcal{A} \text{ is a partition of } X, \lambda_i \geq 0, n \in \mathbb{N}\right\},$$

where χ_{A_i} is the characteristic function of A_i.

Let f, g be real-valued measurable function on $(X, \mathcal{A}, \mu)$. We say that f and g are equal almost everywhere with respect to μ on A, and denoted by $f = g$ μ-a.e. on A (simply, $f = g$ a.e. on A), if $\mu(\{x \in A : f(x) \neq g(x)\}) = 0$.

We present some basic properties of pan-integrals, some of them can be found in [35] (see also [39]).

Proposition 2.2. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and $f, g \in \mathcal{F}^+$. Then we have the following*

(i) if $\mu(A) = 0$ then $\int_A^{pan} f d\mu = 0$;

(ii) if $f = 0$ a.e. on A, then $\int_A^{pan} f d\mu = 0$. Further, if μ is continuous from below, then $\int_A^{pan} f d\mu = 0$ if and only if $f = 0$ a.e. on A;

(iii) if $f \leq g$ on A, then $\int_A^{pan} f d\mu \leq \int_A^{pan} g d\mu$;

(iv) for $A \in \mathcal{A}$, $\int_A^{pan} f d\mu = \int^{pan} f \cdot \chi_A d\mu$;

(v) $\int^{pan} f d\mu = \int_{\{f>0\}}^{pan} f d\mu$, where $\{f > 0\} = \{x \in X : f(x) > 0\}$;

(vi) if μ is null-additive and $f = g$ a.e. on A, then

$$\int_A^{pan} f d\mu = \int_A^{pan} g d\mu.$$

Proof. We only prove (vi). Let $A_1 = \{x \in A, f(x) \neq g(x)\}$, then $\mu(A_1) = 0$ and for any subset B of A we have $\mu(B) = \mu(B \setminus A_1)$. Thus, for any $\sum_{i=1}^n \lambda_i \chi_{B_i} \leq f(x) \cdot \chi_A$, $\sum_{i=1}^n \lambda_i \chi_{B_i \setminus A_1} \leq g(x) \cdot \chi_A$. Moreover,

$$\int_A^{pan} g d\mu \geq \sum_{i=1}^n \lambda_i \mu(B_i \setminus A_1) = \sum_{i=1}^n \lambda_i \mu(B_i),$$

which implies that $\int_A^{pan} g d\mu \geq \int_A^{pan} f d\mu$. In a similar way $\int_A^{pan} f d\mu \geq \int_A^{pan} g d\mu$, and thus (vi) holds. □

Note 2.3. (1) When μ is continuous from below, from the above proposition (ii), we know that for any measurable functions $f, g \in \mathcal{F}^+$ and any real number $p > 0$, $\int_A^{pan} |f - g|^p d\mu = 0$ if and only if $f = g$ a.e. on A.

(2) Due to the above proposition (v), we will not distinguish the partition of X and the partition of $\{f > 0\}$ when we discuss the pan-integral of the function f. Sometimes we may just say that $\{A_i\}_{i=1}^n$ is a partition.

(3) By (vi), if μ is null-additive then, from the pan-integral point of view, we do not distinguish f and g whenever $f = g$ a.e. on X.

3. The additivity of the pan-integral for nonnegative measurable functions

The main task of this section is to prove the following result.

Theorem 3.1. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If μ is subadditive, then the pan-integral is additive w.r.t. the integrands, i.e., for any $f, g \in \mathcal{F}^+$,*

$$\int^{pan} (f+g) d\mu = \int^{pan} f d\mu + \int^{pan} g d\mu \tag{3.1}$$

Proof. Here we only consider the case that $f + g$ is pan-integrable, that is, $\int^{pan} (f + g) d\mu$ is finite. The case of $\int^{pan} (f + g) d\mu = \infty$ will be considered in the next section.

For an arbitrary but fixed $\varepsilon > 0$, there exist a partition $\{A_i\}_{i=1}^n$ and a sequence of nonnegative numbers $\{\lambda_i\}_{i=1}^n$ such that both $\sum_{i=1}^n \lambda_i \chi_{A_i} \leq f + g$ and

$$\int^{pan} (f+g) d\mu < \sum_{i=1}^n \lambda_i \mu(A_i) + \frac{\varepsilon}{2}$$

hold. Without loss of generality, we can assume that $\lambda_i > 0$ for all i (otherwise, the summand $\lambda_i \mu(A_i)$ can be omitted). Since $\mu(A_1) < \infty$ and $\lambda_1 > 0$, there exists a positive number $l_1 < \lambda_1$ such that $(\lambda_1 - l_1)\mu(A_1) < \frac{\varepsilon}{2^2}$. Obviously, $l_1 \chi_{A_1} < \lambda_1 \chi_{A_1} \le (f+g)\cdot\chi_{A_1}$. Let $\delta = \lambda_1 - l_1$ and divide the interval $[0, l_1]$ as

$$0 = r_1^{(0)} < r_1^{(1)} < \ldots < r_1^{(m_1)} = l_1$$

such that $\max_{1\le i\le m_1}(r_1^{(i)} - r_1^{(i-1)}) < \delta$. Denote

$$A_1^{(k)} = \left\{x \in A_1 | r_1^{(k-1)} \le f(x) < r_1^{(k)}\right\}, k = 1, \ldots, m_1,$$

and

$$A_1^{(m_1+1)} = \{x \in A_1 | f(x) \ge l_1\}.$$

Then $A_1^{(i)} \cap A_1^{(j)} = \emptyset$ $(i \ne j)$ and $\cup_{i=1}^{m_1+1} A_1^{(i)} = A_1$. Moreover, we conclude that for each $x \in A_1^{(k)}$, $k = 1, 2, \ldots, m_1$, $g(x) \ge l_1 - r_1^{(k-1)}$. In fact, if $g(x) < l_1 - r_1^{(k-1)}$, then

$$f(x) + g(x) < r_1^{(k)} + (l_1 - r_1^{(k-1)}) = l_1 + (r_1^{(k)} - r_1^{(k-1)}) < l_1 + \delta = \lambda_1,$$

which contradicts with the fact that $\lambda_1 \chi_{A_1} \le (f+g)\cdot\chi_{A_1}$. For $x \in A_1^{(m_1+1)}$, $g(x) \ge 0 = l_1 - r_1^{(m_1)}$ also holds. Denote

$$t_1^{(k)} = l_1 - r_1^{(k-1)}, k = 1, \ldots, m_1 + 1.$$

Then

$$\sum_{k=1}^{m_1+1} r_1^{(k-1)} \chi_{A_1^{(k)}} \le f \cdot \chi_{A_1}$$

and

$$\sum_{k=1}^{m_1+1} t_1^{(k)} \chi_{A_1^{(k)}} \le g \cdot \chi_{A_1}.$$

Observe that the subadditivity of μ implies

$$\mu(A_1) = \mu\left(\bigcup_{k=1}^{m_1+1} A_1^{(k)}\right) \le \sum_{k=1}^{m_1+1} \mu\left(A_1^{(k)}\right).$$

Thus

$$\begin{aligned}
&\sum_{k=1}^{m_1+1} r_1^{(k-1)} \mu\left(A_1^{(k)}\right) + \sum_{k=1}^{m_1+1} t_1^{(k)} \mu\left(A_1^{(k)}\right) \\
&= \sum_{k=1}^{m_1+1} \left(r_1^{(k-1)} + t_1^{(k)}\right) \mu\left(A_1^{(k)}\right) \\
&= \sum_{k=1}^{m_1+1} l_1 \mu\left(A_1^{(k)}\right) \ge l_1 \mu(A_1) \\
&> \lambda_1 \mu(A_1) - \frac{\varepsilon}{2^2}.
\end{aligned}$$

For each $i \le n$, let $l_i \in (0, \lambda_i)$ be such that $(\lambda_i - l_i)\mu(A_i) < \frac{\varepsilon}{2^{i+1}}$. By using the technique used above, we can prove that for each $j \le n$ there exist a partition $\{A_j^{(k)}\}_{k=1}^{m_j+1}$ of A_j and a sequence $\{r_j^{(k-1)}\}_{k=1}^{m_j+1}$ of nonnegative numbers satisfying

$$\sum_{k=1}^{m_j+1} r_j^{(k-1)} \chi_{A_j^{(k)}} \le f \cdot \chi_{A_j}$$

and

$$\sum_{k=1}^{m_j+1} t_j^{(k)} \chi_{A_j^{(k)}} \le g \cdot \chi_{A_j},$$

where $t_j^{(k)} = l_j - r_j^{(k-1)}$, such that

$$\sum_{k=1}^{m_j+1} r_j^{(k-1)} \mu\left(A_j^{(k)}\right) + \sum_{k=1}^{m_j+1} t_j^{(k)} \mu\left(A_j^{(k)}\right) > \lambda_j \mu(A_j) - \frac{\varepsilon}{2^{j+1}}.$$

Thus, we have proven that

$$\sum_{j=1}^{n} \sum_{k=1}^{m_j+1} r_j^{(k-1)} \chi_{A_j^{(k)}} \le \sum_{j=1}^{n} f \cdot \chi_{A_j} \le f,$$

$$\sum_{j=1}^{n} \sum_{k=1}^{m_j+1} t_j^{(k)} \chi_{A_j^{(k)}} \le \sum_{j=1}^{n} g \cdot \chi_{A_j} \le g,$$

and

$$\begin{aligned}
&\int^{pan} f d\mu + \int^{pan} g d\mu \\
&\ge \sum_{j=1}^{n} \sum_{k=1}^{m_j+1} r_j^{(k-1)} \mu\left(A_j^{(k)}\right) + \sum_{j=1}^{n} \sum_{k=1}^{m_j+1} t_j^{(k)} \mu\left(A_j^{(k)}\right) \\
&> \sum_{j=1}^{n} \left(\lambda_j \mu(A_j) - \frac{\varepsilon}{2^{j+1}}\right) \\
&> \sum_{j=1}^{n} \lambda_j \mu(A_j) - \frac{\varepsilon}{2} \\
&> \int^{pan} (f+g) d\mu - \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get $\int^{pan} f d\mu + \int^{pan} g d\mu \ge \int^{pan} (f+g) d\mu$ as desired.

On the other hand, for any $\sum_{i=1}^{n} \alpha_i \chi_{A_i} \le f$ and $\sum_{j=1}^{m} \beta_j \chi_{B_j} \le g$, where $\alpha_i, \beta_j \ge 0$, $\{A_i\}$ and $\{B_j\}$ are two partitions of X, put $C_{ij} = A_i \cap B_j$ ($C_{ij} = \emptyset$ may hold for some i, j) and $\gamma_{ij} = \alpha_i + \beta_j$, $1 \le i \le n, 1 \le j \le m$. Then

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} \chi_{C_{ij}} &= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \chi_{C_{ij}} + \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \chi_{C_{ij}} \\
&= \sum_{i=1}^{n} \alpha_i \chi_{A_i} + \sum_{j=1}^{m} \beta_j \chi_{B_j} \le f + g
\end{aligned}$$

and

$$\begin{aligned}
\int^{pan} (f+g) d\mu &\ge \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} \mu(C_{ij}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \mu(C_{ij}) + \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \mu(C_{ij}) \\
&\ge \sum_{i=1}^{n} \alpha_i \mu\Big(\bigcup_{j=1}^{m} C_{ij}\Big) + \sum_{j=1}^{m} \beta_j \mu\Big(\bigcup_{i=1}^{n} C_{ij}\Big) \\
&= \sum_{i=1}^{n} \alpha_i \mu(A_i) + \sum_{j=1}^{m} \beta_j \mu(B_j),
\end{aligned}$$

from which we can get that

$$\int^{pan} (f+g)d\mu \geq \int^{pan} f d\mu + \int^{pan} g d\mu.$$

Hence it holds that $\int^{pan}(f+g)d\mu = \int^{pan} f d\mu + \int^{pan} g d\mu$. □

Note 3.2. Theorem 3.1 tells us that if μ is subadditive then all nonnegative pan-integrable functions form a convex cone.

Corollary 3.3. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If μ is subadditive, then for any $f \in \mathcal{F}^+$ and any $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, we have*

$$\int_{A\cup B}^{pan} f d\mu = \int_{A}^{pan} f d\mu + \int_{B}^{pan} f d\mu.$$

Proof. Denote $g = f \cdot \chi_A$ and $h = f \cdot \chi_B$. Then, by (iv) of Proposition 2.2 and Theorem 3.1, we have

$$\begin{aligned}\int_{A\cup B}^{pan} f d\mu &= \int^{pan} f \cdot \chi_{A\cup B} d\mu \\ &= \int^{pan} (g+h) d\mu \\ &= \int^{pan} g d\mu + \int^{pan} h d\mu \\ &= \int_{A}^{pan} f d\mu + \int_{B}^{pan} f d\mu. \qquad \square\end{aligned}$$

When X is finite, Theorem 3.1 has obvious meaning. Let $(X, \mathcal{A}, \mu)$ be a monotone measure space with $|X| < \infty$ and $A_1, \ldots, A_k$ be all minimal atoms. Recall that a minimal atom is a measurable set A such that $\mu(A) > 0$ and for each measurable proper subset B of A we have $\mu(B) = 0$. When X is a finite set, each set of positive measure contains at least one minimal atom [24]. If we further assume that μ is subadditive then for each minimal atom A, there are no nonempty proper subsets B of A such that $B \in \mathcal{A}$. In fact, if there is some nonempty proper subset $B \in \mathcal{A}$, then by the definition of minimal atom, $\mu(B) = 0$, $\mu(A \setminus B) = 0$ and the subadditivity of μ implies $\mu(A) = 0$, a contradiction. Observing this fact, to ensure its measurability, a function f must take constant value on each minimal atom. Thus, for any measurable function $f : X \to [0, \infty)$ it holds

$$\int^{pan} f d\mu = \sum_{i=1}^{k} c_i \mu(A_i),$$

where c_i is the constant value f takes on the minimal atom A_i. From this fact it is immediate that the pan-integral is additive.

4. Further discussion of Theorem 3.1

In this section we will complete the remaining proof of Theorem 3.1. That is, we show that the equality (3.1) in Theorem 3.1 remains valid in the case that $f + g$ is not pan-integrable. The following proposition will demonstrate our assertion.

Proposition 4.1. *Under the assumptions of Theorem 3.1, $f + g$ is pan-integrable if and only if both f and g are pan-integrable.*

From the above Proposition 4.1, we know that if $f + g$ is not pan-integrable, then either f or g is not pan-integrable. Therefore, both sides of (3.1) in Theorem 3.1 are equal to infinity, and hence the equality (3.1) holds. The proof of Theorem 3.1 is thereby completed.

Now we prove Proposition 4.1. First, we need two lemmas. In the following the set $\{x \in X : f(x) > 0\}$ will be denoted by $\{f > 0\}$ for short.

Lemma 4.2. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space, and $f, g \in \mathcal{F}^+$. If $\{f > 0\} \cap \{g > 0\} = \emptyset$, then*

$$\int^{pan} (f+g) d\mu \geq \int^{pan} f d\mu + \int^{pan} g d\mu. \tag{4.1}$$

Proof. If one of the two integrals on the right-hand side of Ineq. (4.1) is infinite then, by the monotonicity of the pan-integral, $\int^{pan} (f+g) d\mu$ is also equal to infinity, which implies the validity of (4.1).

So, without loss of generality, we can suppose that both $\int^{pan} f d\mu$ and $\int^{pan} g d\mu$ are finite. For any arbitrary but fixed $\varepsilon > 0$, there are a partition $\{A_i\}_{i=1}^k$ of $\{f > 0\}$, a partition $\{B_j\}_{j=1}^m$ of $\{g > 0\}$, and two sequences of positive numbers $\{\lambda_i\}_{i=1}^k$ and $\{l_j\}_{j=1}^m$ such that $\sum_{i=1}^k \lambda_i \chi_{A_i} \leq f$, $\sum_{j=1}^m l_j \chi_{B_j} \leq g$ and both the following two inequalities hold

$$\int^{pan} f d\mu < \sum_{i=1}^k \lambda_i \mu(A_i) + \frac{\varepsilon}{2}$$

and

$$\int^{pan} g d\mu < \sum_{j=1}^m l_j \mu(B_j) + \frac{\varepsilon}{2}.$$

Since $\{f > 0\} \cap \{g > 0\} = \emptyset$, then $\{A_i\}_{i=1}^k \cup \{B_j\}_{j=1}^m$ is a partition of $\{f + g > 0\}$. Moreover, we have that $\sum_{i=1}^k \lambda_i \chi_{A_i} + \sum_{j=1}^m l_j \chi_{B_j} \leq f + g$, and

$$\begin{aligned}\int^{pan} (f+g) d\mu &\geq \sum_{i=1}^k \lambda_i \mu(A_i) + \sum_{j=1}^m l_j \mu(B_j) \\ &\geq \int^{pan} f d\mu + \int^{pan} g d\mu - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we get the result as desired. $\square$

Notice that we further require that μ is subadditive in Lemma 4.2, then Ineq. (4.1) becomes an equality. That is, the following result holds.

Lemma 4.3. *Let $f, g \in \mathcal{F}^+$ and $\{f > 0\} \cap \{g > 0\} = \emptyset$. If μ is subadditive, then*

$$\int^{pan} (f+g) d\mu = \int^{pan} f d\mu + \int^{pan} g d\mu. \tag{4.2}$$

Proof. By Lemma 4.2, it suffices to prove that

$$\int^{pan} (f+g) d\mu \leq \int^{pan} f d\mu + \int^{pan} g d\mu.$$

For any given partition $\{A_i\}_{i=1}^k$ of $\{f + g > 0\}$ and a sequence of positive numbers $\{\lambda_i\}_{i=1}^k$ such that $\sum_{i=1}^k \lambda_i \chi_{A_i} \leq f + g$. Since $\{f > 0\} \cap \{g > 0\} = \emptyset$, we have that $\{A_i \cap \{f > 0\}\}_{i=1}^k$ (resp. $\{A_i \cap \{g > 0\}\}_{i=1}^k$) is a partition of $\{f > 0\}$ (resp. $\{g > 0\}$), and that

$$\sum_{i=1}^k \lambda_i \chi_{A_i \cap \{f>0\}} \leq f \quad \text{and} \quad \sum_{i=1}^k \lambda_i \chi_{A_i \cap \{g>0\}} \leq g.$$

Moreover, the subadditivity of μ implies that

$$\begin{aligned}\mu(A_i) &= \mu\left(A_i \bigcap (\{f > 0\} \cup \{g > 0\})\right) \\ &\leq \mu(A_i \cap \{f > 0\}) + \mu(A_i \cap \{g > 0\}).\end{aligned}$$

Therefore

$$\sum_{i=1}^{k}\lambda_i\mu(A_i)\le\sum_{i=1}^{k}\lambda_i\mu(A_i\cap\{f>0\})+\sum_{i=1}^{k}\lambda_i\mu(A_i\cap\{g>0\})$$
$$\le\int^{pan} f d\mu+\int^{pan} g d\mu.$$

Thus,

$$\int^{pan}(f+g)d\mu=\sup\{\sum_{i=1}^{n}\lambda_i\mu(A_i):\sum_{i=1}^{n}\lambda_i\chi_{A_i}\le f+g,\{A_i\}_{i=1}^{n}\subset\mathcal{F}\text{ is a partition of }X,\lambda_i\ge 0,n\in\mathbb{N}\}$$
$$\le\int^{pan} f d\mu+\int^{pan} g d\mu.\qquad\square$$

Proof of Proposition 4.1. Denote $A=\{x|f(x)\le g(x)\}$ and $B=\{x|f(x)>g(x)\}$. Then $A\cap B=\emptyset$ and $f=f\cdot\chi_A+f\cdot\chi_B$, $g=g\cdot\chi_A+g\cdot\chi_B$. Moreover, $f\cdot\chi_A\le g\cdot\chi_A$ and $g\cdot\chi_B\le f\cdot\chi_B$. Thus,

$$f+g\le 2g\cdot\chi_A+2f\cdot\chi_B.$$

Combining the monotonicity and positive homogeneity of the pan-integral, and Lemma 4.3, we conclude that

$$\int^{pan}(f+g)d\mu\le\int^{pan}(2g\cdot\chi_A+2f\cdot\chi_B)d\mu$$
$$=\int^{pan}2g\cdot\chi_A d\mu+\int^{pan}2f\cdot\chi_B d\mu$$
$$=2\left(\int^{pan}g\cdot\chi_A d\mu+\int^{pan}f\cdot\chi_B d\mu\right)$$
$$\le 2\left(\int^{pan}g d\mu+\int^{pan}f d\mu\right).$$

Thus if $f+g$ is not pan-integrable then either f or g is not pan-integrable. On the other hand, if one of f,g is not pan-integrable, then $f+g$ is not pan-integrable. Hence $f+g$ is pan-integrable if and only if both f and g are pan-integrable. $\square$

Note 4.4. The following example shows that Ineq. (4.1) can be violated if $\{f>0\}\cap\{g>0\}\ne\emptyset$.

Example 4.5. Let $X=\{1,2,3,\dots\}$, $\mathcal{A}=2^X$ and the monotone measure $\mu:\mathcal{A}\to[0,\infty]$ be defined as $\mu(A)=\frac{1}{|X\setminus A|+1}$, where $|A|$ stands for the cardinality of A. Suppose that $f,g:X\to[0,\infty]$ are defined as $f(x)=x-1$ and $g(x)=1$ for all $x\in X$. Then, we can show that

$$\int^{pan}(f+g)d\mu<\int^{pan} f d\mu+\int^{pan} g d\mu.$$

Observe first that for any subset A, $\mu(A)>0$ if and only if $X\setminus A$ is a finite set. So, for any partition $\{A_i\}_{i=1}^{k}$ of X, there is at most one set A_i with positive measure. For simplicity, we suppose that $\mu(A_1)>0$ and $t=\min\{x|x\in A_1\}$. Then $\mu(A_1)\le\frac{1}{t}$. To ensure $\sum_{i=1}^{k}\lambda_i\chi_{A_i}\le f$, there must be $\lambda_1\le t-1$. Thus, it holds that

$$\sum_{i=1}^{k}\lambda_i\mu(A_i)=\lambda_1\mu(A_1)\le\frac{t-1}{t}<1.$$

For the arbitrariness of the partition $\{A_i\}$, we conclude that $\int^{pan} f d\mu\le 1$. In a similar way, we can prove that $\int^{pan} g d\mu\le 1$ and $\int^{pan}(f+g)d\mu\le 1$. On the other hand, $(t-1)\chi_{\{t,t+1,t+2,\dots\}}\le f$ and $(t-1)\mu(\{t,t+1,t+2,\dots\})=\frac{t-1}{t}$ for each $t\in X$. Thus we have that $\int^{pan} f d\mu\ge\sup\{\frac{t-1}{t}|t\in X\}=1$, and hence $\int^{pan} f d\mu=1$. Similarly, we have that $\int^{pan} g d\mu=1$ and $\int^{pan}(f+g)d\mu=1$.

5. The pan-integral for real-valued functions

The definition of the pan-integral given in [37] (see also [35,39]) is restricted to nonnegative measurable functions. In the previous two sections, we have seen that if μ is subadditive then all nonnegative pan-integrable functions form a convex cone. Now we extend the definition of the pan-integral (Definition 2.1) to general real-valued measurable functions. Similar to the definition of Lebesgue integral for real-valued measurable functions ([1,5,6]), we decompose a real-valued measurable function to be the difference of two nonnegative measurable functions, and then consider the difference of integrals of these two nonnegative measurable functions.

Let $\mathcal{F}$ denote the class of all real-valued measurable functions on $(X, \mathcal{A})$. Let $f \in \mathcal{F}$, then $f = f^+ - f^-$ where f^+ and f^- are the positive and negative parts of f, that is $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Both f^+ and f^- are nonnegative measurable functions, i.e., $f^+, f^- \in \mathcal{F}^+$. Now we give the following the definition of pan-integral for general real-valued measurable functions.

Definition 5.1. Let $(X, \mathcal{A}, \mu)$ be a monotone measure space, and $f \in \mathcal{F}$. If at least one of $\int^{pan} f^+ d\mu$ and $\int^{pan} f^- d\mu$ takes finite value, then we define the pan-integral of f (with respect to μ) via

$$\int^{pan} f d\mu = \int^{pan} f^+ d\mu - \int^{pan} f^- d\mu. \tag{5.1}$$

If both f^+ and f^- are pan-integrable in the sense of Definition 2.1, i.e., $\int^{pan} f^+ d\mu < \infty$ and $\int^{pan} f^- d\mu < \infty$, then we say that f is pan-integrable (or simply, f is integrable).

When $A \subset X$, define $\int_A^{pan} f d\mu = \int^{pan} f \cdot \chi_A d\mu$.

Obviously, if f is integrable, then $\int^{pan} f d\mu$ is finite.

Example 5.2. Let $X = \{x_1, x_2, x_3, x_4\}$, $\mathcal{A} = \mathcal{P}(X)$ and the monotone measure μ be defined as follows:

$$\begin{aligned}
&\mu(\{x_1\}) = \mu(\{x_2\}) = 1, \mu(\{x_3\}) = 2, \mu(\{x_4\}) = \mu(\{x_1, x_2\}) = 1.5,\\
&\mu(\{x_1, x_3\}) = \mu(\{x_2, x_4\}) = \mu(\{x_3, x_4\}) = 4, \mu(\{x_1, x_4\}) = 2.5,\\
&\mu(\{x_2, x_3\}) = 3.5, \mu(\{x_1, x_2, x_3\}) = \mu(\{x_1, x_2, x_4\}) = 5,\\
&\mu(\{x_1, x_3, x_4\}) = 4.5, \mu(\{x_2, x_3, x_4\}) = 6, \mu(X) = 6.5.
\end{aligned}$$

Let $f : X \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 2 & \text{if } x = x_1,\\ -2 & \text{if } x = x_2,\\ 1 & \text{if } x = x_3,\\ -1 & \text{if } x = x_4. \end{cases}$$

Then $\int^{pan} f^+ d\mu = \mu(\{x_1, x_3\}) = 4$ and $\int^{pan} f^- d\mu = \mu(\{x_2, x_4\}) = 4$. Thus $\int^{pan} f d\mu = \int^{pan} f^+ d\mu - \int^{pan} f^- d\mu = 0$.

In the rest of this section, we investigate some basic properties of the generalized pan-integral.

Proposition 5.3. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space, $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ be a constant. Then we have the following:*

(i) $\int^{pan} cf d\mu = c \int^{pan} f d\mu$; *(homogeneity)*

(ii) $f \le g$ implies $\int^{pan} f d\mu \le \int^{pan} f d\mu$; *(monotonicity)*

(iii) If $|f|$ is integrable then f is also integrable. Moreover, if μ is subadditive then f is integrable if and only if $|f|$ is integrable.

Proof. The proofs of (i) and (ii) are standard and thus omitted. For (iii), notice first that $0 \le \max(f^+, f^-) \le |f|$. If $\int^{pan} |f| d\mu < \infty$, by the monotonicity of pan-integral, we then have that both $\int^{pan} f^+ d\mu < \infty$ and $\int^{pan} f^- d\mu < \infty$, which imply the integrability of f.

By definition, the integrability of f implies that both f^+ and f^- are integrable. Suppose now μ is subadditive, Theorem 3.1 holds. Thus

$$\begin{aligned}
\int^{pan} |f| d\mu &= \int^{pan} (f^+ + f^-) d\mu\\
&= \int^{pan} f^+ d\mu + \int^{pan} f^- d\mu < \infty. \qquad \square
\end{aligned}$$

The following result shows that the generalized pan-integral is linear whenever the monotone measure μ is subadditive.

Theorem 5.4. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If μ is subadditive, then for any pan-integrable functions $f, g \in \mathcal{F}$,*

$$\int^{pan} (f+g)d\mu = \int^{pan} f d\mu + \int^{pan} g d\mu.$$

Proof. Similar to the proof of classical Lebesgue integral.

Remark 5.5. For general measurable functions (not necessarily nonnegative) the result of Lemma 4.2 fails. Let us reconsider Example 5.2. Let $g = f \cdot \chi_{\{x_1, x_4\}}$ and $h = f \cdot \chi_{\{x_2, x_3\}}$. Then $\{g \neq 0\} \cap \{h \neq 0\} = \emptyset$. It is easy to see that $\int^{pan} g d\mu = 2\mu(\{x_1\}) - \mu(\{x_4\}) = 0.5$, $\int^{pan} h d\mu = \mu(\{x_3\}) - 2\mu(\{x_2\}) = 0$. That is,

$$\int^{pan} (g+h)d\mu = \int^{pan} f d\mu = 0 < 0.5 = \int^{pan} g d\mu + \int^{pan} h d\mu.$$

Remark 5.6. In Definition 5.1 we got a symmetric and fully homogeneous integral, the generalized pan-integral. Recall the case of the Choquet integral, there are two kinds of extensions, the asymmetric Choquet integral and the symmetric Choquet integral. It is well-known that the asymmetric Choquet integral is comonotonic additive and positively homogeneous, while the symmetric Choquet integral is homogeneous but lacks comonotonic additivity [3]. The asymmetric pan-integral, another extension of the pan-integral, is for us a topic for the further study.

6. Pan-integrable functions space

In this section we will establish an analogue of classical L^p space. Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and $1 \le p < \infty$. Let $\mathcal{L}^p_{pan}(X, \mu)$ be the set of all measurable functions $f \in \mathcal{F}$ such that

$$\int_X^{pan} |f|^p d\mu < \infty.$$

Then $\mathcal{L}^p_{pan}(X, \mu)$ is a linear space in natural linear structure.

Let $f \in \mathcal{L}^p_{pan}(X, \mu)$. We define the symbol $\|f\|_{\mu,p}$ by

$$\|f\|_{\mu,p} = \left(\int_X^{pan} |f|^p d\mu \right)^{\frac{1}{p}}.$$

For a subadditive and continuous from below monotone measure, the Hölder and Minkowski inequalities for classical integrals remain valid for pan-integrals. The following result is due to [38].

Proposition 6.1. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and μ be subadditive and continuous from below. Then,*

(i) (Hölder) If $p, q \ge 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in \mathcal{L}^p_{pan}(X, \mu)$ and $g \in \mathcal{L}^q_{pan}(X, \mu)$, then

$$\|fg\|_{\mu,1} \le \|f\|_{\mu,p} \|g\|_{\mu,q}.$$

(ii) (Minkowski) For $p \ge 1$ and $f, g \in \mathcal{L}^p_{pan}(X, \mu)$, then

$$\|f+g\|_{\mu,p} \le \|f\|_{\mu,p} + \|g\|_{\mu,p}.$$

Let us observe that, in general, $\|\cdot\|_{\mu,p}$ is not a norm on the linear space $\mathcal{L}^p_{pan}(X, \mu)$.

As discussed in the classical L^p space theory, we write $f \sim g$ for $f, g \in \mathcal{L}^p_{pan}(X, \mu)$ if $f = g$ μ-a.e. Note that if μ is subadditive, then the relation $\sim$ is an equivalence relation. We define the space $L^p_{pan}(X, \mu)$ to be the collection of all equivalence classes in $\mathcal{L}^p_{pan}(X, \mu)$. Similar to the discussion of classical L^p space and by using Theorem 3.1, we can get the following result.

Theorem 6.2. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If μ is subadditive, then $L^p_{pan}(X, \mu)$ is a linear space in the usual manner.*

When μ is continuous from below, it follows from Proposition 2.2 (ii) that for any $f \in \mathcal{L}^p_{pan}(X, \mu)$, $\|f\|_{\mu,p} = 0$ if and only if $f = 0$ a.e. on X, i.e., f is the zero element in $L^p_{pan}(X, \mu)$. Combining Proposition 5.3 and 6.1, then $\|\cdot\|_{\mu,p}$ is a norm on $L^p_{pan}(X, \mu)$ and hence $L^p_{pan}(X, \mu)$ is a normed linear space.

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Note: It is common to say or to write that some function f is an element of $L^p_{pan}(X,\mu)$, although the element of that space are actually equivalence classes of functions.

Using Levi's theorem (the monotone convergence theorem) and Fatou's lemma for the pan-integrals ([35,37], see also [39]) and the usual arguments as in [1,6] we can prove the following result (the details are omitted).

Theorem 6.3. *Let $(X,\mathcal{A},\mu)$ be a monotone measure space. If μ is finite, subadditive and continuous, then $L^p_{pan}(X,\mu)$ is a Banach space.*

7. Conclusions

In this paper, we have studied the linearity of $(+,\cdot)$-based pan-integrals. For a subadditive monotone measure the pan-integral is a positively homogeneous linear functional on the convex cone which consists of all nonnegative pan-integrable functions (Theorem 3.1, Note 3.2 and Proposition 4.1). We have also extended the pan-integral to arbitrary real-valued measurable functions (Definition 5.1). The generalized pan-integral has been shown to be symmetric and fully homogeneous (Proposition 5.3), and to be linear for subadditive monotone measures (Theorem 5.4). Further assuming that the monotone measure is finite and continuous, then we have obtained an analogue of classical L^p space, i.e., a complete normed linear space $L^p_{pan}(X,\mu)$ consisting of all p-*th* order pan-integrable functions (Theorem 6.3).

In the follow-up study we will investigate other properties of the space $L^p_{pan}(X,\mu)$, almost along the same lines as in the familiar case of classical Lebesgue space L^p, such as the separability and the dual space of $L^p_{pan}(X,\mu)$, etc.

We point out that the concave integral coincides with the pan-integral when μ is subadditive (see [24,25]), thus the results obtained in this paper also hold for concave integrals.

Note that an outer measure is subadditive. Thus we can define a Lebesgue-like integral (possesses linearity) from an outer measure, and the L^p theory holds for this integral. Moreover, the domain of an outer measure is the power set, measurability restrictions of functions are thus unnecessary for this integral.

In future work, we will investigate similar results for general $(\oplus,\otimes)$-based pan-integral. We stress that this is not a trivial generalization, since we even do not know whether or not the positive homogeneity holds for general pan-integrals. On the other hand, the subadditivity is a rather strong restriction for monotone measure. It is of interest to examine the results of this paper under some weaker conditions, such as the autocontinuity from below of monotone measures ([35]).

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