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ERRATUM: ON THE AUBIN PROPERTY OF CRITICAL POINTS TO PERTURBED SECOND-ORDER CONE PROGRAMS*

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Abstract. Two gaps were found in the proof of the main theorems (Theorems 21 and 26) of the paper "On the Aubin property of critical points to perturbed second-order cone programs" [*SIAM J. Optim.* 21 (2011), 3, pp. 798–823] by J. V. Outrata and H. Ramírez C. In this note both these gaps will be filled. As to the second one, a new technical result will be employed which may possibly be used also in other situations.

Key words. second-order cone programming, strong regularity, Aubin property, strong second-order sufficient optimality conditions, nondegeneracy

AMS subject classifications. 90C, 90C31, 90C46

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1. Introduction.

In [1], the authors consider the following nonlinear *second-order cone programming problem (SOCP)*:

$$\text{(SOCP)} \qquad \operatorname*{Min}_{x\in\mathbb{R}^n, s^j\in\mathbb{R}^{m_j+1}} f(x)\,;\, g^j(x) = s^j,\ (s^j)_0 \geq \|\bar{s}^j\|,\ \ j = 1,\dots,J,$$

where f and g^j, $j = 1,\dots,J$, are twice continuously differentiable mappings from $\mathbb{R}^n$ into $\mathbb{R}$ and $\mathbb{R}^{m_j+1}$, respectively. Here we use the standard convention of indexing components of vectors of $\mathbb{R}^{m_j+1}$ from 0 to m_j, and given $s \in \mathbb{R}^{m_j+1}$, $\bar{s}$ denotes the subvector $(s_1,\dots,s_{m_j})^\top$. The vectors in $\mathbb{R}^n$ are indexed in the standard way from 1 to n, and by $\|\cdot\|$ we denote the Euclidean norm. The *second-order cone* (or *ice-cream cone*, or *Lorentz cone*) of dimension $m+1$ is defined to be

$$\mathcal{Q}_{m+1} := \{s \in \mathbb{R}^{m+1} \,|\, s_0 \geq \|\bar{s}\|\}.$$

The following definitions and results appear in [1] and are relevant for the purpose of this note.

DEFINITION 1.1. *We say that* y *is a* Lagrange multiplier *for* x *(denoted* $y \in \Lambda(x)$*) if it satisfies the standard KKT system associated to* (SOCP)*:*

$$\begin{aligned} 0 &= D_x L(x,y), \\ 0 &\in g(x) + N_{\mathcal{Q}}(y), \end{aligned} \tag{1.1}$$

where $L(x,y) := f(x) + g(x)^\top y$ *is the Lagrangian and* $\mathcal{Q} := \prod_{j=1}^J \mathcal{Q}_{m_j+1}$.

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Under the assumptions posed in [1] this KKT system can be cast as the *generalized equation* (GE)

$$0 \in Df(x) + (Dg(x))^\top N_{\mathcal{Q}}(g(x)). \tag{1.2}$$

Consequently, we define the associated *solution map* as follows:

$$S(\eta) := \{x | \eta \in Df(x) + (Dg(x))^\top N_{\mathcal{Q}}(g(x))\}. \tag{1.3}$$

Definition 1.2. *Let x^* be a feasible point of SOCP. We say that x^* is* nondegenerate *if*

$$Dg(x^*)\mathbb{R}^n + \mathrm{lin}(T_{\mathcal{Q}}(g(x^*))) = \Pi_{j=1}^J \mathbb{R}^{m_j+1}, \tag{1.4}$$

where lin$(\cdot)$ denotes the greatest linear subspace contained in the respective set.

To introduce the following conditions, we define first $\mathcal{H}(x,y) := \sum_{j=1}^J \mathcal{H}^j(x,y^j)$, where we set

$$\mathcal{H}^j(x,y^j) := \begin{cases} -\dfrac{y_0^j}{(g^j(x))_0} Dg^j(x)^\top R_{m_j} Dg^j(x) & \text{if } s^j \in \partial\mathcal{Q}_{m_j+1} \setminus \{0\}, \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

and

$$R_{m_j} := \begin{pmatrix} 1 & 0^\top \\ 0 & -I_{m_j} \end{pmatrix}.$$

Definition 1.3. *Let x^* be a critical point of SOCP and $y^* \in \Lambda(x^*)$. We say that the* second-order necessary condition (SONC) *holds at (x^*,y^*), provided*

$$Q_0(h) := h^\top D^2_{xx}L(x^*,y^*)h + h^\top \mathcal{H}(x^*,y^*)h \ge 0 \quad \forall h \in C(x^*). \tag{1.6}$$

We say that the strong second-order sufficient condition (SSOSC) *holds at (x^*,y^*), provided*

$$Q_0(h) > 0 \quad \forall h \in Sp(C(x^*)) \setminus \{0\}. \tag{1.7}$$

Here, $C(x^*) := Df(x^*)^\perp \cap Dg(x^*)^{-1}T_{\mathcal{Q}}(g(x^*))$ is the cone of critical directions *at x^*, and $Sp(C)$ denotes the smallest linear space which contains the set C.*

The next relations are relevant for the main theorem:

$$0 = D^2_{xx}L(x^*,y^*)v + (Dg(x^*))^\top(b - Dg(x^*)v) \tag{1.8a}$$

$$-Dg(x^*)v \in D^*P(g(x^*) - y^*, g(x^*))(-b), \tag{1.8b}$$

where $P(\cdot)$ denotes the projection operator onto $\mathcal{Q}$.

The main results in [1] are stated below.

Theorem 1.4. *Consider SOCP with $J = 1$. Let x^* be a local solution of the problem and y^* be a corresponding Lagrange multiplier. Then the following assertions are equivalent:*

(i) *x^* is nondegenerate (Definition 1.2) and SOCP fulfills the strong second-order sufficient condition (1.7) at (x^*,y^*).*
(ii) *The GE (1.2) (KKT conditions) is strongly regular at (x^*,y^*).*

(iii) *x^* is nondegenerate, and S has the Aubin property around* $(0, x^*)$.
(iv) *x^* is nondegenerate, and in any solution pair* (v^*, b^*) *of* (1.8) *one has* $v^* = 0$.

The proof of this theorem reduces to showing the implication (iv) ⇒ (i) via a contraposition, which is done separately in six cases specified by the position of the considered pair $(g(x^*), y^*)$. In case 1 ($g(x^*) = 0,\ y^* \in \operatorname{int} Q_{m+1}$) the authors claim that, since (1.6) ((2.44) in [1]) is fulfilled, condition (1.7) ((2.41) in [1]) is violated if and only if there is a nonzero vector $h \in \mathbb{R}^n$ such that

$$Dg(x)^* h = 0 \text{ and } D^*_{xx} L(x^*, y^*)h = 0.$$

However, from the comparison of second-order necessary and sufficient conditions we get only the existence of a nonzero h such that

$$h^\top D^2_{xx} L(x^*, y^*)h = 0, \quad Dg(x^*)h = 0.$$

It follows that this h is a (global) minimum in the optimization problem

$$\begin{array}{ll} \text{minimize} & h^\top D^2_{xx} L(x^*, y^*)h \\ \text{subject to} & Dg(x^*)h = 0. \end{array}$$

Hence, there is a Lagrange multiplier μ such that

$$D^2_{xx} L(x^*, y^*)h + Dg(x^*)^\top \mu = 0.$$

We can now put $v = h$ and $b = \mu$. Then (1.8a) ((3.3a) in [1]) holds true, and it remains to verify that

$$0 \in D^* P(-y^*, 0)(-b).$$

This is, however, fulfilled because in this case one has

$$D^* P(-y^*, 0)(-b) = DP(-y^*)(-b) = 0 \ \forall b.$$

This completes the proof of Theorem 1.4, case 1.

Before we fill the second gap in the case 6 we will now explain, for the sake of completeness, the derivation of relation

$$D^2_{xx} L(x^*, y^*)h = \frac{y_0^*}{s_0} Dg(x^*)^\top R_m Dg(x^*)h \tag{1.9}$$

(cf.(3.4) in [1]) in case 3 ($g(x^*),\ y^* \in \partial Q_{m+1} \setminus \{0\}$) in more detail. For this case, since second-order necessary condition (1.6) ((2.44) in [1]) is fulfilled, second-order sufficient condition (1.7) ((2.41) in [1]) is violated if and only if there exists a nonzero direction h such that $\langle d(h), y^* \rangle = (y^*)^\top Dg(x^*)h = 0$ and $Q_0(h) = 0$, with $d(h) := Dg(x^*)h$. For the sake of simplicity, let us denote by P the symmetric matrix such that $Q_0(h) = h^\top P h$. Thus, in order to proceed, it is enough to find a nonnegative value $\gamma \geq 0$ for which the matrix

$$Q := P + \gamma Dg(x^*)^\top y^* (y^*)^\top Dg(x^*)$$

is positive semidefinite. Indeed, since it holds that

$$h^\top Q h = h^\top P h + \gamma [(y^*)^\top Dg(x^*)h]^2 = h^\top P h = 0,$$

we obtain $Qh = 0$, which implies that $Ph = 0$. The latter coincides with (1.9). Note that if P is positive semidefinite, our assertion is trivially true with $\gamma = 0$. We thus suppose that the smallest eigenvalue of P, denoted by λ, is negative.

Then, since second-order necessary condition (1.6) says that $Q_0(h) = h^\top Ph \geq 0$ over the linear space defined by directions h such that $\langle d(h), y^* \rangle = h^\top Dg(x^*)^\top y^* = 0$, the eigenvector(s) corresponding to λ (which is negative) should belong to the orthogonal space to this one, that is, to the space generated by $Dg(x^*)^\top y^*$. The latter space has of course dimension 1. So, $Dg(x^*)^\top y^*$ generates the eigenspace associated with λ. Consequently, it is an eigenvector; that is,

$$PDg(x^*)^\top y^* = \lambda Dg(x^*)^\top y^*. \tag{1.10}$$

Notice that $Dg(x^*)^\top y^* \neq 0$ because otherwise (1.6) is equivalent to saying that P is positive semidefinite.

Finally, fix $\gamma = -\lambda$. Then, for any $x \in \mathbb{R}^n$, we decompose it as $x = u + v$ with u such that $\langle u, Dg(x^*)^\top y^* \rangle = 0$ and $v = \alpha Dg(x^*)^\top y^*$ for some $\alpha \in \mathbb{R}$. It follows from second-order necessary condition (1.6) and from (1.10) that

$$\begin{aligned}
x^\top Qx &= x^\top Px + \gamma[(y^*)^\top Dg(x^*)x]^2 = x^\top Px + \gamma\alpha^2 \|Dg(x^*)^\top y^*\|^2 \\
&= u^\top Pu + 2u^\top Pv + v^\top Pv + \gamma\alpha^2 \|Dg(x^*)^\top y^*\|^2 \\
&\geq 2u^\top Pv + v^\top Pv + \gamma\alpha^2 \|Dg(x^*)^\top y^*\|^2 \\
&= 2\alpha\lambda\langle u, Dg(x^*)^\top y^* \rangle + \lambda\alpha^2 \|Dg(x^*)^\top y^*\|^2 + \gamma\alpha^2 \|Dg(x^*)^\top y^*\|^2 \\
&= (\lambda + \gamma)\, \alpha^2 \|Dg(x^*)^\top y^*\|^2 = 0.
\end{aligned}$$

Relation (1.9) follows.

In case 6 ($g(x^*) = y^* = 0$), subcase (a), the authors claim that, since (1.7) ((2.41) in [1]) is violated, there exist a nonzero vector h and $\gamma > 0$ such that $h^\top D^2_{xx}L(x^*, y^*)h < 0$, the matrix $C := D^2_{xx}L(x^*, y^*) - \gamma Dg(x^*)^\top RDg(x^*)$ is positive semidefinite, and h belongs to the kernel of C. However, this assertion is not true.

Additionally, Theorem 1.5 of [1] generalizes Theorem 1.4 from $J = 1$ to several second-order cones provided that at most one of them does not belong to cases 4, 5, and 6 therein (which correspond to the cases when the strict complementarity condition does not hold).

THEOREM 1.5. *Let x^* be a local solution of the problem SOCP, and let y^* be a corresponding Lagrange multiplier. Suppose that there is at most one block j such that either $g^j(x^*) = 0$ and $y^{*j} \in \partial\mathcal{Q}_{m_j+1} \setminus \{0\}$ or $g^j(x^*) \in \partial\mathcal{Q}_{m_j+1} \setminus \{0\}$ and $y^{*j} = 0$ or $g^j(x^*) = 0 = y^{*j}$. Then the following assertions are equivalent:*

(i) *x^* is nondegenerate (Definition* 1.2*) and SOCP fulfills the strong second-order sufficient condition* (1.7) *at (x^*, y^*).*

(ii) *The GE* (1.2) *(KKT system) is strongly regular at (x^*, y^*).*

(iii) *x^* is nondegenerate, and S has the Aubin property around $(0, x^*)$.*

(iv) *x^* is nondegenerate, and in any solution pair (v^*, b^*) of* (1.8) *one has $v^* = 0$.*

Regarding the proof of this theorem, in the case $|J_6| = 1$ ($J_6 := \{j \in J : y^{*j} = g^j(x^*) = 0\}$), subcase (a), the authors claim that, since (1.7) (2.41 in [1]) is violated, there exist a nonzero vector h and $\gamma > 0$ such that $Q_0(h) < 0$, the quadratic form $Q_0 - \gamma Q_1$ is positive semidefinite, and h belongs to the kernel of $Q_0 - \gamma Q_1$. Again, this assertion is not true.

In the next section we present corrections both to Theorem 1.4, case 6, as well as to Theorem 1.5. In this way all gaps arising in [1] will be filled.

2. Filling the gap. To remove the remaining gaps in the proof of Theorems 21 and 26 from [1], the next auxiliary lemma will be employed.

LEMMA 2.1 (auxiliary lemma). *Let A, B be symmetric matrices which satisfy the following conditions:*

1. $A \not\succeq 0$,
2. B *is indefinite,*
3. $\forall x \in \mathbb{R}^n,\ x^\top Bx \geq 0 \implies x^\top Ax \geq 0$.

Then there exists $\delta > 0$ such that $Ker(A - \delta B) \cap \{x :\ x^\top Bx \leq 0\} \neq \{0\}$.

Proof. Note that a direct application of the S-lemma [3] implies the existence of a $\gamma > 0$ such that $A - \gamma B \succeq 0$. Moreover, there exists the minimal γ, say $\bar{\gamma}$, for which this condition is fulfilled (this is due to the continuity of the lowest eigenvalue function). We claim that

$$\bar{\gamma} = 1/m \quad \text{with} \quad m := \inf\left\{\frac{x^\top Bx}{x^\top Ax} :\ x^\top Ax < 0\right\}. \tag{2.1}$$

To prove this relationship, we observe first that m is finite. Indeed, it follows from condition 1 that $m < +\infty$, and from condition 3 that $m \geq 0$. In fact, it holds that $m > 0$. By contradiction, in the opposite case, we can consider a minimizing sequence $\{x_n\}_n$ such that $x_n^\top Ax_n < 0$ and $\frac{x_n^\top Bx_n}{x_n^\top Ax_n} \searrow 0$. Without loss of generality we can take $x_n^\top Ax_n = -1\ \forall n$. Let us define

$$F : \mathbb{R}^n \to \mathbb{R}^2,\ F(x) = \begin{pmatrix} x^\top Ax \\ x^\top Bx \end{pmatrix},$$

and consider, by virtue of condition 2, a vector $y \in \mathbb{R}^n$ such that $y^\top By = 1$. Hence, by condition 3, $y^\top Ay \geq 0$. By using the Dines theorem [2], we know that $F(\mathbb{R}^n)$ is a convex set. This implies the existence of a sequence $\{y_n\}_n$ such that $F(y_n) = tF(y) + (1-t)F(x_n)\ \forall n,$ for any $t \in [0,1]$.

It can be checked that $y_n^\top By_n \to t$. Indeed, from $x_n^\top Ax_n = -1\ \forall n$ and $\frac{x_n^\top Bx_n}{x_n^\top Ax_n} \searrow 0$, we deduce that $x_n^\top Bx_n \nearrow 0$ and then

$$y_n^\top By_n = ty^\top By + (1-t)x_n^\top Bx_n = t \cdot 1 + (1-t)x_n^\top Bx_n \to t.$$

Consequently, if we choose $t > 0$, we deduce that $y_n^\top By_n > 0$ for any n sufficiently large. More specifically, if we choose $t = \frac{1}{2(y^\top Ay+1)} \in (0,1)$, the equality $x_n^\top Ax_n = -1\ \forall n$ yields

$$\begin{aligned} y_n^\top Ay_n &= ty^\top Ay + (1-t)x_n^\top Ax_n = ty^\top Ay - (1-t) \\ &= t(y^\top Ay + 1) - 1 = \frac{y^\top Ay + 1}{2(y^\top Ay + 1)} - 1 = -\frac{1}{2}\ \forall n, \end{aligned}$$

thus giving a contradiction with condition 3. Hence, $m > 0$. See Figure 1 for a geometric visualization of this proof.

Now, we prove that $\frac{1}{m} \in \mathcal{A} := \{\gamma > 0 : A - \gamma B \succeq 0\}$. By contradiction, we assume the existence of $z \in \mathbb{R}^n$ such that

$$z^\top Az - \frac{1}{m}z^\top Bz < 0. \tag{2.2}$$

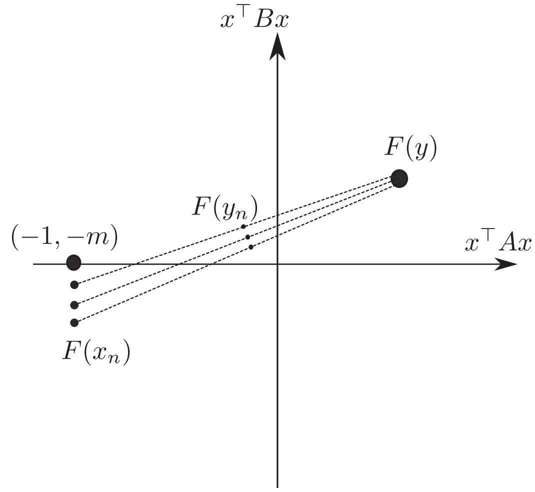

FIG. 1. *Geometric visualization of* $m > 0$.

If $z^\top Bz \leq 0$, then $z^\top Az < \frac{1}{m} z^\top Bz \leq 0$, and so

$$m = \inf \left\{ \frac{x^\top Bx}{x^\top Ax} : \; x^\top Ax < 0 \right\} \leq \frac{z^\top Bz}{z^\top Az} < m.$$

This is a contradiction, and so $z^\top Bz > 0$ and, by virtue of condition 3, $z^\top Az \geq 0$. Without loss of generality we may thus assume that $z^\top Bz = 1$.

Once again, due to the definition of m there is a minimizing sequence $\{x_n\}_n$ such that

$$F(x_n) = \begin{pmatrix} -1 \\ x_n^\top Bx_n \end{pmatrix} \to \begin{pmatrix} -1 \\ -m \end{pmatrix}. \tag{2.3}$$

So, we can clearly assume that $x_n^\top Bx_n < 0 \; \forall n$. From the Dines theorem [2], for any $t \in [0,1]$, there exists a sequence $\{z_n\}_n$ such that

$$F(z_n) = tF(z) + (1-t)F(x_n) \; \forall n.$$

Now, we show that there exists $t \in [0,1]$ such that for $n \in \mathbb{N}$ large enough we have that $z_n^\top Az_n < 0$ and $z_n^\top Bz_n \geq 0$ or, equivalently, $F(z_n) \in \mathbb{R}_{--} \times \mathbb{R}_+$. This would of course contradict condition 3. Indeed, we have the following equivalences:

$$\begin{aligned} F(z_n) \in \mathbb{R}_{--} \times \mathbb{R}_+ &\iff t(z^\top Az + 1) - 1 < 0 \;\text{ and } t(1 - x_n^\top Bx_n) + x_n^\top Bx_n \geq 0 \\ &\iff \alpha := \frac{x_n^\top Bx_n}{x_n^\top Bx_n - 1} \leq t < \frac{1}{1 + z^\top Az} =: \beta, \end{aligned}$$

which can be fulfilled provided n is large enough. Note that the interval $(\alpha, \beta] \subset [0,1]$ is nonempty because α is arbitrarily close to $m/(1+m)$ and $\beta > m/(1+m)$. This is the announced contradiction with condition 3 and so we conclude that $\frac{1}{m} \in \mathcal{A}$. This proof is illustrated in Figure 2.

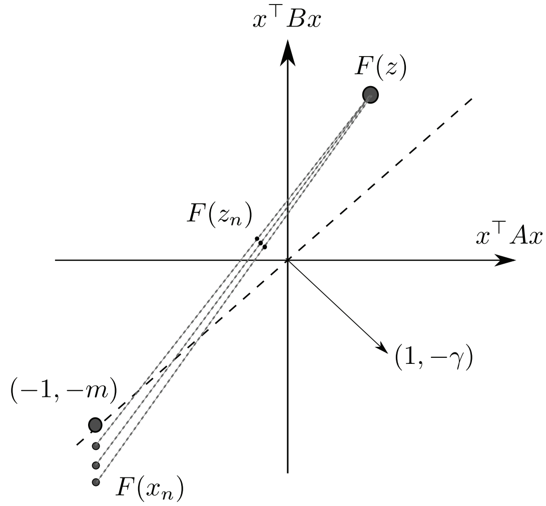

FIG. 2. *Geometric intuition of* $A - \gamma B \succeq 0$.

Now, it is easy to check that for any $\gamma' > 0$ such that $A - \gamma' B \succeq 0$ it follows that $\gamma' \geq \frac{1}{m}$. Indeed, for all x *with* $x^\top Ax < 0$ it follows that

$$x^\top Ax - \gamma' x^\top Bx \geq 0 \iff \frac{1}{\gamma'} \leq \frac{x^\top Bx}{x^\top Ax},$$

which implies the desired inequality. To summarize, $\frac{1}{m}$ amounts to the lower bound of $\mathcal{A}$ and our initial claim $\gamma = 1/m$ follows.

In the last step, keeping in mind that $A - \gamma B$ is positive semidefinite, we prove that $Ker(A - \gamma B) \cap \{x : \ x^\top Bx \leq 0\} \neq \{0\}$. We argue by contradiction and assume that the above intersection amounts to $\{0\}$. Since $F(x) = 0$ implies that $x^\top Bx = 0$ and $x^\top Ax - \gamma x^\top Bx = 0 - \gamma \cdot 0 = 0$, it follows from our contradictory assumption that the implication

$$F(x) = 0 \implies x = 0$$

holds. We may thus invoke [2, Theorem 2] and conclude that the set $F(\mathbb{R}^n)$ is closed. Consequently, for any minimizing sequence $\{x_n\}_n$ for (2.1) satisfying $x_n^\top Ax_n = -1$, it holds that

$$\operatorname*{Lim}_{n\to\infty} F(x_n) = \begin{pmatrix} -1 \\ -m \end{pmatrix} \in F(\mathbb{R}^n).$$

Let w be such that $F(w) = \binom{-1}{-m}$. Then, clearly, $w \neq 0$. Furthermore, one has

$$w^\top Aw - \bar{\gamma} w^\top Bw = -1 + m\bar{\gamma} = 0 \ \text{ and } \ w^\top Bw = -m < 0.$$

It follows that $w \in Ker(A - \gamma B) \cap \{x : \ x^\top Bx \leq 0\}$, which contradicts the posed assumption. It suffices thus to put $\delta = \bar{\gamma}$, and the lemma has been proved. ∎

Now we are in position to fix the proofs of the mentioned results. First we present a corrected proof for the mentioned part of Theorem 1.4.

Proof of Theorem 1.4 *in* [1]*, case* 6*, subcase a.* Let us suppose that (1.7) ((2.41) in [1]) is violated due to the existence of a nonzero vector h^* satisfying

$$(h^*)^\top D^2_{xx} L(x^*,\ y^*) h^* < 0.$$

Let $A = D^2_{xx} L(x^*,\ y^*),\ B = Dg(x^*)^\top R Dg(x^*)$. Then all the hypotheses of the *auxiliary lemma* are satisfied. In fact, condition 1 is true thanks to the existence of h^*, condition 2 holds because R is indefinite, and $Dg(x^*)$ is a surjective operator (using the nondegeneracy of x^*), and condition 3 is a reformulation of the necessary condition (1.6) ((2.44) in [1]).

Then, there exist a positive number $\gamma > 0$ and a vector $h \neq 0$ such that

$$D^2_{xx} L(x^*,\ y^*) h - \gamma Dg(x^*)^\top R Dg(x^*) h = 0, \tag{2.4}$$

$$h^\top Dg(x^*)^\top R Dg(x^*) h \leq 0. \tag{2.5}$$

We claim that the vector $-d = -Dg(x^*)h$ belongs to the set

$$\bar\partial_B P(0)(-d + \gamma R d) = \bar\partial_B P(0) \begin{pmatrix} -(1-\gamma) d_0 \\ -(1+\gamma)\bar d \end{pmatrix}. \tag{2.6}$$

Indeed, it suffices to select in the definition of $\bar\partial_B P(0)$ ((2.15) in [1]) a matrix specified by a unit vector w such that $d^\top(1,\ -w) = 0$ and $\alpha = 1/(1+\gamma)$. Note that the existence of such w is ensured due to inequality $h^\top Dg(x^*)^\top R Dg(x^*) h \leq 0$, which is the same as $||\bar d|| \geq |d_0|$. This condition ensures the existence of a unit vector w such that $\langle \bar d, w\rangle = d_0$. Now, since $D^*P(0)u^*$ contains $\bar\partial_B P(0) u^*$ for all u^*, we conclude that $-d$ belongs to $D^*P(0)(-d + \gamma R d)$. Our claim is proved.

Finally, we can see that the relations (1.8) ((3.3) in [1]) are solved by the vectors $v = h$ and $b = d - \gamma R d = (I_{m+1} - \gamma R) Dg(x^*) h$. This contradicts the statement (iv). ∎

The last thing we need to do is to fix the proof for the mentioned case of *Theorem* 1.5.

Proof of Theorem 1.5*, case* $|J_6| = 1$*, subcase a.* Let $j \in J_6$. If (1.7) ((2.41) in [1]) is violated because there is a vector h^* such that $Q_0(h^*) < 0$, then all the hypotheses of the auxiliary lemma are satisfied for the matrices associated with Q_0, Q_1, say A, B. In fact, condition 1 is true thanks to the existence of h^*, condition 2 holds because R_{m_j} is indefinite, and $Dg^j(x^*)$ is a surjective operator (using the nondegeneracy of x^*), and condition 3 is a reformulation of the necessary condition.

Then, there exist $\gamma > 0$ and a vector $h \neq 0$ such that

$$D^2_{xx} L(x^*,\ y^*) h + \mathcal{H}(x^*, y^*) h - \gamma Dg^j(x^*)^\top R_{m_j} Dg^j(x^*) h = 0, \tag{2.7}$$

$$h^\top Dg^j(x^*)^\top R Dg^j(x^*) h \leq 0. \tag{2.8}$$

It can be proved that $-d = -d^j(h) = -Dg^j(x^*)h$ belongs to the set

$$\bar\partial_B P^j(0)(-d + \gamma R_{m_j} d) = \bar\partial_B P^j(0) \left(- \begin{bmatrix} (1-\gamma) d_0 \\ (1+\gamma)\bar d \end{bmatrix} \right).$$

Indeed, it suffices to choose a unit vector w such that $d^\top(1, -w) = 0$ and $\alpha = 1/(1+\gamma)$ in the definition of $\bar\partial_B P(0)$ ((2.15) in [1]). As in the proof of case 6 of Theorem 1.4, the existence of such w is ensured due to inequality $Q_0(h) \leq 0$, which, together with

$Q_0 - \gamma Q_1 \succeq 0$, implies that $\|\bar{d}\|^2 > d_0^2$ or, equivalently, $\|\bar{d}\| \geq |d_0|$. This condition clearly ensures the existence of a unit vector w such that $\langle \bar{d}, w\rangle = d_0$. For the case when $m_j = 1$, see Remark 27 in [1].

Now, since $D^*P^j(0)(u^*)$ contains $\bar{\partial}_B P^j(0)(u^*)$ for all u^* (see the definition of D^*P ((2.14) in [1])), we conclude that $-d$ belongs to $D^*P^j(0)(-d+\gamma R_{m_j} d)$. Consequently, (v, b^j) with $v = h$ and $b^j = d - \gamma R_{m_j} d = (I_{m_j+1} - \gamma R_{m_j}) Dg^j(x^*)h$ solves (1.8) ((4.1) in [1]) for the block $j \in J_6$. ◻

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