Journal of Convex Analysis
Volume 24 (2017), No. 2, xxx–xxx

Computations of Quasiconvex Hulls of Isotropic Sets

Sebastian Heinz*
Weierstraß-Institut für Analysis und Stochastik, Mohrenstr. 39, 10117 Berlin, Germany

Martin Kružík†
Institute of Information Theory and Automation of the ASCR, Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic

Received: December 8, 2014
Accepted: September 4, 2015

We design an algorithm for computations of quasiconvex hulls of isotropic compact sets in in the space of 2×2 real matrices. Our approach uses a recent result by the first author [17] on quasiconvex hulls of isotropic compact sets in the space of 2×2 real matrices. We show that our algorithm has the time complexity of $\mathcal{O}(N \log N)$ where N is the number of orbits of the set. Finally, we outline some applications of our results to relaxation of L^∞ variational problems.

1. Introduction

Generalized convexity notions play an important role in the modern calculus of variations as conditions ensuring sequential weak lower semicontinuity (swlsc) of integral functionals $J : W^{1,p}(\Omega; \mathbb{R}^n) \to \mathbb{R}$

$$J(u) := \int_\Omega f(\nabla u(x)) \,\mathrm{d}x\,,$$

where $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous function satisfying $0 \le f(A) \le C(1+|A|^p)$ for some $1 < p < +\infty$ and Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. It is well known [13] that swlsc of I is equivalent to (Morrey's) quasiconvexity of f. We say that f is quasiconvex if for all $A \in \mathbb{R}^{n\times n}$ and all $\varphi \in W_0^{1,\infty}([0,1]^n; \mathbb{R}^n)$

$$f(A) \le \int_{[0,1]^n} f(A + \nabla\varphi(x)) \,\mathrm{d}x\,.$$

Quasiconvex functions are necessarily rank-one convex which means that

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B)$$

*S. H. was supported by the DFG through FOR 797 *Microplast* under Mie 459/5-2.
†M. K. was partly supported by the grants 14-15264S and P201/12/0671(GA ČR).

ISSN 0944-6532 / $ 2.50

for all $0 \leq \lambda \leq 1$ and all $A, B \in \mathbb{R}^{n\times n}$ with $\mathrm{rank}(A-B) = 1$. It is well-known, that rank-one convexity does not imply quasiconvexity at least if $n > 2$; [27]. Finite rank-one convex functions are continuous. A stronger condition than quasiconvexity is polyconvexity [3]. We say that f is polyconvex if there is a convex function h such that for all $A \in \mathbb{R}^{n\times n}$ $f(A) = h(T(A))$ where $T(A)$ is a vector of all subdeterminants of A. Quasiconvexity, rank-one convexity, and polyconvexity are implied by convexity and are equivalent to it for $n = 1$. It is, however, very difficult to decide whether a given function is quasiconvex. Moreover, in many applications to mathematical elasticity we know that a given integrand f is not quasiconvex. A prominent example are mathematical models of shape memory materials. Then we search for the largest quasiconvex minorant of f which is generically extremely difficult. Therefore its upper and lower bounds represented by the largest rank-one convex and polyconvex minorants bring an important piece of information and there is extensive literature on the subject [6, 7, 8, 9, 14, 15, 21, 22, 25, 26].

Analogously to the convex hull of a compact set we can define quasiconvex, rank-one convex and polyconvex hulls.

If $K \subset \mathbb{R}^{n\times n}$ is compact we define its quasiconvex hull K^{qc} as follows:

$$K^{\mathrm{qc}} := \{A \in \mathbb{R}^{n\times n};\ f(A) \leq \sup_{X\in K} f(X),\ \forall f : \mathbb{R}^{n\times n} \to R \text{ quasiconvex}\}\,.$$

Analogously, one defines the rank-one convex (polyconvex) hull of K denoted K^{rc} (K^{pc}) by replacing quasiconvex functions by rank-one convex (polyconvex) ones. We have $K^{\mathrm{rc}} \subset K^{\mathrm{qc}} \subset K^{\mathrm{pc}} \subset K^{\mathrm{c}}$ where K^{c} denotes the convex hull of K.

Rank-one convex and quasiconvex hulls are generically very difficult (if not impossible) to compute for a particular choice of K. The subset of K^{rc}, easier to calculate, is the so-called lamination convex hull K^{lc} which is defined recursively as follows:

$$K^{\mathrm{lc}} := \cup_{i=0}^{\infty} K^{\mathrm{lc},i}\,,$$

where $K^{\mathrm{lc},0} := K$ and for $i \geq 0$

$$\begin{aligned} K^{\mathrm{lc},i+1} := \{&X \in \mathbb{R}^{n\times n};\ X = \lambda A + (1-\lambda)B;\\ &\mathrm{rank}(A-B) = 1,\ 0 \leq \lambda \leq 1,\ A, B \in K^{\mathrm{lc},i}\}\,. \end{aligned}$$

If $K^{\mathrm{lc},i} = K^{\mathrm{lc},i+1}$ for some i then the lamination hull is of the order i.

We say that $K \subset \mathbb{R}^{n\times n}$ is lamination convex if $K = K^{\mathrm{lc}}$, i.e., if K contains every line segment $[A, B]$ with $A, B \in K$ and $\mathrm{rank}(A - B) = 1$. In fact, it is not difficult to see that allowing rank-one convex functions to take also the value $+\infty$ we have

$$\begin{aligned} K^{\mathrm{lc}} = \{&A \in \mathbb{R}^{n\times n};\ f(A) \leq \sup_{X\in K} f(X)\,,\\ &\forall f : \mathbb{R}^{n\times n} \to R \cup \{+\infty\} \text{ rank-one convex}\}\,. \end{aligned} \tag{1}$$

Contrary to quasiconvex and rank-one convex hulls which are always compact for compact sets, the lamination convex hull of K can be non-compact even if K is compact; cf. [20]. One can, however, make it compact if we allow only for lower semicontinuous functions in (1).

Denoting by $\mathrm{SO}(n)$ rotation matrices in $\mathbb{R}^{n\times n}$, i.e., orthogonal matrices with unit determinants, we call a set $K\subset\mathbb{R}^{n\times n}$ isotropic if $A\in K$ implies that the orbit $QAR\in K$ for all $Q,R\in\mathrm{SO}(n)$. We will often write $\mathrm{SO}(n)A\mathrm{SO}(n)$ instead of $\{QAR \mid Q,R\in\mathrm{SO}(n)\}$. Accordingly, we call a function $f\colon\mathbb{R}^{n\times n}\to\mathbb{R}\cup\{+\infty\}$ isotropic if $f(A)=f(QAR)$ for every $A\in\mathbb{R}^{n\times n}$ and $Q,R\in\mathrm{SO}(n)$.

Finally, we denote by $\mathrm{cc}(K)$ the set of all connected components (meaning maximal connected subsets with respect to "$\subset$") of K.

Knowing the quasiconvex hull of a set is useful in many situations. For example, if $f\geq 0$ denotes strain energy density of a hyperelastic material and $K:=\{A|\, f(A)=0\}$ denotes the set of microscopically stress-free states then K^{qc} is the set of macroscopically stress-free states [4]. Another application is (sequential) weak* lower semicontinuity of the functional $I: W^{1,\infty}(\Omega;\mathbb{R}^n)\to\mathbb{R}$

$$I(u):=\mathrm{esssup}_{x\in\Omega}\, f(\nabla u)\,,$$

for $f\geq 0$ continuous. As it is proved in [5] quasiconvexity of sublevel sets of f (called weak Morrey quasiconvexity of f in [5, Def. 2.2] is a necessary condition for weak* lower semicontinuity of I. In other words, if $c\in\mathbb{R}$ and $E_c:=\{A\in\mathbb{R}^{n\times n}|\, f(A)\leq c\}$ and I is weak* lower semicontinuous then for all $c\in\mathbb{R}$ we have $E_c^{\mathrm{qc}}=E_c$. A sufficient and also necessary condition is the so-called strong Morrey quasiconvexity of f [5, Def. 2.1].

If I above is not weak* lower semicontinuous, we can search for the largest weak* lower semicontinuous envelope of I called the relaxation of I and defined as

$$I^{\mathrm{rlx}}(u):=\inf\{\liminf_{k\to\infty} I(u_k)|\, u_k\stackrel{*}{\rightharpoonup}u \text{ in } W^{1,\infty}(\Omega;\mathbb{R}^n)\}\,.$$

To our best knowledge [24], however, an explicit formula for I^{rlx} is not known.

The aim of the present work is to exploit our knowledge of the structure of the quasiconvex hull of isotropic sets if $n=2$. First we propose an algorithm (see Thm. 4.1, Cor. 4.2) for computing the quasiconvex hull with the comlexity $\mathcal{O}(N\log N)$ where N is the number of orbits. This means that $\limsup_{N\to\infty}$ (number of operations)/$(N\log N)<+\infty$. Secondly, assuming that the strong Morrey quasiconvexity coincides with the weak one, we give an explicit formula for $I^{\mathrm{rlx}}(u)$ as long as f is isotropic and $u\in W^{1,\infty}(\Omega;\mathbb{R}^n)$ is piecewise affine. This is done in Corollary 5.3 giving thus a partial answer to the open problem. Namely, we show that in this case $I^{\mathrm{rlx}}(u)=\mathrm{esssup}_{x\in\Omega}\, f^{\mathrm{qqc}}(\nabla u)$, where $f^{\mathrm{qqc}}(A):=\inf_c\{A\in E_c^{\mathrm{qc}}\}$. Therefore our algorithm can be used to approximate f^{qqc}.

In what follows, we focus on the case where $n=2$ and, thus, we work with

$2{\times}2$ matrices only. Our starting point is the following result, which was proved in [17].

Proposition 1.1. *Let $K \subset \mathbb{R}^{2\times 2}$ be compact and isotropic. Then its quasiconvex hull coincides with its lamination convex hull of order 2, that is $K^{\mathrm{qc}} = K^{\mathrm{lc},2}$.*

The plan of the paper is as follows. We first start with some description of the notation and the description of isotropic sets in Section 2. Useful facts about hulls of isotropic sets are collected in Section 3. Our algorithm is stated in Section 4 and the relaxation results in Section 5. If $f \in \mathbb{R}^{n\times n} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ then we use the notation $\{f{\leq}\alpha\} := \{X \in \mathbb{R}^{n\times n} |\, f(X){\leq}\alpha\}$. Otherwise, we use a standard notation $W^{1,p}(\Omega;\mathbb{R}^n)$ or $W^{1,p}_0(\Omega;\mathbb{R}^n)$, $1 < p \leq +\infty$, for Sobolev spaces.

2. Coordinates

Let $A \in \mathbb{R}^{2\times 2}$ be a matrix. We denote by $\sigma(A) = (\sigma_1(A), \sigma_2(A)) \in \mathbb{R}^2$ the ordered vector of singular values of A, meaning the eigen values of the matrix $\sqrt{A^tA}$ such that $0 \leq \sigma_1(A) \leq \sigma_2(A)$. In addition, we consider the vector $(\lambda_1(A), \lambda_2(A)) \in \mathbb{R}^2$ (sometimes called the signed singular values) where we set $\lambda_1(A) = \sigma_1(A)$ if $\det(A) \geq 0$ and $\lambda_1(A) = -\sigma_1(A)$ if $\det(A) < 0$ as well as $\lambda_2(A) = \sigma_2(A)$. Note that there exist rotations $Q_1, Q_2 \in \mathrm{SO}(2)$ such that Q_1AQ_2 is nothing but the diagonal matrix $\mathrm{diag}(\lambda_1(A), \lambda_2(A))$. We consider the coordinate transformation $(\lambda_1, \lambda_2) \mapsto (\gamma, \delta)$ given by

$$\gamma = \mathrm{sign}(\lambda_1)\sqrt{|\lambda_1|\lambda_2}, \qquad \delta = \lambda_2 - |\lambda_1|. \tag{2}$$

Consequently, the inverse transformation reads

$$\lambda_1 = \mathrm{sign}(\gamma)\big(-\delta/2 + \sqrt{\delta^2/4{+}\gamma^2}\big), \qquad \lambda_2 = \delta/2 + \sqrt{\delta^2/4{+}\gamma^2}, \tag{3}$$

where we always dropped the dependence on A. This is related to the transformations Φ and Ψ introduced by [10, 11]. We will use the coordinates (λ_1, λ_2) as well as (γ, δ) within the paper.

3. Quasiconvex hull

We recall known results and begin with some remarks on isotropic sets in $\mathbb{R}^{2\times 2}$.

Remark 3.1 (Proposition 3.1 in [26]). For given matrices $A, B \in \mathbb{R}^{2\times 2}$ the isotropic sets $\mathrm{SO}(2)A\mathrm{SO}(2)$ and $\mathrm{SO}(2)B\mathrm{SO}(2)$ are rank-one connected if and only if both $|\lambda_1(A)| \leq \lambda_2(B)$ and $|\lambda_1(B)| \leq \lambda_2(A)$ hold.

Remark 3.2 (Lemma 3.2 in [17]). Let $\alpha, \beta \geq 0$ be non-negative numbers. Then the following three sets are closed, isotropic and lamination convex:

$$\{A{\in}\mathbb{R}^{2\times 2} \mid \alpha{\leq} \pm \lambda_1(A)\}, \quad \{A{\in}\mathbb{R}^{2\times 2} \mid \lambda_2(A){\leq}\beta\}.$$

Remark 3.3 (Remark 2 in [12]). Let $A \in \mathbb{R}^{2\times 2}$ be given. Consider $A_+, A_- \in \mathbb{R}^{2\times 2}$ defined via

$$A_\pm = \begin{pmatrix} |\det(A)|^{1/2} & \pm\sqrt{|A|^2 - 2|\det(A)|} \\ 0 & |\det(A)|^{-1/2}\det(A) \end{pmatrix}.$$

Then every matrix $B \in \mathbb{R}^{2\times 2}$ with $\det(A) = \det(B)$ and $\lambda_2(B) \le \lambda_2(A)$ lies in the set $(\mathrm{SO}(2)A\mathrm{SO}(2))^{\mathrm{lc},1}$.

We are going to introduce additional notation. Let $K \subseteq \mathbb{R}^{2\times 2}$ be compact. We consider the compact sets

$$F_\pm = \big\{\big(\det(B), y\big) \in \mathbb{R}^2 \mid B{\in}K \wedge \lambda_2(B){\pm}\lambda_1(B){\ge}y\big\}. \tag{4}$$

With the help of the convex hulls F_+^{c} and F_-^{c}, we define the sets

$$K_\pm = \big\{B \in \mathbb{R}^{2\times 2} \mid \big(\det(B), \lambda_2(B){\pm}\lambda_1(B)\big) \in F_\pm^{\mathrm{c}}\big\}. \tag{5}$$

The following two propositions give a characterization of the quasiconvex hull. In Section 4, we will use this characterization to analyze the time-complexity of an algorithm that computes the quasiconvex hull. The first result is in the spirit of [10, 11], where the polyconvex hull was given in a similar form. The second result is contained in [17].

Proposition 3.4. *Let $K \subseteq \mathbb{R}^{2\times 2}$ be isotropic and compact. Assume that K^{pc} is connected. Then the polyconvex hull is given by $K^{\mathrm{pc}} = K_+{\cap}K_-$.*

Proof. We know that $A \notin K^{\mathrm{pc}}$ holds if and only if there exists a polyconvex function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi(A) > \max\{\varphi(B) \mid B{\in}K\}$. In order to prove [12, Theorem 2.1], Conti et al. show that it is sufficient to consider polyconvex φ which are given by $\varphi(X) = \pm\det(X)$ or

$$\varphi(X) = \lambda_2(X){\pm}\lambda_1(X){-}\det(X)/c \text{ for some } c \in \mathbb{R}\setminus\{0\}. \tag{6}$$

They assume that K is connected, while we just assume it for K^{pc}. Nevertheless, we can use their arguments in view of $K^{\mathrm{pc}} = (K^{\mathrm{pc}})^{\mathrm{pc}}$. Since the set K is compact, we just have to deal with φ like in (6), where $|c| \ne 0$ may be very small. Hence, $A \notin K^{\mathrm{pc}}$ holds if and only if there exists an affine function $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$\forall B \in K \quad h(\det(B)) > \lambda_2(B){\pm}\lambda_1(B) \;\wedge\; h(\det(A)) = \lambda_2(A){\pm}\lambda_1(A) \tag{7}$$

where we have to read (7) as an alternative: it is true if it holds for $+$ or $-$. In fact, h and φ are linked by $h(x) = \varphi(A){+}x/c$ for $x \in \mathbb{R}$. Note that the part on the left-hand side of (7) means nothing but $F_\pm \subseteq H$ where we set $H = \{(x, y){\in}\mathbb{R}^2 \mid h(x){>}y\}$. Since H is an open half plane and F_+ as well as F_- are compact, we also have $F_\pm^{\mathrm{c}} \subseteq H$. As a consequence, we conclude that $A \notin K^{\mathrm{pc}}$ holds if and only if $A \notin K_+$ or $A \notin K_-$. $\square$

Proposition 3.5. *Let $K \subseteq \mathbb{R}^{2\times 2}$ be isotropic and compact. Then the quasiconvex hull is given by*

$$K^{\mathrm{qc}} = \bigcup \{ (Z \cap K)^{\mathrm{pc}} \mid Z \in \mathrm{cc}(K^{\mathrm{lc},1}) \}.$$

Proof. A similar characterization of K^{qc} is given in [17]. In particular, it is shown that K^{qc} equals $\bigcup \{ Z^{\mathrm{pc}} \mid Z \in \mathrm{cc}(K^{\mathrm{lc},1}) \}$. In addition, for every $Z \in \mathrm{cc}(K^{\mathrm{lc},1})$ we have that Z is equal to $(Z \cap K)^{\mathrm{lc},1}$ and, hence, $Z^{\mathrm{pc}} = (Z \cap K)^{\mathrm{pc}}$. □

4. Computation of the quasiconvex hull

Proposition 3.5 characterizes the quasiconvex hull of an isotropic and compact set in $\mathbb{R}^{2\times 2}$ with the help of the polyconvex hull and the lamination hull of order 1. This indicates that the computation of the quasiconvex hull is possible. The aim of this section is to show that there is an efficient way to do that. Let $N > 0$ be a nonnegative integer and $A_1, \ldots, A_N \in \mathbb{R}^{2\times 2}$ be matrices ordered such that $\det(A_i) \leq \det(A_{i+1})$ holds for every $i = 1, \ldots, N-1$. In what follows, we consider the isotropic and compact set

$$K = \mathrm{SO}(2)A_1\mathrm{SO}(2) \cup \cdots \cup \mathrm{SO}(2)A_N\mathrm{SO}(2). \tag{8}$$

Sets of the form $\mathrm{SO}(2)A_i\mathrm{SO}(2)$ are sometimes called *wells*. In the remainder of this section, we will prove

Theorem 4.1. *Let K be as above. Then the quasiconvex hull K^{qc} can be computed with a time-complexity that lies in $\mathcal{O}(N)$.*

Proof. We are going to show that there is an algorithm of the time-complexity class $\mathcal{O}(N)$ which computes the following objects:

(i) partition $K = K^{<} \cup K^{>}$ such that $K^{<} \subseteq \{\det < 0\}$ and $K^{>} \subseteq \{\det \geq 0\}$,
(ii) connected components of $(K^{<})^{\mathrm{lc},1}$ and $(K^{>})^{\mathrm{lc},1}$,
(iii) connected components of $K^{\mathrm{lc},1}$,
(iv) polyconvex hull for each connected component of $K^{\mathrm{lc},1}$.

In view of Proposition 3.5, the steps (i) to (iv) suffice to compute the quasiconvex hull of K. The step (i) is trivial, since the matrices $A_1, \ldots, A_N$ are ordered by the determinant. The time-complexities of (ii), (iii) as well as (iv) are linear in N, which is shown in Lemma 4.4, Lemma 4.6 and Lemma 4.9 below. □

Now assume that the matrices $A_1, \ldots, A_N$ may not be ordered by the determinant. Recall that the time necessary to sort N elements grows like $N \log N$. As a direct consequence of Theorem 4.1, we get

Corollary 4.2. *Let K be as above, but the matrices $A_1, \ldots, A_N$ may not be ordered by the determinant. Then the quasiconvex hull K^{qc} can be computed with a time-complexity that lies in $\mathcal{O}(N \log N)$.*

The time-complexity classes given in Theorem 4.1 and Corollary 4.2 are both optimal.

4.1. Step (ii) - The case of nonnegative determinant

Before we come to the first ingredient of the proof for Theorem 4.1, we make a simplifying observation. Let us use the coordinates (λ_1, λ_2) which are given in Section 2. Assume that K is such that $\lambda_1(A_1) \geq 0$ for every $A \in K$. This means that we are in the case of nonnegative determinant. We consider the set $\Lambda \subseteq \mathbb{R}$ given by the union of closed intervals

$$\Lambda = [|\lambda_1(A_1)|, \lambda_2(A_1)] \cup \cdots \cup [\lambda_1(A_N), \lambda_2(A_N)]. \tag{9}$$

Lemma 4.3. *Let $A, B \in K$ be two matrices. Then the following conditions are equivalent: (i) A and B belong to the same connected component of $K^{\mathrm{lc},1}$ and (ii) the intervals $[|\lambda_1(A)|, \lambda_2(A)]$ and $[|\lambda_1(B)|, \lambda_2(B)]$ belong to the same connected component of Λ.*

Proof. We assume $\det A \leq \det B$, otherwise we exchange A and B. First, assume that (ii) holds. Then there are matrices $X_1, \ldots, X_M \in K$ with $X_1 = A$ and $X_M = B$ such that for every $i = 1, \ldots, M-1$ the intervals $[|\lambda_1(X_i)|, \lambda_2(X_i)]$ and $[|\lambda_1(X_{i+1})|, \lambda_2(X_{i+1})]$ overlap. In particular, $\mathrm{SO}(2)X_i\mathrm{SO}(2)$ and $\mathrm{SO}(2)X_{i+1}\mathrm{SO}(2)$ are rank-one connected by Remark 3.1. Since SO(2) is connected, we have (i). Second, assume that (ii) fails. Then we find real numbers $\alpha > \beta \geq 0$ such that for every $i = 1, \ldots, N$ either $\lambda_2(A_i) \leq \beta$ or $\lambda_1(A_i) \geq \alpha$ holds and, at the same time, $\lambda_2(A) \leq \beta$ and $\lambda_1(B) \geq \alpha$. Here we have also used the assumption $\lambda_1(X_i), \lambda_1(X_{i+1}) \geq 0$. The sets $\{X \in \mathbb{R}^{2\times 2} \mid \lambda_2(X) \leq \beta\}$ and $\{X \in \mathbb{R}^{2\times 2} \mid \lambda_1(X) \geq \alpha\}$ are disjoint and, in view of Remark 3.2, both lamination convex. Hence, (i) must fail. □

Lemma 4.4. *The connected components of $(K^{<})^{\mathrm{lc},1}$ and the connected components of $(K^{>})^{\mathrm{lc},1}$ can be computed with a time-complexity that lies in $\mathcal{O}(N)$.*

Proof. We concentrate on $(K^{<})^{\mathrm{lc},1}$. Note that the connected components of $(K^{>})^{\mathrm{lc},1}$ can be handled in a similar way exploiting the isomorphism on $\mathbb{R}^{2\times 2}$ which is given by $A \mapsto \mathrm{diag}(1, -1)A$. As an application of Lemma 4.3 with $K^{<}$ instead of K, it is sufficient to compute the connected components of Λ in order to get the connected components of $(K^{<})^{\mathrm{lc},1}$. This is done with the help of

Algorithm 4.5.

```
i ← 1;
t ← 0;
while i ≤ N do
begin
   t ← t+1;
```

6: $(\Lambda_t, I_t) \leftarrow ([|\lambda_1(A_i)|, \lambda_2(A_i)], \{i\})$;
7: **while** $t \geq 2$ **and** $\Lambda_{t-1} \cap \Lambda_t \neq \emptyset$ **do**
8: **begin**
9: $(\Lambda_{t-1}, I_{t-1}) \leftarrow (\Lambda_{t-1} \cup \Lambda_t, I_{t-1} \cup I_t)$;
10: $t \leftarrow t-1$;
11: **end;**
12: $i \leftarrow i+1$;
13: **end;**

After Algorithm 4.5 halts, the sets $\Lambda_1, \dots, \Lambda_t$ are the connected components of Λ. Clearly, these sets are of the form $\Lambda_1 = [\alpha_1, \beta_1], \dots, \Lambda_t = [\alpha_t, \beta_t]$. Recall that $\det(A_i) \leq \det(A_{i+1})$ holds for every $i = 1, \dots, N-1$. As a consequence of this ordering, the condition $\Lambda_{t-1} \cap \Lambda_t = \emptyset$ in the line 7 implies that $\beta_{t-1} < \alpha_t$ holds ($\beta_t < \alpha_{t-1}$ being impossible). By induction, we get $\beta_{\tau-1} < \alpha_\tau$ for every $\tau = 2, \dots, t$. Hence, the sets $\Lambda_1, \dots, \Lambda_t$ are pairwise disjoint. The rest directly follows from the design of Algorithm 4.5. The time-complexity of Algorithm 4.5 is linear in N. In particular, we enter the inner loop (lines 9 and 10) less than N times during the whole computation. □

4.2. Step (iii) - Connected components of $K^{\mathrm{lc},1}$

Assume that for some integer $M \in \{1, \dots, N-1\}$ we have $\det(A_1) \leq \cdots \leq \det(A_M) < 0$ as well as $0 \leq \det(A_{M+1}) \leq \cdots \leq \det(A_N)$. Let us consider the sets

$$K^< = \bigcup \{\mathrm{SO}(2) A_i \mathrm{SO}(2) \mid 1 \leq i \leq M\},$$
$$K^> = \bigcup \{\mathrm{SO}(2) A_i \mathrm{SO}(2) \mid M < i \leq N\}$$

and, accordingly, $\Lambda^<$ as well as $\Lambda^>$, see (9). Let $\Lambda_1^<, \dots, \Lambda_s^<$ be the connected components of $\Lambda^<$ and let $K_1^<, \dots, K_s^<$ be the corresponding subsets of $K^<$. More precisely, for every matrix $A \in K^<$ and every index $\sigma \in \{1, \dots, s\}$ we have that $A \in K_\sigma^<$ holds if and only if $[|\lambda_1(A)|, \lambda_2(A)] \subseteq \Lambda_\sigma^<$. In a similar way, we introduce $\Lambda_1^>, \dots, \Lambda_t^>$ and $K_1^>, \dots, K_t^>$ for $\Lambda^>$ and $K^>$. Assume that the ordering is exactly the one coming from Algorithm 4.5: $\beta_{\sigma-1}^< < \alpha_\sigma^<$ holds for every $\sigma = 2, \dots, s$ and $\beta_{\tau-1}^> < \alpha_\tau^>$ for every $\tau = 2, \dots, t$. Here we set $[\alpha_\sigma^<, \beta_\sigma^<] = \Lambda_\sigma^<$ and $[\alpha_\tau^>, \beta_\tau^>] = \Lambda_\tau^>$.

Lemma 4.6. *The connected components of $K^{\mathrm{lc},1}$ can be computed with a time-complexity that lies in $\mathcal{O}(N)$.*

Proof. Using the above notation, we consider

Algorithm 4.7.

$\sigma \leftarrow s$;
$\tau \leftarrow t$;

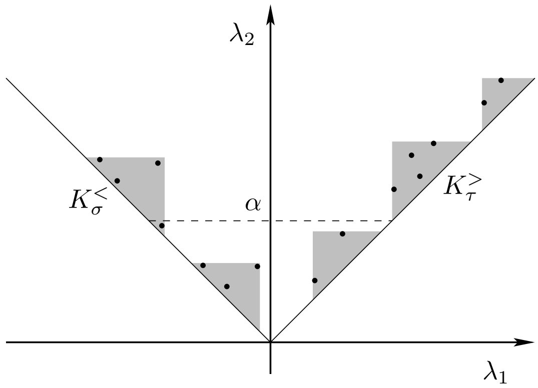


Figure 4.1: The black dots form K. In this example, the subsets $K_\sigma^<$ and $K_\tau^>$ are rank-one connected. Components of K which completely lie below the dashed line ($\lambda_2 = \alpha$) can be dropped without changing the hull K^{qc}.

3: **while** $\sigma \geq 1$ **and** $\tau \geq 1$ **and** $\Lambda_\sigma^< \cap \Lambda_\tau^> = \emptyset$ **do**
4: **begin**
5:   **if** $\alpha_\sigma^< > \beta_\tau^>$ **then** $\sigma \leftarrow \sigma - 1$;
6:   **if** $\alpha_\tau^> > \beta_\sigma^<$ **then** $\tau \leftarrow \tau - 1$;
7: **end;**

After Algorithm 4.7 halts, we distinguish two cases. First, assume that $\sigma = 0$ or $\tau = 0$ holds. Then for every $\tilde{\sigma} = 1, \ldots, s$ and every $\tilde{\tau} = 1, \ldots, t$ the intersection $\Lambda_{\tilde{\sigma}}^< \cap \Lambda_{\tilde{\tau}}^>$ is empty. This is guaranteed by the tests made in line 3 and the ordering of the intervals coming from Algorithm 4.5. In view of Remark 3.1, $K^<$ and $K^>$ cannot be rank-one connected. The connected components of $K^{\mathrm{lc},1}$ are, thus, characterized by

$$\{Z \cap K \mid Z \in \mathrm{cc}(K^{\mathrm{lc},1})\} = \{K_s^<, \ldots, K_1^<, K_1^>, \ldots, K_t^>\}.$$

Second, assume that σ and τ are both greater than 0. Then σ and τ are the largest indices such that $\Lambda_\sigma^< \cap \Lambda_\tau^>$ is non-empty and, in particular, the sets $K_\sigma^<$ and $K_\tau^>$ are rank-one connected. In this case, we consider the set $K_* \subseteq K$ given by

$$K_* = K_s^< \cup \cdots \cup K_\sigma^< \cup K_\tau^> \cup \cdots \cup K_t^>.$$

As a consequence of Lemma 4.8 below, we can replace K by K_*, meaning, we drop the whole set $K \setminus K_*$, without changing the quasiconvex hull $(K_*)^{\mathrm{qc}} = K^{\mathrm{qc}}$. By the design of Algorithm 4.7, the connected components are characterized by

$$\{Z \cap K_* \mid Z \in \mathrm{cc}(K_*^{\mathrm{lc},1})\} = \{K_s^<, \ldots, K_{\sigma+1}^<, K_\sigma^< \cup K_\tau^>, K_{\tau+1}^> \ldots, K_t^>\}.$$

Clearly, the time-complexity of Algorithm 4.7 is linear in N. □

Lemma 4.8. *Let the sets K and K_* be as in the proof of Lemma* 4.6*. Then $(K_*)^{\mathrm{qc}} = K^{\mathrm{qc}}$ holds.*

Proof. Set $\alpha = \max\{\alpha_\sigma^<, \alpha_\tau^>\}$, see Figure 4.1. Then there exist matrices $A^< \in K_\sigma^<$ and $A^> \in K_\tau^>$ such that α lies in $[|\lambda_1(A^<)|, \lambda_2(A^<)]$ as well as in $[|\lambda_1(A^>)|, \lambda_2(A^>)]$ and, in addition, $A^<$ and $A^>$ are rank-one connected. Fix a matrix $A \in K \setminus K_*$. It is not hard to see that $\lambda_2(A) < \alpha$ holds as well as $\det(A^<) < \det(A) < \det(A^>)$. There is a matrix B on the rank-one line between $A^<$ and $A^>$ such that $\det(B)$ and $\det(A)$ are the same. We can assume that $\alpha = \alpha_\tau^> = |\lambda_1(A^>)|$, otherwise the argument is similar. Since $A^>$ and B are rank-one connected, we conclude that $\lambda_2(B) \geq |\lambda_1(A^>)| = \alpha$, see Remark 3.1. This means that $\lambda_2(B) > \lambda_2(A)$, which implies that A lies in $(\mathrm{SO}(2)B\mathrm{SO}(2))^{\mathrm{lc},1}$, see Remark 3.3. But then A lies in $(K_*)^{\mathrm{lc},2}$. We conclude that K is a subset of $(K_*)^{\mathrm{lc},2}$ and, hence, $(K_*)^{\mathrm{lc},2} = K^{\mathrm{lc},2}$ because of Proposition 1.1. $\square$

4.3. Step (iv) - Polyconvex hull

Recall that $K \subseteq \mathbb{R}^{2\times 2}$ is defined by

$$K = \mathrm{SO}(2)A_1\mathrm{SO}(2) \cup \cdots \cup \mathrm{SO}(2)A_N\mathrm{SO}(2)$$

where the matrices $A_1, \ldots, A_N$ are ordered by the determinant. Let $K_1, \ldots, K_L \subseteq K$ be the output of Algorithm 4.5 and Algorithm 4.7, which characterizes the connected components of $(K_*)^{\mathrm{lc},1}$ such that

$$\{Z \cap K_* \mid Z \in \mathrm{cc}(K_*^{\mathrm{lc},1})\} = \{K_1, \ldots, K_L\}.$$

Lemma 4.9. *The polyconvex hulls of the sets $K_1, \ldots, K_L$ can be computed with a total time-complexity that lies in $\mathcal{O}(N)$.*

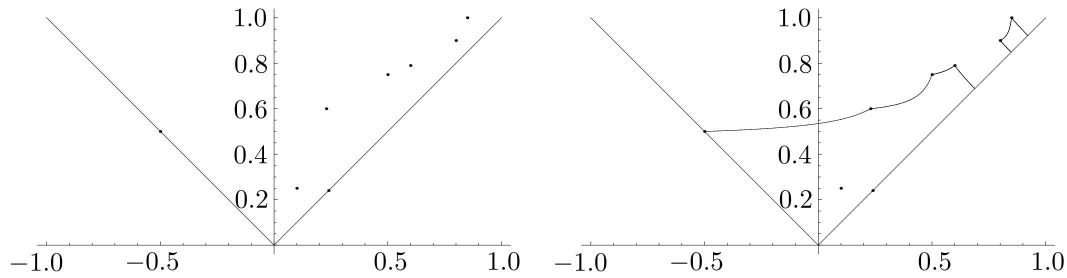
Proof. Fix an index $1 \leq i \leq L$. Determine the numbers $1 \leq i_0 < i_1 \leq N$ such that

$$K_i = \mathrm{SO}(2)A_{i_0}\mathrm{SO}(2) \cup \cdots \cup \mathrm{SO}(2)A_{i_1}\mathrm{SO}(2)$$

for matrices $A_{i_0}, \ldots, A_{i_1} \in \{A_1, \ldots, A_N\}$. Proposition 3.4 gives us the polyconvex hull of K_i once we have computed $(K_i)_+$ and $(K_i)_-$. Following (5), we have to compute the convex hulls of $(F_i)_+$ and $(F_i)_-$. The computation of the (poly-)convex hull, basically, means sorting out points which lie in the interior and, hence, are superfluous for the hull, see Figure 4.2. The matrices $A_{i_0}, \ldots, A_{i_1}$, which define $(F_i)_+$ and $(F_i)_-$, are ordered by the determinant. They form a so-called *simple polyline*. Therefore, the computation of $(F_i)_+^{\mathrm{c}}$ and $(F_i)_-^{\mathrm{c}}$ can be done, for example, using the algorithm proposed in [23] with a time-complexity linear in $i_1 - i_0 + 1$. As a result, we have access to $(K_i)_+$ and $(K_i)_-$ and, in particular, to the boundaries $\partial[(K_i)_+]$ and $\partial[(K_i)_-]$. Following Proposition 3.4, we can effectively characterize the boundary of $(K_i)^{\mathrm{pc}}$ (and, hence, $(K_i)^{\mathrm{pc}}$ itself) with the help of the matrices $A_{i,1}, \ldots, A_{i,N(i)}$ given by

$$\{A_{i,1}, \ldots, A_{i,N_i}\} = \{A_1, \ldots, A_N\} \cap \partial[(K_i)_+] \cap \partial[(K_i)_-]. \tag{10}$$

Now if i varies between 1 and L, we end up with a total time-complexity that is linear in N. $\square$



(a) Eight wells (b) Quasiconvex hull

Figure 4.2: The example is drawn in the λ_1-λ_2-plane: (a) a set of the form (8) with eight wells and (b) the boundary of its quasiconvex hull, showing two connected components. The two wells near the origin lie in the interior and, hence, can be seen as "superfluous".

The sets in (10) are considered to be the output of the whole algorithm, once the matrices $A_{i,1}, \ldots, A_{i,N(i)}$ have been collected for every $i = 1, \ldots, L$. Note that these matrices inherit the ordering from $A_1 \ldots, A_N$, meaning,

$$(i_1{<}i_2) \vee (i_1{=}i_2 \wedge j_1{\le}j_2) \Rightarrow \det(A_{i_1,j_1}){\le}\det(A_{i_2,j_2}). \tag{11}$$

In view of Proposition 3.5, we get a simple characterization for the quasiconvex hull

$$K^{\mathrm{qc}} = \bigcup_{i=1}^{L} \big(\mathrm{SO}(2)A_{i,1}\mathrm{SO}(2) \cup \cdots \cup \mathrm{SO}(2)A_{i,N(i)}\mathrm{SO}(2)\big)^{\mathrm{pc}}. \tag{12}$$

The following remark indicates that this characterization can be used to compute K^{qc}. It works with the coordinates (γ, δ), see (2) in Section 2.

Remark 4.10. Let K as well as the matrices $A_{i,j}$ with $i = 1, \ldots, L$ and $j = 1, \ldots, N(i)$ be as above. In addition, let $\tilde{\gamma}_1 \le \cdots \le \tilde{\gamma}_M$ be real numbers. Then the time-complexity to compute the M values

$$\delta_k = \sup\{\delta(B) \mid B{\in}K^{\mathrm{qc}} \wedge \gamma(B){=}\tilde{\gamma}_k\} \quad \text{for } k = 1, \ldots, M$$

lies in $\mathcal{O}(\max\{N, M\})$. Note that δ_k is equal to $-\infty$ if the supremum is taken over the empty set and that a matrix $B \in \mathbb{R}^{2\times 2}$ with $\gamma(B) = \tilde{\gamma}_k$ lies in K^{qc} if and only if $\delta(B) \le \delta_k$.

Proof. For every index $k \in \{1, \ldots, M\}$ find, if possible, $i_k \in \{1, \ldots, L\}$ and $j_k \in \{1, \ldots, N(i){-}1\}$ such that $\gamma(A_{i_k,j_k}) \le \tilde{\gamma}_k \le \gamma(A_{i_k,j_k+1})$ holds. This can be done with a time-complexity that lies in $\mathcal{O}(\max\{N, M\})$ because the matrices are ordered by the determinant, see (11). If such indices $i(k)$ and $j(k)$ do not exist, then the characterization (12) implies that δ_k has to be equal to $-\infty$.

If such indices exist, set $\gamma_1 = \gamma(A_{i_k,j_k})$, $\gamma_2 = \tilde{\gamma}_k$ and $\gamma_3 = \gamma(A_{i_k,j_k+1})$. We use Proposition 3.4 in order to compute δ_k. Let $B \in K^{\mathrm{qc}}$ be any matrix such that $\gamma(B)$ lies between γ_1 and γ_3. Then the largest possible value of $\delta(B)$ is given by convex interpolation. In fact, choose matrices $B_+, B_- \in \mathcal{B}$ such that $\gamma(B_\pm) = \gamma_2$ holds and, simultaneously,

$$(\lambda_2 \pm \lambda_1)(B_\pm) = \frac{\gamma_3 - \gamma_2}{\gamma_3 - \gamma_1}(\lambda_2 \pm \lambda_1)(A_{i_k,j_k}) + \frac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_1}(\lambda_2 \pm \lambda_1)(A_{i_k,j_k+1}).$$

Then we have $\delta_k = \min\{\delta(B_+), \delta(B_-)\}$. □

following way. Start with K and cut out every subset $K \cap Z_1$ where we find the conditions above fulfilled. Then [17, Lemma 6.1] holds if K is replaced by K_*, see Lemma 4.8. By construction, the sets $K^{\mathrm{lc},2}$ and $(K_*)^{\mathrm{lc},2}$ are the same. Hence, the rest of the arguments in [17] remain unaltered.

5. Relaxation

Before we present our relaxation result, we prove a nice property for first order laminates which is of independent interest and might not be well-known. The lemma holds in any matrix space $\mathbb{R}^{n\times d}$ with $n, d \geq 1$. Yet, we deal with the case $\mathbb{R}^{2\times 2}$ only.

Lemma 5.1. *Let $K \subseteq \mathbb{R}^{2\times 2}$ be a compact set and $\delta_0 > 0$ a real number. Moreover, let $\Omega \subseteq \mathbb{R}^2$ be an open and bounded set. Then for every matrix $A \in \mathbb{R}^{2\times 2}$ with $\operatorname{dist}(A, K^{\mathrm{lc},1}) \leq \delta_0$ and every $\delta > 0$ there exists a function $w \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^2)$ vanishing at the boundary of Ω such that*

$$\operatorname{dist}(A + \nabla w(x), K) \leq \delta_0 + \delta \quad \textit{for a.e. } x \in \Omega.$$

Proof. Fix $A \in \mathbb{R}^{2\times 2}$ with $\operatorname{dist}(A, K^{\mathrm{lc},1}) \leq \delta_0$ and fix a real number $\delta > 0$. If even $\operatorname{dist}(A, K) \leq \delta_0$ holds, we can choose $w = 0$ and are done. Otherwise, there exist a scalar $\lambda \in (0, 1)$ and matrices $B_1, C_1 \in \mathbb{R}^{2\times 2}$ such that all the following conditions are fulfilled: $\operatorname{rank}(B_1 - C_1) = 1$, $\operatorname{dist}(B_1, K) \leq \delta_0$, $\operatorname{dist}(C_1, K) \leq \delta_0$ as well as $A = \lambda B_1 + (1-\lambda)C_1$. Choose vectors $a, \nu \in \mathbb{R}^2$ such that $B_1 - C_1 = a \otimes \nu$ holds. We are going to construct functions $w_1, w_2, \ldots \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^2)$ which vanish at the boundary of Ω.

For every $1 > \sigma_1 > 0$ we consider a rectangle $\omega_1 \subseteq \mathbb{R}^2$ with one edge parallel to ν and of length σ_1 and the other edge orthogonal to ν and of length 1. There is a function $v_1 \in \mathrm{W}^{1,\infty}(\omega_1, \mathbb{R}^2)$ vanishing at the boundary of ω_1 such that the quantity $A + \nabla v_1$ takes only four values: B_1, C_1, A_1^+ or A_1^-. In addition, we force the width of the boundary layer (where $A + \nabla v_1 \in \{A_1^+, A_1^-\}$) to be $\sqrt{\sigma_1}$, see Figure 5.1(a). The set Ω can be written as a union of countably many scaled and translated copies of ω_1, see Figure 5.1(b). As a result, we end up with a function $w_1 \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^2)$ vanishing at the boundary of Ω such that $A + \nabla w_1 \in \{B_1, C_1, A^+, A^-\}$ holds almost everywhere. Consider the set

$\Omega_1 = \{A + \nabla w_1 \in \{B_1, C_1\}\}$. By construction, there exists a constant $c > 0$ independent of σ_1 such that the norms $|A_1^+ - A|$ and $|A_1^- - A|$ as well as the volume of the set $\Omega \setminus \Omega_1$ are smaller than $c\sqrt{\sigma_1}$. Choose $\sigma_1 = \delta^2/(2c)^2$, then we have

$$|A_1^+ - A|, |A_1^- - A| \leq \tfrac{\delta}{2}.$$

In addition, the volume of the set $\Omega \setminus \Omega_1$ is smaller than $\delta/2$.

We choose the function w_2 equal to w_1 on Ω_1, but different on $\Omega \setminus \Omega_1$. Let us focus on the case where $A + \nabla w_1 = A_1^+$ holds. The construction is like above. Set $B_2^+ = B_1 - A + A_1^+$ as well as $C_2^+ = C_1 - A + A_1^+$. Then all the following conditions are fulfilled: $B_2^+ - C_2^+ = a \otimes \nu$, $\operatorname{dist}(B_2^+, K) \leq \delta_0 + \delta/2$, $\operatorname{dist}(C_2^+, K) \leq \delta_0 + \delta/2$ as well as $A_1^+ = \lambda B_2^+ + (1-\lambda) C_2^+$. For every $1 > \sigma_2 > 0$ we consider a rectangle $\omega_2 \subseteq \mathbb{R}^2$ with one edge parallel to ν and of length σ_2 and the other edge orthogonal to ν and of length 1. There is a function $v_2^+ \in \mathrm{W}^{1,\infty}(\omega_2, \mathbb{R}^2)$ vanishing at the boundary of ω_2 such that the quantity $A_1^+ + \nabla v_2^+$ takes only four values: B_2^+, C_2^+, A_2^{++} or A_2^{+-}. The set where $A + \nabla w_1 = A_1^+$ holds can be written as a union of countably many scaled and translated copies of ω_2. The case where $A + \nabla w_1 = A_1^-$ holds can be handled in a similar way. We end up with a function $w_2 \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^2)$. Consider the set $\Omega_2 = \{A + \nabla w_2 \in \{B_1, C_1, B_2^+, C_2^+, B_2^-, C_2^-\}\}$. Choose $\sigma_2 = \delta^2/(2^2 c)^2$, then we have

$$\begin{aligned} &|A_2^{++} - A|,\ |A_2^{+-} - A|,\ |A_2^{-+} - A|,\ |A_2^{--} - A| \\ &\leq \tfrac{\delta}{4} + \max\{|A_1^+ - A|, |A_1^- - A|\} \leq \tfrac{\delta}{2} + \tfrac{\delta}{4}. \end{aligned}$$

In addition, the volume of the set $\Omega \setminus \Omega_2$ is smaller than $\delta^2/8$.

In this spirit, we construct the functions $w_1, w_2, \ldots \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^2)$. There is a function $w \in \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^2)$ such that $w_k \to w$ holds pointwise almost everywhere in Ω. The values of the quantity $A + \nabla w$ are contained in the set $\{B_1, C_1, B_2^+, C_2^+, B_2^-, C_2^-, \ldots\}$ almost everywhere, since the volume of $\Omega \setminus \Omega_k$ tends to zero. We compute

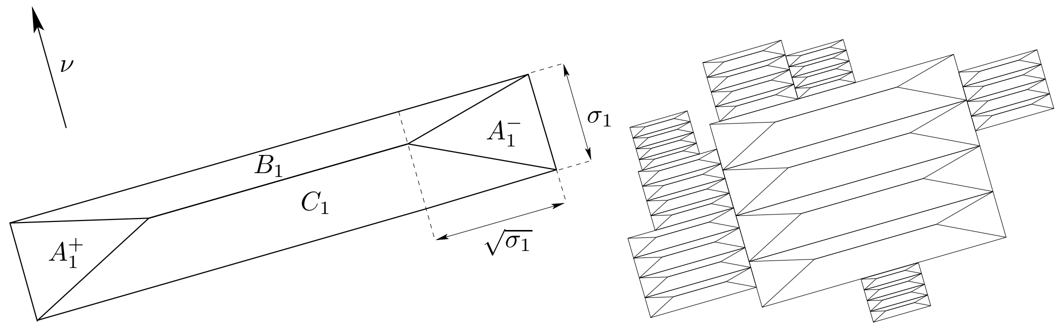
$$\operatorname{dist}(A + \nabla w(x), K) \leq \delta_0 + \tfrac{\delta}{2} + \tfrac{\delta}{4} + \cdots = \delta_0 + \delta \quad \text{for a.e. } x \in \Omega.$$

As a consequence, w has the desired property. $\square$

Let $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ be isotropic, continuous and coercive, meaning, we have $f(A) \to \infty$ whenever $|A| \to \infty$. In addition, let $\Omega \subseteq \mathbb{R}^{2\times 2}$ be a domain. We study the weak* lower semicontinuity of the functional $I\colon \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^2) \to \mathbb{R}$ given by

$$I(u) = \operatorname{esssup} f(\nabla u).$$

In particular, we want to find the weak* lower semicontinuous envelope I^{rlx} (sometimes called relaxation) whenever I fails to be weak* lower semicontinuous. Strongly connected to this is quasi-quasiconvexity and the quasi-quasiconvex envelope. For every continuous and coercive function $g\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$



(a) Basic structure (b) Filled domain

Figure 5.1: The construction of w_1 is sketched. Scaled copies of the basic structure are used to fill an arbitrary domain.

the *quasi-quasiconvex envelope* $g^{\mathrm{qqc}}\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is defined via

$$g^{\mathrm{qqc}}(A) = \min\big\{\alpha{\in}\mathbb{R} \mid A{\in}\{g{\le}\alpha\}^{\mathrm{qc}}\big\}.$$

Here $\{g{\le}\alpha\}^{\mathrm{qc}}$ denotes the quasiconvex hull of the set $\{g{\le}\alpha\}$. Moreover, g is called *quasi-quasiconvex* if $g = g^{\mathrm{qqc}}$ holds. Clearly, we have always $g^{\mathrm{qqc}} \le g$. We obtain the lower estimate

$$I^{\mathrm{rlx}}(u) \ge \operatorname{esssup} f^{\mathrm{qqc}}(\nabla u) \quad \text{for every } u \in \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^2)\,.$$

If $u \mapsto \operatorname{esssup} f^{\mathrm{qqc}}(\nabla u)$ is weak* lower semicontinuous we have the following two result.

Lemma 5.2. *Let $u \in \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^2)$ be affine and $u \mapsto \operatorname{esssup} f^{\mathrm{qqc}}(\nabla u)$ be weak* lower semicontinuous. Then $I^{\mathrm{rlx}}(u) = \operatorname{esssup} f^{\mathrm{qqc}}(\nabla u)$.*

Proof. Fix a matrix $A \in \mathbb{R}^{2\times 2}$. It is sufficient to show that for every $\epsilon > 0$ there exists a function $w \in \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^2)$ vanishing at the boundary of Ω such that

$$\operatorname{esssup} f(A{+}\nabla w) \le f^{\mathrm{qqc}}(A){+}\epsilon.$$

Since f is continuous, there exists a number $\delta = \delta(\epsilon) > 0$ such that the above condition is implied by

$$\operatorname{dist}\big(A{+}\nabla w(x), \{f{\le}f^{\mathrm{qqc}}(A)\}\big) \le \delta \quad \text{for a.e. } x \in \Omega. \tag{13}$$

By definition, we must have $A \in \{f{\le}f^{\mathrm{qqc}}(A)\}^{\mathrm{qc}}$ and, in view of Proposition 1.1, $A \in \{f{\le}f^{\mathrm{qqc}}(A)\}^{\mathrm{lc},2}$ follows. Now two iterations of Lemma 5.1 imply that for every $\delta > 0$ there is a function $w \in \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^2)$ vanishing at the boundary of Ω such that (13) holds. $\square$

A function $u \in \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^2)$ is called *piecewise affine* if for almost every $x \in \Omega$ there exists a nonempty open subset $O \subseteq \Omega$ containing x, a matrix $A \in \mathbb{R}^{2\times 2}$ and a vector $b \in \mathbb{R}^2$ such that for almost every $y \in O$ we have $u(y) = Ay+b$. As an immediate consequence of Lemma 5.2 we get the following corollary.

Corollary 5.3. *Let* $u \mapsto \mathrm{esssup} f^{\mathrm{qqc}}(\nabla u)$ *be weak* lower semicontinuous and let* $\mathrm{Aff} \subseteq \mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^2)$ *be the closure of the set of all piecewise affine functions with respect to strong convergence in* $\mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^2)$*. Then for every function* $u \in \mathrm{Aff}$ *we have equality* $I^{\mathrm{rlx}}(u) = \mathrm{esssup} f^{\mathrm{qqc}}(\nabla u)$*.*

Acknowledgements. This work was conceived and completed during visits of M. K. in WIAS Berlin and of S. H. in ÚTIA AV ČR. The support and hospitality of these two institutions are gratefully acknowledged.

References

[1] E. Acerbi, G. Buttazzo, F. Prinari: The class of functionals which can be represented by a supremum, J. Convex Analysis 9 (2002) 225–236.

[2] G. Aronsson, E. N. Barron: L^∞ variational problems with running costs and constraints, Appl. Math. Optim. 65 (2012) 53–90.

[3] J. M. Ball: Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (1977) 337–403.

[4] J. M. Ball, R. D. James: Fine phase mixtures as minimizers of energy, Arch. Ration. Mech. Anal. 100 (1987) 15–52.

[5] E. N. Barron, R. R. Jensen, C. Y. Wang: Lower semicontinuity of L^∞ functionals, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 18 (2001) 495–517.

[6] S. Bartels: Linear convergence in the approximation of rank-one convex envelopes, M2AN, Math. Model. Numer. Anal. 38 (2004) 811–820.

[7] S. Bartels: Reliable and efficient approximation of polyconvex envelopes, SIAM J. Numer. Anal. 43 (2005) 363–385.

[8] S. Bartels, M. Kružík: An efficient approach to the numerical solution of rate-independent problems with nonconvex energies, Multiscale Model. Simul. 9 (2011) 1276–1300.

[9] B. Benešová: Global optimization numerical strategies for rate-independent processes, J. Glob. Optim. 50 (2011) 197–220.

[10] P. Cardaliaguet, R. Tahraoui: Equivalence between rank-one convexity and polyconvexity for isotropic sets of $\mathbb{R}^{2\times 2}$, Part I, Nonlinear. Anal. 50 (2002) 1179–1199.

[11] P. Cardaliaguet, R. Tahraoui: Equivalence between rank-one convexity and polyconvexity for isotropic sets of $\mathbb{R}^{2\times 2}$, Part II, Nonlinear. Anal. 50 (2002) 1201–1239.

[12] S. Conti, C. De Lellis, S. Müller, M. Romeo: Polyconvexity equals rank-one convexity for connected isotropic sets in $\mathbb{M}^{2\times 2}$, C. R., Math., Acad. Sci. Paris 337 (2003) 233–238.

[13] B. Dacorogna: Direct Methods in the Calculus of Variations, 2nd Ed., Springer, Berlin (2008).

[14] G. Dolzmann: Numerical computation of rank-one convex envelopes, SIAM J. Numer. Anal. 36 (1999) 1621–1635.

[15] G. Dolzmann, N. Walkington: Estimates for numerical approximations of rank one convex envelopes, Numer. Math. 85 (2000) 647–663.

[16] R. D. Grigorieff: A note on von Neumann's trace inequality, Math. Nachr. 151 (1991) 327–328.

[17] S. Heinz: Quasiconvexity equals lamination convexity for isotropic sets of 2×2 matrices, Adv. Calc. Var. 8 (2015) 43–53.

[18] R. Kohn, G. Strang: Explicit relaxation of a variational problem in optimal design, Bull. Amer. Math. Soc. 9 (1983) 211–214.

[19] R. Kohn, G. Strang: Optimal design and relaxation of variational problems I, II, III, Commun. Pure Appl. Math. 39 (1986) 113–137, 139–182, 353–377.

[20] J. Kolář: Non-compact lamination convex hulls, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 20 (2003) 391–403.

[21] M. Kružík, M. Luskin: The computation of martensitic microstructure with piecewise laminates, J. Sci. Comput. 19 (2003) 293–308.

[22] J. Matoušek, P. Plecháč: On functional separately convex hulls, Discrete Comput. Geom. 19 (1998) 105–130.

[23] A. Melkman: On-line construction of the convex hull of a simple polyline, Inf. Process. Lett. 25 (1987) 11–12.

[24] F. Prinari: Personal communication (March 2014).

[25] M. Šilhavý: Rank 1 convex hulls of isotropic functions in dimension 2 by 2, Math. Bohem. 126 (2001) 521–529.

[26] M. Šilhavý: Rotationally invariant rank 1 convex functions, Appl. Math. Optim. 44 (2001) 1–15.

[27] V. Šverák: Rank-one convexity does not imply quasiconvexity, Proc. R. Soc. Edinb., Sect. A 120 (1992) 185–189.