Weak Lower Semicontinuity of Integral Functionals and Applications*

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This paper is dedicated to Irene Fonseca on the occasion of her 60th birthday

Abstract. Minimization is a recurring theme in many mathematical disciplines ranging from pure to applied. Of particular importance is the minimization of integral functionals, which is studied within the calculus of variations. Proofs of the existence of minimizers usually rely on a fine property of the functional called weak lower semicontinuity. While early studies of lower semicontinuity go back to the beginning of the 20th century, the milestones of the modern theory were established by C. B. Morrey, Jr. [*Pacific J. Math.*, 2 (1952), pp. 25–53] in 1952 and N. G. Meyers [*Trans. Amer. Math. Soc.*, 119 (1965), pp. 125–149] in 1965. We recapitulate the development of this topic from these papers onwards. Special attention is paid to signed integrands and to applications in continuum mechanics of solids. In particular, we review the concept of polyconvexity and special properties of (sub-)determinants with respect to weak lower semicontinuity. In addition, we emphasize some recent progress in lower semicontinuity of functionals along sequences satisfying differential and algebraic constraints that can be used in elasticity to ensure injectivity and orientation-preservation of deformations. Finally, we outline generalizations of these results to more general first-order partial differential operators and make some suggestions for further reading.

Key words. calculus of variations, null Lagrangians, polyconvexity, quasiconvexity, weak lower semicontinuity

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> *Nothing takes place in the world whose meaning is not that of some maximum or minimum.*
>
> LEONHARD PAUL EULER (1707–1783)

Contents

1. Introduction. Many tasks in the world surrounding us can be mathematically formulated as minimization or maximization problems. For example, in physics we minimize the energy, in economy one tries to minimize the cost and maximize the profit, and entrepreneurs may try to minimize their investment risk. In addition, minimization problems appear in many more specific tasks: in a fitting procedure, or more generally in inverse problems, one tries to minimize the deviation of the model prediction from the experimental observation, and the training of a neuronal network is based on minimizing a suitable cost function.

In a very general manner, we may express these problems as

$$\text{(1.1)} \qquad\qquad\qquad \text{minimize } I \text{ over } \mathcal{Y} \, ,$$

where $\mathcal{Y}$ is a set over which the minimum is sought and $I : \mathcal{Y} \to \mathbb{R}$ is a functional which might represent the energy, cost, risk, or loss, for instance. From the mathematical point of view, two questions are immediate when inspecting problem (1.1): first, whether (1.1) *is solvable*, that is, whether I possesses minimizers on $\mathcal{Y}$, and second, *how to find a solution (i.e., a minimizer)* of (1.1).

Calculus of variations is devoted to solving (1.1) when $\mathcal{Y}$ is (a subset) of an infinite-dimensional vector space. Its starting point may have been the question of Johann Bernoulli as to which curve a mass point will descend the fastest in a gravitational field; the so-called *brachistochrone problem*. In the most typical situation (that covers the brachistochone problem, in particular), I in (1.1) is an integral functional

depending on functions $u : \Omega \to \mathbb{R}^m$ with $\Omega \subset \mathbb{R}^n$ and their derivatives. In the easiest case, in which $n = m = 1$, $\Omega = [a, b]$, and $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a suitable integrand, the functional reads

$$(1.2) \qquad I(u) := \int_a^b f(x, u(x), u'(x)) \, \mathrm{d}x \quad \text{ with } u(a) = u_a \text{ and } u(b) = u_b,$$

where u_a and u_b are given boundary data. The task is to either solve (1.1) or at least prove existence of minimizers.

Foundations of the calculus of variations were laid down in the 18th century by L. P. Euler and J. L. Lagrange, who also realized its important connections to physics and mechanics. These early works quite naturally concentrated on the question of how to find (candidates for) solutions of (1.1). The classical method to do so considers so-called *variations*. Indeed, if u_0 is a minimizer of I in (1.2), then

$$(1.3) \qquad\qquad I(u_0) \leq I(u_0 + \varepsilon\varphi) \qquad \text{for all } \varphi \in C_0^\infty([a, b]),$$

where $\varepsilon\varphi$ is called a variation of the minimizer. Now, assume that f is twice continuously differentiable and $u_0 \in C^2([a, b])$; then by the classical calculus (1.3) implies that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon} I(u_0(x) + \varepsilon\varphi(x))\big|_{\varepsilon=0}$ vanishes for all $\varphi \in C_0^\infty([a, b])$. This is equivalent to solving

$$(1.4) \qquad\qquad \frac{\partial f}{\partial r}(x, u_0, u_0') - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial s}(x, u_0, u_0') = 0 \quad \text{on } [a, b],$$

where $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$ denote the partial derivatives of f with respect to the second and third variable, respectively. Equation (1.4) is referred to as the *Euler–Lagrange equation* and solving it is the classical path to finding solutions of (1.1). Of course, any critical point of I (and not only the minimizer) is a solution to (1.4), but solving (1.4) is still an efficient approach to (1.1) at least in a situation in which all critical points are minimizers, for example, if f is convex. For more details, see, for example, the book by Bolza [46].

Nevertheless, solving the Euler–Lagrange equation naturally relies on smoothness properties of f which might not be available. Therefore, it is often advantageous to address existence of solutions to (1.1) in a nonconstructive way by using suitable compactness properties of $\mathcal{Y}$ and continuity properties of I. For example, if $\mathcal{Y}$ is a bounded closed interval of reals and $I : \mathcal{Y} \to \mathbb{R}$ is a function, then (1.1) has a solution whenever I is continuous. This observation goes back to Bernard Bolzano, who proved it in his work *Function Theory* in 1830, and it is called the *extreme value theorem*. Later on, it was independently observed by Karl Weierstrass around 1860. The main ingredient of the proof, namely, the fact that one can extract a convergent subsequence from a closed bounded interval of reals, is nowadays known as the Bolzano–Weierstrass theorem.

The results of Bolzano and Weierstrass easily extend to the situation where $\mathcal{Y}$ is a bounded and closed set of a *finite-dimensional* vector space. However, they cannot be generalized to the situation in which, for example, $\mathcal{Y}$ is a ball in an infinite-dimensional vector space, since the Bolzano–Weierstrass theorem does not hold in this case. In fact, being able to extract a convergent subsequence from a sequence of elements in the unit ball of a normed vector space $\mathcal{X}$ is *equivalent* to $\mathcal{X}$ being finite-dimensional. This is a classical result attributed to F. Riesz.

Thus, the only hope of transferring a variant of the Bolzano–Weierstrass theorem to infinite-dimensional spaces is to seek compactness in a "weaker" topology than the

one induced by the norm. This possibility was opened up by Riesz and Hilbert, who used the *weak topology* on Hilbert spaces from the beginning of the 20th century, and by Stefan Banach, who defined it on other normed spaces around 1929 [192, 237].

DEFINITION 1.1. *Let $\mathcal{X}$ be a Banach space and $\mathcal{X}'$ its dual. We say that a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{X}$ converges* weakly *in $\mathcal{X}$ to $u \in \mathcal{X}$ if*

$$\psi(u_k) \to \psi(u) \quad \text{for all } \psi \in \mathcal{X}', \text{ and we write that} \quad u_k \rightharpoonup u.$$

Similarly, a sequence $\{v_k\}_{k \in \mathbb{N}} \subset \mathcal{X}'$ converges weakly* *in $\mathcal{X}'$ to $v \in \mathcal{X}'$ if*

$$v_k(\varphi) \to v(\varphi) \quad \text{for all } \varphi \in \mathcal{X}, \text{ and we write that} \quad v_k \overset{*}{\rightharpoonup} v.$$

A crucial property of the weak topology is that it allows for a generalization of the Bolzano–Weierstrass theorem to infinite-dimensional vector spaces. Indeed, take $\mathcal{X}'$ to be the dual to a Banach space $\mathcal{X}$. Then, bounded subsets of $\mathcal{X}'$ are *precompact in the weak* topology* by the Banach–Alaoglu theorem, even though they are not generically compact if $\mathcal{X}'$ is infinite-dimensional. As an immediate consequence, we have that bounded subsets of a reflexive Banach space $\mathcal{X}$ are precompact in the weak topology.

Having the weak topology at hand, a generalization of the Bolzano extreme value theorem becomes possible and is today known as the *direct method of the calculus of variations*. This algorithm was proposed by David Hilbert around 1900 to show (in a nonconstructive way) the existence of a solution to the minimization problem (1.1). It consists of three steps:
1. Find a minimizing sequence along which I converges to its infimum on $\mathcal{Y}$.
2. Show that a subsequence of the minimizing sequence converges to an element of $\mathcal{Y}$ in some topology τ.
3. Prove that this limit element is a minimizer.

The first step of the direct method is easily handled if the infimum of I is finite. For the second step, the appropriate choice of the topology τ is crucial. In the most typical situation, the set $\mathcal{Y}$ is a subset of a Banach space or its dual and τ is either the weak or the weak* topology. In this case, if $\mathcal{Y}$ is bounded, the existence of a converging subsequence of the minimizing sequence is immediate from the Banach–Alaoglu theorem. If $\mathcal{Y}$ is not bounded, the usual remedy is to realize that the minimizer can only lie in a bounded subset of $\mathcal{Y}$ due to *coercivity* of I. Coercivity refers to the property of I that it takes arbitrarily large values if the norm of its argument is sufficiently large. More precisely, we say that I is coercive if

$$(1.5) \qquad\qquad\qquad \lim_{\|u\| \to \infty} I(u) = \infty \ .$$

This allows us to say that all minimizers of I are contained in some closed ball centered at the origin.

The third step of the direct method relies on suitable semicontinuity properties of I; a sufficient and widely used condition is the *(sequential) lower semicontinuity* of I with respect to the weak/weak* topology:

DEFINITION 1.2. *Let $\mathcal{Y}$ be a subset of a Banach space. We say that the functional $I : \mathcal{Y} \to \mathbb{R}$ is (sequentially) weakly/weakly* lower semicontinuous on $\mathcal{Y}$ if for every sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{Y}$ converging weakly/weakly* to $u \in \mathcal{Y}$, we have that*

$$I(u) \leq \liminf_{k \to \infty} I(u_k).$$

If I is not weak/weak* lower semicontinuous, solutions to (1.1) need not exist. However, weak lower semicontinuity of I is not a necessary condition for the existence of minimizers. These facts are demonstrated by the following example.

EXAMPLE 1.1. *Consider the following special case of* (1.2):

$$(1.6) \qquad I(u) = \int_0^1 \left(1 - (u'(x))^2\right)^2 + (u(x))^2 \, dx$$

with

$$\mathcal{Y} := \{u \in W^{1,\infty}(0,1); \ -1 \le u' \le 1, \ u(0) = u(1) = 0\} \ .$$

We can see, for example, by the Lebesgue dominated convergence theorem, that I is continuous on $W^{1,\infty}(0,1)$ but is not weakly lower semicontinuous. To show this, define

$$u(x) = \begin{cases} x & \text{if } 0 \le x \le 1/2, \\ -x + 1 & \text{if } 1/2 \le x \le 1, \end{cases}$$

and extend it periodically to the whole $\mathbb{R}$. Let $u_k(x) := k^{-1}u(kx)$ for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}$. Notice that $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{Y}$.

The sequence of "zig-zag" functions $\{u_k\}_{k \in \mathbb{N}}$ converges weakly* to zero in $W^{1,\infty}(0,1)$. It is not hard to see that $I(u_k) \to 0$ for $k \to \infty$ but

$$1 = I(0) > \lim_{k \to \infty} I(u_k) = 0,$$

so that I is not weakly* lower semicontinuous on $W^{1,\infty}(0,1)$ and, in fact, no minimizer exists in this case.

Indeed, $0 = \inf_{\mathcal{Y}} I \neq \min_{\mathcal{Y}} I$ because $I \ge 0$ and $I(u_k) \to 0$, so that $0 = \inf_{\mathcal{Y}} I$. However, $I(u) > 0$ for every $u \in \mathcal{Y}$, for otherwise we could find a Lipschitz function whose derivative is ± 1 a.e. on $(0,1)$ but whose function value is identically zero.

If, however, we consider a slight modification of $\mathcal{Y}$ by changing the boundary condition at $x = 1$, and we consider

$$\mathcal{Y}_1 := \{u \in W^{1,\infty}(0,1); \ -1 \le u' \le 1, \ u(0) = 0, \ u(1) = 1\},$$

then $\min_{\mathcal{Y}_1} I = 1/3$ and the unique minimizer is $u(x) = x$ for $x \in (0,1)$.

First, this shows that weak/weak* lower semicontinuity of I is not necessary for the existence of a minimizer, and second, it stresses the influence of boundary conditions on the solvability of (1.1)*. This phenomenon is even more pronounced in higher dimensions.*

Although the study of weak lower semicontinuity is motivated by the desire to understand minimization problems, it has become an independent subject in mathematical literature that is studied in its own right. In the case of integral functionals as in (1.2), further properties of the integrand besides continuity are needed to assure weak/weak* lower semicontinuity: the right additional property is always some type of *convexity* of f. Indeed, notice that I in Example 1.1 is not convex.

The importance of convexity for weak/weak* lower semicontinuity for integral functionals was discovered in 1920 by Tonelli [233], who pioneered the study of lower semicontinuity of an integral functional rather than studying the associated Euler–Lagrange equation. Tonelli considered a function $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in (1.2) that is twice continuously differentiable and showed that I is lower semicontinuous sub-

ject to a "convergence of curves"[1] if and only if f is convex in its last variable, i.e., in the derivative u'. Later, several authors generalized this result to functions in $W^{1,\infty}(\Omega;\mathbb{R})$ with $\Omega \subset \mathbb{R}^n$ and $n > 1$; see, for example, Serrin [215], where differentiability properties of f were removed from assumptions and f was only assumed to be continuous, and Marcellini and Sbordone [168], who allowed for Carathéodory integrands.[2] Similar to this one-dimensional situation, relaxing smoothness/continuity assumptions of f will be a recurring theme throughout this review in which we focus on the higher-dimensional case.

Let us now allow the function u to be vector-valued, i.e., $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$ with $\Omega \subset \mathbb{R}^n$ and $n > 1$ as well as $m > 1$, and consider an integral functional of the form

$$(1.7) \qquad I(u) := \int_\Omega f(x, u(x), \nabla u(x))\, \mathrm{d}x \ .$$

In this case, the convexity hypothesis turns out to be sufficient for weak/weak* lower semicontinuity, but unnecessary. A suitable condition, termed quasiconvexity, was introduced by Morrey [178].

DEFINITION 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with the Lebesgue measure $\mathcal{L}^n(\Omega)$. A function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is quasiconvex at $A \in \mathbb{R}^{m \times n}$ if for every $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$,*

$$(1.8) \qquad f(A)\mathcal{L}^n(\Omega) \le \int_\Omega f(A + \nabla \varphi(x))\, \mathrm{d}x \ .$$

The function f is termed quasiconvex if it is quasiconvex in all $A \in \mathbb{R}^{m \times n}$.

Quasiconvexity is implied by convexity and can be understood as, roughly speaking, convexity over gradients. Indeed, take a convex function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$. Then for some arbitrary $A \in \mathbb{R}^{m \times n}$ fixed and every $B \in \mathbb{R}^{m \times n}$, we know that $f(A + B) \ge f(A) + g(A) \cdot B$; i.e., we can find an affine function that touches f at A and whose values are not greater than f (in fact, this can be found by taking $g(\cdot)$ to be one element of the subdifferential of f). Let us now take some arbitrary $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$ and plug in $\nabla \varphi(x)$ for B and take an average of the inequality over Ω to obtain that

$$\frac{1}{\mathcal{L}^n(\Omega)}\int_\Omega f(A + \nabla\varphi(x))\,\mathrm{d}x \ge \frac{1}{\mathcal{L}^n(\Omega)}\int_\Omega f(A)\,\mathrm{d}x + \frac{1}{\mathcal{L}^n(\Omega)}\int_\Omega g(A)\cdot\nabla\varphi(x)\,\mathrm{d}x \ ,$$

where the last integral vanishes due to integration by parts because $\varphi = 0$ on $\partial\Omega$, so that we truly obtain (1.8). We also note that quasiconvex functions are continuous [67].

Morrey showed, under strong regularity assumptions on f, that I from (1.7) is weakly lower semicontinuous in $W^{1,\infty}(\Omega;\mathbb{R}^m)$ if and only if f is quasiconvex in the last variable (i.e., in the gradient). To see how quasiconvexity is used in the proof of lower semicontinuity, let us consider the following simplified example.

EXAMPLE 1.2. *Assume that $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ is such that $u_k \overset{*}{\rightharpoonup} u$ with $u(x) = Ax$ for some fixed matrix $A \in \mathbb{R}^{m \times n}$. We show how in this case weak* lower semicontinuity on $W^{1,\infty}(\Omega;\mathbb{R}^m)$ is obtained for*

$$\tilde{I}(u) := \int_\Omega f(\nabla u(x))\mathrm{d}x$$

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[1]Notice that the notion of weak topology was invented later than Tonelli's studies.

[2]In other words, $f(x,\cdot,\cdot)$ is continuous for almost all $x \in \Omega$ and $f(\cdot, s, A)$ is measurable for all $(s, A) \in \mathbb{R} \times \mathbb{R}$.

for $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ *quasiconvex. To this end, let us take a smooth cut-off function* $\eta_\ell : \Omega \to \mathbb{R}$ *such that* $\eta_\ell = 1$ *on* Ω_ℓ *and* $\eta_\ell = 0$ *on* $\partial\Omega$, *where* $\Omega_\ell \subset \Omega$ *is a Lipschitz domain satisfying* $\mathcal{L}^n(\Omega \setminus \Omega_\ell) \leq \frac{1}{\ell}$. *We may find* η_ℓ *in such a way that* $|\nabla\eta_\ell| \leq C\ell$ *uniformly on* Ω, *where* C *is a constant that depends only on* Ω. *Let us now define*

$$u_{k,\ell}(x) = \eta_\ell u_k + (1-\eta_\ell)Ax \quad \text{so that} \quad \nabla u_{k,\ell}(x) = \eta_\ell \nabla u_k + (1-\eta_\ell)A + (u_k - Ax) \otimes \nabla\eta_\ell;$$

notice that $u_{k,\ell}$ *coincides with* u_k *on* Ω_ℓ. *Now, since* $u_k \to u$ *strongly in* $L^\infty(\Omega; \mathbb{R}^m)$, *we may choose a subsequence of* k's, *labeled* $k(\ell)$, *such that* $(u_{k(\ell)} - Ax) \otimes \nabla\eta_\ell$ *stays uniformly bounded (whence* $u_{k(\ell),\ell}$ *is bounded in* $W^{1,\infty}(\Omega; \mathbb{R}^m)$). *Due to the fact that* $u_{k(\ell),\ell}(x) = Ax$ *on* $\partial\Omega$ *we find from* (1.8) *that*

$$
\begin{aligned}
(1.9) \qquad f(A)\mathcal{L}^n(\Omega) &\leq \int_\Omega f(\nabla u_{k(\ell),\ell}(x))\mathrm{d}x \\
&= \int_\Omega f(\nabla u_{k(\ell)}(x))\mathrm{d}x + \int_{\Omega \setminus \Omega_\ell} f(\nabla u_{k(\ell),\ell}(x)) - f(\nabla u_{k(\ell)}(x))\mathrm{d}x.
\end{aligned}
$$

As f *is continuous and* $\{\nabla u_{k(\ell),\ell}\}_{\ell \in \mathbb{N}}$ *is uniformly bounded on* Ω, *so is* $f(\nabla u_{k(\ell),\ell}) - f(\nabla u_{k(\ell)})$, *and thus the last integral in* (1.9) *vanishes as* $\ell \to \infty$. *Therefore, taking the limit* $\ell \to \infty$ *yields the claim.*

The results of Morrey were generalized more than fifty years ago, in 1965, by Norman G. Meyers in his seminal paper [171]. Taking $k \in \mathbb{N}$ and $1 \leq p \leq +\infty$ he investigated the $W^{k,p}$-weak (weak* if $p = +\infty$) lower semicontinuity of integral functionals of the form

$$(1.10) \qquad I(u) := \int_\Omega f(x, u(x), \nabla u(x), \ldots, \nabla^k u(x)) \, \mathrm{d}x \;,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u : \Omega \to \mathbb{R}^m$ is a mapping possessing (weak) derivatives up to the order $k \in \mathbb{N}$. The function f was supposed to be continuous in all its arguments. Since higher gradients (and not only the first ones) are now considered, the definition of quasiconvexity also needs to be generalized accordingly; see section 3 for details.

Moreover, more generally than in Morrey's work, the function f is not necessarily bounded from below in [171]. This leads to additional difficulties and, in fact, quasiconvexity is no longer a sufficient condition for weak lower semicontinuity (cf. section 3). In addition, the regularity assumptions on the integrand in (1.10) were weakened in Meyers' work.

The motivation for studying functionals of the type (1.10) is twofold: from the point of view of applications in continuum mechanics it is reasonable to let f depend also on higher-order gradients since their appearance in the energy usually models interfacial energies or multipolar elastic materials [111]. Another reason might be to consider deformation gradient dependent surface loads [19]. On the other hand, not assuming a constant lower bound on f is an important consideration for mathematical completeness. Additionally, integrands of the type $f(A) := \det A$, which are unbounded from below, are of crucial importance in continuum mechanics.

Meyers' main results are necessary and sufficient conditions on f so that I is weakly lower semicontinuous on $W^{k,p}(\Omega; \mathbb{R}^m)$. We review these results in section 3. He first discusses the problem on $W^{k,\infty}(\Omega; \mathbb{R}^m)$, where quasiconvexity in the highest-order gradient (cf. Theorem 3.2) turns out to be a necessary and sufficient condition for weak*-lower semicontinuity. Lower semicontinuity on $W^{k,p}(\Omega; \mathbb{R}^m)$ with $1 < p < +\infty$

is, however, much more subtle, and an additional condition (cf. Theorem 3.4 and section 3.1) is needed.

Since the appearance of Meyers' work, significant progress has been achieved with respect to the characterization of weak lower semicontinuity of functionals of the type (1.10). In particular, for $k = 1$ in (1.10) the additional condition for sequential weak lower semicontinuity was characterized more explicitly and results relaxing Meyers' continuity assumptions were obtained for functionals bounded from below; cf. section 3.

Moreover, those functions f whose functional I in (1.10) is even weakly continuous (see section 4) have been identified: the so-called null Lagrangians; this knowledge led to the notion of polyconvexity (see section 6) that is sufficient for weak lower semicontinuity and is of particular importance in mathematical elasticity. In fact, quasiconvexity, which is, for a large class of integrands, the necessary and sufficient condition for weak lower semicontinuity, is not well suited for elasticity. We explain this issue in section 7 and review some recent progress in this field. Null Lagrangians have also been identified for functionals defined on the boundary (see section 5). Finally, we review weak lower semicontinuity results for functionals depending on maps that satisfy general differential constraints in section 8 and conclude with some suggestions for further reading in section 9.

2. Notation. In this section, we summarize the notation that is used throughout the article. It largely coincides with that in [19]. In what follows, $\Omega \subset \mathbb{R}^n$ is a bounded domain whose boundary is Lipschitz or smoother. This domain is mapped to a set in $\mathbb{R}^m$ by means of a mapping $u : \Omega \to \mathbb{R}^m$.

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $J := (j_1, \ldots, j_n) \in \mathbb{N}_0^n$ and $K := (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ are two multi-indices, we define $J \pm K := (j_1 \pm k_1, \ldots, j_n \pm k_n)$; further, $|J| = \sum_{i=1}^n j_i$, $J! := \Pi_{i=1}^n j_i!$ and we say that $J \leq K$ if $j_i \leq k_i$ for all i. We also define $\binom{J}{K} := \frac{J!}{K!(K-J)!}$, $\partial u_K^j := \frac{\partial^{k_1} \ldots \partial^{k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} u^j$, $x_K = x^K := x_1^{k_1} \ldots x_n^{k_n}$, and $(-D)^K := \frac{(-\partial)^{k_1} \ldots (-\partial)^{k_n}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$.

We will work with the space of higher-order matrices $X = X(n, m, k)$ with the dimension $m\binom{n+k-1}{k}$. This space consists of (higher-order) matrices $M = (M_K^i)$ for $1 \leq i \leq m$ and $|K| = k$. Similarly, $Y = Y(n, m, k)$ is the space of (higher-order) matrices $M = (M_K^i)$ for $1 \leq i \leq m$ and $|K| \leq k$. Its dimension is $m\binom{n+k}{k}$. We denote the elements of $X(n, m, k)$ by A^k, while $A^{[k]} = (A, A^2, \ldots, A^k)$ is an element of $Y(n, m, k)$. We use an analogous notation for gradients; thus, if $x \in \Omega$, then $\nabla^k u(x) \in X(n, m, k)$, while $\nabla^{[k]} u(x) \in Y(n, m, k)$.

Integrands which define the integral functional will be denoted by f. They will depend on x, u, and (higher-order) gradients of u, in general. Occasionally, we will work with integrands independent of u or x; however, this will be clear from the context and will not cause any ambiguity. We denote by $B(x_0, r)$ the ball of origin x_0 with the radius r, while $D_\varrho(x_0, r)$ is the half-ball with ϱ being the normal of the planar component of its boundary; i.e.,

$$D_\varrho(x_0, r) := \{x \in B(x_0, r) : (x - x_0) \cdot \varrho < 0\},$$

and we write $D_\varrho := D_\varrho(0, 1)$.

For this review, we will assume that the reader is familiar with functional analysis and measure theory, in particular, the theory of Lebesgue and Sobolev spaces, and we refer, for example, to the books by Evans and Gariepy [84], Rudin [205], Leoni [162], and Roubíček [204], for an introduction. We shall use the standard notation for the

Lebesgue spaces $L^p(\Omega; \mathbb{R}^m)$ and Sobolev spaces $W^{k,p}(\Omega; \mathbb{R}^m)$. Moreover, $\mathrm{BV}(\Omega; \mathbb{R}^m)$ is the space of functions of a bounded variation. If $m = 1$, we may omit the target space. If Ω is a bounded open domain, we denote by $\mathcal{M}(\Omega)$ the space of Radon measures on Ω and by $\mathcal{L}^n(\Omega)$ the n-dimensional Lebesgue measure of Ω; cf., e.g., Halmos [116]. Further, $\mathcal{M}^1_+(\mathbb{R}^{m \times n})$ is the set of probability measures on $\mathbb{R}^{m \times n}$. Moreover, $\mathcal{D}(\Omega)$ is the space of infinitely differentiable functions with compact support in Ω and its dual $\mathcal{D}'(\Omega)$ is the space of distributions.

If $n = m = 3$ and $F \in \mathbb{R}^{3 \times 3}$, the cofactor matrix $\mathrm{Cof}\, F \in \mathbb{R}^{3 \times 3}$ is a matrix whose entries are signed subdeterminants of 2×2 submatrices of F. More precisely, $[\mathrm{Cof}\, F]_{ij} := (-1)^{i+j} \det F'_{ij}$, where F'_{ij} for $i, j \in \{1, 2, 3\}$ is a submatrix of F obtained by removing the ith row and jth column. If F is invertible, we have $\mathrm{Cof}\, F = (\det F) F^{-\top}$. Rotation matrices with determinants equal to one are denoted $\mathrm{SO}(n)$, while orthogonal matrices with determinants ± 1 are denoted $\mathrm{O}(n)$. Additional notation needed locally in the text will be explained as necessary.

3. A Review of Meyers' Results. Within this section we review the results of Meyers' seminal paper [171] and give generalizations of his results that have been proved since the appearance of his work. As highlighted above, Meyers generalized Morrey's results [178] in two particular respects: First, he considers integral functionals of the type (1.10)

$$I(u) := \int_\Omega f(x, u(x), \nabla u(x), \dots, \nabla^k u(x)) \, \mathrm{d}x \ ,$$

i.e., those that also depend on higher gradients, and second, he allows for f to be unbounded from below. Now if (1.10) depends on higher gradients, the definition of quasi-convexity also needs to be generalized accordingly.

DEFINITION 3.1. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain. We say that a function* $f : X(n, m, k) \to \mathbb{R}$ *is* k-*quasiconvex[3] if for every* $A \in X(n, m, k)$ *and any* $\varphi \in W^{k,\infty}_0(\Omega; \mathbb{R}^m)$,

(3.1) $$f(A) \mathcal{L}^n(\Omega) \leq \int_\Omega f(A + \nabla^k \varphi(x)) \, \mathrm{d}x \ .$$

Thus, more precisely, k-quasiconvexity of f (i.e., quasiconvexity with respect to the kth gradient) means that $A^k \mapsto f(x, A^{[k-1]}, A^k)$ is quasiconvex for all fixed $(x, A^{[k-1]}) \in \Omega \times Y(m, n, k-1)$; cf. section 2 for notation.

REMARK 3.1. *In fact, it was shown in [70] that if* $k = 2$ *and if* f *satisfies a (slightly) stronger version of* 2-*quasiconvexity, then* 2-*quasiconvexity coincides with* 1-*quasiconvexity. See [58] for an analogous result with general* k.

With this definition at hand, Meyers proves an analogous result to the one found in the original work of Morrey for $k = 1$ [178].

THEOREM 3.2 (from [171]). *Let* Ω *be a bounded domain and* f *a continuous function. Then* I *from (1.10) is weakly* lower semicontinuous on* $W^{k,\infty}(\Omega; \mathbb{R}^m)$ *if and only if it is* k-*quasiconvex in the last variable.*

Nevertheless, when it comes to the case of $W^{k,p}(\Omega; \mathbb{R}^m)$ with $1 < p < +\infty$, the situation is substantially more involved, particularly because the considered integrands are not from below. In fact, as can be seen from the definition of the class $\mathcal{F}_p(\Omega)$

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[3] In the original paper [171], quasiconvexity with respect to the kth gradient is also referred to as quasiconvexity.

below, Meyers studies weak lower semicontinuity of (1.10) on a fairly general class of integrands including those with critical negative growth.

DEFINITION 3.3 (class $\mathcal{F}_p(\Omega)$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. A continuous integrand $f : \Omega \times Y(n,m,k) \to \mathbb{R}$ is said to be in the class $\mathcal{F}_p(\Omega)$ for $1 \le p < +\infty$ if ($C > 0$ is a constant depending only on f)*
  (i) $f(x, A^{[k]}) \le C\big(1 + |A^{[k]}|\big)^p$,
  (ii) $|f(x, A^{[k]} + B^{[k]}) - f(x, A^{[k]})| \le C\big(1 + |A^{[k]}| + |B^{[k]}|\big)^{p-\gamma}|B^{[k]}|^\gamma$ *for some* $0 < \gamma \le 1$,
  (iii) $|f(x+y, A^{[k]}) - f(x, A^{[k]})| \le (1 + |A^{[k]}|)^p\eta(|y|)$ *with* $\eta : [0; +\infty) \to [0; +\infty)$ *continuous, increasing, and vanishing at zero.*

REMARK 3.2 (class $\mathcal{F}_p(\Omega)$ for $k = 1$). *Let us, for clarity, repeat the conditions given in Definition 3.3 for the case $k = 1$. In this case, the notation is much simpler so that the important features of functions in the class $\mathcal{F}_p(\Omega)$ can be seen more easily.*
   We say that $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ is in the class $\mathcal{F}_p(\Omega)$ for $1 \le p < +\infty$ if
  (i) $f(x, s, A) \le C\big(1 + |s| + |A|\big)^p$,
  (ii) $|f(x, s+r, A+B) - f(x, s, A)| \le C\big(1 + |s| + |r| + |A| + |B|\big)^{p-\gamma}(|r| + |B|)^\gamma$ *for some $0 < \gamma \le 1$,*
  (iii) $|f(x+y, s, A) - f(x, s, A)| \le (1 + |s| + |A|)^p\eta(|y|)$ *with* $\eta : [0; +\infty) \to [0; +\infty)$ *continuous, increasing, and vanishing at zero.*
Above, $C > 0$ is a constant depending on f.

When setting $A^{[k]} = 0$ in (ii) in Definition 3.3 (or alternatively $s = 0$ and $A = 0$ in (ii) of Remark 3.2) we find that $|f(x, B^{[k]})| \le C(1 + |B^{[k]}|)^p$ and thus the class $\mathcal{F}_p(\Omega)$ also contains noncoercive integrands and, in particular, those which decay as $A \mapsto -|A|^p$. Quasiconvexity is not sufficient to prove sequential weak lower semicontinuity for such integrands. We shall devote section 3.1 to a detailed discussion of this issue and we state at this point Meyers' original theorem, which handles the noncoercivity of f by introducing an additional condition (item (ii) in Theorem 3.4).

THEOREM 3.4. *Let Ω be a bounded domain and $f \in \mathcal{F}_p(\Omega)$. Then I from (1.10) is weakly lower semicontinuous on $W^{k,p}(\Omega; \mathbb{R}^m)$ with $1 \le p < \infty$ if and only if the following two conditions hold simultaneously:*
  (i) *$f(x, A^{[k-1]}, \cdot)$ is k-quasiconvex for all values of $(x, A^{[k-1]})$;*
  (ii) *$\liminf_{j \to \infty} I(u_j, \Omega') \ge -\mu(\mathcal{L}^n(\Omega'))$ for every subdomain $\Omega' \subset \Omega$ and every sequence $\{u_j\}_{j \in \mathbb{N}} \subset W^{k,p}(\Omega; \mathbb{R}^m)$ such that $u_j = u$ on $\Omega \setminus \Omega'$ and $u_j \rightharpoonup u$ in $W^{k,p}(\Omega; \mathbb{R}^m)$. Here, μ is an increasing continuous function with $\mu(0) = 0$ which only depends on u and on $\limsup_{j \to \infty} \|u_j\|_{W^{k,p}(\Omega; \mathbb{R}^m)}$.*

Above, $I(\cdot, \Omega')$ denotes the functional I when the integration domain Ω is replaced by Ω'. We immediately see that condition (ii) is satisfied if f has a lower bound; for example, if $f \ge 0$. This is a very common case, in which Theorem 3.4 can be sharpened; we refer to section 3.2 where this situation is handled in detail.

REMARK 3.3 (Theorem 3.4 for $k = 1$). *If $k = 1$ and $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R}$ is in $\mathcal{F}_p(\Omega)$ in the sense of Remark 3.2, then Theorem 3.4 assures that the functional I from (1.7) is lower semicontinuous in $W^{1,p}(\Omega; \mathbb{R}^m)$ if f is quasiconvex in the last variable and condition (ii) in Theorem 3.4 is fulfilled with $k = 1$.*

3.1. Understanding Condition (ii) in Theorem 3.4. Condition (ii) in Theorem 3.4 is rather implicit and thus hard to verify. Nevertheless, in this section, we will suggest and demonstrate that it should be linked to concentrations on the boundary

of the domain. To our best knowledge, this link has been fully developed only in the case $k = 1$ and for integrands $f(x, u, \nabla u) := f(x, \nabla u)$ in (1.7). Thus, we will limit our scope to this particular case and then describe some possible extensions at the end of the section.

In essence, (ii) in Theorem 3.4 must cope with the potential nonequi-integrability of the negative part of the integrand f. To explain this statement in more detail, let us start with the definition of equi-integrability.

DEFINITION 3.5. *We say that a sequence of functions $\{\varphi_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ is equi-integrable if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $\omega \subset \Omega$ with $\mathcal{L}^n(\omega) \leq \delta$ it holds that*

$$\sup_{k\in\mathbb{N}} \int_\omega |\varphi_k(x)|\mathrm{d}x \leq \varepsilon.$$

As $L^1(\Omega)$ is not reflexive, a bounded sequence in $L^1(\Omega)$ does not necessarily contain a weakly convergent subsequence in $L^1(\Omega)$ (though it will always contain a subsequence weakly* convergent in measures), but it follows from the Dunford–Pettis criterion [79, 92] that this measure is an L^1 function and the convergence improves from weak* in measures to the weak one in L^1 if and only if the sequence is equi-integrable. Since the failure of equi-integrability is caused by concentrations of the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$, we say that a sequence bounded in $L^1(\Omega)$ is concentrating if it converges weak* in measures but not weakly in $L^1(\Omega)$.

Recall that two effects may cause a sequence $\{u_n\}_{n\in N} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ to converge weakly but not strongly to some limit function u: *oscillations* and *concentrations*. Here, concentrations are understood in the sense that $|u_n|^p$ is a concentrating sequence. In fact, it can be seen by Vitali's convergence theorem that if $|u_n|^p$ is equi-integrable (i.e., concentrations are excluded) and $u_n \to u$ a.e. in Ω (i.e., oscillations are excluded), $\{u_n\}_{n\in N}$ actually converges strongly to u in $W^{1,p}(\Omega;\mathbb{R}^m)$.

Concentrations and oscillations in a sequence $\{u_n\}_{n\in N} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ can be separated from each other by the so-called decomposition lemma due to Kristensen [144] and Fonseca, Müller, and Pedregal [93].

LEMMA 3.6 (decomposition lemma). *Let $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be bounded. Then there is a subsequence $\{u_j\}_{j\in\mathbb{N}}$ and a sequence $\{z_j\}_{j\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$(3.2) \qquad \lim_{j\to\infty} \mathcal{L}^n(\{x \in \Omega;\ z_j(x) \neq u_j(x)\ or\ \nabla z_j(x) \neq \nabla u_j(x)\}) = 0$$

and $\{|\nabla z_j|^p\}_{j\in\mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$.

This lemma, proved by means of the notion of maximal functions [220], allows us to find, for a general sequence bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$, another one, called $\{z_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, whose gradients are p-equi-integrable, i.e., for which $\{|\nabla z_j|^p\}$ is relatively weakly compact in $L^1(\Omega)$, making it a *purely oscillating* sequence. Thus, we decompose $u_j = z_j + w_j$ and $\{|\nabla w_j|^p\}_{j\in\mathbb{N}}$ tends to zero in measure for $j \to \infty$; i.e., it is a *purely concentrating* sequence. Roughly speaking, this means that for every weakly converging sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$, $p > 1$, we can decompose the sequence of gradients into a purely oscillating and a purely concentrating sequence. Note, however, that due to (3.2), this decomposition is very special. Notice that Lemma 3.6 is named after this decomposition.

Moreover, for *quasiconvex integrands* in (1.7) the effect of concentrations and oscillations also splits additively for the (nonlinear) functional I; i.e., we find for

(nonrelabeled) subsequences that

(3.3)

$$\lim_{j\to\infty}\int_\Omega f(x,\nabla u_j(x))\,\mathrm{d}x = \lim_{j\to\infty}\int_\Omega f(x,\nabla z_j(x))\,\mathrm{d}x + \lim_{j\to\infty}\int_\Omega f(x,\nabla w_j(x))\,\mathrm{d}x,$$

with $\{u_j\}, \{w_j\}$, and $\{z_j\}$ as introduced in Lemma 3.6 and the discussion thereafter. Relation (3.3) can be proved by exploiting the so-called p-Lipschitz continuity of quasiconvex functions by a straightforward technical calculation (see, e.g., [149]). The p-Lipschitz continuity asserts that if $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex and $|f| \le C(1+|\cdot|^p)$ for some $C > 0$ and $1 \le p < +\infty$, then there is a constant $\alpha \ge 0$ such that for all $A, B \in \mathbb{R}^{m\times n}$,

(3.4) $$|f(A) - f(B)| \le \alpha(1 + |A|^{p-1} + |B|^{p-1})|A - B| \ .$$

REMARK 3.4. *The p-Lipschitz continuity holds even if f is only separately convex, i.e., convex along the Cartesian axes in $\mathbb{R}^{m\times n}$. Various variants of this statement are proven, e.g., in [101, 167] and in [67]; an analogous result for k-quasiconvex functions also holds and can be found, e.g., in [113, 207]. It follows from (3.4) that quasiconvex functions satisfying the mentioned bound are locally Lipschitz.*

Owing to the decomposition lemma and the split (3.3), we may inspect lower semicontinuity of I in (1.7) along a sequence $\{u_j\}_{j\in\mathbb{N}}$ separately for the oscillating and the concentrating parts. Roughly speaking, the oscillating part is handled by quasiconvexity itself, while additional conditions are needed for the concentrating part. This statement is formalized via the following theorem.

THEOREM 3.7 (adapted from Kałamajska and Kružík [132]). *Let $f \in C(\overline{\Omega} \times \mathbb{R}^{m\times n})$, $|f| \le C(1+|\cdot|^p)$, $C > 0$, $f(x,\cdot)$ quasiconvex for all $x \in \overline{\Omega}$, and $1 < p < +\infty$. Then the functional*

(3.5) $$I(w) := \int_\Omega f(x,\nabla w(x))\,\mathrm{d}x$$

is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if for every bounded sequence $\{w_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\nabla w_j \to 0$ in measure we have $\liminf_{j\to\infty} I(w_j) \ge I(0)$.

Thus, let us study weak lower semicontinuity of I only along purely concentrating sequences, i.e., along a sequence $\{w_j\}_{j\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\nabla w_j\rightharpoonup 0$ and $\mathcal{L}^n(x \in \operatorname{supp} w_j) \to 0$ as $j \to \infty$. For simplicity, we set $f(\cdot,0) = 0$. Then, we can write

$$\int_\Omega f(x,\nabla w_j)\mathrm{d}x = \int_\Omega f^+(x,\nabla w_j)\mathrm{d}x - \int_\Omega f^-(x,\nabla w_j)\mathrm{d}x$$

$$\ge \int_\Omega f(x,0)\mathrm{d}x - \int_\Omega f^-(x,\nabla w_j)\mathrm{d}x,$$

where f^- and f^+ are the negative and the positive part of f, respectively. So, we see that lower semicontinuity of I along the sequence $\{w_j\}_{j\in\mathbb{N}}$ is obtained if $\int_\Omega f^-(x,\nabla w_j)\mathrm{d}x \to 0$. Recall from Definition 3.5 that this is always the case once the sequence $\{f^-(\cdot,\nabla w_j)\}_j$ is equi-integrable and we conclude that for quasiconvex integrands only the fact that $\{f^-(\cdot,\nabla w_j)\}_{j\in\mathbb{N}}$ *is a concentrating sequence might harm weak lower semicontinuity.* Notice that equi-integrability of $\{f^-(\cdot,\nabla u_j)\}_{j\in\mathbb{N}}$ can, for

example, be achieved if the negative part of f is of subcritical growth (cf. Theorem 3.8 below).

However, not all concentrations of $\{|\nabla u_j|^p\}_{j\in\mathbb{N}}$ affect the weak lower semicontinuity of I. In fact, we show in Remark 3.7 that concentrations inside the domain Ω are ineffectual for weak lower semicontinuity of I in (3.5) if $f(x,\cdot)$ is quasiconvex for all $x \in \overline{\Omega}$ and $f(\cdot, A)$ is continuous for all $A \in \mathbb{R}^{m\times n}$. Therefore, only concentrations *at the boundary* need to be excluded by further requirements, since along concentrating sequences of gradients, energy may be *gained* and hence the lower semicontinuity might be destroyed. The following examples show that such a situation does occur.

EXAMPLE 3.5 (following [148], [12]). *Choose* $\Omega = (0,1)$ *and a smooth, nonnegative function* $\Phi : \mathbb{R} \to \mathbb{R}$ *with compact support in* $(0,1)$ *and such that* $\int_0^1 \Phi(y)\mathrm{d}y = 1$. *Let us now define the sequence* $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,1}(0,1)$ *through*

$$u_n(x) = 1 - \int_0^x n\Phi\left(nt\right)\mathrm{d}t \qquad \text{so that} \qquad u_n'(x) = -n\Phi\left(nx\right).$$

It can be seen that $\{u_n\}_{n\in\mathbb{N}}$ *is a concentrating sequence that converges to* 0 *pointwise and in measure on* $(0,1)$. *Further, let us choose* $f(x,r,s) := s$ *in* (1.2)*; i.e.,* f *is a linear function and so quasiconvex. Then the functional* (1.2) *fulfills* $I(u_n) = -1$ *for all* n*, but* $u_n' \overset{*}{\rightharpoonup} 0$ *in measure and* $I(0) = 0 > -1$.

The example illustrates the above-mentioned effect that a sequence concentrating on the boundary (such as $\{u_n\}_{n\in\mathbb{N}}$) may actually lead to an energy gain in the limit. However, the failure of weak lower semicontinuity is shown with respect to the weak topology in measure for the derivative and not the weak convergence in $W^{1,1}(0,1)$. The reason is that this allows us to take a linear, and thus a particularly easy, integrand in (1.7), which is, however, of critical negative growth only in $W^{1,1}(0,1)$. But any sequence converging weakly in $W^{1,1}(0,1)$ is also equi-integrable, so the concentration effect could not be seen. Let us point to Example 3.6 below for appropriate nonlinear integrands that lead to the same effect in $W^{1,p}(\Omega;\mathbb{R}^m)$ with $p > 1$.

Let us also mention that the above example allows an easy adaptation to $\mathrm{BV}(0,1)$ that avoids the mollification kernel Φ. Take a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{BV}(0,1)$ defined through $u_n := \chi_{(0,\frac{1}{n})}$, i.e., the characteristic function of $(0,\frac{1}{n})$ in $(0,1)$, so that $Du_n = -\delta_{\frac{1}{n}}$. Then

$$I(u) = \int_{(0,1)} \mathrm{d}Du(x),$$

which is a BV-equivalent of (1.2) with $f(x,r,s) := s$, is not weakly* lower semicontinuous on $\mathrm{BV}(0,1)$ because $I(u_n) = -1$ for all n, but $u_n \overset{*}{\rightharpoonup} 0$ in $\mathrm{BV}(0,1)$ and $I(0) = 0 > -1$.

EXAMPLE 3.6 (see [25]). *Let* $n = m = p = 2$, $0 < a < 1$, $\Omega := (0,a)^2$, *and for* $x \in \Omega$ *define*

$$u_j(x_1, x_2) = \frac{1}{\sqrt{j}}(1 - |x_2|)^j\left(\sin(jx_1), \cos(jx_1)\right).$$

We see that $\{u_j\}_{j\in\mathbb{N}}$ *converges weakly in* $W^{1,n}(\Omega;\mathbb{R}^2)$ *as well as pointwise to zero. Moreover, we calculate for* $j \to \infty$

$$\int_0^a \int_0^a \det \nabla u_j(x)\,\mathrm{d}x \to \frac{-a}{2} < 0 \ .$$

Hence, we see that $I(u) := \int_\Omega \det \nabla u(x)\,\mathrm{d}x$ *is not weakly lower semicontinuous in* $W^{1,n}(\Omega;\mathbb{R}^2)$. *This example can be generalized to arbitrary dimensions* $m = n \geq 2$.

Indeed, take $u \in W_0^{1,n}(B(0,1); \mathbb{R}^n)$ and extend u by zero to the whole $\mathbb{R}^n$. We find that $\int_{B(0,1)} \det \nabla u(x) \, dx = 0$ because of the zero Dirichlet boundary conditions on $\partial B(0,1)$. Take $\varrho \in \mathbb{R}^n$, a unit vector, such that $\int_{D_\varrho} \det \nabla u(x) \, dx < 0$; here recall from section 2 that $D_\varrho := \{x \in \mathbb{R}^n; \ x \cdot \varrho < 0\}$. Notice that this condition can be fulfilled if we define u suitably.

Denote $u_j(x) := u(jx)$ for all $j \in \mathbb{N}$; then $u_j \rightharpoonup 0$ in $W^{1,n}(B(0,1); \mathbb{R}^n)$ but also $\int_{D_\varrho} \det \nabla u_j(x) \, dx \to \int_{D_\varrho} \det \nabla u(x) \, dx < 0$ by construction. The same conclusion can be drawn if we take $\Omega \subset \mathbb{R}^n$ with arbitrarily smooth boundary and such that $0 \in \partial\Omega$. Let ϱ be the outer unit normal to $\partial\Omega$ at zero. Then we have for the same sequence as before,

$$\lim_{j\to\infty} \int_\Omega \det \nabla u_j(x) \, dx = \lim_{j\to\infty} \int_{B(0,1)\cap\Omega} \det \nabla u_j(x) \, dx$$

$$= \lim_{j\to\infty} \int_{B(0,1)\cap\Omega} j^n \det \nabla u(jx) \, dx = \int_{D_\varrho} \det \nabla u(y) \, dy < 0 \ .$$

REMARK 3.7. *In this remark, we indicate why quasiconvexity is capable of preventing concentrations in the domain Ω from breaking weak lower semicontinuity. Indeed, let $\zeta \in \mathcal{D}(\Omega)$, $0 \le \zeta \le 1$ and take a quasiconvex function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ such that $|f(A)| \le C(1 + |A|^p)$ for some $C > 0$ and all $A \in \mathbb{R}^{m\times n}$ with $p > 1$. Moreover, let $\{w_j\}_{j\in\mathbb{N}}$ be a purely concentrating sequence. From Definition 1.3 for $A := 0$, we have that*

$$\mathcal{L}^n(\Omega) f(0) \le \int_\Omega f(\nabla(\zeta(x) w_j(x))) \, dx$$

and by using the chain rule, the p-Lipschitz property (3.4), and the facts that $w_j \to 0$ strongly in $L^p(\Omega; \mathbb{R}^n)$ and $\{\nabla w_j\}_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega; \mathbb{R}^{m\times n})$, we find that

$$(3.6) \qquad \mathcal{L}^n(\Omega) f(0) \le \liminf_{j\to\infty} \int_\Omega f(\zeta(x) \nabla w_j(x)) \, dx \ .$$

Let $|\nabla w_j|^p \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{M}(\overline{\Omega})$ for a (nonrelabeled) subsequence. Given the assumption that all concentrations appear inside the domain Ω, we have that $\sigma(\partial\Omega) = 0$, whence we continue with the estimate

$$\lim_{j\to\infty} \int_\Omega f(\zeta(x)\nabla w_j(x)) \, dx$$

$$\le \lim_{j\to\infty} \int_\Omega f(\nabla w_j(x)) + \alpha(1 - \zeta(x))(1 + \zeta^{p-1}(x))|\nabla w_j(x)|^p + \alpha(1 - \zeta(x))|\nabla w_j(x)| \, dx$$

$$= \lim_{j\to\infty} \int_\Omega f(\nabla w_j(x)) \, dx + \alpha \int_\Omega (1 - \zeta(x))(1 + \zeta^{p-1}(x)) \sigma(dx),$$

$$(3.7)$$

where we have again used the p-Lipschitz property. Now, we choose a sequence $\{\zeta_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\Omega)$, satisfying $0 \le \zeta_j \le 1$ that tends pointwise to the characteristic function of Ω, χ_Ω, σ-a.e. Taking into account (3.6) and (3.7), we have by Lebesgue's dominated convergence theorem,

$$\mathcal{L}^n(\Omega) f(0) \le \lim_{j\to\infty} \int_\Omega f(\nabla w_j(x)) \, dx \ .$$

Hence, weak lower semicontinuity is preserved.

The reasoning of Remark 3.7, however, clearly breaks down if $\partial\Omega$ is not a σ-null set, hence concentrations at the boundary appear. Nevertheless, not every boundary concentration is fatal for weak lower semicontinuity. Arguing heuristically, concentrations at $\partial\Omega$ are influenced by interior concentrations coming from Ω and exterior ones coming from the complement. If exterior concentrations can be excluded, then the interior ones cannot spoil weak lower semicontinuity, which is, roughly speaking, why Dirichlet boundary conditions suffice to ensure (ii) in Theorem 3.4 at least if $k = 1$. If periodic boundary conditions are applicable, then they will do, as well, because exterior and interior concentrations mutually compensate due to periodicity.

The next theorem formalizes the discussion concerning equi-integrability of the negative part of f and Dirichlet boundary conditions.

THEOREM 3.8 (taken from Kałamajska and Kružík [132]). *Let the assumptions of Theorem* 3.7 *hold. Further, let* $\{u_j\} \subset W^{1,p}(\Omega; \mathbb{R}^m)$, $u_j \rightharpoonup u$ *in* $W^{1,p}(\Omega; \mathbb{R}^m)$, *and let at least one of the following conditions be satisfied:*
  (i) *for every subsequence of* $\{u_j\}_{j\in\mathbb{N}}$ *(not relabeled) such that* $|\nabla u_j|^p \overset{*}{\rightharpoonup} \sigma$ *in* $\mathcal{M}(\overline{\Omega})$, *where* $\sigma \in \mathcal{M}(\overline{\Omega})$ *depends on the particular subsequence, it holds that* $\sigma(\partial\Omega) = 0$;
  (ii) $\lim_{|A|\to\infty} \frac{f^-(x,A)}{1+|A|^p} = 0$ *for all* $x \in \overline{\Omega}$, *where* $f^- := \max\{0, -f\}$;
  (iii) $u_j = u$ *on* $\partial\Omega$ *for every* $j \in \mathbb{N}$ *and* Ω *is Lipschitz.*
Then $I(u) \leq \liminf_{j\to\infty} I(u_j)$.

Notice that (ii) is satisfied, for example, if $f \geq 0$ or if $f^- \leq C(1 + |\cdot|^q)$ for some $1 \leq q < p$, in which case $-C(1 + |A|^q) \leq f(x, A) \leq C(1 + |A|^p)$, $C > 0$, and $x \in \Omega$. This result can be found, e.g., in [67].

It follows from the discussion in this section that condition (ii) in Theorem 3.4 is connected with concentrations on the boundary. This must have been clear to Meyers, who conjectured [171, p. 146] that it can be dropped if $\partial\Omega$ is "smooth enough" or a "smooth enough" function is prescribed on the boundary *as the datum*. The second part of the conjecture turned out to be true in the following special cases: for $k = 1$ in (1.10) (see [171, Thm. 5] and Thm. 3.8) or if the integrand in (1.10) depends only on the highest gradient (see the end of section 8). However, the general case is still an open problem.

OPEN PROBLEM 3.9. *Is the functional* (1.10) *weakly lower semicontinuous along sequences with fixed Dirichlet boundary data if* f *is a general function in the class* $\mathcal{F}_p(\Omega)$ *that is* k-*quasiconvex?*

The first part of the conjecture of Meyers turned out *not* to hold, as is illustrated by Example 3.6 in which weak lower semicontinuity breaks down independently of the smoothness of $\partial\Omega$.

Let us return to the issue of making condition (ii) in Theorem 3.4 more explicit. It was identified in [149] that a suitable *growth* from below of the whole functional in (1.10) (which does not necessarily imply a lower bound on the integrand f itself) equivalently replaces this condition. First, let us illustrate that some form of boundedness from below is indeed necessary for weak lower semicontinuity.

EXAMPLE 3.8. *Take* $u \in W_0^{1,p}(B(0,1); \mathbb{R}^m)$ *(*$1 < p < \infty$*) and extend it by zero to the whole of* $\mathbb{R}^n$. *Define, for* $x \in \mathbb{R}^n$ *and* $j \in \mathbb{N}$, $u_j(x) = j^{\frac{n-p}{p}} u(jx)$ *and consider a smooth domain* $\Omega \subset \mathbb{R}^n$ *such that* $0 \in \partial\Omega$; *denote by* ϱ *the outer unit normal to* $\partial\Omega$ *at* 0. *Notice that* $u_j \rightharpoonup 0$ *in* $W^{1,p}(\Omega; \mathbb{R}^m)$ *and* $\{|\nabla u_j|^p\}_{j\in\mathbb{N}}$ *concentrates at zero. Moreover, take a function* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *that is positively* p-*homogeneous, i.e.,*

$f(\alpha\xi) = \alpha^p f(\xi)$ *for all* $\alpha \geq 0$. *If*

$$I(u) = \int_\Omega f(\nabla u(x)) \, \mathrm{d}x$$

is weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^m)$, *then*

$$(3.8) \quad 0 = I(0) \leq \liminf_{j\to\infty} \int_\Omega f(\nabla u_j(x)) \, \mathrm{d}x = \liminf_{j\to\infty} \int_{B(0,1/j)\cap\Omega} f(\nabla u_j(x)) \, \mathrm{d}x$$

$$= \liminf_{j\to\infty} \int_{B(0,1/j)\cap\Omega} j^n f(\nabla u(jx)) \, \mathrm{d}x = \int_{D_\varrho} f(\nabla u(y)) \, \mathrm{d}y \ .$$

Thus, we see that

$$(3.9) \qquad\qquad 0 \leq \int_{D_\varrho} f(\nabla u(y)) \, \mathrm{d}y$$

for all $u \in W_0^{1,p}(B(0,1); \mathbb{R}^m)$ *forms a necessary condition for weak lower semicontinuity of* I *whenever* f *is positively* p-*homogeneous.*

For functions that are not p-homogeneous, S. Krömer [149] generalized (3.9) as follows.

DEFINITION 3.10 (following [149][4]). *Assume that* $\Omega \subset \mathbb{R}^n$ *has a smooth boundary and let* $\varrho(x)$ *be the unit outer normal to* $\partial\Omega$ *at* x. *We say that a function* $f : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *is* p-quasi-subcritical growth from below *if for every* $x \in \partial\Omega$ *and for every* $\varepsilon > 0$, *there exists* $C_\varepsilon \geq 0$ *such that*

$$(3.10) \qquad \int_{D_{\varrho(x)}(x,1)} f(x, \nabla u(z)) \mathrm{d}z \geq -\varepsilon \int_{D_{\varrho(x)}(x,1)} |\nabla u(z)|^p \mathrm{d}z - C_\varepsilon$$
$$\textit{for all } u \in W_0^{1,p}(B(0,1); \mathbb{R}^m).$$

It was proved in [149] that the p-quasi-subcritical growth from below of the function $f := f(x, \nabla u)$ equivalently replaces (ii) in Theorem 3.4.

Notice that (3.10) is expressed only in terms of f and that it is local in x. Moreover, it shows again that, at least in the case when f depends only on the first gradient of u but not on u itself, only *concentrations at the boundary* may interfere with weak lower semicontinuity of functionals involving quasiconvex functions.

REMARK 3.9. *Let us realize that* (3.10) *implies* (3.9) *if it holds that* f *is positively* p-*homogeneous and independent of* x. *To this end, we use, for* $t \geq 0$, $u = t\tilde{u}$ *in* (3.10) *to show that*

$$0 \leq \frac{1}{t^p} \left(\int_{D_\varrho(x_0,1)} f(t\nabla\tilde{u}(x)) \, dx + \varepsilon |t\nabla\tilde{u}(x)|^p \mathrm{d}x + C_\varepsilon \right).$$

Letting $t \to \infty$ *gives that* $C_\varepsilon = 0$; *we may also let* $\varepsilon \to 0$ *to obtain* (3.9).

Since only concentration effects play a role for (ii) in Theorem 3.4, it is natural to expect that weak lower semicontinuity can be linked to properties of the so-called

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[4]In [149] this condition is actually not referred to as p-quasi-subcritical growth from below, but is introduced in Theorem 1.6 (ii).

recession function of the function f, if it admits one. Recall that we say that the function $f_\infty : \overline{\Omega} \times \mathbb{R}^{m \times n} \to \mathbb{R}$ is a recession function for $f : \overline{\Omega} \times \mathbb{R}^{m \times n} \to \mathbb{R}$ if for all $x \in \overline{\Omega}$,

$$\lim_{|A| \to \infty} \frac{f(x, A) - f_\infty(x, A)}{|A|^p} = 0.$$

Thus, informally speaking, the recession function describes the behavior of f at "infinitely large matrices." Note that f_∞ is necessarily positively p-homogeneous; i.e., $f_\infty(x, \lambda A) = \lambda^p f_\infty(x, A)$ for all $\lambda \geq 0$, all $x \in \overline{\Omega}$, and all $A \in \mathbb{R}^{m \times n}$.

It follows from Remark 3.9 in [149] that if f admits a recession function, then quasi-subcritical growth from below is equivalent to (3.9) for f_∞.

Since weak lower semicontinuity is connected to quasiconvexity and to condition (ii) in Theorem 3.4, which is connected to effects at the boundary, it is reasonable to ask whether the two ingredients can be combined. Indeed, so-called *quasiconvexity at the boundary* was introduced by Ball and Marsden [22] to study necessary conditions satisfied by local minimizers of variational problems—we also refer the reader to [108, 109, 174, 219, 222] where this condition is analyzed. In order to define quasiconvexity at the boundary, we put for $1 \leq p \leq +\infty$

$$(3.11) \qquad W^{1,p}_{\partial D_\varrho \setminus \Gamma_\varrho}(D_\varrho; \mathbb{R}^m) := \{u \in W^{1,p}(D_\varrho; \mathbb{R}^m); \ u = 0 \text{ on } \partial D_\varrho \setminus \Gamma_\varrho\},$$

where Γ_ϱ is the planar part of ∂D_ϱ.

DEFINITION 3.11 (taken from [174][5]). *Let $\varrho \in \mathbb{R}^n$ be a unit vector. A function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is called quasiconvex at the boundary at the point $A \in \mathbb{R}^{m \times n}$ with respect to ϱ if there is $q \in \mathbb{R}^m$ such that for all $\varphi \in W^{1,\infty}_{\partial D_\varrho \setminus \Gamma_\varrho}(D_\varrho; \mathbb{R}^m)$ it holds that*

$$(3.12) \qquad \int_{\Gamma_\varrho} q \cdot \varphi(x) \, \mathrm{d}S + f(A)\mathcal{L}^n(D_\varrho) \leq \int_{D_\varrho} f(A + \nabla\varphi(x)) \, \mathrm{d}x .$$

Let us remark that, analogously to quasiconvexity, we may generalize quasiconvexity at the boundary to $W^{1,p}$-quasiconvexity at the boundary (for $1 < p < \infty$) by using all $\varphi \in W^{1,p}_{\partial D_\varrho \setminus \Gamma_\varrho}(D_\varrho; \mathbb{R}^m)$ as test functions in (3.12). For functions with p-growth these two notions coincide.

REMARK 3.10. *Let us give some intuition on the above definition. Take a convex function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ and $\varphi \in W^{1,\infty}_{\partial D_\varrho \setminus \Gamma_\varrho}(D_\varrho; \mathbb{R}^m)$; then we know that*

$$f(A + \nabla\varphi(x)) \geq f(A) + g(A) \cdot \nabla\varphi(x),$$

where $g(A)$ is a subgradient of f evaluated at A; see, e.g., Rockafellar and Wets [202] for details about this notion. Integrating this expression over Ω then gives

$$\int_\Omega f(A + \nabla\varphi(x))\mathrm{d}x \geq \int_\Omega \big(f(A) + g(A) \cdot \nabla\varphi\big)\mathrm{d}x = \mathcal{L}^n(\Omega)f(A) + \int_{\partial\Omega} \big(g(A)\varrho\big) \cdot \varphi \, \mathrm{d}S,$$

where ϱ is the outer normal to $\partial\Omega$. Now, when setting $q := g(A)\varrho$ we obtain the definition of the quasiconvexity at the boundary.

REMARK 3.11. *It is possible to work with more general domains than half-balls in Definition 3.11, namely, with so-called* standard boundary domains. *We say that $\tilde{D}_\varrho$ is a standard boundary domain with the normal ϱ if there is $a \in \mathbb{R}^n$ such that*

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[5]The original definition in [22] considers the case $q := 0$.

$\tilde{D}_\varrho \subset H_{a,\varrho} := \{x \in \mathbb{R}^n;\ \varrho \cdot x < a\}$ and the $(n-1)$-dimensional interior of $\partial \tilde{D}_\varrho \cap \partial H_{a,\varrho}$, called Γ_ϱ, is nonempty. Roughly speaking, this means that the boundary of $\tilde{D}_\varrho$ should contain a planar part.

As with standard quasiconvexity, if (3.12) holds for one standard boundary domain, it holds for other standard boundary domains, too.

REMARK 3.12. If $p > 1$, and $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is positively p-homogeneous, continuous, and $W^{1,p}$-quasiconvex at the boundary at $(0, \varrho)$, then $q = 0$ in (3.12). Indeed, we have $f(0) = 0$ and suppose, by contradiction, that $\int_{D_\varrho} f(\nabla \varphi(x))\, dx < 0$ for some $\varphi \in W^{1,\infty}_{\partial D_\varrho \backslash \Gamma_\varrho}(D_\varrho; \mathbb{R}^m)$. By (3.12), we must have for all $\lambda > 0$

$$0 \le \lambda^p \int_{D_\varrho} f(\nabla \varphi(x))\, dx - \lambda \int_{\Gamma_\varrho} q \cdot \varphi(x)\, dS \ .$$

However, this is not possible for $\lambda > 0$ large enough and therefore for all $\varphi \in W^{1,\infty}_{\partial D_\varrho \backslash \Gamma_\varrho}(D_\varrho; \mathbb{R}^m)$ it has to hold that $\int_{D_\varrho} f(\nabla \varphi(x))\, dx \ge 0$. Thus, we can take $q = 0$.

From the above remark and from (3.9), we have the following lemma.

LEMMA 3.12. If a function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is $W^{1,p}$-quasiconvex at the boundary at zero and every $\varrho \in \mathbb{R}^n$, a unit normal vector to $\partial \Omega$, then it is also of p-subcritical growth from below. The two notions become equivalent if f is positively p-homogeneous. Here Ω must have a smooth boundary, so that the outer unit normal to it is defined everywhere.

All the results presented so far just concern the case $k = 1$ and integrands $f = f(x, \nabla u)$ in (1.7). In fact, in the general case in which $f = f(x, u, \nabla u)$ only a few results are available. One of them is, of course, Meyers' original Theorem 3.4 that applies to a general class of integrands. Another result is due to Ball and Zhang [27], who considered the following bound on a Carathéodory integrand f:

$$(3.13) \qquad\qquad |f(x, s, A)| \le a(x) + C(|s|^p + |A|^p) \ ,$$

where $C > 0$ and $a \in L^1(\Omega)$. Under (3.13), we cannot expect weak lower semi-continuity of I along generic sequences. Indeed, they proved the following weaker result.

THEOREM 3.13 (Ball and Zhang [27]). Let $1 \le p < +\infty$, $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$, let $f(x, s, \cdot)$ be quasiconvex for all $s \in \mathbb{R}^m$ and almost all $x \in \Omega$, and let (3.13) hold. Then there exists a sequence of sets $\{\Omega_j\}_{j \in \mathbb{N}} \subset \Omega$ satisfying $\Omega_{j+1} \subseteq \Omega_j$ for all $j \ge 1$ and $\lim_{j \to \infty} \mathcal{L}^n(\Omega_j) = 0$ such that for all $j \ge 1$,

$$(3.14) \qquad \int_{\Omega \backslash \Omega_j} f(x, u(x), \nabla u(x))\, dx \le \liminf_{k \to \infty} \int_{\Omega \backslash \Omega_j} f(x, u_k(x), \nabla u_k(x))\, dx \ .$$

The sets $\{\Omega_j\}$ that must be removed (or bitten off) from Ω are sets where possible concentration effects of the bounded sequence $\{|f(x, u_k, \nabla u_k)|\}_{k \in \mathbb{N}} \subset L^1(\Omega)$ take place. Thus, $\{\Omega_j\}$ depends on the sequence $\{u_k\}$ itself and Ω_j are not known a priori. Nevertheless, in fact, Ω_j depends just on the sequence of gradients. Indeed, (3.13) and the strong convergence of $\{u_k\}_{k \in \mathbb{N}}$ in $L^p(\Omega; \mathbb{R}^m)$ imply that whenever $\{|\nabla u_k|^p\}_{k \in \mathbb{N}}$ is equi-integrable, then the same holds for $\{|f(x, u_k(x), \nabla u_k(x))|\}_{k \in \mathbb{N}}$. The main tool of the proof of Theorem 3.13 is the biting lemma due to Chacon [55, 26].

LEMMA 3.14 (biting lemma). *Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable set. Let $\{z_k\}_{k \in \mathbb{N}} \subset L^1(\Omega; \mathbb{R}^m)$ be bounded. Then there is a (nonrelabeled) subsequence of $\{z_k\}_{k \in \mathbb{N}}$, $z \in L^1(\Omega; \mathbb{R}^m)$, and a sequence of sets $\{\Omega_j\}_{j \in \mathbb{N}} \subset \Omega$, $\Omega_{j+1} \subset \Omega_j$, $j \in \mathbb{N}$, with $\mathcal{L}^n(\Omega_j) \to 0$ for $j \to \infty$, such that $z_k \rightharpoonup z$ in $L^1(\Omega \setminus \Omega_j; \mathbb{R}^m)$ for $k \to \infty$ and every $j \in \mathbb{N}$.*

Finally, let us remark that concentration effects do not appear if we study lower semicontinuity of functionals with linear growth with respect to the weak $W^{k,1}(\Omega, \mathbb{R}^m)$-topology (see Remark 3.13 below). Nevertheless, this topology is too strong when it comes to the study of the existence of minimizers for such functionals; cf. the discussion at the end of section 3.2.

REMARK 3.13 (case $p = 1$). *Let us remark that if examining weak lower semicontinuity of integral functionals with linear growth along sequences converging weakly in $W^{k,1}(\Omega, \mathbb{R}^m)$, condition* (ii) *in Theorem 3.4 is also satisfied automatically. This follows from the fact that such sequences are already equi-integrable.*

3.2. Integrands Bounded from Below. In the previous section, we saw that characterizing weak lower semicontinuity of integral functionals with the integrand unbounded from below brings along many peculiarities if the negative part of the integrand is not equi-integrable. Naturally, all difficulties disappear if the integrand is bounded from below; notice, for example, that condition (ii) in Theorem 3.4 is automatically satisfied. Thus, all the results from the previous section are readily applicable in this situation, too. Yet, as the case $f \geq 0$ for an integrand in (1.10) is the most typical one found in applications, it is worth studying it independently. In fact, it is natural to expect that if f in (1.10) has a lower bound, one can strengthen Theorem 3.4 by relaxing the continuity assumptions stated in Definition 3.3. We review the available results in this section.

In the case $k = 1$ in (1.10), the following result due to Acerbi and Fusco [1] shows that the continuity assumption on the integrand can be replaced by the Carathéodory property.

THEOREM 3.15 (Acerbi and Fusco [1]). *Let $k = 1$, $\Omega \subset \mathbb{R}^n$ be an open, bounded set, and let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to [0; +\infty)$ be a Carathéodory integrand, i.e., $f(\cdot, s, A)$ is measurable for all $(s, A) \in \mathbb{R}^m \times \mathbb{R}^{m \times n}$ and $f(x, \cdot, \cdot)$ is continuous for almost all $x \in \Omega$. Further, let $f(x, s, \cdot)$ be quasiconvex for almost all $x \in \Omega$ and all $s \in \mathbb{R}^m$, and suppose that for some $C > 0$, $1 \leq p < +\infty$, and $a \in L^1(\Omega)$, we have that*[6]

$$(3.15) \qquad 0 \leq f(x, s, A) \leq a(x) + C(|s|^p + |A|^p) .$$

Then $I : W^{1,p}(\Omega; \mathbb{R}^m) \to [0; +\infty)$ given in (1.10) is weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^m)$.

Interestingly, the paper by Acerbi and Fusco [1] implicitly contains a version of the decomposition lemma, Lemma 3.6.

Marcellini [167] proved, by a different technique of constructing a suitable nondecreasing sequence of approximations, a very similar result to Theorem 3.15 allowing also for a slightly more general growth,

$$(3.16) \qquad -c_1|A|^r - c_2|s|^t - c_3(x) \leq f(x, s, A) \leq g(x, s)(1 + |A|^p),$$

where $c_1, c_2 \geq 0$, $c_3 \in L^1(\Omega)$; g is an arbitrary Carathéodory function and the expo-

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[6]This bound is often called "natural growth conditions."

nents satisfy that $p \geq 1$, $1 \leq r < p$ (but $r = 1$ if $p = 1$), and $1 \leq t < np/(n-p)$ if $p < n$ and $t \geq 1$ otherwise.

Note that the growth condition (3.16) actually allows for integrands unbounded from below, but the exponent r determining this growth is strictly smaller than p. Such integrands are of *subcritical growth* and for integrands of the class $\mathcal{F}_p(\Omega)$ weak lower semicontinuity under this growth also follows from Theorem 3.8(ii).

Acerbi and Fusco [1, p. 127] remarked that "*using more complicated notations as in* [19], [171], *our results can be extended to the case of functionals of the type* (1.10)." This extension was considered by Fusco [101] for the case $p = 1$ and later by Guidorzi and Poggilioni [113], who rewrote functional (1.10) as (using the notation from section 2)

$$(3.17) \qquad I(u) = \int_\Omega f(x, \nabla^{[k-1]} u(x), \nabla^k u(x)) \mathrm{d}x$$

and proved the following proposition.

PROPOSITION 3.16 (Guidorzi and Poggilioni [113]). *Let $f : \Omega \times Y(n, m, k-1) \times X(n, m, k) \to \mathbb{R}$ be a Carathéodory k-quasiconvex function satisfying, for all $H \in Y(n, m, k-1)$ and all $A \in X(n, m, k)$,*

$$0 \leq f(x, H, A) \leq g(x, H)(1 + |A|)^p,$$
$$|f(x, H, A) - f(x, H, B)| \leq C(1 + |A|^{p-1} + |B|^{p-1})|A - B|,$$

where g is a Carathéodory function and $C \geq 0$. Then the functional from (3.17) is weakly lower semicontinuous in $W^{k,p}(\Omega; \mathbb{R}^n)$ for $1 \leq p < \infty$ and $k \in \mathbb{N}$.

Note that in this result the continuity of the integrand in the space variable x could be omitted, which is, roughly speaking, due to the fact that quasiconvexity is sufficient to handle the concentration effects. On the other hand, the continuity assumption from Definition 3.3(ii) still remains (with $\gamma = 1$). A similar result can be drawn from the more general setting of $\mathcal{A}$-quasiconvexity (which we review in section 8 below) considered in [52].

Let us end this section with some remarks on weak lower semicontinuity of integral functionals on $W^{k,1}(\Omega; \mathbb{R}^m)$. We refer, e.g., to [102, 132] as well as section 9 for various further weak lower semicontinuity results with integrands bounded from below. While the above results also handle weak lower semicontinuity on $W^{k,1}(\Omega; \mathbb{R}^m)$ with respect to the standard weak convergence in this space, it is more suitable to investigate lower semicontinuity with respect to the strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^m)$. This is due to the fact that $W^{k,1}(\Omega; \mathbb{R}^m)$ is not reflexive and therefore coercivity of (1.10) does not allow us to select a minimizing sequence that would be weakly convergent in $W^{k,1}(\Omega; \mathbb{R}^m)$, but the strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^m)$ can be assured.

The case for $k = 1$ was treated by Fonseca and Müller [90], who considered continuous integrands under mild growth conditions. The result was later generalized by Fonseca et al. [89] with respect not only to the continuity of the integrand that could be partially dropped, but also to arbitrary k. We give the result for $k = 1$ in Theorem 3.17, while the general case is given in Theorem 3.18.

THEOREM 3.17 (due to Fonseca et al. [89]). *Let f in (1.7) be a Borel integrand that is continuous in the following sense: For all $\varepsilon > 0$ and $(x_0, s_0) \in \Omega \times \mathbb{R}^m$ there exist $\delta > 0$ and a modulus of continuity ω with the property that, for some $C > 0$, $\omega(t) \leq C(1 + t), t > 0$, such that*

$$f(x_0, s_0, A) - f(x, s, A) \leq \varepsilon(1 + f(x, s, A)) + \omega(|s_0 - s|)$$

for all $x \in \Omega$ satisfying $|x - x_0| \leq \delta$ and for all $s \in \mathbb{R}^m$ and all $A \in \mathbb{R}^{m \times n}$. Suppose further that f is quasiconvex and satisfies

$$0 \leq f(x_0, s, A) \leq c(1 + |A|) \qquad \forall A \in \mathbb{R}^{m \times n}$$

for some $c > 0$ or that f is convex in the last variable. Then, (1.7) is lower semicontinuous with respect to the strong convergence in $L^1(\Omega; \mathbb{R}^m)$.

THEOREM 3.18 (due to Fonseca et al. [89]). *Let f in (1.10) be a Borel integrand that is continuous in the following sense: For all $\varepsilon > 0$ and $(x_0, H_0) \in \Omega \times Y(n, m, k - 1)$ there exist $\delta > 0$ and a modulus of continuity ω with the property that, for some $C > 0$, $\omega(s) \leq C(1 + s), s > 0$, such that*

$$f(x_0, H_0, A) - f(x, H, A) \leq \varepsilon(1 + f(x, H, A)) + \omega(|H_0 - H|)$$

for all $x \in \Omega$ satisfying $|x - x_0| \leq \delta$ and for all $H \in Y(n, m, k - 1)$ and all $A \in X(n, m, k)$. Suppose further that f is k-quasiconvex and satisfies

$$\frac{1}{c}|A| - c \leq f(x_0, H_0, A) \leq c(1 + |A|),$$

for some $c > 0$ and all $A \in X(n, m, k)$. Then (1.10) is lower semicontinuous with respect to the strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^m)$.

For the functions $f : X(m, n, k) \to \mathbb{R}$, i.e., those depending only on the highest gradient, an analogous result has been obtained in [4]. We point the reader to the suggested further reading on integrals with linear growth in section 9.

4. Null Lagrangians. Having studied weak lower semicontinuity, let us turn our attention to conditions under which the functional (1.10) is actually *weakly continuous* on $W^{k,p}(\Omega; \mathbb{R}^m)$. As it will turn out, (1.10) is weakly continuous only for a small, special class of integrands f, the so-called *null Lagrangians* (cf. Theorem 4.3 below). Null Lagrangians are known explicitly and consist of, roughly speaking, minors of the highest-order gradient; we review their characterization in this section. Null Lagrangians play an important role in the calculus of variations, and notably they are at the heart of the definition of polyconvexity that is sufficient for weak lower semicontinuity (cf. section 6 for more details).

We start the discussion by presenting definitions of null Lagrangians of the first and higher orders.

DEFINITION 4.1. *We say that a continuous map $L : \mathbb{R}^{m \times n} \to \mathbb{R}$ is a null Lagrangian of the first order if for every $u \in C^1(\overline{\Omega}; \mathbb{R}^m)$ and every $\varphi \in C_0^1(\Omega; \mathbb{R}^m)$ it holds that*

$$(4.1) \qquad \int_\Omega L(\nabla u(x) + \nabla \varphi(x)) \, \mathrm{d}x = \int_\Omega L(\nabla u(x)) \, \mathrm{d}x \ .$$

Notice that the definition is independent of the particular Lipschitz domain Ω. In fact, if (4.1) holds for one domain Ω, it also holds for all other (Lipschitz) domains.

REMARK 4.1. *The name "null Lagrangians" comes from the fact that if L is smooth enough that the variations of $J(u) := \int_\Omega L(\nabla u(x)) \, \mathrm{d}x$ can be evaluated, it easily follows from (4.1) that J satisfies $J'(u) = 0$ for all $u \in C^1(\overline{\Omega}; \mathbb{R}^m)$. In other words, the Euler–Lagrange equations of J are fulfilled identically in the sense of distributions.*

REMARK 4.2. *Notice that if L is a null Lagrangian, the value of $J(u) = \int_\Omega L(\nabla u(x)) \, \mathrm{d}x$ is only dependent on the boundary values of u. This can be seen from (4.1) as the value remains unchanged even if we add arbitrary functions vanishing on the boundary.*

It is straightforward to generalize (4.1) to higher-order problems.

DEFINITION 4.2. *Let $k \geq 2$. We say that $L : X(n, m, k) \to \mathbb{R}$ is a (higher-order) null Lagrangian if*

$$(4.2) \qquad \int_\Omega L(\nabla^k u(x) + \nabla^k \varphi(x)) \, \mathrm{d}x = \int_\Omega L(\nabla^k u(x)) \, \mathrm{d}x$$

for all $u \in C^k(\overline{\Omega}; \mathbb{R}^m)$ and all $\varphi \in C_0^k(\Omega; \mathbb{R}^m)$.

Similar to the first-order gradient case, the definition is independent of the particular (Lipschitz) domain Ω. In the same way as in the first-order case, given sufficient smoothness, it follows that the Euler–Lagrange equations

$$(4.3) \qquad \sum_{|K| \leq l} (-D)^K \frac{\partial L}{\partial u_I^i}(\nabla^l u) = 0$$

are satisfied identically in the sense of distributions for arbitrary $u \in C^k(\overline{\Omega}; \mathbb{R}^m)$.

REMARK 4.3. *It is natural to generalize the notion of null Lagrangians to functionals of the type (1.10), i.e., those depending also on lower-order gradients, in the following way: We say that the function $L : \Omega \times Y(n, m, k) \to \mathbb{R}$ is a null Lagrangian for the functional (1.10) if for all $u \in C^k(\Omega; \mathbb{R}^m)$ and all $\varphi \in C_0^k(\Omega; \mathbb{R}^m)$ it holds that*

$$J(u + \varphi) = J(u) \qquad and \qquad J(u) = \int_\Omega L(x, u(x), \nabla u(x), \ldots, \nabla^k u(x)) \, \mathrm{d}x.$$

We shall see at the end of this section that null Lagrangians for these types of functionals are actually determined by null Lagrangians at least if $k = 1$.

The following result highlights some of the remarkable properties of null Lagrangians L of first and higher order. In particular, it shows that null Lagrangians, are the only integrands for which $u \mapsto \int_\Omega L(\nabla^k u(x)) \, \mathrm{d}x$ is continuous in the weak topology of suitable Sobolev spaces. It is due to Ball, Curie, and Olver [19].

THEOREM 4.3 (characterization of (higher-order) null Lagrangians). *Let $L : X(n, m, k) \to \mathbb{R}$ be continuous. Then the following statements are mutually equivalent:*
 (i) *L is a null Lagrangian;*
 (ii) *$\int_\Omega L(A + \nabla^k \varphi(x)) \, \mathrm{d}x = \int_\Omega L(A) \, \mathrm{d}x$ for every $\varphi \in C_0^\infty(\Omega; \mathbb{R}^m)$, every $A \in X(n, m, k)$, and every open subset $\Omega \subset \mathbb{R}^n$;*
 (iii) *L is continuously differentiable and (4.3) holds in the sense of distributions;*
 (iv) *the map $u \mapsto L(\nabla^k u)$ is sequentially weakly* continuous from $W^{k,\infty}(\Omega; \mathbb{R}^m)$ to $L^\infty(\Omega)$. This means that if $u_j \overset{*}{\rightharpoonup} u$ in $W^{k,\infty}(\Omega; \mathbb{R}^m)$, then $L(\nabla^k u_j) \overset{*}{\rightharpoonup} L(\nabla^k u)$ in $L^\infty(\Omega)$;*
 (v) *L is a polynomial of degree p and the map $u \mapsto L(\nabla^k u)$ is sequentially weakly continuous from $W^{k,p}(\Omega; \mathbb{R}^m)$ to $\mathcal{D}'(\Omega)$. This means that if $u_j \rightharpoonup u$ in $W^{k,p}(\Omega; \mathbb{R}^m)$, then $L(\nabla^k u_j) \rightharpoonup L(\nabla^k u)$ in $\mathcal{D}'(\Omega)$.*

While Theorem 4.3 provides us with very useful properties of null Lagrangians, it is interesting to note that they are known *explicitly* in the first as well as the higher order. In fact, null Lagrangians are formed by minors or subdeterminants of the gradient entering the integrand in J.

4.1. Explicit Characterization of Null Lagrangians of the First Order. Let us start with the first-order case: If $A \in \mathbb{R}^{m \times n}$, we denote by $\mathbb{T}_i(A)$ the vector of all subdeterminants of A of order i for $1 \leq i \leq \min(m, n)$. Notice that the dimension of $\mathbb{T}_i(A)$ is $d(i) := \binom{m}{i}\binom{n}{i}$, hence the number of all subdeterminants of A is $\sigma := \binom{m+n}{n} - 1$. Finally, we write $\mathbb{T} := (\mathbb{T}_1, \ldots, \mathbb{T}_{\min(m,n)})$. For example, if $m = 1$ or $n = 1$, then $\mathbb{T}(A)$ consists only of entries of A; if $m = n = 2$, then $\mathbb{T}(A) = (A, \det A)$; and for $m = n = 3$ we obtain $\mathbb{T}(A) = (A, \operatorname{Cof} A, \det A)$.

Clearly, linear maps are weakly continuous, yet it has been known at least since [178, 199, 14] that minors also have this property (see Theorem 4.4 below). This result, usually called *(sequential) weak continuity of minors*, is unexpected because if $i > 1$, then $A \mapsto \mathbb{T}_i(A)$ is a nonlinear polynomial of the ith degree. As is well known, weak convergence generically does not commute with nonlinear mappings.

THEOREM 4.4 (weak continuity of minors (see, e.g., [67])). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $1 \leq i \leq \min(m, n)$. Let $\{u_k\}_{k \in \mathbb{N}} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ be such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ for $p > i$. Then $\mathbb{T}_i(\nabla u_k) \rightharpoonup \mathbb{T}_i(\nabla u)$ in $L^{p/i}(\Omega; \mathbb{R}^{d(i)})$.*

The proof of Theorem 4.4 uses the structure of null Lagrangians, namely that they can be written in the divergence form. To explain this idea briefly, we restrict ourselves to $m = n = 2$. We have for $u \in C^2(\Omega; \mathbb{R}^2)$

$$(4.4) \qquad \det \nabla u = \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(u_1 \frac{\partial u_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(u_1 \frac{\partial u_2}{\partial x_1}\right).$$

Hence, if $\varphi \in \mathcal{D}(\Omega)$ is arbitrary we obtain

$$(4.5) \qquad \int_\Omega \det \nabla u(x) \varphi(x)\, \mathrm{d}x = -\int_\Omega \left(u_1 \frac{\partial u_2}{\partial x_2}\right)\frac{\partial \varphi(x)}{\partial x_1} - \left(u_1 \frac{\partial u_2}{\partial x_1}\right)\frac{\partial \varphi(x)}{\partial x_2} \ .$$

If $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ for $p > 2$, then the right-hand side of (4.5) written for u_k in the place of u allows us to pass easily to the limit for $k \to \infty$ to obtain Theorem 4.4 for $m = n = i = 2$. Notice that the right-hand side of (4.5) is defined in the sense of distributions even if $p \geq 4/3$; however, the integral identity (4.5) fails to hold if $p < 2$. Inspired by a conjecture of Ball [14], Müller [182] showed that if $u \in W^{1,p}(\Omega; \mathbb{R}^2)$, $p \geq 4/3$, then the *distributional determinant*

$$\operatorname{Det}\nabla u := \frac{\partial}{\partial x_1}\left(u_1 \frac{\partial u_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(u_1 \frac{\partial u_2}{\partial x_1}\right)$$

belongs to $L^1(\Omega)$ and $\det \nabla u = \operatorname{Det}\nabla u$. Generalizations to higher dimensions are possible, defining the distributional determinant with the help of the cofactor matrix. We refer the reader to [182] for details.

Minors are the *only* mappings depending exclusively on ∇u which are weakly continuous, and thus in view of Theorem 4.3 they are the only null Lagrangians of the first order. We make the statement more precise in the following theorem.

THEOREM 4.5 (see [19] or [67]). *Let $L \in C(\mathbb{R}^{m \times n})$. Then L is a null Lagrangian if and only if it is an affine combination of elements of $\mathbb{T}$, i.e., for every $A \in \mathbb{R}^{m \times n}$,*

$$(4.6) \qquad\qquad\qquad L(A) = c_0 + c \cdot \mathbb{T}(A) \ ,$$

where $c_0 \in \mathbb{R}$ and $c \in \mathbb{R}^\sigma$ are arbitrary constants.

Let us note, however, that it has been realized independently in, e.g., [81, 82] that minors are the only maps for which the Euler–Lagrange equation of $J(u) = \int_\Omega L(\nabla u)\mathrm{d}x$ is satisfied identically.

As we saw in Example 3.6, Theorem 4.4 fails if $p = i$. Nevertheless, the results can be much improved if we additionally assume that, for every $k \in \mathbb{N}$, $\mathbb{T}_i(\nabla u_k) \geq 0$ (elementwise) a.e. in Ω. Indeed, Müller [180, 181] proved the following result.

PROPOSITION 4.6 (higher integrability of determinant). *Assume that* $\omega \subset \Omega \subset \mathbb{R}^n$ *is compact,* $u \in W^{1,n}(\Omega;\mathbb{R}^n)$, *and that* $\det \nabla u \geq 0$ *a.e. in* Ω. *Then*

$$(4.7) \qquad \|(\det \nabla u)\ln(2 + \det \nabla u)\|_{L^1(\omega)} \leq C(\omega, \|u\|_{W^{1,n}(\Omega;\mathbb{R}^n)})$$

for some $C(\omega, \|u\|_{W^{1,n}(\Omega;\mathbb{R}^n)}) > 0$ *a constant depending only on* ω *and the Sobolev norm of* u *in* Ω.

This proposition results in the following corollary:

COROLLARY 4.7 (uniform integrability of determinant). *If* $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,n}(\Omega;\mathbb{R}^n)$ *is bounded and* $\det \nabla u_k \geq 0$ *a.e. in* Ω *for all* $k \in \mathbb{N}$, *then* $\det \nabla u_k \rightharpoonup \det \nabla u$ *in* $L^1(\omega)$ *for every compact set* $\omega \subset \Omega$.

A related statement was made by Kinderlehrer and Pedregal in [134]. It says that under the assumptions of Corollary 4.7, and if $u_k = u$ on $\partial\Omega$ for all $k \in \mathbb{N}$, the claim of Corollary 4.7 holds for $\omega := \Omega$. See also [239].

REMARK 4.4. *Proposition* 4.6 *can be strengthened if* $\det \nabla u$ *of a mapping* $u \in W^{1,n}(\Omega;\mathbb{R}^n)$ *is nonnegative and, additionally, the following inequality is valid for some* $K \geq 1$:

$$(4.8) \qquad |\nabla u(x)|^n \leq K \det \nabla u(x) \qquad a.e. \ in \ \Omega.$$

Such mappings are called quasiregular *(and if* u *is additionally a homeomorphism,* quasiconformal*) and we shall encounter them again in section* 7. *In the case of quasiregular mappings, we even have that* $\det \nabla u \in L^{1+\varepsilon}(\Omega)$ *with* $\varepsilon > 0$ *depending only on* K *and the dimension* n *(see, e.g.,* [119], *where generalizations of this result for* K *depending on* x *are also discussed). In the quasiconformal case in dimension* 2, *this observation goes back to Bojarski* [45]; *in this case even the precise value of* $\varepsilon < \frac{1}{K-1}$ *was established by Astala* [6].

4.2. Explicit Characterization of Null Lagrangians of Higher Order. Null Lagrangians of higher order are of the same structure as those of the first order. Indeed, they also correspond to minors. In order to make the statement more precise, we assume that $K := (k_1, \ldots, k_r)$ is such that $1 \leq k_i \leq n$ and denote $\alpha := (\nu_1, J_1; \nu_2, J_2; \ldots; J_r, \nu_r)$ with $|J_i| = k - 1$ and where $1 \leq \nu_i \leq m$. We define the kth-order Jacobian determinant $J_K^\alpha : X \to \mathbb{R}$ by the formula

$$J_K^\alpha(\nabla u) = \det\left(\frac{\partial u_{J_i}^{\nu_i}}{\partial x^{k_j}}\right).$$

Then any null Lagrangian of higher order is just an affine combination of J_K^α, i.e., we have the following theorem.

THEOREM 4.8 (see Ball, Currie, and Olver [19]). *Let* $L \in C(X(n,m,k))$. *Then* L *is a null Lagrangian if and only if it is an affine combination of* kth-*order Jacobian determinants, i.e.,*

$$L = C_0 + \sum_{\alpha,K} C_K^\alpha J_K^\alpha$$

for some constants C_0 *and* C_K^α.

4.3. Null Lagrangians with Lower-Order Terms. As is pointed out in Remark 4.3, the notion of null Lagrangians can also be generalized to functionals of the type (1.10), i.e., those also containing lower-order terms. A characterization of these null Lagrangians is due to Olver and Sivaloganathan [190], who considered the first-order case, i.e., null Lagrangians for those functionals which can also depend on x and u. Based on Olver's results [189], they showed in [190] that such null Lagrangians are given by the formula

$$L(x, u, \nabla u) = C_0(x, u) + \sum_i C_i(x, u) \cdot \mathbb{T}_i(\nabla u) ,$$

where C_0 and C_1 are C^1-functions. This means that null Lagrangians with lower-order terms are determined by the already known null Lagrangians of the first order. Let us remark that it is noted in [190] that the result generalizes analogously to the higher-order case.

5. Null Lagrangians at the Boundary. We have seen that null Lagrangians of the first order are exactly those functions that fulfill (1.8) in the definition of quasiconvexity with an equality. This, of course, ensures that null Lagrangians are weakly* continuous with respect to the $W^{1,\infty}(\Omega; \mathbb{R}^m)$ weak* topology; in addition, due to Theorem 4.4, they are weakly continuous with respect to the $W^{1,p}(\Omega; \mathbb{R}^m)$ weak topology if $p > \min(m, n)$ with $\Omega \subset \mathbb{R}^n$.

However, in the critical case when $p = \min(m, n)$, the weak continuity fails. In fact, as we have seen in Example 3.6, for $n = m = p = 2$ the functional (1.10) with $k = 1$ and $f(x, u, \nabla u) = \det(\nabla u)$ is not even weakly lower semicontinuous, even though the determinant itself is definitely a null Lagrangian. Once again, the reason for the failure of weak continuity is concentrations on the boundary combined with the fact that null Lagrangians are unbounded from below.

Nevertheless, as we have seen in section 3.1, at least for p-homogeneous functions, weak lower semicontinuity can be assured for functionals with integrands that are quasiconvex at the boundary, i.e., fulfill (3.12). Thus, a proper equivalent of null Lagrangians in this case is those functions that fulfill (3.12) with an equality—these functions are referred to as *null Lagrangians at the boundary.* We study them in this section.

Clearly, null Lagrangians at the boundary form a subset of null Lagrangians of the first order. Moreover, they have exactly the sought properties: We know from Theorem 4.3 that if $\mathcal{N}$ is a null Lagrangian at the boundary, then it is a polynomial of degree p for some $p \in [1, \min(m, n)]$. If, additionally, $\{u_k\}_{k \in \mathbb{N}} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ converges weakly to $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, then $\{\mathcal{N}(\nabla u_k)\}_{k \in \mathbb{N}} \subset L^1(\Omega)$ weakly* converges to $\mathcal{N}(\nabla u)$ in $\mathcal{M}(\overline{\Omega})$, i.e., in measures on the closure of the domain. This means that the L^1-bounded sequence $\{\mathcal{N}(\nabla u_k)\}$ converges to a Radon measure whose singular part vanishes. Thus, functionals with integrands that are null Lagrangians at the boundary are weakly continuous even in the critical case. Null Lagrangians at the boundary can also be used to construct functions quasiconvex at the boundary; cf. Definition 3.11.

We first give a formal definition of null Lagrangians at the boundary.

DEFINITION 5.1. *Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $L : \mathbb{R}^{m \times n} \to \mathbb{R}$ be a given function.*
  (i) *L is called a null Lagrangian at the boundary at given $A \in \mathbb{R}^{m \times n}$ if both L and $-L$ are quasiconvex at the boundary of A in the sense of Definition 3.11; cf. [222]. This means that there is $q \in \mathbb{R}^m$ such that for all*

$\varphi \in W^{1,\infty}_{D_\varrho \setminus \Gamma_\varrho}(D_\varrho; \mathbb{R}^m)$ *it holds that*

$$(5.1) \qquad \int_{\Gamma_\varrho} q \cdot \varphi(x) \, \mathrm{d}S + L(A) \mathcal{L}^n(D_\varrho) = \int_{D_\varrho} L(A + \nabla \varphi(x)) \, \mathrm{d}x \ .$$

(ii) *If L is a null Lagrangian at the boundary at every $F \in \mathbb{R}^{m \times n}$, we call it a null Lagrangian at the boundary.*

The following theorem explicitly characterizes all possible null Lagrangians at the boundary. It was first proved by Sprenger in his thesis [219, Satz 1.27]. Later, the proof was slightly simplified in [131]. Before stating the result we recall that $\mathrm{SO}(n) := \{R \in \mathbb{R}^{n \times n}; \ R^\top R = RR^\top = \mathbb{I}, \ \det R = 1\}$ denotes the set of orientation-preserving rotations and that if we write $A = (B|\varrho)$ for some $B \in \mathbb{R}^{n \times (n-1)}$ and $\varrho \in \mathbb{R}^n$, then $A \in \mathbb{R}^{n \times n}$, its last column is ϱ, and $A_{ij} = B_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq n-1$. We also recall that $\mathbb{T}_i(A)$ denotes the vector of all subdeterminants of A of order i.

THEOREM 5.2. *Let $\varrho \in \mathbb{R}^n$ be a unit vector and let $L : \mathbb{R}^{m \times n} \to \mathbb{R}$ be a given continuous function. Then the following three statements are equivalent:*
  (i) $\mathcal{N}$ *satisfies* (5.1) *for every $F \in \mathbb{R}^{m \times n}$;*
  (ii) $\mathcal{N}$ *satisfies* (5.1) *for $F = 0$;*
  (iii) *there are constants $\tilde{\beta}_s \in \mathbb{R}^{\binom{m}{s} \times \binom{n-1}{s}}$, $1 \leq s \leq \min(m, n-1)$, such that for all $H \in \mathbb{R}^{m \times n}$,*

$$(5.2) \qquad \mathcal{N}(H) = \mathcal{N}(0) + \sum_{i=1}^{\min(m,n-1)} \tilde{\beta}_i \cdot \mathbb{T}_i(H\tilde{R}),$$

  where $\tilde{R} \in \mathbb{R}^{n \times (n-1)}$ is a matrix such that $R = (\tilde{R}|\varrho)$ belongs to $\mathrm{SO}(n)$;
  (iv) $\mathcal{N}(F + a \otimes \varrho) = \mathcal{N}(F)$ *for every $F \in \mathbb{R}^{m \times n}$ and every $a \in \mathbb{R}^m$.*

If $m = n = 3$, the only nonlinear null Lagrangian at the boundary with the normal ϱ is

$$\mathcal{N}(F) = \mathrm{Cof} \, F \cdot (a \otimes \varrho) = a \cdot \mathrm{Cof} \, F \varrho,$$

where $a \in \mathbb{R}^3$ is some fixed vector; see Šilhavý [222].

In the following theorem, we let ϱ freely move along the boundary, which introduces an x-dependence to the problem. Then the vector a may depend on x as well.

THEOREM 5.3 (due to [152]). *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Let $\{u_k\} \subset W^{1,2}(\Omega; \mathbb{R}^3)$ be such that $u_k \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^3)$. Let $\tilde{L}(x, F) := \mathrm{Cof} \, F \cdot (a(x) \otimes \varrho(x))$, where $a, \varrho \in C(\overline{\Omega}; \mathbb{R}^3)$ and ϱ coincides at $\partial\Omega$ with the outer unit normal to $\partial\Omega$. Then for all $g \in C(\overline{\Omega})$,*

$$(5.3) \qquad \lim_{k \to \infty} \int_\Omega g(x) \tilde{L}(x, \nabla u_k(x)) \, \mathrm{d}x = \int_\Omega g(x) \tilde{L}(x, \nabla u(x)) \, \mathrm{d}x \ .$$

If, moreover, for all $k \in \mathbb{N}$, $\tilde{L}(\cdot, \nabla u_k) \geq 0$ a.e. in Ω, then $\tilde{L}(\cdot, \nabla u_k) \rightharpoonup \tilde{L}(\cdot, \nabla u)$ in $L^1(\Omega)$.

Notice that even though $\{\tilde{L}(\cdot, \nabla u_k)\}_{k \in \mathbb{N}}$ is bounded merely in $L^1(\Omega)$, its weak* limit in measures is $\mathcal{N}(\cdot, \nabla u) \in L^1(\Omega)$, i.e., a measure which is absolutely continuous with respect to the Lebesgue measure on Ω. This holds independently of $\{\nabla u_k\}$.

Therefore, the fact that $\tilde{L}$ is a null Lagrangian at the boundary automatically improves regularity of the limit measure, namely, its singular part vanishes. In order to understand why this happens, denote by $\mathbb{P}(x) := \mathbb{I} - \varrho(x) \otimes \varrho(x)$ the orthogonal projector on the plane with the normal $\varrho(x)$, i.e., a tangent plane to $\partial\Omega$ at $x \in \partial\Omega$. Then

$$\mathrm{Cof}(F\mathbb{P}) = \mathrm{Cof}F\mathrm{Cof}\mathbb{P} = (\mathrm{Cof}F)(\varrho \otimes \varrho) \ .$$

Consequently,

$$\mathrm{Cof}(F\mathbb{P})\varrho = (\mathrm{Cof}F)\varrho \ ,$$

and if we plug in ∇u for F, we see that $\tilde{L}(x, \cdot)$ only depends on the surface gradient of u. In other words, concentrations in the sequence of normal derivatives, $\{\nabla u_k \cdot (\varrho \otimes \varrho)\}_{k\in\mathbb{N}}$, are filtered out.

The following two statements describing weak sequential continuity of null Lagrangians at the boundary can be found in [131]. Here, an effect similar to the one in Theorem 4.6 and Corollary 4.7 is observed: the nonnegativity of the null Lagrangian allows us to prove weak continuity.

THEOREM 5.4 (see [131]). *Let $m, n \in \mathbb{N}$ with $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be open and bounded with a boundary of class C^1, and let $L : \overline{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be a continuous function. In addition, suppose that for every $x \in \Omega$, $L(x, \cdot)$ is a null Lagrangian and for every $x \in \partial\Omega$, $L(x, \cdot)$ is a null Lagrangian at the boundary with respect to $\varrho(x)$, the outer normal to $\partial\Omega$ at x. Hence, by Theorem 5.2, $L(x, \cdot)$ is a polynomial, the degree of which we denote by $d_{\tilde{L}}(x)$. Finally, let $p \in (1, \infty)$ with $p \geq d_f(x)$ for every $x \in \overline{\Omega}$ and let $\{u_k\} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ be a sequence such that $u_k \rightharpoonup u$ in $W^{1,p}$. If*

$$L(x, \nabla u_k(x)) \geq 0 \ \ \textit{for every } k \in \mathbb{N} \textit{ and a.e. } x \in \Omega,$$

then $L(\cdot, \nabla u_n) \rightharpoonup L(\cdot, \nabla u)$ weakly in $L^1(\Omega)$.

The above theorem allows us to prove a weak lower semicontinuity result for convex functions of null Lagrangians at the boundary which relates to the concept of polyconvexity introduced in section 6.

THEOREM 5.5 (see [131]). *Let $h : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be such that $h(\cdot, s)$ is measurable for all $s \in \mathbb{R}$ and $h(x, \cdot)$ is convex for almost all $x \in \Omega$. Let L and d_L be as in Theorem 5.4. Then $\int_\Omega h(x, L(x, \nabla u(x)))\, \mathrm{d}x$ is weakly lower semicontinuous on the set $\{u \in W^{1,p}(\Omega; \mathbb{R}^m); L(\cdot, \nabla u) \geq 0 \text{ in } \Omega\}$.*

Let us finally point out that $A \mapsto h(L(x, A))$ for a convex function h is quasiconvex at the boundary with respect to the normal $\varrho(x)$. Therefore, null Lagrangians at the boundary allow us to construct functions which are quasiconvex at the boundary.

6. Polyconvexity and Applications to Hyperelasticity. We have seen that, at least for integrands bounded from below and satisfying (i) in Definition 3.3, quasiconvexity is an equivalent condition for weak lower semicontinuity. This presents an *explicit* characterization of the latter since it is not necessary to examine all weakly converging sequences. Nevertheless, in practice quasiconvexity is almost impossible to verify since, in a sense, its verification calls for solving a minimization problem itself. Therefore, it is desirable to find at least *sufficient* conditions for weak lower semicontinuity that can be easily verified. Such a notion, called *polyconvexity*, was introduced by J.M. Ball and can be designed by employing the null Lagrangians introduced in the last section.

We start with the definition of polyconvexity suitable for first-order functionals.

DEFINITION 6.1 (due to Ball [14]). *We say that* $f : \mathbb{R}^{m \times n} \to \mathbb{R} \cup \{+\infty\}$ *is polyconvex if there exists a convex function* $h : \mathbb{R}^{\sigma} \to \mathbb{R} \cup \{+\infty\}$ *such that* $f(A) = h(\mathbb{T}(A))$[7] *for all* $A \in \mathbb{R}^{m \times n}$.

REMARK 6.1. *Interestingly, Morrey in* [178, Thm. 5.3] *proved that* 1-*homogeneous convex functions depending on minors are quasiconvex.*

If h is affine in the above definition, we call f polyaffine. In this case, $f(A)$ is a linear combination of all minors of A plus a real constant. Consequently, any polyconvex function is bounded from below by a polyaffine function. Similarly, as in the convex case, a polyconvex function is found by forming the supremum of all polyaffine functions lying below it; see, e.g., [67, Rem. 6.7]; i.e., we have the following lemma.

LEMMA 6.2. *The function* $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ *is polyconvex if and only if*

$$f(A) = \sup\{\varphi(A); \varphi \text{ polyaffine and } \varphi \leq f\}.$$

It is straightforward to generalize polyconvexity to higher-order variational problems, i.e., those that depend on higher-order gradients of a mapping. The attractiveness of such problems for applications is clear. Suitably chosen terms depending on higher-order gradients allow for compactness of a minimizing sequence in some stronger topology on $W^{1,p}(\Omega; \mathbb{R}^m)$, which enables us to pass to a limit in lower-order terms without restrictive assumptions on their convexity properties. Thus, for example, models of shape memory alloys (see section 7) can be treated by this approach; cf., e.g., [183, 184].

We extend the notion of polyconvexity to higher-order problems (1.10) by employing the notion of null Lagrangians of higher order due to Ball, Currie, and Olver [19].

DEFINITION 6.3 (higher-order polyconvexity). *Let* $1 \leq r \leq n$. *Let* $U \subset X(n, m, k)$ *be open. A function* $G : U \to \mathbb{R}$ *is* r-*polyconvex if there exists a convex function* $h : \mathrm{Co}(J^{[r]}(U)) \to \mathbb{R}$ *such that* $f(A) = h(J^{[r]}(A))$ *for all* $A \in U$; *here,* $\mathrm{Co}(J^{[r]}(U))$ *is the convex hull of* $J^{[r]}(U)$. G *is polyconvex if it is* R-*polyconvex. Here,* $J^r(H) := (J^{r,1}(H), \dots, J^{r,N_r}(H))$ *is an* N_r-*tuple with the property that any Jacobian determinant of degree* r *can be written as a linear combination of elements of* J^r. *Consequently,* $J^{[r]} := (J^1, \dots, J^r)$. *If* h *is affine, then we call* f r-*polyaffine.*

Since polyconvexity implies quasiconvexity, we may deduce by the results in section 3 that integral functions with polyconvex functions in the class $\mathcal{F}_p(\Omega)$ (from Definition 3.3) are weakly lower semicontinuous. However, weak lower semicontinuity can be proved for a wider class of polyconvex functions than those in $\mathcal{F}_p(\Omega)$; in particular, the functions do not have to be of p-growth. This is of great importance in elasticity as is explained later in this section.

The proof of weak lower semicontinuity of polyconvex functions can be actually based on *convexity* and weak continuity of null Lagrangians. Thus, because weak lower semicontinuity can be shown for arbitrarily growing convex functions, this generalizes to polyconvex ones, too. The following result for convex functions can be found in [19, Thm. 5.4] and is based on results by Eisen [80], who proved this theorem for Φ finite-valued.

THEOREM 6.4 (weak lower semicontinuity). *Let* $\Phi : \Omega \times \mathbb{R}^s \times \mathbb{R}^{\sigma} \to \mathbb{R} \cup \{+\infty\}$ *satisfy the following properties:*

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[7]Recall that $\mathbb{T}(A)$ denotes the vector of all minors of A.

(i) $\Phi(\cdot, z, a) : \Omega \to \mathbb{R} \cup \{+\infty\}$ is measurable for all $(z, a) \in \mathbb{R}^s \times \mathbb{R}^\sigma$;
(ii) $\Phi(x, \cdot, \cdot) : \mathbb{R}^s \times \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ is continuous for almost every $x \in \Omega$;
(iii) $\Phi(x, z, \cdot) : \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ is convex.

    Assume further that there is $\phi \in L^1(\Omega)$ such that $\Phi(\cdot, z, a) \geq \phi$ for all $(z, a) \in \mathbb{R}^s \times \mathbb{R}^\sigma$. Let $\{z_k\}_{k \in \mathbb{N}} \subset L^1(\Omega; \mathbb{R}^s)$, $\{a_k\}_{k \in \mathbb{N}} \subset L^1(\Omega; \mathbb{R}^\sigma)$, and let $z_k \to z$ a.e. in Ω as well as $a_k \rightharpoonup a$ in $L^1(\Omega; \mathbb{R}^\sigma)$. Then

$$\int_\Omega \Phi(x, z(x), a(x)) \, \mathrm{d}x \leq \liminf_{k \to \infty} \int_\Omega \Phi(x, z_k(x), a_k(x)) \, \mathrm{d}x \ .$$

Using this theorem, we may easily deduce weak lower semicontinuity of polyconvex functions. For the sake of clarity, let us start with first-order problems. Then, consider $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ as $k \to \infty$ where $p > \min(m, n)$. Then $u_k \to u$ in $L^p(\Omega; \mathbb{R}^m)$, so, for a (nonrelabeled) subsequence, even $u_k \to u$ a.e. in Ω. Hence, we can apply Theorem 6.4 with $z_k := u_k$, $a_k := \mathbb{T}(\nabla u_k)$ and $f(x, u, \nabla u) := \Phi(x, u, \mathbb{T}(\nabla u))$ to obtain the following corollary.

    COROLLARY 6.5. *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R} \cup \{+\infty\}$ satisfy the following properties:*
  (i) *$f(\cdot, r, A) : \Omega \to \mathbb{R} \cup \{+\infty\}$ is measurable for all $(r, A) \in R^m \times \mathbb{R}^{m \times n}$;*
  (ii) *$f(x, \cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^{m \times n} \to \mathbb{R} \cup \{+\infty\}$ is continuous for almost every $x \in \Omega$;*
 (iii) *$f(x, r, A) = \Phi(x, r, \mathbb{T}(A))$, where Φ satisfies (i)–(iii) from Theorem 6.4.*
    If $u_k \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$ as $k \to \infty$ where $p > \min(m, n)$, then

$$\int_\Omega f(x, u(x), \nabla u(x)) \, \mathrm{d}x \leq \liminf_{k \to \infty} \int_\Omega f(x, u_k(x), \nabla u_k(x)) \, \mathrm{d}x \ .$$

Similarly as in the case of first-order problems, we can exploit (v) of Theorem 4.3 and Theorem 6.4 to show the existence of minimizers to energy functionals (1.10). Let us present the result just for functionals (1.10) with $k = 2$; generalizations for higher k are straightforward and can be found in [19].

    COROLLARY 6.6 (after [19]). *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and that $1 \leq r \leq n$. Let $f : \Omega \times Y(n, m, 2) \to \mathbb{R} \cup \{+\infty\}$ and*

$$I(u) = \int_\Omega f(x, u, \nabla u, \nabla^2 u) \mathrm{d}x$$

satisfy the following assumptions:
  (i) *$f(x, H, A) = h(x, H, J^{[r]}(A))$, where $h(x, \cdot, \cdot) : (\mathbb{R}^m \times \mathbb{R}^{m \times n}) \times J^{[r]}(X(n, m, 2)) \to \mathbb{R} \cup \{+\infty\}$ is continuous for almost every $x \in \Omega$;*
  (ii) *$h(\cdot, H, J^{[r]}(A)) : \Omega \to \mathbb{R} \cup \{+\infty\}$ is measurable for all $(H, J^{[r]}(A)) \in (\mathbb{R}^m \times \mathbb{R}^{m \times n}) \times J^{[r]}(X(n, m, 2))$;*
 (iii) *$h(x, H, \cdot) : J^{[r]}(X) \to \mathbb{R} \cup \{+\infty\}$ is convex for almost all $x \in \Omega$ and all $H \in (\mathbb{R}^m \times \mathbb{R}^{m \times n})$;*
 (iv) *$f(x, H, A) \geq C(-1 + |A|^p)$ for some $C > 0$, $p > n$, almost all $x \in \Omega$, and all $A \in \mathbb{R}^{m \times n}$.*
Then I is weakly lower semicontinuous on $W^{2,p}(\Omega; \mathbb{R}^m)$.

    It cannot be stressed enough that the main strength of polyconvexity consists in the fact that convexity in subdeterminants can be advantageously combined with the Mazur lemma to show weak lower semicontinuity in a similar way to the proof for mere convex and lower semicontinuous integrands. This contrasts with proofs available for

quasiconvex integrands, where manipulations with boundary conditions are needed to prove the result. This was already clearly visible in Meyers' paper [171]. These manipulations typically destroy any pointwise constraints on the determinant of ∇y, which, however, are crucial in elasticity. We shall return to this issue in section 7.

It is well known that minimizers of integral functionals with strictly convex integrands are unique. However, the same is not true for variational problems with polyconvex integrands, even if h in Definition 6.1 is strictly convex. Examples were provided by Spadaro [218] if $m = n = 2$.

6.1. Rank-One Convexity. Since polyconvexity is an explicit sufficient condition for quasiconvexity, we might ask if a simpler necessary condition can be similarly found. This is indeed so, and the resulting notion of convexity is the so-called *rank-one convexity*:

DEFINITION 6.7 (due to [179]). *We say that $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is rank-one convex if*

$$(6.1) \qquad f(\lambda A_1 + (1 - \lambda)A_2) \leq \lambda f(A_1) + (1 - \lambda)f(A_2)$$

for all $\lambda \in [0, 1]$ and all A_1, A_2 such that $\operatorname{rank}(A_1 - A_2) \leq 1$.

The relations among the introduced notions of convexity are as follows:

$$\text{convexity} \Rightarrow \text{polyconvexity} \Rightarrow \text{quasiconvexity} \Rightarrow \text{rank-one convexity};$$

however, none of the converse implications holds if $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ and $m \geq 3$ and $n \geq 2$. To see that polyconvexity does not imply convexity (even for $m, m > 1$), just consider the function $f(F) := \det(F)$, which is even polyaffine but not convex. Also, quasiconvexity does not imply polyconvexity even for $m, n > 1$, as was shown in, e.g., [2, 232]. Šverák's important counterexample [228] is a construction of a function that is rank-one convex, but not quasiconvex, and it holds for $m \geq 3$ and $n \geq 2$. For $m = 2$ and $n \geq 2$ the question of equivalence between quasiconvexity and rank-one convexity is still unsolved (see Open Problem 6.8). Notice that if $m = 1$ or $n = 1$, all the generalized notions of convexity trivially coincide with standard convexity itself.

OPEN PROBLEM 6.8. *Let $m = 2$ and $n \geq 2$. Does rank-one convexity imply quasiconvexity for $f : \mathbb{R}^{m \times n} \to \mathbb{R}$?*

Many attempts can be found in the literature at the solution of Open Problem 6.8 and indications in both the positive and the negative exist. Morrey conjectured in his original paper [178] that the solution to Open Problem 6.8 is negative. On the other hand, in several special cases it has been shown that rank-one convexity indeed implies quasiconvexity. For example, if f is a quadratic form, i.e., if there exists $\mathbb{C} \in \mathbb{R}^{m \times n \times m \times n}$ such that $f(A) = \mathbb{C}A \cdot A := \sum_{ijkl} \mathbb{C}_{ijkl} A_{ij} A_{kl}$, then f is rank-one convex if and only if it is quasiconvex [236]. Additionally, if $\min(m, n) = 2$, then any quasiconvex (or rank-one convex) quadratic form is even polyconvex [166, 214, 232]. Also, as has been realized by Müller [185], rank-one convexity is equivalent to quasiconvexity on diagonal matrices. Very recently, Open Problem 6.8 has been answered positively for isotropic, objective, and isochoric elastic energies [169]. On top of that, even in the case $m \geq 3$ and $n \geq 2$ only very few examples of functions that are rank-one convex but not quasiconvex are known. Besides the already mentioned Šverák's counterexample, Grabovsky [107] very recently provided a new example of a rank-one convex function which is not quasiconvex in dimensions $m = 8$ and $n = 2$.

On the other hand, let us also mention that if $m = n = 2$, a necessary condition for the equivalence of rank-one convexity and quasiconvexity is that every $f : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ quasiconvex satisfies the inequality

$$f(A) \leq \int_{(0,1)^2} f(A + (\nabla \varphi(x))^\top) \, \mathrm{d}x \quad \text{for every } \varphi \in W_0^{1,\infty}((0,1)^2; \mathbb{R}^2)$$

$$\text{and every } A \in \mathbb{R}^{2 \times 2}.$$

An analogous implication for $m = n > 2$ turns out to be false [151, 186]; however, the two-dimensional case remains open.

Before continuing to higher-dimensional equivalents of rank-one convexity, let us point out one important function that may form a counterexample to Open Problem 6.8: the so-called Burkholder function

$$B_p(A) = \frac{p}{2} \det(A)|A|^{p-2} + \left(1 - \frac{p}{2}\right)|A|^p \quad \text{with } p \geq 2,$$

with the $|\cdot|$ denoting the operator norm. The Burkholder function emerged in the study of stochastic integrals and martingales [56, 57], and is rank-one concave [126] (which means that $-B_p$ is rank-one convex). Nevertheless, it is a standing problem whether it is also quasiconcave (i.e., $-B_p$ quasiconvex).

OPEN PROBLEM 6.9. *Is the Burkholder function $B_p(\cdot)$ quasiconcave?*

Naturally, Open Problem 6.9 is answered affirmatively if the answer to Open Problem 6.8 is "yes." However, Open Problem 6.9 is interesting in its own right due to its implications for harmonic and quasiconformal analysis. In fact, if one could prove that the Burkholder function is quasiconcave, this would give a precise bound on the operator norm of the Beurling–Ahlfors transform $S : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ (here $\mathbb{C}$ denotes the complex plane), which plays an important role in complex analysis since it converts the complex partial derivative $\partial_{\bar{z}}$ into ∂_z. Indeed, the standing conjecture due to Iwaniec [125] is that this norm is equal to $p^\star - 1 := \max\{p-1, 1/(p-1)\}$; cf. the reviews [28, 11]. It is classical that $p^\star - 1$ is the lower bound for the norm of S [161]; as for the upper bound, the attempts in the literature have come progressively closer to the number $p^\star - 1$ (see the review [28] and also, e.g., the current improvements [29, 48]), but it has not been reached as of today. Reaching this upper bound would, for example, imply the distortion result by Astala mentioned in Remark 4.4.

In the context of Open Problems 6.9 and 6.8 let us also point to the recent work of Astala et al. [8] in which the authors show that the Burkholder function is quasiconcave in the identity for quasiconform perturbations[8]; in other words, in the language of section 7, the Burkholder function is quasiconformally quasiconcave at the identity.

An equivalent to rank-one convexity can also be defined for higher-order problems: the corresponding notion is called Λ-convexity. Following [19], we define a nonconvex cone $\Lambda \subset X(n, m, k)$ as $\Lambda := \{a \otimes^k b : a \in \mathbb{R}^m, b \in \mathbb{R}^n\}$, where $(a \otimes^k b)_K^i = a^i b_K$. Here, recall from section 2 that $b_K = (b_1)^{k_1}(b_2)^{k_2} \ldots (b_n)^{k_n}$ and $K = (k_1, \ldots, k_n)$ with $1 \leq k_i \leq k$ for $i = 1, \ldots, n$ is a multi-index.

DEFINITION 6.10. *A function $f : X \to \mathbb{R}$ is called Λ-convex if $t \mapsto f(A + tB) : \mathbb{R} \to \mathbb{R}$ is convex for every $A \in X(n, m, k)$ and any $B \in \Lambda$.*

---

[8]By a *perturbation* we mean the function φ in Definition 1.3.

Notice that for $k = 1$ Λ-convexity coincides with rank-one convexity. If f is twice continuously differentiable, then Λ-convexity is equivalent to the Legendre–Hadamard condition

$$\sum_{j,i=1}^{m} \sum_{|J|=|K|=k} \frac{\partial^2 f(A)}{\partial A_J^j \partial A_K^i} a^j a^i b_J b_K \geq 0$$

for all $A \in X(n, m, k)$, $a \in \mathbb{R}^m$, and $b \in \mathbb{R}^n$.

PROPOSITION 6.11 (see [19]). *Continuous and k-quasiconvex functions $f : X(n, m, k) \to \mathbb{R}$ are Λ-convex.*

Hence, Λ-convexity forms a necessary condition for k-quasiconvexity. This proposition was first proved by Meyers [171, Thm. 7] for smooth functions and then generalized by Ball, Currie, and Olver [19] to the continuous case. The opposite assertion does not hold. Indeed, if $n = k = 2$ and $m = 3$, then we have the following example due to Ball, Currie, and Olver for $f : X(2, 3, 2) \to \mathbb{R}$:

$$f(\nabla^2 u) = \sum_{i,j,l=1}^{3} \varepsilon_{ijk} \frac{\partial^2 u^i}{\partial x_1^2} \frac{\partial^2 u^j}{\partial x_1 \partial x_2} \frac{\partial^2 u^l}{\partial x_2^2} \ ,$$

where ε_{ijk} is the Levi-Civita symbol. This function is even Λ-affine (i.e., both $\pm f$ are Λ-convex) but not a null Lagrangian and not quasiconvex. As Λ-convexity replaces rank-one convexity in the current setting, we see that this example is reminiscent of Šverák's example mentioned above.

6.2. Applications to Hyperelasticity in the First-Order Setting. In elasticity, one is interested in modeling the response of a rubber-like material exposed to the action of applied forces. This response is obtained by solving a minimization problem; to be more specific, we are to minimize the free energy of the material. We will see that polyconvexity is perfectly fitted to the elasticity setting and that existence of minimizers can be assured for polyconvex energies. We give a short introduction to this matter in this section and refer the reader, e.g., to the monographs [114, 115, 222] for more details on the physical modeling.

Take a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ which, for $n = 3$, plays the role of a reference configuration of an elastic material. For given applied loads, we search for a mapping $u : \Omega \to \mathbb{R}^m$, the *deformation* of the material, which describes the new "shape" $u(\Omega)$ of the body. The mapping u is found by solving the following system of equations:

$$
\begin{align}
-\operatorname{div} S &= b &&\text{in } \Omega, \tag{6.2}\\
S\varrho &= g &&\text{on } \Gamma_{\mathrm{N}}, \tag{6.3}\\
u &= u_0 &&\text{on } \Gamma_{\mathrm{D}}. \tag{6.4}
\end{align}
$$

Here, (6.2) is the reduced version of Newton's law of motion for the (quasi)static case, and b is the applied volume force. Further, (6.3) represents the action of applied surface forces g (ϱ denotes the outer unit normal vector to Γ_{N}) and (6.4) models that the body may be clamped at some part of the boundary to a prescribed shape u_0. We shall require that $\Gamma_{\mathrm{D}} \subset \partial\Omega$ is disjoint from $\Gamma_{\mathrm{N}} \subset \partial\Omega$ and of positive $(n-1)$-dimensional Lebesgue measure.

The material properties of the specimen are encoded in the first Piola–Kirchhoff stress tensor $S : \Omega \to \mathbb{R}^{m \times n}$ in (6.2) and (6.3). The form of the Piola–Kirchhoff

stress tensor cannot be deduced from first principles within continuum mechanics but has to be prescribed phenomenologically. The prescription for S is called the *constitutive relation* of the given material. In the easiest case, we assume the form $S(x) = \hat{S}(x, \nabla u(x))$ for some given $\hat{S}$. Materials for which this assumption is adequate are sometimes referred to as *simple* materials as opposed to nonsimple materials for which $\hat{S}$ may depend also on higher gradients of u. Later, in section 6.3, we will also consider these sophisticated constitutive relations.

Hyperelasticity is a part of elasticity where an *additional assumption* is made; namely, that S has a potential $W : \mathbb{R}^{m \times n} \to [0; +\infty]$ such that

$$S_{ij}(x) = \frac{\partial W(F)}{\partial F_{ij}}\big|_{F = \nabla u(x)} \; .$$

This assumption emphasizes the idea that there are no energy losses in elasticity and all work, made by external forces and/or Dirichlet boundary conditions, stored in the material can be fully exploited.

In what follows, let us restrict our attention to *deformations of bulks*, i.e., we do not treat plates and rods, and thus set $m = n$. In order to fulfill the basic physical requirements, W has to satisfy the following relations:

(6.5) $W(RF) = W(F)$ for all $F \in \mathbb{R}^{n \times n}$ and for all $R \in \mathrm{SO}(n)$,

(6.6) $W(F) = +\infty$ if $\det F \leq 0$, and

(6.7) $W(F) \to +\infty$ if $\det F \to 0_+$.

Indeed, assumption (6.5) is a consequence of the *axiom of frame indifference* [62]; in other words, the assumption ensures that material properties are independent of the position of the observer. Conditions (6.6) and (6.7) ensure, respectively, that the material does not locally penetrate itself and that compression of a finite volume of the specimen into zero volume is not possible. These conditions, however, do not yet ensure that the body does not penetrate itself, which is also natural to assume from a physical point of view. Nevertheless, we shall see at the end of this section that with additional assumptions on the growth of the energy and, e.g., the boundary conditions, even complete noninterpenetration can be assured.

The assumptions (6.5)–(6.7) rule out the possibility that W can be convex. Indeed, there are matrices $A, B \in \mathbb{R}^{n \times n}$, both with positive determinants, such that the line segment $[A, B]$ contains a matrix C with zero determinant. However, no convex function can be finite at A, B and infinite at C.

Moreover, due to (6.6)–(6.7), even if W was quasiconvex, we could not apply theorems in section 3 since W cannot be an element of the class $\mathcal{F}_p(\Omega)$. Nevertheless, *polyconvexity* is fully compatible with these assumptions.

The mechanical model is that stable states of the system are found by minimizing the overall free energy

(6.8) $\mathcal{E}(y) = \displaystyle\int_\Omega W(\nabla u(x)) \, \mathrm{d}x - \int_\Omega b(x) \cdot u(x) \, \mathrm{d}x - \int_{\Gamma_N} g(x) \cdot u(x) \, \mathrm{d}S,$

subject to (6.4). Smooth minimizers fulfill the balance equations (6.2)–(6.3); however, even in the smooth case there might exist solutions to (6.2)–(6.3) that are not minimizers of (6.8). Nevertheless, such solutions are thought to be metastable and are hence left after a small perturbation. Thus, minimizing (6.8) is indeed the proper way to find stable states.

REMARK 6.2. *Let us note that since the minimizers of* (6.8) *might be non-smooth, it is not guaranteed that they will satisfy the Euler–Lagrange equations in either strong or weak form. Indeed, in* [23] *even one-dimensional examples of smooth W were given such that the minimizer does not fulfill the Euler–Lagrange equation.*

One of the reasons why deducing the Euler–Lagrange equation might be difficult is that even the calculation of the variation of $\mathcal{E}$ itself can pose difficulties. Indeed, due to (6.6), *the minimizer y might be such that $\mathcal{E}(y + t\varphi)$ is infinite for all small enough $t > 0$ and a large class of φ. Let us refer to* [23] *for explicit examples in which this situation occurs.*

REMARK 6.3. *The method of deriving Euler–Lagrange equations by taking variations of the energy functional is not completely appropriate for elasticity. In particular, if two subsequent deformations are applied to an elastic body, they are not added to each other but are rather composed in order to obtain the final deformation of the body.*

REMARK 6.4. *Let us notice that the condition* (6.6) *is really necessary to be stated explicitly. Indeed, from the physical point of view, the frame-indifference* (6.5) *requires that $W(F) := \tilde{W}(C)$, where $C := F^\top F$ is the so-called right Cauchy–Green strain tensor. Note that $F^\top Q^\top Q F = F^\top F$ for every orthogonal matrix Q. Hence, pointwise minimizers of the energy density W contain the set $\{QF_0 : Q \in \mathrm{O}(n)\}$ for some given matrix F_0 with $\det F_0 > 0$, which is itself a pointwise minimizer. Besides the physically acceptable energy wells $\{RF_0 : R \in SO(n)\}$, other minimizers live on wells $\{RF_0 : R \in \mathrm{O}(n) \setminus \mathrm{SO}(n)\}$ which are not mechanically admissible. Those wells are excluded by* (6.6).

In order to prove existence of stable states, that is, minimizers of (6.8), we assume suitable coercivity of the energy density:

(6.9) $W(F) \geq c(-1 + |F|^p)$ for all $F \in \mathbb{R}^{n \times n}$ and for some $c > 0$.

The existence theorem then follows directly from Corollary 6.5, where we replace f by W.

THEOREM 6.12. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz bounded domain, $p > 3$, $u_0 \in W^{1,p}(\Omega; \mathbb{R}^3)$, and let $\Gamma_D \subset \partial\Omega$ have a finite two-dimensional Lebesgue measure. Let W satisfy the assumptions* (i)–(iii) *from Corollary* 6.5 *posed on f with $m = n = 3$. Further, let* (6.5)–(6.7) *and* (6.9) *hold. If*

$$\mathcal{Y} := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u = u_0 \ on \ \Gamma_D, \det \nabla u > 0\} \neq \emptyset$$

is such that $\inf_{\mathcal{Y}} I < +\infty$, then there is a minimizer of $\mathcal{E}$ on $\mathcal{Y}$.

This result can be generalized for different coercivity conditions like the one considered in (6.13) below. Even more general settings can be found in [62], where various additional requirements on minimizers, such as, e.g., conditions ensuring a frictionless contact (Signorini problem), are also included.

Let us mention a few important examples of polyconvex stored energy densities. Contrary to nontrivial examples of quasiconvex functions, it is relatively easy to design a polyconvex function. For ease of notation we only define the densities for matrices of positive determinant. Otherwise, the energy density is implicitly extended by infinity. We refer to [211, 212, 213] for examples of polyconvex functions with various special symmetries.

EXAMPLE 6.5 (compressible Mooney–Rivlin material). *This material has a stored energy of the form*

$$(6.10) \qquad W(F) = a|F|^2 + \tilde{a}|\mathrm{Cof}\,F|^2 + \gamma(\det F),$$

where $a, \tilde{a} > 0$ and $\gamma(\delta) = c_1\delta^2 - c_2\log\delta$, $c_1, c_2 > 0$.
    It can be shown that for $n = 3$,

$$W(F) = \frac{\lambda}{2}(\mathrm{tr}E)^2 + \mu|E|^2 + \mathcal{O}(|E|^3) \ , \ E = (C - \mathbb{I})/2,$$

where λ and μ are the usual Lamé constants, and $\mathbb{I}$ denotes the identity matrix. Indeed, it is a matter of a tedious computation to show that, given λ, μ, the following equations must be fulfilled by a, b, c_1, c_2: $c_2 := (\lambda + 2\mu)/2$, $2a + 2b = \mu$, and $4b + 4c_1 = \lambda$.

EXAMPLE 6.6 (compressible neo-Hookean material). *This material has a stored energy of the form*

$$(6.11) \qquad W(F) = a|F|^2 + \gamma(\det F)$$

with the same constants as for the compressible Mooney–Rivlin material.

EXAMPLE 6.7 (Ogden material). *This material has a stored energy of the form (recall that $C = F^\top F$)*

$$(6.12) \qquad W(F) = \sum_{i=1}^{M} a_i \mathrm{tr}\, C^{\gamma_i/2} + \sum_{i=1}^{N} \tilde{a}_i \mathrm{tr}(\mathrm{Cof}\,C)^{\delta_i/2} + \gamma(\det F),$$

where $a_i, \tilde{a}_i > 0$, $\lim_{\delta\to 0_+} \gamma(\delta) = +\infty$ for $\gamma : \mathbb{R}_+ \to \mathbb{R}$ convex growing suitably at infinity.

If W satisfies conditions (6.6)–(6.7), then any $u \in C^1(\Omega, \mathbb{R}^3)$ for which $\mathcal{E}(u)$ from (6.8) is finite is also locally invertible. This follows from the standard inverse function theorem. Nevertheless, what is actually desired for a physical deformation is that it is *injective* [62]. Indeed, noninjectivity of the deformation would mean that two material points from the reference configuration would be mapped to just one in the deformed configuration, which means that the specimen has penetrated through itself. Thus, additional assumptions to (6.6)–(6.7) on W are needed to assure *global invertibility of u*. Preferably, these assumptions should be compatible with polyconvexity and weak lower semicontinuity.

Take a diffeomorphism $u : \Omega \to u(\Omega)$ with $\det\nabla u > 0$ on Ω. Then we have by the change of variables formula for $p > 1$,

$$\int_{u(\Omega)} |\nabla u^{-1}(w)|^p \mathrm{d}w = \int_\Omega |\nabla u^{-1}(u(x))|^p \det\nabla u(x)\mathrm{d}x$$

$$= \int_\Omega |(\nabla u(x))^{-1}|^p \det\nabla u(x)\mathrm{d}x = \int_\Omega \frac{|\mathrm{Cof}^\top \nabla u(x)|^p}{(\det\nabla u(x))^{p-1}}\mathrm{d}x,$$

where we have used that $\nabla u^{-1}(u(x)) = (\nabla u(x))^{-1}$ for all x in Ω and that for every invertible matrix the relation $A^{-1} = \frac{\mathrm{Cof}^\top A}{\det A}$ holds.

Therefore, for energies satisfying a stricter coercivity condition than (6.9) in the form of

$$(6.13) \qquad W(F) \geq c\left(-1 + |F|^p + \frac{|\mathrm{Cof}\,F^\top|^p}{(\det F)^{p-1}}\right) \text{ for some } c > 0,$$

one might well expect that deformations on which $\mathcal{E}(u)$ is finite are invertible. This is indeed so, as Theorem 6.13 shows.

Nevertheless, before proceeding to the theorem, let us point out that the new growth condition (6.13) is *fully compatible with polyconvexity*. Indeed, since the function g defined by $g(x, y) := \frac{x^p}{y^{p-1}}$ is convex for $p > 1$ on the set $\{(x, y) \in \mathbb{R}^2; y > 0\}$, $\frac{|\mathrm{Cof}^\top F|^p}{(\det F)^{p-1}}$ is polyconvex on the set of matrices having a positive determinant.

THEOREM 6.13 (due to Ball [15]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $u_0 : \overline{\Omega} \to \mathbb{R}^n$ be continuous in $\overline{\Omega}$ and one-to-one in Ω such that $u_0(\Omega)$ is also bounded and Lipschitz. Let $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ for some $p > n$, $u(x) = u_0(x)$ for all $x \in \partial\Omega$, and let $\det \nabla u > 0$ a.e. in Ω. Finally, assume that for some $q > n$*

$$(6.14) \qquad \int_\Omega |(\nabla u(x))^{-1}|^q \det \nabla u(x) \, \mathrm{d}x < +\infty \ .$$

Then $u(\overline{\Omega}) = u_0(\overline{\Omega})$ and u is a homeomorphism of Ω onto $u_0(\Omega)$. Moreover, the inverse map $u^{-1} \in W^{1,q}(u_0(\Omega); \mathbb{R}^n)$ and $\nabla u^{-1}(w) = (\nabla u(x))^{-1}$ for $w = u(x)$ and a.a. $x \in \Omega$.

Let us note that the Sobolev regularity needed in this theorem was later weakened in [227]. Indeed, in this work it was shown that an inverse to deformation can be defined even for $p > n - 1$ and $q \geq \frac{p}{p-1}$.

Theorem 6.13 ensures injectivity of u under the growth (6.13) if an up-to-the-boundary injective Dirichlet condition is prescribed. This, however, has the disadvantage that we cannot use it to model situations in which hard loads (Dirichlet boundary conditions) are prescribed on only a part on the boundary.

One possible remedy is to minimize $\mathcal{E}$ along with the so-called *Ciarlet–Nečas* condition

$$(6.15) \qquad \int_\Omega \det \nabla u(x) \mathrm{d}x \leq \mathcal{L}^n(u(\Omega)),$$

which was introduced in [63] (for $n = 3$) in order to assure *global injectivity* of deformations. It was shown in [63] that C^1-functions satisfying (6.15) and $\det \nabla u > 0$ are actually injective. The result generalizes to $W^{1,p}$-functions as well, but injectivity is obtained only almost everywhere in the deformed configuration; i.e., almost every point in the deformed configuration has only one preimage.

REMARK 6.8. *Maps that are injective almost everywhere in the deformed configuration still include rather nonphysical situations. For example, a dense, countable set of points could be mapped to one point. This can be prevented if the deformation is injective everywhere.*

Using condition (6.15), this can be achieved for finite deformations of the energy $\mathcal{E}$ with a density W satisfying (6.13) for $p = m = n = 2$. This setting is the one most explored due to its relationship to quasiconformal maps (see section 7). Such deformations are open (that is, they map open sets to open sets) and discrete (the set of preimages for every point does not accumulate) and, moreover, they satisfy the Lusin N-condition (i.e., they map sets of zero measure again to sets of zero measure); cf., e.g., [119]).

Then, we have by the area formula

$$\int_\Omega \det \nabla u \, \mathrm{d}x = \int_{\mathbb{R}^n} N(u, \Omega, z) \, \mathrm{d}z = \int_{u(\Omega)} N(u, \Omega, z) \, \mathrm{d}z,$$

where $N(u, \Omega, z)$ is defined as the number of preimages of $z \in u(\Omega)$ in Ω. So the Ciarlet–Nečas condition is satisfied if and only if $N(u, \Omega, z) = 1$ a.e. on $u(\Omega)$. Also, we can immediately see that the reverse inequality to (6.15) always holds.

Further, if there existed $z \in u(\Omega)$ that had at least two preimages x_1 and x_2, then we could find an $\varepsilon > 0$ such that $B(x_1, \varepsilon) \cap B(x_2, \varepsilon) = \emptyset$ and $B(x_j, \varepsilon) \subset \Omega$ for $j = 1, 2$. On the other hand, for the images we have that $u(B(x_1, \varepsilon)) \cap u(B(x_2, \varepsilon)) \neq \emptyset$. In fact, $u(B(x_1, \varepsilon)) \cap u(B(x_2, \varepsilon))$ is of positive measure since both $u(B(x_j, \varepsilon))$ are open. Therefore, there exists a set of positive measure where $N(u, \Omega, z)$ is at least two, which is a contradiction to (6.15).

6.3. Applications to Hyperelasticity in the Higher-Order Setting. Let us now turn our attention to models of hyperelastic materials depending on higher-order gradients. Such materials are called nonsimple of grade k, where k refers to the highest derivatives appearing in the stored energy density. The concept of such materials has been developing for a long time, since the work by R.A. Toupin [234], under various names such as nonsimple materials in, e.g., [98, 140, 196, 221] or multipolar materials (in particular, in fluids).

Here, we will consider only second-grade nonsimple materials, i.e., those for which second-order deformation gradients (first-order strain gradients) are involved. The main mathematical advantage of nonsimple materials is that higher-order deformation gradients bring additional regularity of deformations and, possibly, also compactness in a stronger topology. Moreover, taking the stored energy to be even convex in the highest derivatives of the deformation does not contradict the basic physical requirements, which is helpful in proving existence of minimizers. The downside of this approach is that there are very few physically justified models of nonsimple materials and material constants are rarely available.

For nonsimple materials of the second grade, we define the energy functional

$$
\mathcal{E}(u) := \int_\Omega W(\nabla u(x), \nabla^2 u(x)) \, dx - \int_\Omega b(x) \cdot u(x) \, dx
$$

(6.16)

$$
- \int_{\Gamma_N} \left(g(x) \cdot u(x) + \hat{g}_1(x) \cdot \frac{\partial u(x)}{\partial \varrho} \right) dS \, ,
$$

where ϱ is the outer unit normal to Γ_N and $\hat{g}_1 : \Gamma_N \to \mathbb{R}^n$ is the surface density of (hypertraction) forces balancing the *hyperstress*

(6.17)
$$
x \mapsto \frac{\partial}{\partial G_{ijk}} W(F, G)|_{F = \nabla u(x), \, G = \nabla^2 u(x)} \, .
$$

The corresponding first Piola–Kirchhoff stress tensor is constructed as follows. Denote for $i, j \in \{1, \ldots, n\}$

$$
H_{ij}(F, G) := \sum_{k=1}^n \frac{\partial}{\partial G_{ijk}} W(F, G) \, .
$$

Then for $x \in \Omega$, $F := \nabla y(x)$, and $G := \nabla^2 y(x)$ we evaluate the first Piola–Kirchhoff stress tensor as

$$
S_{ij}(x) = \frac{\partial W(\nabla u(x), \nabla^2 u(x))}{\partial F_{ij}} - H_{ij}(\nabla u(x), \nabla^2 u(x)) \, .
$$

We will assume that

$$(6.18) \qquad u \mapsto \int_\Omega b(x) \cdot u(x) \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \left(g(x) \cdot u(x) + \hat{g}_1(x) \cdot \frac{\partial u(x)}{\partial \varrho} \right) \mathrm{d}S$$

is a linear continuous functional evaluating the work of external forces on the specimen. Here, however, we assume for simplicity that b, $\hat{g}_1$, and g depend only on $x \in \Omega$ and $x \in \Gamma_{\mathrm{N}}$, respectively. Notice that the existence of minimizers of $\mathcal{E}$ is guaranteed by Corollary 6.6.

Similarly, as in the case of simple materials, we may formally derive the Euler–Lagrange equations for minimizers of $\mathcal{E}$. Interestingly, second-grade materials together with suitable integrability of $1/\det \nabla u$ imply a strictly positive lower bound of $\det \nabla u$ on the whole closure of Ω. This enables us to show that minimizers of the energy functionals are weak solutions to the corresponding Euler–Lagrange equations; cf. [117]. Unlike the simple material situation, here the smoothness of $\partial \Omega$ is important because the mean curvature of the boundary enters the equations. Details on surface differential operators can be found, for example, in [195].

7. Weak Lower Semicontinuity in General Hyperelasticity. We have seen in the last section that polyconvexity is relatively easy to verify and ensures weak lower semicontinuity of the corresponding energy functional. Nevertheless, there are materials that cannot be modeled by polyconvex energy densities.

A prototypical example is that of systems featuring phase transition with each phase characterized by some specific deformation of the underlying atomic lattice. This setup is, for example, found in *shape-memory alloys* (see, e.g., the monographs [41, 77, 96, 97, 194], [20], or the recent review [128]). Shape-memory alloys are intermetallic materials which have a high temperature highly symmetric phase called austenite and a low temperature phase called martensite, they can, however, exist in several variants. Such systems are (for a suitable temperature range) typically modeled by a multiwell stored energy of the form

$$(7.1) \qquad W(F) \begin{cases} = 0 & \text{if } F = QU_i \text{ for some } i = 1, \ldots, M \text{ and some } Q \ \in \mathrm{SO}(n), \\ > 0, & \text{otherwise.} \end{cases}$$

where $U_1, \ldots, U_M \in \mathbb{R}^{n \times n}$ are given matrices representing the phases found in the material and $\mathrm{SO}(n)$ is the set of rotations in $\mathbb{R}^{n \times n}$. These materials form complicated patterns (microstructures) composed from different variants of martensite; cf. Figure 1.

Energy densities satisfying (7.1) are generically neither polyconvex nor quasiconvex and their construction is a modeling issue [240]. Therefore, in order to design an appropriate model one has to find the *weakly lower semicontinuous envelope* of (1.7) with an energy density given by (7.1); in other words, one seeks the supremum of weakly lower semicontinuous functionals lying below the given energy. We refer also to section 7.1 for more details on how this *relaxation* of the problem might be performed. Let us remark that the need to perform a relaxation persists even if we use a geometrically linear description of energy wells; see [60, 61, 106, 136, 217], for instance.

In order to find the weakly lower semicontinuous envelope of (7.1), a precise characterization of weak lower semicontinuity in terms of convexity conditions on W is needed. We found these conditions in section 3; though only under the growth condition (i) in Definition 3.3. However, this is incompatible with the physical assumptions formulated in (6.6)–(6.7). Indeed, notice that (6.6) requires the stored energy to be infinite if $\det F \leq 0$, while growth condition (i) in Definition 3.3 corresponds to p-
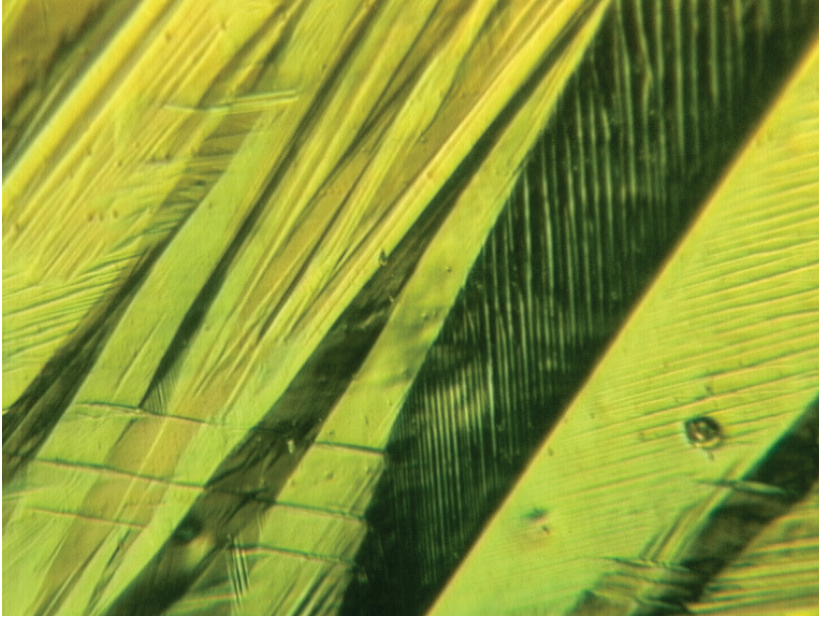
Fig. 1 *Laminated microstructure in CuAlNi. Courtesy of P. Šittner (Institute of Physics, CAS, Prague).*

growth which, roughly speaking, means that the stored energy is bounded from above by a polynomial of the pth order and so is finite for all its arguments.

For energies taking infinite values it is no longer known that quasiconvexity implies weak lower semicontinuity. Indeed, this is one of the outstanding problems in elasticity, which was formulated by J.M. Ball in the following manner.

OPEN PROBLEM 7.1 (Problem 1 in [17]). *"Prove the existence of energy minimizers for elastostatics for quasiconvex stored-energy functions satisfying* (6.7).*"*

While Problem 7.1 is motivated by elasticity, it leads to a study of weak lower semicontinuity that is interesting in its own right. Therefore, we shall denote the integrands by f in this section as is done elsewhere in the article and use the notation W only if the stored energy of an elastic material is specifically meant.

In order to better see why the methods from section 3 might fail once one considers integrands taking infinite values, let us revisit Example 1.2 given in the introduction.

EXAMPLE 7.1 (Example 1.2 revisited). *In Example* 1.2*, we studied weak lower semicontinuity of the functional*

$$\tilde{I}(u) = \int_\Omega f(\nabla u(x)) \mathrm{d}x,$$

along the sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $u_k \overset{*}{\rightharpoonup} Ax$ with A some matrix in $\mathbb{R}^{m\times n}$, and we saw that if f is quasiconvex and finite on $\mathbb{R}^{n\times m}$, then $\tilde{I}$ is weakly lower semicontinuous. A key step in the proof was the cut-off *procedure, i.e., the construction of a modified sequence*

$u_{k,\ell}(x) = \eta_\ell u_k + (1-\eta_\ell)Ax \quad such\ that \quad \nabla u_{k,\ell}(x) = \eta_\ell \nabla u_k + (1-\eta_\ell)A + (u_k - Ax) \otimes \nabla \eta_\ell,$

which fulfills $u_{k,\ell}(x) = Ax$ on $\partial\Omega$, as well as the realization that once $u_{k,\ell}(x)$ is used

as a test function in (1.8), we have that

$$(7.2) \quad \begin{aligned} f(A)\mathcal{L}^n(\Omega) &\leq \int_\Omega f(\nabla u_{k(\ell),\ell}(x))\mathrm{d}x \\ &= \int_\Omega f(\nabla u_{k(\ell)})\mathrm{d}x + \int_{\Omega\setminus\Omega_\ell} f(\nabla u_{k(\ell),\ell}(x)) - f(\nabla u_{k(\ell)}(x))\mathrm{d}x, \end{aligned}$$

where the last integral vanishes if f is continuous on $\mathbb{R}^{m\times n}$ because the measure of $\Omega \setminus \Omega_\ell$ converges to zero.

Now, if f is allowed to take infinite values, the last step is no longer justified. In fact, even if we assume that the matrix A as well as the sequence $\{u_k\}_{k\in\mathbb{N}}$ are such that $f(A) < \infty$ as well as $f(\nabla u_k(x)) < \infty$ for almost every $x \in \Omega$, there is, a priori, no guarantee that $f(\nabla u_{k,\ell}(x)) = +\infty$ on a set of positive measure in $\Omega \setminus \Omega_\ell$ can be excluded. However, in this case, $\int_{\Omega\setminus\Omega_\ell} f(\nabla u_{k(\ell),\ell}(x)) - f(\nabla u_{k(\ell)}(x))\mathrm{d}x = +\infty$ so that, in particular, this term cannot vanish.

Example 7.1 presents a simplified setting that, however, contains all the difficulty in proving that quasiconvex integrands taking infinite values are weakly lower semicontinuous. In fact, to the best of our knowledge, all proofs available in the literature showing that quasiconvexity implies weak lower semicontinuity, starting from the works of Morrey [178] and Meyers [171], are built upon the same principle as the one presented in Examples 1.2 and 7.1 (also see the monograph [67]). In particular, the cut-off method plays a crucial role, which, as seen in Example 7.1, might result in the consequence that the method of proof becomes unusable for integrands taking values in $\mathbb{R}\cup\{+\infty\}$. Here, the strength of polyconvexity should be again highlighted, since for functionals with polyconvex integrands weak lower semicontinuity is proved by a different method; see section 6.

REMARK 7.2. *Let us note that if the set on which f is finite is* convex, *the procedure in Example 7.1 might still work. Indeed, in this case, if we assume that A as well as $\{u_k\}_{k\in\mathbb{N}}$ are such that $f(A) < \infty$ as well as $f(\nabla u_k(x)) < \infty$ for almost every $x \in \Omega$, then $f(\eta_\ell\nabla u_k + (1-\eta_\ell)A) < \infty$ a.e. on Ω due to convexity. This still does not mean that $f(\nabla u_{k,\ell}(x)) < \infty$ due to the small error $(u_k - Ax)\otimes\nabla\eta_\ell$, because of which $\nabla u_{k,\ell}(x)$ might no longer lie in the set where f takes finite values. Nevertheless, by suitable scaling and continuity of f on its domain, this difficulty can be overcome, at least if the domain of f is a ball; see [133]. On the other hand, if the domain of f is not convex, we can expect that $f(\eta_\ell\nabla u_k(x) + (1-\eta_\ell)A) = \infty$ for some $x \in \Omega$ even if $f(A), f(\nabla u_k(x))$ is finite, which is the scenario discussed in Example 7.1.*

The physical requirements dictated by (6.6)–(6.7) force the stored energy to be finite only on matrices with a positive determinant, which is a *nonconvex* set. In view of the above remark, this means that one faces exactly the situation from Example 7.1, in which the available techniques of proving that quasiconvex energies are weakly lower semicontinuous reach their limit. This is also what makes Problem 7.1 so challenging.

REMARK 7.3. *Notice that if (7.1) is additively enriched by a convex term of the form $\varepsilon \int_\Omega |\nabla^2 y|^p\mathrm{d}x$, which is usually interpreted as some kind of interfacial energy of the microstructure, Corollary 6.6 can be readily applied to show the existence of minimizers for $\mathcal{E}$.*

Let us also point out that a different approach was proposed recently [223, 224]. There, a new notion of interface polyconvexity was introduced which enables the proof

of the existence of minimizers for simple materials with an additional phase field variable.

REMARK 7.4. *It has been pointed out in* [17, 18] *that one of the reasons why Open Problem* 7.1 *is hard to solve is the fact that quasiconvexity possesses no local characterization* [146].

Let us stress that Problem 7.1 is an important attempt toward combining quasi-convexity and elasticity, but additional steps are still required. Namely, if $u : \Omega \to \mathbb{R}^m$ entering (1.7) should represent a deformation of a physical body, it should be *injective* and *orientation-preserving*. It is even natural to require that the deformation is a *homeomorphism*, at least if one restricts attention to the case in which the formation of cavities and cracks is excluded; we refer to, e.g., [118] for problems connected to the appearance of cavities. Notice that this is not automatically satisfied for all maps on which the functional (1.7) is finite, even if f fulfills (6.6)–(6.7), but we can rely on Theorem 6.13 to assure it, provided suitable coercivity of the energy.

An alternative (and related) approach is to study directly weak lower semicontinuity along sequences found in a suitable class of mappings that are injective and orientation-preserving. As a first step, one may study classes of functions that fulfill some constraint on the Jacobian, e.g., that $\det\nabla u > 0$.

REMARK 7.5. *Clearly, the positivity of the Jacobian* $\det\nabla u > 0$ *does not imply that the mapping u is a homeomorphism. However, surprisingly, the converse implication also does not necessarily hold, even though a homeomorphism on a domain is necessarily sense-preserving (or sense-reversing) in the topological sense. We refer to an example by Goldstein and Hajłasz* [105]*, where a sense-preserving homeomorphism with* $\det\nabla u = -1$ *a.e. in* Ω *was constructed. This phenomenon concerns functions with a low Sobolev differentiability. Indeed, it was shown by Hencl and Malý* [120] *that any sense-preserving homeomorphism in* $W^{1,p}(\Omega;\mathbb{R}^m)$ *satisfies* $\det\nabla u > 0$ *a.e. on* Ω *if* $p > \left[\frac{n}{2}\right]$.[9] *Recently, it was shown by Campbell, Hencl, and Tengvall* [59] *that the exponent is critical in the sense that if* $n \geq 4$ *and* $p < \left[\frac{n}{2}\right]$*, then there exists a homeomorphism in* $W^{1,p}(\Omega;\mathbb{R}^m)$ *the Jacobian of which is positive and negative on a set of positive measure, respectively.*

Even though Problem 7.1 remains widely open to date, it has recently been approached from different perspectives. We review the results within this section, all of which are based on changing the cut-off technique introduced in Examples 1.9 and 7.1 in order to avoid the convex averaging.

In [141, 142], Koumatos, Rindler, and Wiedemann study weak lower semicontinuity along sequences in $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ with $p < n$ satisfying $\det\nabla u_k > 0$. They prove that (1.7) with $f = f(x, \nabla u)$ is weak lower semi-continuous along such sequences if and only if it is $W^{1,p}$-*orientation-preserving quasiconvex*, i.e., for almost all $x \in \Omega$,

$$f(x, A) \leq \frac{1}{\mathcal{L}^n(\Omega)} \int_{\Omega} f(x, \nabla\varphi(z))\mathrm{d}z$$

for all A with $\det(A) > 0$ and all $\varphi \in W^{1,p}(\Omega;\mathbb{R}^m)$ satisfying that $\varphi(z) = Az$ on $\partial\Omega$ and $\det\nabla\varphi(z) > 0$ for a.a. $z \in \Omega$.

However, in [142] the authors also show that, in fact, for $p < n$ no $W^{1,p}$-orientation-preserving quasiconvex integrands exist that satisfy the natural coercivity/

---

[9]Here $[\cdot]$ denotes the integer part.

growth condition

$$\frac{1}{c}\big(|A|^p + \kappa(\det A)\big) \le f(x,A) \le c\big(|A|^p + \kappa(\det A)\big)$$

for almost all $x \in \Omega$. Here $c > 0$ is a constant and $\kappa > 0$ is a convex function satisfying $\lim_{s \to 0} \kappa(s) = +\infty$, $\kappa(s) = +\infty$ for $s \le 0$ and $\limsup_{s \to \infty} \frac{\kappa(s)}{s^{p/n}} < +\infty$. Notice that this growth condition is compatible with (6.6)–(6.7).

The proof in [141] is based on so-called *convex integration*, a technique for solving differential inclusions. It goes back to Nash [188], Kuiper [155, 156], and later Gromov [112], and it has found applications in various problems including continuum mechanics (interestingly in continuum mechanics of solids as well as fluid dynamics) and regularity theory; see, e.g., [75, 216, 209, 187]. We refer also to the monograph [68], where solutions to partial differential inclusions by means of Baire category methods are introduced.

To the best of our knowledge, the only works in which the authors actually consider equivalent characterization of weak lower semicontinuity for *injective maps* are [38] and [36], where bi-Lipschitz and quasiconformal maps *in the plane* are studied, respectively.

Here, by bi-Lipschitz maps the following set is meant:

$$W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2) = \Big\{ u : \Omega \mapsto u(\Omega) \text{ an orientation-preserving homeomorphism;}$$

$$(7.3) \qquad\qquad u \in W^{1,\infty}(\Omega;\mathbb{R}^2) \text{ and } u^{-1} \in W^{1,\infty}(u(\Omega);\mathbb{R}^2) \text{ is Lipschitz}\Big\},$$

while quasiconformal maps are introduced as follows:

$$\mathcal{QC}(\Omega;\mathbb{R}^2) = \Big\{ u \in W^{1,2}(\Omega;\mathbb{R}^2) : \ u \text{ is a homeomorphism and } \exists K \ge 1 \text{ such that}$$

$$(7.4) \qquad\qquad |\nabla u|^2 \le K \det \nabla u \text{ a.e. in } \Omega \Big\}.$$

It is natural to expect that weak lower semicontinuity of the functional

$$I(u) = \int_\Omega f(\nabla u)\mathrm{d}x,$$

along sequences in $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega;\mathbb{R}^2)$ is connected with a suitable notion of quasiconvexity of f. One even expects a *weaker* notion than the one from Definition 1.3, since the set of possible sequences along which semicontinuity is studied is restricted. Indeed, the most appropriate notion for this setting seems to be an alternation of Definition 1.3 in which only functions from $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega;\mathbb{R}^2)$ enter as test functions. Precisely this was achieved in [38] and [36]; we review the result in Proposition 7.3.

First, let us introduce a notion of weak convergence on $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ and $\mathcal{QC}(\Omega;\mathbb{R}^2)$. We say that $u_k \overset{*}{\rightharpoonup} u$ in $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ if the sequence has uniformly bounded bi-Lipschitz constants[10] and $u_k \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega;\mathbb{R}^2)$. Note that the weak limit is bi-Lipschitz, too.

---

[10]Notice that a function $u \in W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ satisfies for all $x_1, x_2 \in \Omega$

$$(7.5) \qquad\qquad \frac{1}{\ell}|x_1 - x_2| \le |u(x_1) - u(x_2)| \le \ell|x_1 - x_2|$$

for some $\ell \ge 1$. This ℓ is then called the bi-Lipschitz constant of u.

For a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{QC}(\Omega;\mathbb{R}^2)$, we say that it converges weakly to $u \in W^{1,2}(\Omega;\mathbb{R}^2)$ in $\mathcal{QC}(\Omega;\mathbb{R}^2)$ if for $u_k \rightharpoonup u$ in $W^{1,2}(\Omega;\mathbb{R}^2)$, there exists a $K \geq 1$ such that the u_k are all K-quasiconformal and u is nonconstant. Here it is important to assume that the limit function is nonconstant, for otherwise the limit function might not be quasiconformal.[11]

Moreover, let us introduce the notions of *bi-quasiconvexity* and *quasiconformal quasiconvexity*.

DEFINITION 7.2. *We say that a Borel measurable and bounded from below function* $f : \mathbb{R}^{2\times2} \to \Omega$ *is bi-quasiconvex if*

$$(7.6) \qquad \mathcal{L}^2(\Omega)f(A) \leq \int_\Omega f(\nabla\varphi(x))\,\mathrm{d}x$$

for all $\varphi \in W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$, $\varphi = Ax$ *on* $\partial\Omega$, *and all* A *with* $\det A > 0$.
We say that f *is quasiconformally quasiconvex if* (7.6) *holds for all* A *with* $\det(A) > 0$ *and all* $\varphi \in \mathcal{QC}(\Omega;\mathbb{R}^2)$ *such that* $\varphi(x) = Ax$ *on* $\partial\Omega$.

Then we have the following result:

PROPOSITION 7.3 (due to Benešová, Kampschulte, and Kružík [38, 36]). *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded Lipschitz domain. Let* f *be continuous on the set of matrices that are* 2×2 *with a positive determinant. Then* f *is bi-quasiconvex if and only if*

$$u \mapsto I(u) = \int_\Omega f(\nabla u(x))\,\mathrm{d}x$$

is sequentially weakly* lower semicontinuous on $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$.
Moreover, let f *satisfy*

$$0 \leq f(A) \leq c(1 + |A|^2) \qquad \text{with } c > 0$$

on the set of matrices with a positive determinant. Then f *is quasiconformally quasiconvex if and only if* I *is weakly lower semicontinuous on* $\mathcal{QC}(\Omega;\mathbb{R}^2)$.

Let us briefly comment on the proof of Proposition 7.3 given in [38] and [36]; it is based on finding a cut-off technique that can cope with the nonconvexity of the set of homeomorphisms. The idea is that constructing a cut-off is very much related to understanding the trace operator. Indeed, generally speaking, we may formulate the cut-off problem as follows: given a Lipschitz domain Ω, find another domain $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| \leq \delta$ and a function (or *deformation*) in the considered set (here $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega;\mathbb{R}^2)$) such that it takes some prescribed values on $\partial\Omega$ and in Ω_δ. Reformulating this once again, we might ask to find a function from the considered set of functions on $\Omega \setminus \Omega_\delta$ (here a function in $W_+^{1,\infty,-\infty}(\Omega \setminus \Omega_\delta;\mathbb{R}^2)$ or $\mathcal{QC}(\Omega \setminus \Omega_\delta;\mathbb{R}^2)$) that has some prescribed boundary values on $\partial(\Omega \setminus \Omega_\delta)$, i.e., on $\partial\Omega$ and $\partial\Omega_\delta$. It is clear that not all boundary data will admit such an extension (even not all smooth data); to see this recall that the Jacobian is a null Lagrangian (cf. section 4) which, for example, immediately excludes all affine mappings with negative Jacobian as boundary data.[12]

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[11]This is because a sequence of uniformly K-quasiconformal maps converges locally uniformly either to a K-quasiconformal function or a constant [7], and the locally uniform convergence is implied by the notion of weak convergence in $\mathcal{QC}(\Omega;\mathbb{R}^2)$.

[12]Let us note here the contrast to the situation considered by Koumatos, Rindler, and Wiedemann [141]. In fact, it was shown in [141] that every smooth function on the boundary of a Lipschitz domain admits an extension in $y \in W^{1,p}(\Omega;\mathbb{R}^n)$ with $\det \nabla y > 0$ a.e. in Ω if $p < n$.

The characterization of the trace operator on sets $W_+^{1,\infty,-\infty}(\Omega;\mathbb{R}^2)$ as well as $\mathcal{QC}(\Omega;\mathbb{R}^2)$ is Ω due to [73, 235] and [40], respectively. However, the works [73, 235, 40] consider only special geometries of Ω (for example, Ω can be chosen as a square) but they are not appropriate for a doubly-connected domain like $\Omega \setminus \Omega_\delta$. This difficulty was overcome in [38, 36] by suitable meshing $\Omega \setminus \Omega_\delta$ and by defining the cut-off on the grid of the mesh. Even though Proposition 7.3 provides us with a weak lower semicontinuity result, this is not yet enough to prove existence of minimizers for functionals with densities from some suitable class. This holds because bi-Lipschitz as well as quasiconformal maps include a L^∞-type constraint which can be enforced by letting the stored energy density be finite only on a suitable subset of $\mathbb{R}^{2\times2}$; yet, this subset is usually left when employing cut-off methods—this happens even in the standard cases where, however, the issue can be solved by scaling (see Remark 7.2). Thus, letting f become infinite on a subset of $\mathbb{R}_+^{2\times2}$ is incompatible with the proof of Proposition 7.3.

The usual remedy for proving existence of minimizers or relaxation results is to work with L^p-type (with p finite) constraints only. In the setting above, this would mean working with so-called *bi-Sobolev* classes (see, e.g., [122]) for $1 < p < \infty$:

$$W_+^{1,p,-p}(\Omega;\mathbb{R}^2) = \Big\{ u : \Omega \mapsto u(\Omega) \text{ an orientation-preserving homeomorphism;}$$
$$u \in W^{1,p}(\Omega;\mathbb{R}^2) \text{ and } u^{-1} \in W^{1,p}(u(\Omega);\mathbb{R}^2) \Big\}.$$

However, for these classes of functions, the approach from [38] and [36] cannot be adopted since, as we explained above, it relies on having a complete characterization of the trace operator, which has not yet been achieved for these classes. In fact, we have the following open problem.

OPEN PROBLEM 7.4. *Characterize the class of functions* $\mathcal{X}(\partial\Omega;\mathbb{R}^2)$ *such that*

$$\mathrm{Tr} : W_+^{1,p,-p}(\Omega;\mathbb{R}^2) \xrightarrow{onto} \mathcal{X}(\partial\Omega;\mathbb{R}^2)$$

with $1 < p < \infty$, *at least for* Ω *the unit square.*

Let us note that the above problem may also play a role in constructing smooth approximations (by diffeomorphisms) of deformations in elasticity. Indeed, the standard techniques of smoothing Sobolev functions (by a mollification kernel) fail under the injectivity requirement since they essentially rely on convex averaging.

Recently, several results on smoothing even under these constraints appeared [127, 72, 177, 121, 123] that use completely different techniques and limit their scope to planar deformations. In particular, Iwaniec, Koskela, and Onninen prove in [127] that a homeomorphism in $W^{1,p}(\Omega;\mathbb{R}^2)$ can be strongly approximated by diffeomorphisms in the $W^{1,p}$-norm for $p > 1$. For $p = 1$, this result has recently been extended in [123].

Nevertheless, in elasticity, one might rather be interested in approximating a function in $W_+^{1,p,-p}(\Omega;\mathbb{R}^2)$ *together with its inverse*. To our knowledge, the only results in this direction are by Daneri and Pratelli [72], who showed that bi-Lipschitz maps can be strongly approximated together with their inverse in the $W^{1,p}$-norm for every finite p, and Pratelli [197], who proved that diffeomorphic approximation is possible in $W^{1,-1}(\Omega;\mathbb{R}^2)$. Yet, for functions in $W_+^{1,p,-p}(\Omega;\mathbb{R}^2)$ with $1 < p < \infty$ the problem remains largely open, as is also mentioned in [127].

To end this section, let us remark (by formulating several open problems) that the relationship between bi-quasiconvexity and the standard notions of quasiconvexity

mentioned in this article is still unexplored. We focus here only on bi-quasiconvexity, but similar problems could also be formulated for quasiconformal quasiconvexity.

It is clear from the definitions that any function that is quasiconvex on the set of matrices with a positive determinant is also bi-quasiconvex. Moreover, bi-quasiconvexity implies, at least in the plane, rank-one convexity on the set of matrices with a positive determinant.

REMARK 7.6. *To see why bi-quasiconvexity implies rank-one convexity on the set of matrices with a positive determinant, we proceed as follows. First, notice that the determinant changes affinely on rank-one lines due to the formula*

$$(7.7) \qquad \det(A + \lambda a \otimes n) = \det A\big(1 + \lambda n{\cdot}(A^{-1}a)\big),$$

where a and n are some arbitrary vectors. Therefore, rank-one convexity on the set of matrices with a positive determinant is really meaningful, since all matrices on a rank-one line between two matrices with a positive determinant have this property, too.

Next we mimic the proof from [67, Lemma 3.11 and Theorem 5.3] *showing that quasiconvexity implies rank-one convexity. Without loss of generality, we suppose that Ω is the unit square and that we want to show rank-one convexity along the line $A + a \otimes e_1$ with e_1 the unit vector in the first coordinate. Then we consider the sequence of mappings*

$$u_n(x) = u_n(x_1, x_2) = \begin{cases} Ax & \text{for } x_1 \in \left[\frac{k}{n}, \frac{k}{n} + \lambda\frac{1}{n}\right) \text{ for } k = 0, \dots, n-1, \\ (A + a \otimes e_1)x & \text{for } x_1 \in \left[\frac{k}{n} + \lambda\frac{1}{n}, \frac{k+1}{n}\right) \text{ for } k = 0, \dots, n-1, \end{cases}$$

with some $\lambda \in [0,1]$. Notice that $\{u_n\}_{n\in\mathbb{N}}$ are Lipschitz and injective and that $(\nabla y)^{-1}$ is uniformly bounded and $\det\nabla u$ is bounded away from zero. Thus, $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of uniformly bi-Lipschitz maps that converges weakly to $\lambda Ax + (1-\lambda)(A + a \otimes e_1)x$. We may therefore use the cut-off technique from [38] *to modify the sequence in such a way that it attains exactly the value of the weak limit at the boundary. Then, the same procedure as in* [67, Theorem 5.3] *gives rank-one convexity.*

In summary, we have the series of implications

quasiconvexity on $\mathbb{R}^{2\times 2}_+ \Rightarrow$ bi-quasiconvexity $\Rightarrow$ rank-one convexity on $\mathbb{R}^{2\times 2}_+$,

where we denote by $\mathbb{R}^{2\times 2}_+$ the 2×2 matrices with positive determinant. However, it is unclear whether some of the converse implications also hold. We have the following open problems.

OPEN PROBLEM 7.5. *Does rank-one convexity on $\mathbb{R}^{2\times 2}_+$ imply bi-quasiconvexity?*

OPEN PROBLEM 7.6. *Does bi-quasiconvexity imply quasiconvexity on $\mathbb{R}^{2\times 2}_+$?*

7.1. Relaxation of Non(quasi)convex Variational Problems. As we have already seen, mathematical (hyper)elasticity is one area of analysis where mechanical requirements are beyond the scope of the current tools and results available in the calculus of variations. Orientation-preservation and injectivity for simple nonpolyconvex materials are prominent examples. Resorting to nonsimple materials depending on second-order deformation gradients might seem to be a way out. The definition of a physically acceptable form of the higher-order energy density is, however, a largely open problem. See, e.g., [24] for a discussion on this topic.

Another approach is to accept the fact that our minimization problem may have *no solution* and to trace out the behavior of minimizing sequences driving the elastic energy functional to its infimum on a given set of deformations and deduce some effective material properties from their patterns. This is the idea of *relaxation* in variational calculus. We explain the main ideas using the following simplified example, for which f in (1.10) depends only on the first gradient. Assume we want to

$$(7.8) \qquad \text{minimize } \mathcal{E}(u) := \int_\Omega f(\nabla u(x)) \, \mathrm{d}x \qquad \text{for } u \in \mathcal{Y}.$$

Here, $\mathcal{Y}$ represents an admissible set of deformations equipped with some topology. In typical situations, $\mathcal{Y}$ is a subset of a Sobolev space and the topology is the weak one on this space. If no minimizer exists but the infimum is finite, we want to find a new functional $\mathcal{E}_R$ defined on some set $\mathcal{Y}_R$ such that the following properties hold:

(i) $\min_{\mathcal{Y}_R} \mathcal{E}_R = \inf_\mathcal{Y} \mathcal{E}$;

(ii) if $\{y_k\}_{k\in\mathbb{N}} \subset \mathcal{Y}$ is a minimizing sequence of $\mathcal{E}$, then its convergent subsequences converge (in the topology of $\mathcal{Y}_R$) to minimizers of $\mathcal{E}_R$ on $\mathcal{Y}_R$; and

(iii) any minimizer of $\mathcal{E}_R$ on $\mathcal{Y}_R$ is a limit of a minimizing sequence of $\mathcal{E}$.

Notice that it is already implicitly assumed in (i) that minimizers of $\mathcal{E}_R$ do exist on $\mathcal{Y}_R$. Conditions (ii) and (iii) state that, roughly speaking, there is a "one-to-one" correspondence between minimizing sequences of $\mathcal{E}$ and minimizers of $\mathcal{E}_R$. If (i)–(iii) hold, we say that $\mathcal{E}_R$ is the relaxation of $\mathcal{E}$ and that $\mathcal{E}_R$ is the *relaxed functional*. The concept of relaxation is also very closely related to Γ-convergence and Γ-limits introduced by E. de Giorgi. We refer to Braides [49] and Dal Maso [69] for a modern exposition and section 9 for further references.

If $\mathcal{Y} \subset W^{1,p}(\Omega; \mathbb{R}^n)$ and the continuous stored energy $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ fulfills

$$(7.9) \qquad c_0(-1 + |F|^p) \le f(F) \le c_1(1 + |F|^p)$$

with $c_1 > c_0 > 0$ and $1 < p < +\infty$ then Dacorogna [65] showed[13] that $\mathcal{Y}_R = \mathcal{Y}$ equipped with the weak convergence and

$$(7.10) \qquad \mathcal{E}_R(u) := \int_\Omega Qf(\nabla u(x)) \, \mathrm{d}x \ ,$$

where $Qf : \mathbb{R}^{m\times n} \to \mathbb{R}$ is the *quasiconvex envelope* (or quasiconvexification) of f, being the largest quasiconvex function not exceeding f. It can also be evaluated at any $A \in \mathbb{R}^{m\times n}$ as

$$(7.11) \qquad Qf(A) := \mathcal{L}^n(\Omega)^{-1} \inf_{\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \int_\Omega f(A + \nabla\varphi(x)) \, \mathrm{d}x \ .$$

Notice that the above formula (7.11) generally holds only for f locally finite (see [67] and also [21] for its regularity). The definition of Qf does not depend on the (Lipschitz) domain Ω, but as we see, the calculation of Qf requires us to again solve a minimization problem. Not surprisingly, there are only a few cases where Qf is known in closed form. We mention the works by DeSimone and Dolzmann [76], where

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[13]In fact, Dacorogna's result is stated for more general integrands, namely, $|f(F)| \le c(1 + |F|^p)$ with $Qf > -\infty$, In this case, however, fixed Dirichlet boundary conditions must inevitably be assigned on the whole of $\partial\Omega$. This is again strongly related to condition (ii) in Meyers' Theorem 3.4 and concentrations on the boundary discussed in section 3.1.

the authors calculate the quasiconvex envelope of the stored energy density arising in the modeling of nematic elastomers in three dimensions, and by LeDret and Raoult [158, 160, 198], where the quasiconvex envelope of an isotropic homogeneous Saint Venant–Kirchhoff energy density ($m = n = 3$)

$$W(F) := \frac{\mu}{4}|C - \mathbb{I}|^2 + \frac{\lambda}{8}(\operatorname{tr} C - 3)^2$$

is derived. Here, λ, μ are Lamé constants of the material and $C = F^\top F$ is the right Cauchy–Green strain tensor. Notice that W is convex in C, but it is not even rank-one convex in F.

As for relaxation of multivariant materials, we refer to [136], where a geometrically linear two-well problem is considered such that elasticity tensors corresponding to both wells are equal. It was later extended in [60], where nonequal elastic moduli are admitted. See also [137, 138, 139] for quasiconvexification in optimal design problems.

Recently, Conti and Dolzmann [64] proved that the expression of the quasiconvex envelope via (7.11) is valid even for functions taking infinite values and satisfying the growth condition

$$\begin{cases} c_0(-1 + |F|^p + \theta(\det F)) \le f(F) \le c_1(1 + |F|^p + \theta(\det F)) & \text{if } \det F > 0, \\ f(F) = +\infty & \text{otherwise.} \end{cases}$$

Here, $c_1 > c_0 > 0$, $p \ge 1$, and $\theta : (0; +\infty] \to [0; +\infty)$ is a suitable convex function. They, however, require that Qf is polyconvex for the result to hold. Needless to say, this assumption is extremely hard to verify. Results applicable to a generic situation have not yet been discovered.

As we pointed out in the introduction, when solving a general minimization problem (1.1) one is interested not only in the value of the minimum but, often more importantly, in the minimizer. If the minimizer does not exist, we are still interested in the minimizing sequence. For example, if the minimization problem aims to find the stable states of an elastic material—and these do not exist—it is still reasonable to argue that the material will be found in a state "near" the actual infimum, i.e., that the minimizing sequence contains the information on physically relevant states. Actually, patterns of minimizing sequences of functionals with density of the type (7.1) can be linked to observed microstructures as shown in Figure 1.

The downside of the relaxation via the quasiconvex envelope is that it does not "store" too much information on the minimizing sequence. Therefore, it would be valuable to have a tool for relaxation that retains some important features of the minimizing sequence by defining "generalized functions" for which the limits $\lim_{k\to\infty} \mathcal{E}(y_k)$ where $\{y_k\} \subset \mathcal{Y}$ are evaluated. This is the basic idea of *Young measures* [238]. Instead of replacing the original integrand by its quasiconvex envelope, we extend the original problem defined on vector-valued functions to a new problem defined on parametrized measures. These measures enable us to describe the limit of a weakly converging sequence composed with a nonlinear function and effectively describe the "patterns" of the minimizing sequence.

Before giving an example (see Example 7.8) of how Young measures can be employed, and before discussing their properties further, let us start with the so-called fundamental theorem on Young measures asserting their existence. This result is originally due to L.C. Young [238] for L^∞-bounded sequences; various versions of the theorem below valid for L^p can be found in [13, 16, 88, 154, 210, 229, 230], for instance.

THEOREM 7.7 (L^p-Young measures). *If* $\Omega \subset \mathbb{R}^n$ *is bounded and* $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;$ $\mathbb{R}^{m\times n})$, $1 \le p < +\infty$, *is a bounded sequence, then there exists a (nonrelabeled) subsequence and a family of parametrized (by* $x \in \Omega$*) probability measures* $\nu = \{\nu_x\}_{x\in\Omega}$ *supported on* $\mathbb{R}^{m\times n}$ *such that for every Carathéodory integrand* $f : \Omega \times \mathbb{R}^{m\times n} \to$ $\mathbb{R}\cup\{+\infty\}$ *which is bounded from below and such that* $\{f(\cdot, Y_k)\}_{k\in\mathbb{N}}$ *is relatively weakly compact in* $L^1(\Omega)$, *it holds that*

$$(7.12) \qquad \lim_{k\to\infty} \int_\Omega f(x, Y_k(x))\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{m\times n}} f(x, A)\mathrm{d}\nu_x(A)\,\mathrm{d}x \ .$$

Conversely, if $\nu = \{\nu_x\}_{x\in\Omega}$ *is a weakly* measurable family of probability measures supported on* $\mathbb{R}^{m\times n}$ *and either*

$$(7.13) \qquad \int_\Omega \int_{\mathbb{R}^{m\times n}} |A|^p \mathrm{d}\nu_x(A)\,\mathrm{d}x < \infty \quad \textit{for some } 1 < p < +\infty \qquad \textit{or}$$

$$(7.14) \qquad \operatorname{supp}\nu_x \subset B(0, r) \textit{ for almost all } x \in \Omega \textit{ and some } r > 0,$$

then there is a sequence $\{Y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{m\times n})$ $(p = +\infty$ *if* (7.14) *holds) such that* (7.12) *holds. Moreover, if* (7.13) *holds, then* $\{Y_k\}_{k\in\mathbb{N}}$ *can be chosen such that* $\{|Y_k|^p\}_{k\in\mathbb{N}}$ *is relatively weakly compact in* $L^1(\Omega)$.

Here the adjective "weakly* measurable" means that $x \mapsto \int_{\mathbb{R}^{m\times n}} f(A)\mathrm{d}\nu_x(A)$ is measurable for all $f \in C_0(\mathbb{R}^{m\times n})$. The measure ν from Theorem 7.7 is called an L^p-Young measure generated by $\{Y_k\}$. It easily follows from Theorem 7.7 that a weakly* measurable map $\nu = \{\nu_x\} : \Omega \to \mathcal{M}_+^1(\mathbb{R}^{m\times n})$ ($\mathcal{M}_+^1(\mathbb{R}^{m\times n})$ denotes the set of probability measures supported on $\mathbb{R}^{m\times n}$) is an L^p-Young measure for $1 \le p < +\infty$ or $p = +\infty$ if and only if (7.13) or (7.14), respectively, holds.

We noted in section 3.1 that weak convergence can essentially be caused by concentrations of oscillations that can be separated from each other by Decomposition Lemma 3.6. As for the Young measure, only the oscillating part of the sequence is important. Indeed, two sequences that differ only on a set of vanishing measure generate the same Young measure [191]; this happens exactly for the original sequence $\{y_k\}_{k\in\mathbb{N}}$ and the sequence $\{z_k\}_{k\in\mathbb{N}}$ constructed in Lemma 3.6.

REMARK 7.7. *There are finer tools than Young measures that have the ability to capture both oscillations and concentrations in a generating sequence. These are, for example, Young measures and varifolds* [93]. *A detailed treatment of such generalized Young measures can be found in Roubíček* [203].

In summary, we see that Young measures are an effective tool to capture the asymptotic behavior of a nonlinear functional along an oscillating sequence. Moreover, the Young measure carries information about the oscillations in the sequences themselves. To see this, let us return to Example 1.1 from the introduction.

EXAMPLE 7.8 (Example 1.1 revisited). *In Example* 1.1 *we constructed a sequence of "zig-zag" functions* $\{u_k\}_{k\in\mathbb{N}}$ *defined by setting*

$$u(x) = \begin{cases} x & \text{if } 0 \le x \le 1/2, \\ -x + 1 & \text{if } 1/2 \le x \le 1, \end{cases}$$

extending it periodically to the whole of $\mathbb{R}$ *and letting* $u_k(x) := k^{-1}u(kx)$ *for all* $k \in \mathbb{N}$ *and all* $x \in \mathbb{R}$.

The sequence of derivatives $\{u_k'\}_{k\in\mathbb{N}}$ *converges in* $L^\infty(0, 1)$ *weakly* to zero. This is so because the derivatives oscillate between* 1 *and* -1 *with zero being the mean. We*

thus see that the weak limit does carry information about the "mean" of the oscillating sequence, but not the values between which the oscillations happened. This information, however, is encoded in the corresponding Young measure; in fact, $\{u'_k\}_{k \in \mathbb{N}}$ generates the sum of two Dirac measures $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$.

The above example illustrates a recurring theme in Young measures: while the weak limit carries information on average values in a oscillating sequence, the Young measure contains more information on "where" the oscillations have taken place.

Usually, Theorem 7.7 is applied to $f(x, A) := \tilde{f}(A)g(x)$, where $\tilde{f} \in C(\mathbb{R}^{m \times n})$, $\lim_{|A| \to \infty} \tilde{f}(A)/|A|^p = 0$, and $g \in L^\infty(\Omega)$. These conditions make $\{f(\cdot, Y_k)\}_{k \in \mathbb{N}}$ relatively weakly compact in $L^1(\Omega)$ so that (7.12) holds, i.e.,

$$(7.15) \qquad \lim_{k \to \infty} \int_\Omega \tilde{f}(Y_k(x))g(x)\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^{m \times n}} \tilde{f}(A)g(x)\mathrm{d}\nu_x(A)\,\mathrm{d}x \ .$$

The original functional introduced in (7.8) is then extended by continuity to obtain its relaxed version expressed in terms of Young measures. However, there are a few important issues which need to be addressed. First of all, in taking a minimizing sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{Y} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ for (7.8) we must ensure that $\{f(\nabla u_k)\}_{k \in \mathbb{N}}$ is weakly relatively compact in $L^1(\Omega)$. This is possible, for example, if (7.9) holds. Indeed, if $\{u_k\} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ is a bounded minimizing sequence for $\mathcal{E}$, we can assume that $\{|\nabla u_k|^p\}_{k \in \mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$ due to the Decomposition Lemma 3.6.[14] Then applying Theorem 7.7 to $Y_k := \nabla u_k$ we obtain

$$\inf \mathcal{E} = \lim_{k \to \infty} \mathcal{E}(u_k) = \int_\Omega \int_{\mathbb{R}^{m \times n}} f(A)\mathrm{d}\nu_x(A)\,\mathrm{d}x \ .$$

However, an important issue is that the resulting Young measure is generated by $\{\nabla u_k\}_{k \in \mathbb{N}}$, i.e., by gradients of Sobolev maps. Such measures are called *gradient Young measures* and clearly form a subset of Young measures. Nevertheless, an explicit characterization of this subset is far more involved than with the characterization of mere L^p-Young measures by means of (7.13). Indeed, the only known characterization of admissible measures, called gradient Young measures, again involves quasiconvex functions [42, 135, 133, 145, 191, 184, 203], which makes the goal of obtaining a closed formula of relaxation by means of parametrized measures largely unreachable. The following theorem is a characterization of gradient Young measures (also called $W^{1,p}$-Young measures) due to Kinderlehrer and Pedregal [135, 133]. Before we state the theorem we define the set

$$Q(p) := \begin{cases} \{f : \mathbb{R}^{m \times n} \to \mathbb{R}; \ f \text{ is quasiconvex } \& \ |f| \leq c(f)(1 + |\cdot|^p), \ c(f) > 0\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } 1 < p < +\infty, \\ \{f : \mathbb{R}^{m \times n} \to \mathbb{R}; \ f \text{ is quasiconvex}\} \qquad\quad \text{if } p = +\infty. \end{cases}$$

THEOREM 7.8 ($W^{1,p}$-Young measures). *Let $1 < p \leq +\infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. An L^p-Young measure ν is a gradient Young measure if and only if there is $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and a set $\omega \subset \Omega$, $\mathcal{L}^n(\omega) = 0$ such that for all $x \in \Omega \setminus \omega$,*

$$(7.16) \qquad\qquad f(\nabla u(x)) \leq \int_{\mathbb{R}^{m \times n}} f(A)\mathrm{d}\nu_x(A)$$

for all $f \in Q(p)$.

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[14]Here, recall that the gradients of the two sequences introduced in Lemma 3.6 generate the same Young measure.
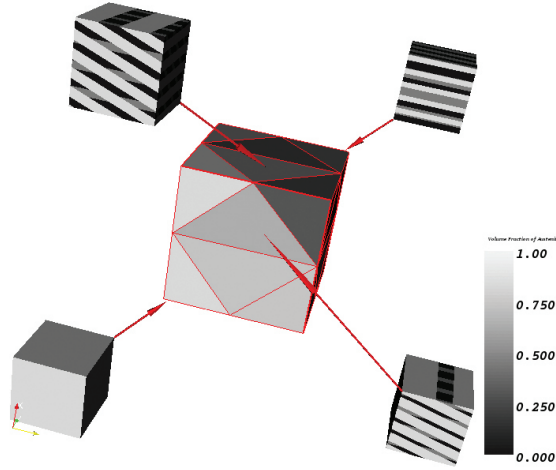
Fig. 2 *An illustration of the calculation of a relaxed energy of a cube under loading. The cube in the middle is the specimen and on the sides a few elements show the calculated microstructure in the form of laminates. The gray scale indicates volume fractions of the phases involved.*

Taking $f(A) := \pm A_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ in (7.16) we find that $\nabla u(x) = \int_{\mathbb{R}^{m \times n}} A \nu_x(\mathrm{d}A)$, i.e., that ∇u is the first moment (the expectation) of the Young measure ν. Notice that ∇u is also the weak limit of every generating sequence of the Young measure ν.

If (7.9) holds, then the relaxed problem is as follows:

$$(7.17) \qquad \text{minimize } \mathcal{E}_R(\nu) := \int_\Omega \int_{\mathbb{R}^{m \times n}} f(A) \nu_x(\mathrm{d}A) \, \mathrm{d}x$$

over all $W^{1,p}$-Young measures generated by $\{\nabla u_k\}_{k \in \mathbb{N}}$ for some arbitrary bounded $\{u_k\} \subset \mathcal{Y}$. These admissible Young measures then form $\mathcal{Y}_R$.

The Jensen-like inequality (7.16) puts gradient Young measures into duality with quasiconvexity. As an efficient description of the set of quasiconvex functions is not available, it is practically impossible to decide whether a given ν is generated by gradients.

Nevertheless, one can restrict attention to a subset of gradient Young measures consisting of so-called *laminates* [191]. Laminates are those gradient Young measures that satisfy the Jensen-like inequality (7.16) even for all functions from the set

$$R(p) := \begin{cases} \{f : \mathbb{R}^{m \times n} \to \mathbb{R}; \ f \text{ is rank-one convex} \ \& \ |f| \leq c(f)(1 + |\cdot|^p), \ c(f) > 0\} \\ \hfill \text{if } 1 < p < +\infty, \\ \{f : \mathbb{R}^{m \times n} \to \mathbb{R}; \ f \text{ is rank-one convex}\} \hfill \text{if } p = +\infty. \end{cases}$$

This subset can be advantageously exploited in the numerical minimization of (7.17); see, e.g., [10, 30, 35, 77, 150] as well as the review paper by Luskin [164] on different finite-element approaches for such a treatment and the illustration in Figure 2, which shows computations of material microstructures by means of laminates.

Another possibility is to minimize (7.17) over a superset of gradient Young measures. For example, one can require (7.16) to hold only for null Lagrangians to obtain a lower bound on the value of the minimizer. This approach corresponds to replacing f in (7.8) by its polyconvex envelope, i.e., the biggest polyconvex function not exceeding

f. We refer to [31, 32] for details on this approach. We emphasize that the two mentioned bounds are not sharp in general because rank-one convexity, quasiconvexity, and polyconvexity are all different.

Nevertheless, most of the results on relaxation presented above are applicable only to energies with p-growth. However, as we explained in detail at the beginning of this section, this clearly prevents us from applying the requirements (6.6) and (6.7) on f. Thus, we see that even cutting-edge tools and techniques of mathematical analysis and calculus of variations need to be suitably tailored to cope with deep problems in elasticity, and new ideas and approaches are needed to solve them.

A natural question is whether one can extend weak lower semicontinuity results known for integrands depending on gradients to more a general framework. We will investigate this question in the following section.

8. $\mathcal{A}$-quasiconvexity. In this section, we summarize results about weak lower semicontinuity of integral functionals along sequences which satisfy a first-order linear differential constraint. Clearly, gradients as curl-free fields are included in this setting but, as emphasized by L. Tartar, besides curl-free fields there are also other important differential constraints on possible minimizers. Such a setting naturally arises in electromagnetism, linearized elasticity, or even higher-order gradients, to name a few. Tartar's program was materialized by Dacorogna in [66] and then studied by many other authors; see, e.g., [12, 52, 94, 224] and references therein.

The problem studied in this section can be formulated as follows: Given a sequence $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$, such that each member satisfies a linear differential constraint $\mathcal{A}u_k = 0$ ($\mathcal{A}$-free sequence), or $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^n)$ (asymptotically $\mathcal{A}$-free sequence), what conditions on f precisely ensure weak lower semicontinuity of integral functionals in the form

$$(8.1) \qquad \mathcal{I}(u) := \int_\Omega f(x, u(x))\,\mathrm{d}x?$$

Here $\mathcal{A}$ is a first-order linear differential operator.

To the best of our knowledge, the first result of this type was proved in [92] for nonnegative integrands. In this case, the crucial necessary and sufficient condition ensuring weak lower semicontinuity of $\mathcal{I}$ in (8.1) is the so-called $\mathcal{A}$-quasiconvexity; cf. Definition 8.1 below. However, if we refrain from considering only nonnegative integrands, this condition is not necessarily sufficient, as we have already observed in the case $\mathcal{A} :=$ curl.

8.1. The Operator $\mathcal{A}$ and $\mathcal{A}$-quasiconvexity. Following [92], we consider the linear operators $A^{(i)} : \mathbb{R}^m \to \mathbb{R}^d$, $i = 1, \ldots, n$, and we define $\mathcal{A} : L^p(\Omega;\mathbb{R}^m) \to W^{-1,p}(\Omega;\mathbb{R}^d)$ by

$$\mathcal{A}u := \sum_{i=1}^n A^{(i)} \frac{\partial u}{\partial x_i} \ , \ \text{ where } \ u : \Omega \to \mathbb{R}^m \ ,$$

i.e., for all $w \in W_0^{1,p'}(\Omega;\mathbb{R}^d)$,

$$\langle \mathcal{A}u, w \rangle = -\sum_{i=1}^n \int_\Omega A^{(i)} u(x) \cdot \frac{\partial w(x)}{\partial x_i}\,\mathrm{d}x \ .$$

For $w \in \mathbb{R}^n$ we define the linear map

$$\mathbb{A}(w) := \sum_{i=1}^{n} w_i A^{(i)} \ : \ \mathbb{R}^m \to \mathbb{R}^d \ .$$

In this review, we assume that there exists $r \in \mathbb{N} \cup \{0\}$ such that

(8.2)   rank $\mathbb{A}(w) = r$ for all $w \in \mathbb{R}^n$, $|w| = 1$,

i.e., $\mathcal{A}$ has the so-called *constant-rank property*. In what follows we use ker $\mathcal{A}$ to denote the set of all locally integrable functions u such that $\mathcal{A}u = 0$ in the sense of distributions, i.e., $\langle \mathcal{A}u, w \rangle = 0$ for all $w \in \mathcal{D}(\Omega)$. Of course, ker $\mathcal{A}$ depends on the considered domain Ω.

DEFINITION 8.1 (cf. [92, Def. 3.1, 3.2]). *We say that a continuous function* f : $\mathbb{R}^m \to \mathbb{R}$, *satisfying* $|f(s)| \le C(1 + |s|^p)$ *for some* $C > 0$, *is* $\mathcal{A}$-*quasiconvex if for all* $s_0 \in \mathbb{R}^m$ *and all* $\varphi \in L^p_{\#}(Q; \mathbb{R}^m) \cap ker \, \mathcal{A}$ *with* $\int_Q \varphi(x) \, \mathrm{d}x = 0$ *it holds that*

$$f(s_0) \le \int_Q f(s_0 + \varphi(x)) \, \mathrm{d}x \ .$$

In the above definition, we used the space of Q-periodic Lebesgue integrable functions

$$L^p_{\#}(\mathbb{R}^n; \mathbb{R}^m) := \{u \in L^p_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m) : \ u \text{ is } Q\text{-periodic}\},$$

where Q denotes the unit cube $(-1/2, 1/2)^n$ in $\mathbb{R}^n$, and we say that $u : \mathbb{R}^n \to \mathbb{R}^m$ is Q-*periodic* if for all $x \in \mathbb{R}^n$ and all $z \in \mathbb{Z}^n$ it holds that $u(x + z) = u(x)$.

Fonseca and Müller [92] proved the following result linking $\mathcal{A}$-quasiconvexity and weak lower semicontinuity. Notice that the integrand is more general than that in (8.1).

THEOREM 8.2. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded and let* $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0; +\infty)$ *be a Carathéodory integrand. Let*

$$0 \le f(x, z, u) \le a(x, z)(1 + |u|^p)$$

for almost every $x \in \Omega$ *and all* $(z, u) \in \mathbb{R}^d \times \mathbb{R}^m$, $1 < p < +\infty$, *and some* $0 \le a \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$. *Assume that* $\{z_k\}_{k \in \mathbb{N}} \subset L^\infty(\Omega; \mathbb{R}^m)$, $z_k \to z$ *in measure, and that* $u_k \rightharpoonup u$ *in* $L^p(\Omega; \mathbb{R}^d), \|\mathcal{A}u_k\|_{W^{-1,p}(\Omega; \mathbb{R}^m)} \to 0$.
Then

$$\liminf_{k \to \infty} \int_\Omega f(x, z_k, u_k) \, \mathrm{d}x \ge \int_\Omega f(x, z, u) \, \mathrm{d}x$$

if and only if $f(x, z, \cdot)$ *is* $\mathcal{A}$-*quasiconvex for almost all* $x \in \Omega$ *and all* $z \in \mathbb{R}^d$.

The following definition is motivated by our discussion above Theorem 3.8. It first appeared in [87].

DEFINITION 8.3. *Let* $1 < p < +\infty$ *and* $\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m) \cap ker \, \mathcal{A}$. *We say that* $\{u_k\}$ *has an* $\mathcal{A}$-*free p-equi-integrable extension if for every domain* $\tilde{\Omega} \subset \mathbb{R}^n$ *such that* $\Omega \subset \tilde{\Omega}$, *there is a sequence* $\{\tilde{u}_k\}_{k \in \mathbb{N}} \subset L^p(\tilde{\Omega}; \mathbb{R}^m) \cap ker \, \mathcal{A}$ *such that*
  (i) $\tilde{u}_k = u_k$ *a.e. in* Ω *for all* $k \in \mathbb{N}$,
  (ii) $\{|\tilde{u}_k|^p\}_{k \in \mathbb{N}}$ *is equi-integrable on* $\tilde{\Omega} \setminus \Omega$, *and*
  (iii) *there is* $C > 0$ *such that* $\|\tilde{u}_k\|_{L^p(\tilde{\Omega}; \mathbb{R}^m)} \le C \|u_k\|_{L^p(\Omega; \mathbb{R}^m)}$ *for all* $k \in \mathbb{N}$.

Then we have the following result proved in [87].

THEOREM 8.4. *Let $0 \leq g \in C(\overline{\Omega})$, let $|f| \leq C(1 + |\cdot|^p)$ be $\mathcal{A}$-quasiconvex, satisfy (3.4), and have a recession function, and let $1 < p < +\infty$. Let $\{u_k\} \subset L^p(\Omega; \mathbb{R}^m) \cap \ker \mathcal{A}$, $u_k \rightharpoonup u$ weakly, and assume that $\{u_k\}$ has an $\mathcal{A}$-free p-equi-integrable extension. Then $\mathcal{I}(u) \leq \liminf_{k \to \infty} \mathcal{I}(u_k)$, where*

$$(8.3) \qquad \mathcal{I}(u) := \int_\Omega g(x) f(u(x)) \, \mathrm{d}x.$$

Surprisingly, it was shown by Fonseca and Müller [92, p. 1380] that higher-order gradients can also be recast as $\mathcal{A}$-free mappings. They construct $\mathcal{A}$ such that $\mathcal{A}u = 0$ if and only if $u = \nabla^k w$ for some $w \in W^{k,p}(\Omega; \mathbb{R}^m)$. In this situation, $\mathcal{A}$-quasiconvexity coincides with Meyers' k-quasiconvexity. This allows us to study the weak lower semicontinuity of

$$(8.4) \qquad \mathcal{I}(w) := \int_\Omega g(x) f(\nabla^k w(x)) \, \mathrm{d}x,$$

with g and f as in Theorem 8.4, on the set

$$\{w \in W^{k,p}(\Omega; \mathbb{R}^m) : w = w_0 \text{ on } \partial\Omega\},$$

with $w_0 \in W^{k,p}(\Omega; \mathbb{R}^m)$, in the context of $\mathcal{A}$-quasiconvexity. In particular, in this case, we may construct the extension required in Definition 8.3 by extending the Dirichlet boundary condition, and thus Theorem 8.4 is applicable. It follows that k-quasiconvexity of f is sufficient to make (8.4) weakly lower semicontinuous, which affirmatively answers Problem 3.9 in this particular setting.[15]

Let us finally point out that the treatment of $\mathcal{A}$-quasiconvexity for integrands whose negative part grows with the pth power is a very subtle issue which has recently been addressed in [143]. A new condition called $\mathcal{A}$-quasiconvexity at the boundary is introduced in two forms depending on whether or not u can be extended to a larger domain preserving the $\mathcal{A}$-free property. This allows us to remove the assumption of the existence of an $\mathcal{A}$-free p-equi-integrable extension from Theorem 8.4.

9. Suggestions for Further Reading. The above exposition aims to reflect developments in weak lower semicontinuity related to Morrey's [178] and Meyers' [171] papers, with emphasis on applications to static problems in continuum mechanics of solids. We dare to hope that it provides a fairly complete picture of the theory from 1952 to current trends. On the other hand, it is clearly influenced by our personal points of view. We believe we have convinced the reader that the calculus of variations has been a very active research area for the last fifty years with many important results and with many remaining challenging open problems. Below we describe some additional research directions and we invite the interested reader to find more details in the references.

9.1. Applications to Continuum Mechanics and Beyond. We have seen that weak lower semicontinuity serves as a main ingredient in proofs of existence of minimizers to variational integrals, and we have outlined applications in elastostatics. Even in the static case, models of elasticity can be combined with other phenomena,

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[15]In fact, one can consider integrands of the form $f(x, \nabla^k w(x))$ whenever $f(x, \cdot)$ is k-quasiconvex for all $x \in \overline{\Omega}$, $|f(x, A)| \leq C(1 + |A|^p)$, $f(\cdot; A)$ is continuous in $\overline{\Omega}$ for all $A \in X(n, m, k)$, and $f(x, \cdot)$ possesses a recession function.

such as magnetism. This leads to magnetoelasticity (magnetostriction), a property of NiMnGa, for instance. We refer the reader to, e.g., to the monograph [78] or to [83] for a physical background.

The idea of drawing macroscopic properties of composite materials from their microscopic ones lies at the core of *homogenization theory*. We cite the classical books by Jikov, Kozlov, and Olejnik [129], Braides and Defranceschi [51], and Bensoussan, Lions, and Papanicolaou [39] for a thorough overview. Γ-convergence of integral functionals (see monographs by Braides or Dal Maso [49, 69]) plays a key role in this research. The main objective is to study properties for $\varepsilon \to 0$ of $I_\varepsilon(u_\varepsilon) :=$ $\int_\Omega f(x, \frac{x}{\varepsilon}, \nabla u_\varepsilon(x)) \, \mathrm{d}x$, where f is $(0,1)^n$-periodic in the middle variable. Homogenization problems are not only restricted to gradients, but new results in the context of $\mathcal{A}$-quasiconvexity (even with nonconstant coefficients) have recently appeared, e.g., in [74]. Various other generalizations including stochastic features to describe randomness in the distribution of inhomogeneities can be found in the papers by Dal Maso and Modica [71], Messaoudi and Michaille [170], and the recent work of Gloria, Neukamm, and Otto [104]. Γ-convergence is also one of the main tools applied to dimension-reduction problems in mechanics to obtain various plate models; see, for instance, LeDret and Raoult [159], Friesecke, James, and Müller [100], and Hornung, Neukamm, and Velčić [124] and references therein.

Quasistatic evolutionary problems and their treatment by means of the so-called "energetic solution" has been a lively area of mathematical continuum mechanics for many years. This theory is thoroughly discussed and summarized in the book by Mielke and Roubíček [172], where one can find applications to continuum mechanics of solids including plasticity, damage, and mechanics of shape-memory materials including numerical approximations. Various combinations of inelastic processes are also treated, e.g., in [47, 95]. Weak lower semicontinuity of corresponding energy functionals is a key part of the theory, which also includes relaxation and Young measures. We also point out the papers [173] for (evolutionary) Γ-convergence treatment of quasistatic problems or [47] for combinations of damage and plasticity. This opens up many possibilities of applying this type of solution to dimension-reduction problems, linearization, or brittle damage, for instance. Weak lower semicontinuity finds its application in dynamical problems, too. For example, it is the main tool proving existence of solutions in time-discrete approximations of evolution in various models. We refer, e.g., to the book of Braides [50] or [172] for many such instances. We also refer to [103] and references therein for further results concerning the mathematical treatment of nonlinear elasticity.

Derivation of nonlinear continuum models from discrete atomic models and corresponding numerical computations has been a lively field of research in recent years. We refer to Blanc, Le Bris, and Lions [43] for a survey article and to work of Braides and Gelli [53], Allicandro and Cicalese [3], and Braun, Friedrich, and Schmidt [99, 54] and references therein. Various aspects of continuum/atomistic coupling, which combines accurate atomistic models with efficient continuum elasticity, are thoroughly discussed in Luskin and Ortner [165]; see also Lazzaroni, Palombaro, Schäffner, and Schlömerkemper [157, 208] for further analytical results in this direction.

Nonlocal theories of elastodynamics, such as, e.g., peridynamics, have initiated research activities in nonlocal variational problems. Here we wish to mention recent work of Bellido and Mora-Corral [33] and Bellido, Mora-Corral, and Pedregal [34], for instance.

9.2. Functionals with Linear Growth. Contrary to weak lower semicontinuity in $W^{1,1}(\Omega;\mathbb{R}^m)$, where the definition excludes concentrations in gradient, a serious analytical problem arisess if one wants to minimize functionals defined in (1.7) for $f(x,r,\cdot)$ with linear growth at infinity. In this case, we would naturally work in $W^{1,1}(\Omega;\mathbb{R}^m)$ which is, however, not reflexive, and therefore the weak limit of a minimizing (sub)sequence does not necessarily exist. This leads to various extensions of $W^{1,1}(\Omega;\mathbb{R}^m)$ as well as of the functional I. Usually, we embed $W^{1,1}(\Omega;\mathbb{R}^m)$ into the space of functions with bounded variations $\mathrm{BV}(\Omega;\mathbb{R}^m)$ which contains integrable mappings whose gradient is a Radon measure in Ω. Detailed descriptions and properties can be found in references by Ambrosio, Fusco, and Pallara [5] and by Attouch, Buttazzo, and Michaille [9]. Besides the function space, one must also suitably extend the functional I to allow for measure-valued gradients. This uses the notion of the recession function to $f(x,r,\cdot)$. We refer the reader to Fonseca and Müller [90, 91] and references therein for the case of nonnegative integrands. Recently, Kristensensen and Rindler [147, 148] resolved the weak* lower semicontinuity relaxation of the functional if $f = f(x,\nabla u)$ with $|f(x,s)| \leq C(1 + |s|)$ along sequences with prescribed Dirichlet boundary conditions. This result was then generalized by Benešová, Krömer, and Kružík [37] to avoid restrictions on the boundary. A closely related topic is weak* lower semicontinuity in the space of functions with bounded deformations [231], i.e., in the subspace of $L^1(\Omega;\mathbb{R}^m)$ of mappings whose symmetrized gradient is a measure on Ω. This set naturally appears in problems of linearized perfect plasticity [226], for instance. We refer to Rindler [201] for a general lower semicontinuity statement.

Functionals with linear growth and their setting in the space of maps with bounded variations often arise as Γ-limits of various phase-field problems. Here one tries to accommodate two or more phases/materials of a given volume in such a way that the interfacial energy assigned to two mutual interfaces is minimal. Besides classical works of Modica and Mortola [175, 176] on phase transition, versatile applications exist nowadays in many areas of mechanics like topology optimization [44].

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