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Error identities for variational problems with obstacles

S. Repin[1,2] **and J. Valdman**[3,4,*]

[1] University of Jyväskylä, P.O.Box 35, 40014, Finland
[2] St. Petersburg Department of V.A. Steklov Mathematical Institute, 191011, Fontanka 27 and Peter the Great St. Petersburg Polytechnic University, 195251, Politekhnicheskaya 29, St. Petersburg, Russia
[3] Institute of Mathematics and Biomathematics, Faculty of Science, University of South Bohemia, Branišovská 31, 37005 České Budějovice, Czech Republic
[4] The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod vodárenskou věží 4, 18208, Praha 8, Czech Republic

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The paper is devoted to analysis of a class of nonlinear free boundary problems that are usually solved by variational methods based on primal, dual or primal-dual variational settings. We deduce and investigate special relations (error identities). They show that a certain nonlinear measure of the distance to the exact solution (specific for each problem) is equivalent to the respective duality gap, whose minimization is the keystone of all variational numerical methods. Therefore, the identity actually sets the measure that contains maximal quantitative information on the quality of a numerical solution available through these methods. The measure has quadratic terms generated by the linear part of the differential operator and nonlinear terms associated with the free boundary. We obtain fully computable two sided bounds of this measure and show that they provide efficient estimates of the distance between the minimizer and any function (approximation) from the corresponding energy space. Several computational examples show that for different minimization sequences the balance between the quadratic and the nonlinear terms of the overall error measure may be different and essential contribution of nonlinear terms may serve as an indicator that the free boundaries are approximated very roughly.

1 Introduction

Variational inequalities form an important class of nonlinear models that describe free boundary phenomena arising in various applied problems (see, e.g., G. Duvaut and J. L. Lions [8] and other publications cited therein). Usually free boundaries separate regions where solutions possess quite different physical properties. Therefore, any reliable information on the shape and location of such a boundary is very important. Qualitative properties of free boundaries are studied by purely analytical (a priori) methods unlike quantitative information, which in the vast majority of cases can be obtained only by computational methods. In this context, it is necessary to know what quantitative information could be really extracted from a numerical solution.

Differentiability properties of exact solutions to variational inequalities are, in general, restricted even if all external data of a problem are smooth (e.g., see H. Brezis [1], L.A. Caffarelli [7], D. Kinderlehrer and G. Stampacchia [16], A. Friedman [10], N. N. Uraltseva [28]). In H. Brezis and M. Sibony [2], it was proved that there exists a unique solution $u \in W^{2,2}(\Omega)$ of the classical obstacle problem

$$\int_\Omega \nabla u \cdot \nabla w \, dx \geq \int_\Omega f w \, dx, \qquad \forall w \in K$$

where $K := \{w \in H^1(\Omega) \mid w = u_D \; on \; \partial\Omega, \, w \geq \psi\}$ provided that $\psi \in W^{2,2}(\Omega)$, $f \in L^2(\Omega)$, and the function u_D (which defines the Dirichlét boundary condition) belongs to $W^{2,2}(\Omega)$ and satisfies the constrain $u_D \geq \psi$ on $\partial\Omega$. Many researches were focused on clarifying mathematical properties of the coincidence set. In particular, it was proved that if the domain $\Omega \subset R^2$ is strictly convex with a smooth boundary $\partial\Omega$ and if the obstacle $\psi \in C^2(\Omega)$ is strictly concave, then the coincidence set is connected and its boundary is smooth and homeomorphic to the unit circle (see, e.g., D. Kinderlehrer

* Corresponding author, E-mail: jan.valdman@gmail.com

and G. Stampacchia [16]). However, in general, the structure of a coincidence set can be very complicated and for any domain one can find an obstacle such that this set contains any number of disjoint subsets.

Numerical methods for problems with obstacles (and many other problems related to variational inequalities) were systematically studied in R. Glowinski, J.-L. Lions, and R. Tremolieres [11, 12]. Getting the respective a priori rate convergence estimates (in terms of the mesh size h) was the first question studied by many authors. In the context of finite element approximations, such type estimates were derived by R. S. Falk [9] who proved standard a priori convergence error estimates (with the rate h for the L^2 norm of gradients and the rate h^2 for the L^2 norm of the functions) provided that $u \in W^{2,2}$. Convergence of mixed approximations for problems with obstacles was established in F. Brezzi, W. Hager, and P. A. Raviart [3] and numerical methods based on the augmented Lagrangian approach were studied in T. Kärkkäinen, K. Kunisch, and P. Tarvainen [15].

In this paper, we are concerned with another important question arising in quantitative analysis of nonlinear problems - *which measure* $\mathbb{M}$ *of the distance to the exact solution is adequate (natural) for a particular problem?* We study this question in detail for two classes of obstacle type problems - the classical obstacle problem and the two-phase obstacle problem. Our analysis is based on general type *error identities* derived and used in [18,21,22,24] for a wide class of convex variational problems and applied here to problems with obstacles. The error identities establish equivalence of a certain nonlinear measure $\mathbb{M}$ and the duality gap between the primal and dual energy functionals. Since the variational methods are based on minimization of this gap, the measure $\mathbb{M}$ shows limits of quantitative analysis for this class of methods. The newly derived energy identities (primal, dual, or primal-dual) are formulated in Theorem 1 and Theorem 2 for the classical obstacle problem and in Theorem 3 for the two-phase obstacle problem. In the last section, we present several numerical tests aimed to illustrate theoretical results. We show that the measures correctly represent the quality of approximations for various minimizing sequences. In different cases, components of the measure may have different values. Depending on the minimizing sequence, one or the other measure component may dominate. In other cases, the measures may be comparable. Theoretical and computational results give a presentation on the amount of quantitative information contained in a minimizing sequence constructed by means this or other version of the variational approach.

2 General energy identities for variational problems

For convenience of the reader we shortly recall the main items necessary to understand the material. Consider the class of variational problems

$$\inf_{v \in V} J(v), \qquad J(v) = G(\Lambda v) + F(v), \tag{1}$$

where $\Lambda : V \to Y$ is a bounded linear operator, $G : Y \to \mathbb{R}$ is a convex, coercive, and lower semicontinuous functional, $F : V \to \mathbb{R}$ is another convex lower semicontinuous functional, and Y and V are reflexive Banach spaces. The dual spaces are denoted by Y^* and V^*, respectively, and the duality pairings are denoted by (y^*, y) and $\langle v^*, v\rangle$. The dual variational problem consists of finding $p^* \in Y^*$ maximizing the dual functional

$$I^*(y^*) := -G^*(y^*) - F^*(-\Lambda^* y^*) \tag{2}$$

over the space Y^*. Here $G^* : Y^* \to \mathbb{R}$ and $F^* : V^* \to \mathbb{R}$ are the Young-Fenchel transforms (convex conjugates) of G and F, respectively. Henceforth, we use the so called *compound* functionals

$$D_F(v, v^*) := F(v) + F^*(v^*) - \langle v^*, v\rangle,$$
$$D_G(y,\ y^*) \ := G(y) + G^*(y^*) - (y^*, y)$$

generated by the convex functionals F and G, respectively. These functionals are nonnegative and vanish if and only if v and v^* (resp. y and y^*) are joined by special differential relations (see, e.g., [19]). Notice that in the simplest case where V is a Hilbert space and $F(v) = \frac{1}{2}\|v\|^2$, the functional $D_F(v, v^*)$ coincides with the norm $\frac{1}{2}\|v - v^*\|^2$. However, in general $D_F(v, v^*)$ should be viewed as a nonlinear measure, which vanishes if and only if the pair (v, v^*) satisfies certain conditions.

Let $v \in V$ and $y^* \in Y^*$ be the functions (approximations) compared with u and p^*. Introduce the following measure of the distance between $\{u, p^*\}$ and $\{v, y^*\}$:

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) := D_F(u, -\Lambda^* y^*) + D_F(v, -\Lambda^* p^*) + D_G(\Lambda u, y^*) + D_G(\Lambda v, p^*) \geq 0.$$

It vanishes if and only if

$$\Lambda v \in \partial G^*(p^*),\ \ y^* \in \partial G(\Lambda u),\ \ -\Lambda^* y^* \in \partial F(u),\ \ v \in \partial F^*(-\Lambda^* p^*).$$

The above conditions are satisfied if and only if $v = u$ and $y^* = p^*$ (i.e., if approximations coincide with the exact primal and dual solutions). In [18] (Sect. 7.2), [21], and [24], it was proved that

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) = J(v) - I^*(y^*). \tag{3}$$

Hence $\mathbb{M}\{(u, p^*), (v, y^*)\} = 0$ if and only if $J(v) = I^*(y^*)$ (what means that v is a minimizer of the problem $\mathcal{P}$ and y^* is a maximizer of the problem $\mathcal{P}^*$). Two particular forms of (3) arise if we set $v = u$ or $y^* = p^*$. They are

$$\mathbb{M}(u, v) := \mathbb{M}(\{u, p^*\}, \{v, p^*\}) \quad \text{and} \quad \mathbb{M}(p^*, y^*) := \mathbb{M}(\{u, p^*\}, \{u, y^*\}).$$

In view of (3),

$$\mathbb{M}(u, v) = D_F(v, -\Lambda^* p^*) + D_G(\Lambda v, p^*) = J(v) - J(u), \tag{4}$$

$$\mathbb{M}(p^*, y^*) = D_F(u, -\Lambda^* y^*) + D_G(\Lambda u, y^*) = I^*(p^*) - I^*(y^*). \tag{5}$$

Henceforth, we call (3), (4), and (5) the primal-dual, the primal and the dual energy identities, respectively. The functionals $\mathbb{M}(\{u, p^*\}, \{v, y^*\})$, $\mathbb{M}(u, v)$, $\mathbb{M}(y^*, p^*)$ are in fact the error measures used by any energy based numerical procedure designed to solve (1). Since the error measures are equal to the respective duality gaps, they present the strongest (and in a sense the most natural) measure for the class of problems considered.

Below we study energy identities for two classes of nonlinear variational problems and show that they generate specific error measures containing two parts. The first (quadratic) part is presented by a norm equivalent to the H^1 norm and the second (nonlinear) part controls (in a rather weak sense) how accurately the approximate solution recovers the free boundary.

3 Classical obstacle problem

3.1 Variational setting

We begin with the classical obstacle problem (see, e.g. [1, 10, 16]), where admissible functions belong to the set

$$K := \{w \in V_0 \mid \phi(x) \le w(x) \le \psi(x) \text{ a.e. in } \Omega\}.$$

Here, $V_0 := H^1_0(\Omega)$ denotes the Sobolev space of functions vanishing on $\partial\Omega$ (hence we consider the case $u_D = 0$), $\Omega \subset \mathbb{R}^d$ ($d \in \{1, 2, 3\}$) is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$ and $\phi, \psi \in H^2(\Omega)$ are two given functions (lower and upper obstacles) such that

$$\phi(x) \le 0 \text{ on } \partial\Omega, \quad \psi(x) \ge 0 \text{ on } \partial\Omega, \quad \phi(x) \le \psi(x), \quad \forall x \in \Omega.$$

The problem is to find $u \in K$ satisfying the variational inequality

$$a(u, w - u) \ge \int_\Omega f(w - u)\, dx \quad \forall w \in K \tag{6}$$

for a given function $f \in L^2(\Omega)$ and a bilinear form

$$a(u, w) := \int_\Omega A\nabla u \cdot \nabla w\, dx.$$

It is assumed that A is a symmetric matrix subject to the condition

$$A(x)\xi \cdot \xi \ge c_1 |\xi|^2 \qquad c_1 > 0, \qquad \forall \xi \in \mathbb{R}^d \tag{7}$$

almost everywhere in Ω. Under the assumptions made, the problem (6) is uniquely solvable. In general, the solution u divides Ω into three sets:

$$\begin{aligned} \Omega^u_- &:= \{x \in \Omega \mid u(x) = \phi(x)\}, \\ \Omega^u_+ &:= \{x \in \Omega \mid u(x) = \psi(x)\}, \\ \Omega^u_0 &:= \{x \in \Omega \mid \phi(x) < u(x) < \psi(x)\}. \end{aligned} \tag{8}$$

The sets Ω^u_- and Ω^u_+ are the *lower* and *upper coincidence sets* and Ω^u_0 is an open set, where u satisfies the Poisson equation $\operatorname{div}(A\nabla u) + f = 0$. Thus, the problem involves a priori unknown *free boundaries*. Let $v \in K$ be an approximation of u.

It defines the sets

$$\begin{aligned} \Omega_-^v &:= \{x \in \Omega \mid v(x) = \phi(x)\}, \\ \Omega_+^v &:= \{x \in \Omega \mid v(x) = \psi(x)\}, \\ \Omega_0^v &:= \{x \in \Omega \mid \phi(x) < v(x) < \psi(x)\}. \end{aligned} \tag{9}$$

Notice that unlike the sets in (8), the sets (9) are known. The dual component of the exact solution

$$p^* = A\nabla u \tag{10}$$

satisfies the conditions

$$\begin{aligned} \operatorname{div} p^* + f &\le 0 \quad \text{on } \Omega_-^u, \\ \operatorname{div} p^* + f &\ge 0 \quad \text{on } \Omega_+^u, \\ \operatorname{div} p^* + f &= 0 \quad \text{on } \Omega_0^u. \end{aligned} \tag{11}$$

It is well known that the pair $(u, p^*) \in K \times L^2(\Omega, \mathbb{R}^d)$ is a saddle point of the respective minimax formulation. Moreover, p^* has square summable divergence and satisfies the relations (10) and (11) almost everywhere in Ω.

3.2 Error measures

The variational inequality (6) is known to have the equivalent form (1) for

$$\Lambda v = \nabla v, \qquad \Lambda^* y^* = -\operatorname{div} y^*,$$

$$G(\Lambda v) = \tfrac{1}{2}\int\limits_\Omega A\nabla v \cdot \nabla v \, dx, \qquad F(v) = -\int\limits_\Omega f v \, dx + \chi_K(v),$$

where χ_K is the characteristic functional of the set K, i.e.,

$$\chi_K(v) := \begin{cases} 0 & \text{if } \phi \le v \le \psi, \\ +\infty & \text{else.} \end{cases}$$

In this case, $V = V_0$, $V^* = H^{-1}(\Omega)$, $Y = L^2(\Omega, \mathbb{R}^d)$,

$$G^*(y^*) = \frac{1}{2}\int\limits_\Omega A^{-1} y^* \cdot y^* \, dx,$$

$$D_G(\Lambda v, y^*) = \frac{1}{2}\int\limits_\Omega (A\nabla v - y^*) \cdot (\nabla v - A^{-1} y^*) \, dx. \tag{12}$$

Thus, we have

$$D_G(\Lambda v, p^*) = \frac{1}{2}\int\limits_\Omega A\nabla(u - v) \cdot \nabla(u - v) \, dx =: \frac{1}{2}||\nabla(u - v)||_A^2,$$

$$D_G(\Lambda u, y^*) = \frac{1}{2}\int\limits_\Omega A^{-1}(p^* - y^*) \cdot (p^* - y^*) \, dx =: \frac{1}{2}||p^* - y^*||_{A^{-1}}^2.$$

Next, for $v^* \in L^2(\Omega)$,

$$\begin{aligned} F^*(v^*) &= \sup_{v \in K} \int\limits_\Omega v(v^* + f) \, dx \\ &= \sup_{v \in K} \int\limits_\Omega (-v(v^* + f)_- + v(v^* + f)_+) \, dx = \int\limits_\Omega (-\phi(v^* + f)_- + \psi(v^* + f)_+) \, dx. \end{aligned} \tag{13}$$

Here, $(z)_-$ and $(z)_+$ denote the negative and positive parts of the quantity z, i.e., $(z)_- := -\min\{0, z\}$, $(z)_+ := \max\{0, z\}$ and it holds $z = -(z)_- + (z)_+$, $|z| = (z)_- + (z)_+$.

In view of (13), we deduce explicit form of the functional D_F provided that $y^* \in Y^*_{\mathrm{div}}(\Omega) := \{y^* \in Y^* \mid \mathrm{div} y^* \in L^2(\Omega)\}$:

$$\begin{aligned} D_F(v, -\Lambda^* y^*) &= F(v) + F^*(-\Lambda^* y^*) + \langle \Lambda^* y^*, v \rangle \\ &= \int_\Omega \left(-f v - \phi\,(\mathrm{div} y^* + f)_- + \psi\,(\mathrm{div} y^* + f)_+ - \mathrm{div} y^* v\right) dx \\ &= \int_\Omega \left((v - \phi)\,(\mathrm{div} y^* + f)_- + (\psi - v)\,(\mathrm{div} y^* + f)_+\right) dx. \end{aligned}$$

Since p^* belongs to $Y^*_{\mathrm{div}}(\Omega)$ and satisfies the relation (11), we find that

$$\begin{aligned} D_F(v, -\Lambda^* p^*) &= -\int_{\Omega^u_-} (v - \phi)(\mathrm{div} p^* + f)\, dx + \int_{\Omega^u_+} (\psi - v)(\mathrm{div} p^* + f)\, dx \\ &= -\int_{\Omega^u_-} (v - \phi)(\mathrm{div} A \nabla \phi + f)\, dx + \int_{\Omega^u_+} (\psi - v)(\mathrm{div} A \nabla \psi + f)\, dx. \end{aligned}$$

This quantity can be viewed as a certain measure

$$\mu_{\phi\psi}(v) := \int_{\Omega^u_-} \mathsf{W}_\phi (v - \phi)\, dx + \int_{\Omega^u_+} \mathsf{W}_\psi (\psi - v)\, dx, \tag{14}$$

where $\mathsf{W}_\phi := -(\mathrm{div} A \nabla \phi + f)$, $\mathsf{W}_\psi := \mathrm{div} A \nabla \psi + f$ are two nonnegative weight functions generated by the source term f, the obstacles ψ, ϕ and the diffusion A. It is clear that $\mu_{\phi\psi}(v) = 0$ if $\Omega^v_- \subset \Omega^u_-$ and $\Omega^v_+ \subset \Omega^u_+$. In other words, if all points of approximate sets Ω^v_- and Ω^v_+ indeed belong to the coincidence sets, then the measure is zero.

Remark 1. Assume that $A = \mathbb{I}$ (the identity matrix), obstacles ϕ, ψ are harmonic functions ($\triangle \phi = \triangle \psi = 0$ in Ω) satisfying $\phi < 0 < \psi$ a.e. in Ω and $f = const \neq 0$. If $f > 0$ then $\Omega^u_- = \emptyset$ (the lower obstacle ϕ is never active) and

$$\mu_{\phi\psi}(v) = f \int_{\Omega^u_+} (\psi - v)\, dx = f \|\psi - v\|_{L^1(\Omega^u_+)} = f \|\psi - v\|_{L^1(\Omega^u_+ \setminus \Omega^v_+)}.$$

Here, we decomposed

$$\Omega^u_+ = (\Omega^u_+ \setminus \Omega^v_+) \cup (\Omega^u_+ \cap \Omega^v_+)$$

and applied the equality $\|\psi - v\|_{L^1(\Omega^u_+)} = \|\psi - v\|_{L^1(\Omega^u_+ \setminus \Omega^v_+)}$ (which holds because $\|\psi - v\|_{L^1(\Omega^u_+ \setminus \Omega^v_+)} = 0$). Analogously, if $f < 0$ then $\Omega^u_+ = \emptyset$ (the upper obstacle ψ is never active) and

$$\mu_{\phi\psi}(v) = -f \int_{\Omega^u_-} (v - \phi)\, dx = -f \|v - \phi\|_{L^1(\Omega^u_-)} = -f \|v - \phi\|_{L^1(\Omega^u_- \setminus \Omega^v_-)}.$$

We see that $\mu_{\phi\psi}(v)$ represents a certain measure, which controls (in a weak integral sense) whether or not the function v coincides with obstacles ψ, ϕ on true coincidence sets Ω^u_- and Ω^u_+.

Analogously, the quantity

$$D_F(u, -\Lambda^* y^*) = -\int_{\Omega^{y^*}_-} (u - \phi)(\mathrm{div} y^* + f)\, dx + \int_{\Omega^{y^*}_+} (\psi - u)(\mathrm{div} y^* + f)\, dx$$

forms another measure

$$\mu^*_{\phi\psi}(y^*) := -\int_{\Omega^{y^*}_-} (u - \phi)(\mathrm{div} y^* + f)\, dx + \int_{\Omega^{y^*}_+} (\psi - u)(\mathrm{div} y^* + f)\, dx, \tag{15}$$

where the sets

$$\begin{aligned} \Omega^{y^*}_- &:= \{x \in \Omega \mid \mathrm{div} y^* + f < 0\}, \\ \Omega^{y^*}_+ &:= \{x \in \Omega \mid \mathrm{div} y^* + f > 0\}, \\ \Omega^{y^*}_0 &:= \{x \in \Omega \mid \mathrm{div} y^* + f = 0\} \end{aligned}$$

are approximations of Ω^u_-, Ω^u_+, Ω^u_0 generated on the basis of the dual solution y^*. It is clear that this measure is zero if $\Omega^{y^*}_- \subset \Omega^u_-$ and $\Omega^{y^*}_+ \subset \Omega^u_+$. Hence, the measure $\mu^*_{\phi\psi}(y^*)$ is positive if the sets $\Omega^{y^*}_-$ and $\Omega^{y^*}_+$ contain parts which do not belong to true coincidence sets. We summarize properties of $\mu_{\phi\psi}(v)$ and $\mu^*_{\phi\psi}(y^*)$ as follows:

$$\Omega^u_- \subset \Omega^v_- \quad \text{and} \quad \Omega^u_+ \subset \Omega^v_+ \quad \Rightarrow \quad \mu_{\phi\psi}(v) = 0, \tag{16}$$

$$\Omega^{y^*}_- \subset \Omega^u_- \quad \text{and} \quad \Omega^{y^*}_+ \subset \Omega^u_+ \quad \Rightarrow \quad \mu^*_{\phi\psi}(y^*) = 0. \tag{17}$$

Now we use (3), (4), and (5) and deduce primal and dual error identities in terms of the primal and dual settings.

Theorem 1 (primal and dual energy identities for the classical obstacle problem)**.** *Let v and y^* be approximations of u and p^*, respectively. Then,*

$$\mathbb{M}(u, v) = \frac{1}{2}||\nabla(u - v)||^2_A + \mu_{\phi\psi}(v) = J(v) - J(u), \tag{18}$$

$$\mathbb{M}(p^*, y^*) = \frac{1}{2}||p^* - y^*||^2_{A^{-1}} + \mu^*_{\phi\psi}(y^*) = I^*(p^*) - I^*(y^*). \tag{19}$$

In view of the relation between the primal and dual functionals, the identities (18) and (19) yield the form of the primal-dual energy identity

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) = \mathbb{M}(u, v) + \mathbb{M}(p^*, y^*) = J(v) - I^*(y^*). \tag{20}$$

This error identity decomposes the primal-dual measure $\mathbb{M}(\{u, p^*\}, \{v, y^*\})$ additively to the primal and the dual measures. It shows that the duality gap consists of four nonnegative quantities. Two of them are quadratic terms $\frac{1}{2}||\nabla(u - v)||^2_A$ and $\frac{1}{2}||p^* - y^*||^2_{A^{-1}}$ associated with energy errors. Two others are nonlinear terms $\mu_{\phi\psi}(v)$ and $\mu^*_{\phi\psi}(y^*)$ defined by (14) and (15).

Remark 2. By neglecting nonlinear terms $\mu_{\phi\psi}(v)$, $\mu^*_{\phi\psi}(y^*)$ in (18) and (19) we obtain known inequalities

$$\frac{1}{2}||\nabla(u - v)||^2_A \le J(v) - J(u), \qquad \frac{1}{2}||p^* - y^*||^2_{A^{-1}} \le I^*(p^*) - I^*(y^*).$$

3.3 Computable bounds of error measures

First we show that the primal-dual measure $\mathbb{M}(\{u, p^*\}, \{v, y^*\})$ can be directly computed for any pair of approximate solutions $\{v, y^*\}$ provided that y^* possesses an additional regularity.

Theorem 2 (primal-dual error identity for the classical obstacle problem)**.** *Let $\{v, y^*\} \in K \times Y^*_{\mathrm{div}}(\Omega)$. Then,*

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) = \frac{1}{2}||A\nabla v - y^*||^2_{A^{-1}} + \Upsilon(v, y^*) = J(v) - I^*(y^*), \tag{21}$$

where

$$\Upsilon(v, y^*) := \int\limits_{\Omega^{y^*}_- \setminus \Omega^v_-} (\phi - v)(\mathrm{div} y^* + f)\, dx + \int\limits_{\Omega^{y^*}_+ \setminus \Omega^v_+} (\psi - v)(\mathrm{div} y^* + f)\, dx. \tag{22}$$

P r o o f . In view of (2), (12), and (13), we have

$$J(v) = \frac{1}{2}||\nabla v||^2_A - \int\limits_\Omega f v\, dx,$$

$$I^*(y^*) = -\frac{1}{2}||y^*||^2_{A^{-1}} + \int\limits_\Omega (\phi\, (\mathrm{div} y^* + f)_- - \psi\, (\mathrm{div} y^* + f)_+)\, dx. \tag{23}$$

According to (20),

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) = \frac{1}{2}||\nabla v||^2_A - \int\limits_\Omega f v\, dx + \frac{1}{2}||y^*||^2_{A^{-1}}$$

$$+ \int\limits_\Omega (-\phi\, (\mathrm{div} y^* + f)_- + \psi\, (\mathrm{div} y^* + f)_+)\, dx. \tag{24}$$

Since

$$\frac{1}{2}||\nabla v||_A^2 + \frac{1}{2}||y^*||_{A^{-1}}^2 - \int_\Omega f v\,dx = \frac{1}{2}||A\nabla v - y^*||_{A^{-1}}^2 + \int_\Omega (y^* \cdot \nabla v - f v)\,dx$$
$$= \frac{1}{2}||A\nabla v - y^*||_{A^{-1}}^2 - \int_\Omega (\mathrm{div} y^* + f) v\,dx$$

and

$$-\int_\Omega (\mathrm{div} y^* + f) v\,dx + \int_\Omega (-\phi\,(\mathrm{div} y^* + f)_- + \psi\,(\mathrm{div} y^* + f)_+)\,dx$$
$$= -\int_{\Omega_-^{y^*}} (v-\phi)(\mathrm{div} y^* + f)\,dx + \int_{\Omega_+^{y^*}} (\psi - v)(\mathrm{div} y^* + f)\,dx$$
$$= -\int_{\Omega_-^{y^*}\setminus\Omega_-^v} (v-\phi)(\mathrm{div} y^* + f)\,dx + \int_{\Omega_+^{y^*}\setminus\Omega_+^v} (\psi - v)(\mathrm{div} y^* + f)\,dx$$

the substitution of last two equalities in (24) yields (21). □

Remark 3. Assume that the right hand side of (21) is equal to zero. Then $y^* = A\nabla v$ and

$$v = \phi \qquad \text{if } \mathrm{div} y^* + f < 0,$$
$$v = \psi \qquad \text{if } \mathrm{div} y^* + f > 0.$$

Hence, $\Omega_-^{y^*} \subset \Omega_-^v$ and $\Omega_+^{y^*} \subset \Omega_+^v$. The sets Ω_+^v and Ω_-^v do not intersect as well as the sets $\Omega_+^{y^*}$ and $\Omega_-^{y^*}$. Therefore, the set $\Omega_0^v = \Omega \setminus (\Omega_+^v \cup \Omega_-^v)$ is contained in the set $\Omega_0^{y^*} = \Omega \setminus (\Omega_+^{y^*} \cup \Omega_-^{y^*})$. Thus, $\mathrm{div} y^* + f = 0$ in Ω_0^v. For any $w \in K$, we have

$$\int_\Omega A\nabla v \cdot \nabla(w - v)\,dx - \int_\Omega f(w-v)\,dx = \int_{\Omega_-^v} (\mathrm{div} y^* + f)(\phi - w)\,dx$$
$$+ \int_{\Omega_+^v} (\mathrm{div} y^* + f)(\psi - w)\,dx + \int_{\Omega_0^v} (\mathrm{div} y^* + f)(v - w)\,dx.$$

The right hand side of the above relation is nonnegative. Indeed, the first two integrals are nonnegative and the last one is equal to zero. This means that v satisfies the variational inequality and, consequently, the pair $\{v, y^*\}$ coincides with $\{u, p^*\}$.

Remark 4. If approximations of the coincidence sets (constructed on the basis of v and y^*) satisfy the relations $\Omega_-^{y^*} \subset \Omega_-^v$ and $\Omega_+^{y^*} \subset \Omega_+^v$, then (21) reads

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) = \frac{1}{2}||A\nabla v - y^*||_{A^{-1}}^2.$$

Moreover, if $\Omega_-^{y^*} \subset \Omega_-^u \subset \Omega_-^v$ and $\Omega_+^{y^*} \subset \Omega_+^u \subset \Omega_+^v$, then both nonlinear terms of $\mathbb{M}(\{u, p^*\}, \{v, y^*\})$ vanish and we arrive at the equality

$$||\nabla(u - v)||_A^2 + ||p^* - y^*||_{A^{-1}}^2 = ||A\nabla v - y^*||_{A^{-1}}^2.$$

However, the sets Ω_-^u and Ω_+^u are unknown, so that in practice it is impossible to verify the conditions that yield this simplest (hypercircle type) form of the error identity.

Remark 5 (nonhomogeneous Dirichlét boundary conditions)**.** Error identities for the classical obstacle problem were derived in this section for the homogeneous Dirichlét boundary conditions only. However, the exact solution u of the forthcoming Example 3 satisfies nonhomogeneous Dirichlét boundary conditions. It is possible to show that all forms of error identities remain unchanged. The only difference is that an extra boundary term

$$\int_{\partial\Omega} (y^* \cdot n) u_D \, dx$$

needs to be added to the dual functional (23).

3.4 Estimates for the primal and dual measures

Theorem 2 provides a way to compute $\mathbb{M}(\{u, p^*\}, \{v, y^*\})$, which is the sum of error measures $\mathbb{M}(u, v)$ and $\mathbb{M}(p^*, y^*)$. These measures separately evaluate deviations of v from u and y^* from p^*. It is desirable to have guaranteed bounds for them as well (notice that in view of (21) two sided bounds of $\mathbb{M}(u, v)$ imply two sided bounds of $\mathbb{M}(p^*, y^*)$ and vise versa). For this purpose, we require knowledge of the exact energy $J(u) = I^*(p^*)$, which is generally unknown. However, there is a way to derive computable bounds of $J(v) - J(u)$ without this knowledge (see [21, 23]). In this subsection, we briefly discuss applications of these results addressing the reader to a more systematic exposition and numerical tests in [6, 13, 14, 18, 20]. First, we have the estimate

$$\begin{aligned} \mathbb{M}(u, v) \leq \mathbb{M}^+(v; \beta, \lambda_1, \lambda_2, y^*) &:= (1 + \beta^{-1}) D_G(\nabla v, y^*) \\ &+ \frac{1}{2} C_\Omega^2 (1 + \beta) \| \operatorname{div} y^* + f + \lambda_1 - \lambda_2 \|_\Omega^2 + \int_\Omega (\lambda_1 (v - \phi) + \lambda_2 (\psi - v)) \, dx. \end{aligned} \tag{25}$$

The majorant $\mathbb{M}^+$ contains free variables: $\beta > 0$, $y^* \in Y^*_{\operatorname{div}}(\Omega)$, and two nonnegative functions (Lagrange multipliers) $\lambda_1, \lambda_2 \in L^2(\Omega)$. The constant $C_\Omega > 0$ is a minimal constant in a Friedrichs type inequality $\|w\| \leq C_\Omega \|\nabla w\|_A$ for all $w \in V_0$. It should be outlined that originally it was proved that $\mathbb{M}^+$ majorates only the quadratic part of the error $\frac{1}{2}\|\nabla(v - u)\|_A^2$, see, e.g., [23]. Now, we can claim that $\mathbb{M}^+$ *is the majorant of the full measure* $\mathbb{M}(u, v)$.

It is not difficult to show that for any v, there exist β, λ_1, λ_2, and y^* such that (25) holds as equality. Indeed, set $y^* = p^*$, and

$$\begin{aligned} &\lambda_1 = -(\operatorname{div} p^* + f), && \lambda_2 = 0 && \text{on } \Omega_-^u, \\ &\lambda_2 = \operatorname{div} p^* + f, && \lambda_1 = 0 && \text{on } \Omega_+^u, \\ &\lambda_1 = 0, && \lambda_2 = 0 && \text{on } \Omega_0^u. \end{aligned}$$

Then, the second term of $\mathbb{M}^+$ vanishes (for any choice of β) and the third term is equal to $\mu_{\phi\psi}(v)$. By taking a limit $\beta \to +\infty$, the first term converges to

$$D_G(\nabla v, p^*) = \int_\Omega \left(\frac{1}{2} A \nabla v \cdot \nabla v + \frac{1}{2} A^{-1} p^* \cdot p^* - \nabla v \cdot p^* \right) dx = \frac{1}{2} \|\nabla(v - u)\|_A^2.$$

This result is quite natural: since we use the exact coincidence sets, the nonlinear measure $\mu_{\phi\psi}(v)$ is zero. In practice, v should be based on approximations of Ω_-^u, Ω_+^u, and Ω_0^u and set

$$\begin{aligned} &\lambda_1 = (\operatorname{div} y^* + f)_-, && \lambda_2 = 0 && \text{on } \Omega_-^v, \\ &\lambda_2 = (\operatorname{div} y^* + f)_+, && \lambda_1 = 0 && \text{on } \Omega_+^v, \\ &\lambda_1 = 0, && \lambda_2 = 0 && \text{on } \Omega_0^v. \end{aligned}$$

The third term of (25) vanishes and we obtain another majorant

$$\begin{aligned} \mathbb{M}(u, v) \leq \mathbb{M}_1^+(v; \beta, y^*) &:= \frac{1}{2}(1 + \beta^{-1}) D_G(\nabla v, y^*) \\ &+ \frac{1}{2} C_\Omega^2 (1 + \beta) \| [f + \operatorname{div} y^*]_v \|^2, \end{aligned} \tag{26}$$

where

$$[f + \operatorname{div} y^*]_v := \begin{cases} (f + \operatorname{div} y^*)_- & \text{in } \Omega_+^v, \\ (f + \operatorname{div} y^*)_+ & \text{in } \Omega_-^v, \\ f + \operatorname{div} y^* & \text{in } \Omega_0^v. \end{cases}$$

Since $J(v) - J(w) \leq J(v) - J(u)$ holds for all $w \in K$, we always have a computable lower bound

$$\mathbb{M}^-(v, w) := J(v) - J(w) \leq \mathbb{M}(u, v). \tag{27}$$

In practice, a suitable w can be constructed by local (e.g., patch wise) improvement of v and ideas of hierarchical basis methods. Hence, (26) and (27) provide two-sided bounds of $\mathbb{M}(u, v)$. In view of identity (20), we thus obtain computable two-sided bounds of the dual measure $\mathbb{M}(p^*, y^*)$.

4 Two-phase obstacle problem

4.1 Variational setting

The following two-phase obstacle problem was studied in H. Shahgholian, N. N. Uraltseva, and G. S.Weiss [26], N.N. Uraltseva [29], G. S. Weiss [27] and some other papers cited therein. Here the variational (energy) functional $J(v)$ is defined by the relation

$$J(v) := \int_\Omega \left(\frac{1}{2} A\nabla v \cdot \nabla v - fv + \alpha_-(v)_- + \alpha_+(v)_+ \right) dx.$$

The functional $J(v)$ is minimized on the set

$$V_0 + u_D := \{v = v_0 + u_D \ : \ v_0 \in V_0, \ u_D \in H^1(\Omega)\}.$$

Here u_D is a given bounded function that defines the boundary condition (u_D may attain both positive and negative values on different parts of the boundary $\partial\Omega$). It is assumed that the coefficients $\alpha_+, \alpha_- : \Omega \to \mathbb{R}$ are positive constants (without essential difficulties the consideration and main results can be extended to the case where they are positive Lipschitz continuous functions). Also, it is assumed that $f \in L^\infty(\Omega)$, $A \in L^\infty(\Omega, \mathbb{R}^{d\times d})$, and the condition (7) holds. Since the functional $J(v)$ is strictly convex and continuous on V, existence and uniqueness of a minimizer $u \in K$ is guaranteed by well known results of the calculus of variations (see, e.g., [19]). Analysis of the corresponding Euler-Lagrangian equation leads to the nonlinear problem ([26,27,29])

$$\operatorname{div}(A\nabla u) + f = -\alpha_- \chi_{\{u<0\}} + \alpha_+ \chi_{\{u>0\}}, \qquad u = u_D \text{ on } \partial\Omega, \tag{28}$$

where χ denotes the characteristic function of a set (attaining values 1 and 0 inside and outside the set, respectively). A physical interpretation of the problem (28) is presented by an elastic membrane touching the planar phase boundary between two liquid/gaseous phases (see, e.g., [26]).

We introduce two decompositions of Ω associated with the minimizer u and an approximation v:

$$\begin{aligned}
\Omega_-^u &:= \{x \in \Omega \mid u(x) < 0\}, & \Omega_-^v &:= \{x \in \Omega \mid v(x) < 0\},\\
\Omega_+^u &:= \{x \in \Omega \mid u(x) > 0\}, & \Omega_+^v &:= \{x \in \Omega \mid v(x) > 0\},\\
\Omega_0^u &:= \{x \in \Omega \mid u(x) = 0\}, & \Omega_0^v &:= \{x \in \Omega \mid v(x) = 0\}.
\end{aligned}$$

These decompositions generate exact and approximate free boundaries. Using the above notation we can rewrite (28) as follows

$$\operatorname{div}(A\nabla u) + f = \begin{cases} -\alpha_- & \text{in } \Omega_-^u, \\ \alpha_+ & \text{in } \Omega_+^u, \\ 0 & \text{in } \Omega_0^u. \end{cases} \tag{29}$$

4.2 Error measures

The problem is reduced to (1) if $V = V_0 := H_0^1(\Omega)$, $Y = L^2(\Omega, \mathbb{R}^d)$, $\Lambda w = \nabla w$, $\Lambda^* y^* = -\operatorname{div} y^*$, and the functionals

$$\widehat{G}(y) = \frac{1}{2} \int_\Omega A(y + y_D) \cdot (y + y_D)\, dx, \quad y_D = \nabla u_D,$$

$$\widehat{F}(v_0) := \int_\Omega (-f(v_0 + u_D) + \alpha_+(v_0 + u_D)_+ + \alpha_-(v_0 + u_D)_-) \, dx$$

stand for G and F, respectively. The problem is to find $u_0 \in V_0$ such that the functional $\widehat{J}(v_0) = \widehat{G}(\nabla v_0) + \widehat{F}(v_0)$ attains its infimum on the space V_0.

$$\widehat{G}^*(y^*) = \sup_{y\in Y} \int_\Omega \left(y^* \cdot y - \frac{1}{2} A(y + y_D) \cdot (y + y_D) \right) dx$$

$$= \sup_{y\in Y} \int_\Omega \left(y^* \cdot (y - y_D) - \frac{1}{2} Ay \cdot y \right) dx = \int_\Omega \left(\frac{1}{2} A^{-1} y^* \cdot y^* - y^* \cdot y_D \right) dx$$

Hence,

$$D_{\widehat{G}}(\Lambda v_0, y^*) = \int_\Omega \left(\frac{1}{2} A\nabla v \cdot \nabla v + \frac{1}{2} A^{-1} y^* \cdot y^* - y^* \cdot \nabla v \right) dx = \frac{1}{2} \| A\nabla v - y^* \|^2_{A^{-1}}, \tag{30}$$

for any $v = v_0 + u_D$. Computation of $\widehat{F}^*(v^*)$ is more sophisticated.

Lemma 1. *Let $v^* \in L^\infty(\Omega)$. Then,*

$$\widehat{F}^*(v^*) = \begin{cases} -\int_\Omega v^* u_D \, dx \; \text{if} \; v^* + f \in [-\alpha_-, \alpha_+], \\ +\infty \qquad \qquad \text{else}. \end{cases} \tag{31}$$

P r o o f . Assume that $v^* + f > \alpha_+$ on some open subset $\omega \subset \Omega$. Then this inequality holds on a ball $B \subset \omega$. Define two smooth cut off functions λ_1^ϵ and λ_2^ϵ such that

$$\begin{aligned} &\lambda_i^\epsilon(x) \in [0, 1], && i = 1, 2, \\ &\lambda_1^\epsilon = 1 \text{ on } \partial\Omega, && \lambda_1^\epsilon = 0 \text{ if } \operatorname{dist}(x, \partial\Omega) > \epsilon, \\ &\lambda_2^\epsilon = 1 \text{ in } B, && \lambda_2^\epsilon = 0 \text{ if } \operatorname{dist}(x, B) > \epsilon, \quad \operatorname{supp}\lambda_2^\epsilon \subset \omega. \end{aligned}$$

Here ϵ is a positive quantity smaller than $\frac{1}{2}\operatorname{dist}(B, \partial\Omega)$. For any $\rho \in \mathbb{R}$, the function $v^\epsilon := \lambda_1^\epsilon u_D + \rho\lambda_2^\epsilon$ belongs to $V_0 + u_D$. It is not difficult to see that

$$v^\epsilon = \begin{cases} \lambda_1^\epsilon u_D & \text{in } S_1^\epsilon := \operatorname{supp}\lambda_1^\epsilon, \\ \rho\lambda_2^\epsilon & \text{in } S_2^\epsilon := \operatorname{supp}\lambda_2^\epsilon \setminus B, \\ \rho & \text{in } B, \\ 0 & \text{in all other points} \end{cases}$$

and

$$(v^\epsilon)_- = \begin{cases} \lambda_1^\epsilon (u_D)_- & \text{in } S_1^\epsilon, \\ 0 & \text{in all other points} \end{cases}, \quad (v^\epsilon)_+ = \begin{cases} \lambda_1^\epsilon (u_D)_+ & \text{in } S_1^\epsilon, \\ \rho\lambda_2^\epsilon & \text{in } S_2^\epsilon, \\ \rho & \text{in } B, \\ 0 & \text{in all other points}. \end{cases}$$

Therefore,

$$\widehat{F}^*(v^*) = \sup_{v_0\in V_0} \left\{ \int_\Omega (v^* v_0 + f(v_0 + u_D) - \alpha_-(v_0 + u_D)_- - \alpha_+(v_0 + u_D)_+) \, dx \right\}$$

$$= \sup_{v\in V_0+u_D} \left\{ \int_\Omega ((v^* + f)v - \alpha_-(v)_- - \alpha_+(v)_+) \, dx \right\} - \int_\Omega v^* u_D \, dx$$

$$\geq \int_\Omega ((v^* + f)v^\epsilon - \alpha_-(v^\epsilon)_- - \alpha_+(v^\epsilon)_+) \, dx - \int_\Omega v^* u_D \, dx$$

$$= \int_{S_1^\epsilon} ((v^* + f)\lambda_1^\epsilon u_D - \alpha_-(u_D)_- - \alpha_+(u_D)_+) \, dx$$

$$+ \int_{S_2^\epsilon} \rho(v^* + f - \alpha_+)\lambda_2^\epsilon \, dx + \rho \int_{S_2^\epsilon} (v^* + f - \alpha_+) \, dx - \int_\Omega v^* u_D \, dx.$$

Let $\epsilon \to 0$ and $\rho \to +\infty$. Then the first integral in the right hand side vanishes, the second is positive and the third tends to $+\infty$. Hence, $F^*(v^*) = +\infty$.

Quite analogously we prove that $F^*(v^*) = +\infty$ if $v^* + f < \alpha_-$ on some open set $\omega \subset \Omega$. It remains to show that $F^*(v^*) = -\int_\Omega v^* u_D\, dx$ if $-\alpha_- \le v^* + f \le \alpha_+$. For this purpose, we define $v^\epsilon := \lambda_1^\epsilon u_D$. In this case,

$$\widehat{F}^*(v^*) = \sup_{v\in V_0+u_D} \left\{ \int_\Omega ((v^* + f)v - \alpha_-(v)_- - \alpha_+(v)_+)\, dx \right\} - \int_\Omega v^* u_D\, dx$$
$$= \int_{\Omega_-^v} ((v^* + f + \alpha_-)v\, dx + \int_{\Omega_+^v} (v^* + f - \alpha_+)v\, dx - \int_\Omega v^* u_D\, dx.$$

We see that the first two integrals are nonpositive, so that $\widehat{F}^*(v^*) \le -\int_\Omega v^* u_D\, dx$. On the other hand,

$$\int_{\Omega_-^{v^\epsilon}} ((v^* + f + \alpha_-)v^\epsilon\, dx + \int_{\Omega_+^{v^\epsilon}} (v^* + f - \alpha_+)v^\epsilon\, dx \to 0$$

as $\epsilon \to 0$ and we arrive at (31). □

Corollary 1. *If v^* satisfies $-\alpha_- \le v^* + f \le \alpha_+$, then*

$$D_{\widehat{F}}(v_0) = \widehat{F}(v_0) + \widehat{F}^*(v^*) - \langle v^*, v_0\rangle = \int_\Omega (-(f + v^*)v + \alpha_-(v)_- + \alpha_+(v)_+)\, dx,$$

where $v = v_0 + u_D$. Hence, if

$$y^* \in Y^*_{\mathrm{div},[-\alpha_-,\alpha_+]} := \left\{ y^* \in Y^* : \mathrm{div} y^* + f \in [-\alpha_-, \alpha_+]\ a.e.\, in\ \Omega \right\},$$

then

$$D_{\widehat{F}}(v_0, -\Lambda^* y^*) = \int_\Omega (\alpha_-(v)_- + \alpha_+(v)_+ - (\mathrm{div} y^* + f)v)\, dx =$$
$$= \int_{\Omega_-^v} (-\alpha_- - (\mathrm{div} y^* + f))\, v\, dx + \int_{\Omega_+^v} (\alpha_+ - (\mathrm{div} y^* + f))\, v\, dx \tag{32}$$

To obtain error identities, we need to express (32) for two particular cases where either $y^* = p^*$ or $v = u$. For the first case, we have

$$D_{\widehat{F}}(v_0, -\Lambda^* p^*) = \int_{\Omega_-^v} (-\alpha_- - (\mathrm{div} p^* + f))\, v\, dx + \int_{\Omega_+^v} (\alpha_+ - (\mathrm{div} p^* + f))\, v\, dx. \tag{33}$$

Since $p^* = A\nabla u$ the relation (29) guarantees that $\mathrm{div} p^* + f \in [-\alpha_-, -\alpha_+]$ almost everywhere in Ω and, therefore, $p^* \in Y^*_{\mathrm{div},[\alpha_-,\alpha_+]}$.

Let us introduce the set

$$\omega := \omega_+ \cup \omega_- \cup \omega_\pm,$$

where disjoint subsets read

$$\omega_+ : = \Omega_+^v \cap \Omega_0^u,$$
$$\omega_- : = \Omega_-^v \cap \Omega_0^u,$$
$$\omega_\pm : = \left\{ \Omega_+^v \cap \Omega_-^u \right\} \cup \left\{ \Omega_-^v \cap \Omega_+^u \right\}.$$

These subsets characterize the difference between exact coincidence sets and those formed by v. The remaining part $\widehat{\Omega} := \Omega \setminus \omega$ contains the points of Ω which belong to $\Omega_+^v \cap \Omega_+^u$ or $\Omega_-^v \cap \Omega_-^u$ or Ω_0^v. In view of (29), at these points the

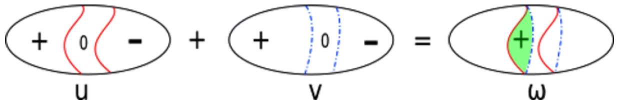

Fig. 1 The exact coincidence sets Ω_+^u, Ω_0^u, Ω_-^u (left), the approximate coincidence sets Ω_+^v, Ω_0^v, Ω_-^v (middle), and the intersection set $\omega_+ := \Omega_+^v \cap \Omega_0^u$ colored green (right).

integrands of (33) vanish. Hence (33) yields a nonnegative functional (measure)

$$\mu_\omega(v) := D_{\widehat{F}}(v_0, -\Lambda^* p^*) = \int_\omega \alpha(x)|v|\,dx, \qquad v = v_0 + u_D,$$

where

$$\alpha(x) := \begin{cases} \alpha_+ & \text{if } x \in \omega_+, \\ \alpha_- & \text{if } x \in \omega_-, \\ \alpha_+ + \alpha_- & \text{if } x \in \omega_\pm. \end{cases}$$

Figure 1 illustrates an example of exact and approximate coincidence sets. Note that in this example only the set ω_+ is nonempty. The sets ω_- and $\omega_\pm$ are empty.

For the second case, we have a nonlinear functional (measure)

$$\begin{aligned} \mu^*_\omega(y^*) &:= D_{\widehat{F}}(u_0, -\Lambda^* y^*) \\ &= \int_{\Omega_-^u} (-\alpha_- - (\operatorname{div} y^* + f))\, u\,dx + \int_{\Omega_+^u} (\alpha_+ - (\operatorname{div} y^* + f))\, u\,dx. \end{aligned} \tag{34}$$

It is clear that

$$\Omega_-^v \subset \Omega_-^u \quad \text{and} \quad \Omega_+^v \subset \Omega_+^u \quad \Rightarrow \quad \mu_\omega(v) = 0, \tag{35}$$

$$\Omega_-^u \subset \Omega_-^{y^*} \quad \text{and} \quad \Omega_+^u \subset \Omega_+^{y^*} \quad \Rightarrow \quad \mu^*_\omega(y^*) = 0, \tag{36}$$

where the sets

$$\Omega_-^{y^*} := \left\{x \in \Omega \mid \operatorname{div} y^* + f = -\alpha_-\right\},$$
$$\Omega_+^{y^*} := \left\{x \in \Omega \mid \operatorname{div} y^* + f = \alpha_+\right\}$$

are approximations of Ω_-^u, Ω_+^u generated on the basis of the dual solution y^*.

Now (3), (4), and (5) imply the following result.

Theorem 3 (primal, dual and primal-dual energy identities for the two-phase obstacle problem)**.** *Let $v \in V_0 + u_D$ and $y^* \in Y^*_{\mathrm{div},[\alpha_-,\alpha_+]}$ be approximations of u and y^*, respectively. Then*

$$\mathbb{M}(u, v) = \frac{1}{2}||\nabla(u - v)||_A^2 + \mu_\omega(v) = J(v) - J(u), \tag{37}$$

$$\mathbb{M}(p^*, y^*) = \frac{1}{2}||p^* - y^*||_{A^{-1}}^2 + \mu^*_\omega(y^*) = I^*(p^*) - I^*(y^*), \tag{38}$$

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) = \frac{1}{2}\|A\nabla v - y^*\|_{A^{-1}}^2 + \Upsilon(v, y^*) = J(v) - I^*(y^*), \tag{39}$$

where

$$\Upsilon(v, y^*) := \int_\Omega (\alpha_+(v)_+ + \alpha_-(v)_- - (f + \operatorname{div} y^*)v)\,dx \tag{40}$$

is a nonnegative functional, which vanishes if $y^* = p^*$ and $v = u$.

P r o o f . We apply (4) and (5). Notice that $\widehat{J}(v_0) = G(\nabla v_0) + F(v_0) = J(v)$. Next,

$$D_{\widehat{F}}(v_0, -\Lambda^* p^*) + D_{\widehat{G}}(\Lambda v_0, p^*) = \mu_\omega(v) + \frac{1}{2}\|A\nabla(u - v)\|_A^2.$$

It is easy to see that for any $v = v_0 + u_D \in V_0 + u_D$, the functional $J(v)$ coincides with $\widehat{J}(v_0)$ and $\widehat{J}(u_0)$ coincides with $J(u)$. Since

$$D_{\widehat{F}}(v_0, -\Lambda^* p^*) + D_{\widehat{G}}(\Lambda v_0, p^*) = \widehat{J}(v_0) - \widehat{J}(u_0) = J(v) - J(u),$$

we arrive at (37).

Since $u_0 = u - u_D$ (where u satisfies the relation $A\nabla u = p^*$), we use (30) and (34) and obtain

$$D_{\widehat{F}}(u_0, -\Lambda^* y^*) + D_{\widehat{G}}(\Lambda u_0, y^*) = \mu^*_\omega(y^*) + \frac{1}{2}||p^* - y^*||^2_{A^{-1}}.$$

Now (5) yields (38), where

$$\begin{aligned} I^*(y^*) &= -\widehat{G}^*(y^*) - \widehat{F}^*(-\Lambda^* y^*) \\ &\quad -\frac{1}{2}\|y^*\|^2_{A^{-1}} + \int\limits_\Omega (y^* \cdot \nabla u_D + \mathrm{div} y^* u_D)\,dx = -\frac{1}{2}\|y^*\|^2_{A^{-1}} + \int\limits_{\partial\Omega} (y^* \cdot n) u_D\,dx. \end{aligned}$$

Finally, summation of (37) and (38) yields

$$\begin{aligned} \mathbb{M}(\{u, p^*\}, \{v, y^*\}) &= \widehat{J}(v_0) - I^*(y^*) = J(v) - \widehat{I}^*(y^*) \\ &= \frac{1}{2}\|A\nabla v - y^*\|^2_{A^{-1}} + \Upsilon(v, y^*), \end{aligned}$$

where

$$\begin{aligned} \Upsilon(v, y^*) &= \int\limits_\Omega (\alpha_+(v)_+ + \alpha_-(v)_- - fv + y^* \cdot \nabla(v - u_D) - \mathrm{div} y^* u_D)\,dx \\ &= \int\limits_\Omega (\alpha_+(v)_+ + \alpha_-(v)_- - (f + \mathrm{div} y^*)v)\,dx. \end{aligned}$$

□

Corollary 2. *From (39) it follows that*

$$\frac{1}{2}||\nabla(u - v)||^2_A + \frac{1}{2}||p^* - y^*||^2_{A^{-1}} \le \frac{1}{2}\|A\nabla v - y^*\|^2_{A^{-1}} + \Upsilon(v, y^*).$$

This inequality has a practical value because it provides a directly computable upper bound of the error.

Remark 6. It is not difficult to show the equivalence

$$\Upsilon(v, y^*) = 0 \quad \Leftrightarrow \quad \Omega^{y^*}_{<>} \subset \Omega^v_0,$$

where

$$\Omega^{y^*}_{<>} := \Omega \setminus \left\{\Omega^{y^*}_- \cup \Omega^{y^*}_+\right\} = \left\{x \in \Omega \;|\; -\alpha_- < \mathrm{div} y^* + f < \alpha_+\right\}.$$

In other words, v must not have positive values in $\Omega^{y^*}_-$ and negative values in $\Omega^{y^*}_+$. To prove this fact we represent $\Upsilon(v, y^*)$ in the form

$$\begin{aligned} \Upsilon(v, y^*) &= \int\limits_{\Omega^{y^*}_-} (\alpha_+(v)_+ + \alpha_-((v)_- + v)\,dx + \int\limits_{\Omega^{y^*}_{<>}} (\alpha_+(v)_+ + \alpha_-(v)_- - (f + \mathrm{div} y^*)v)\,dx \\ &\quad + \int\limits_{\Omega^{y^*}_+} (\alpha_+((v)_+ - v) + \alpha_-(v)_-)\,dx \\ &= \int\limits_{\Omega^{y^*}_-} (\alpha_+ + \alpha_-)(v)_+\,dx + \int\limits_{\Omega^{y^*}_{<>}} (\alpha_+\chi_{\{v>0\}} - \alpha_-\chi_{\{v>0\}} - f - \mathrm{div} y^*)v\,dx \\ &\quad + \int\limits_{\Omega^{y^*}_+} (\alpha_+ + \alpha_-)(v)_-\,dx = \Upsilon_1(v, y^*) + \Upsilon_2(v, y^*) + \Upsilon_3(v, y^*), \end{aligned}$$

where the terms are defined by the relations

$$\Upsilon_1(v, y^*) = \int\limits_{\{\Omega_-^{y^*} \cap \Omega_+^v\} \cup \{\Omega_+^{y^*} \cap \Omega_-^v\}} (\alpha_+ + \alpha_-)|v| \, dx,$$

$$\Upsilon_2(v, y^*) = \int\limits_{\Omega_{<>}^{y^*} \cap \Omega_+^v} W_+(y^*)|v| \, dx, \qquad \Upsilon_3(v, y^*) = \int\limits_{\Omega_{<>}^{y^*} \cap \Omega_-^v} W_-(y^*)|v| \, dx,$$

with the weights $W_+(y^*) := (\alpha_+ - f - \text{div}y^*)$ and $W_-(y^*) := \alpha_- + f + \text{div}y^*$. The term $\Upsilon_1(v, y^*)$ vanishes if $v \leq 0$ in $\Omega_-^{y^*}$ and $v \geq 0$ in $\Omega_+^{y^*}$. In the set $\Omega_{<>}^{y^*}$ the weights $W_+(y^*)$ and $W_-(y^*)$ are positive. Therefore, $\Upsilon_2(v, y^*) = \Upsilon_3(v, y^*) = 0$ implies $v = 0$ almost everywhere in $\Omega_{<>}^{y^*}$, i.e., $\Omega_{<>}^{y^*} \subset \Omega_0^v$. If all the above conditions are satisfied, then $\Upsilon(v, y^*) = 0$ and we arrive at the identity

$$\mathbb{M}(\{u, p^*\}, \{v, y^*\}) = \frac{1}{2}\|A\nabla v - y^*\|_{A^{-1}}^2.$$

It is clear that $\Upsilon(v, y^*) = 0$ if the set Ω_-^v coincides (up to a set of zero measure) with the set $\Omega_-^{y^*}$ and Ω_+^v coincides $\Omega_+^{y^*}$.

4.3 Estimates for the primal and dual measures

Computable upper bound of the primal error measure $\mathbb{M}(u, v)$ was first derived in [25]. It has the form

$$\begin{aligned}\mathbb{M}(u, v) \;&\leq\; \mathbb{M}^+(v; \beta, \lambda_+, \lambda_-, y^*) \\ &:= \frac{1}{2}(1+\beta)||A\nabla v - y^*||_{\Omega, A^{-1}}^2 + \frac{1}{2}(1+\frac{1}{\beta})C_\Omega^2||\text{div}y^* + f - \alpha_+\lambda_+ + \alpha_-\lambda_-||_\Omega^2 \\ &\quad + \int\limits_\Omega (\alpha_+ (v_+ - \lambda_+ v) + \alpha_- (v_- + \lambda_- v)) \, dx.\end{aligned}$$

The majorant $\mathbb{M}^+$ contains free variables: $\beta > 0$, $y^* \in Y_{\text{div}}^*(\Omega)$, and two nonnegative functions $\lambda_+, \lambda_- \in L^2(\Omega)$ satisfying the condition $\lambda_+(x), \lambda_-(x) \in [0, 1]$ a.e. in Ω. In practical computations it is convenient to simplify $\mathbb{M}^+$ and use the majorant

$$\begin{aligned}\mathbb{M}(u, v) \leq \mathbb{M}_1^+(v; \beta, \lambda, y^*) &:= \frac{1}{2}(1+\beta)||A\nabla v - y^*||_{\Omega, A^{-1}}^2 \\ &+ \frac{1}{2}(1+\frac{1}{\beta})C_\Omega^2||\text{div}y^* + f - \lambda||_\Omega^2 + \int\limits_\Omega (\alpha_+ v_+ + \alpha_- v_- - \lambda v) \, dx,\end{aligned}$$

where only one multiplier $\lambda \in L^2(\Omega)$ satisfying $\lambda \in [-\alpha_-, \alpha_+]$ a.e. in Ω is required. For a consequent exposition and numerical examples, we address the reader to [5]. Finally, we note that two-side bounds of the dual measure $\mathbb{M}(p^*, y^*)$ can be deduces by the same arguments as in Subsect. 3.4.

5 Examples

Below we discuss computational examples that illustrate the above presented relations. In these test examples, the exact solutions u and p^* are known. Due to this fact, we are able to verify the theoretical results and study proportions between the quadratic and nonlinear parts of the primal, dual and primal-dual measures (cf. Theorems 1, 2, 3) for various sequences of approximations converging to exact solutions.

5.1 Example 1: the classical obstacle problem in 1D

Let $\Omega = (0, 1)$, $A = 1$, $f = const < 0, \phi = const < 0, \psi = +\infty$, and u satisfy the homogeneous Dirichlét boundary conditions $u(0) = 0, u(1) = 0$. The exact solution has the form (see [13])

$$u(x) = \begin{cases} -\frac{f}{2}x^2 - \sqrt{2f\phi}\,x & \text{if } x \in [0, \frac{1}{2} - r], \\ \phi & \text{if } x \in (\frac{1}{2} - r, \frac{1}{2} + r), \\ -\frac{f}{2}(x-1)^2 + \sqrt{2f\phi}\,(x-1) & \text{if } x \in [\frac{1}{2} + r, 1], \end{cases}$$

where $r := \frac{1}{2} - \sqrt{\frac{2\phi}{f}} \in (0, \frac{1}{2})$. In this case, $J(u) = f\phi(\frac{4}{3}\sqrt{\frac{2\phi}{f}} - 1)$. Consider the approximations

$$v_{\epsilon_1}(x) = \begin{cases} -\frac{f_{\epsilon_1}}{2}x^2 - \sqrt{2f_{\epsilon_1}\phi}\,x & \text{if } x \in [0, \frac{1}{2} - r_{\epsilon_1}], \\ \phi & \text{if } x \in (\frac{1}{2} - r_{\epsilon_1}, \frac{1}{2} + r_{\epsilon_1}), \\ -\frac{f_{\epsilon_1}}{2}(x-1)^2 + \sqrt{2f_{\epsilon_1}\phi}\,(x-1) & \text{if } x \in [\frac{1}{2} + r_{\epsilon_1}, 1], \end{cases}$$

where ϵ_1 is a small positive parameter, $r_{\epsilon_1} := r - \epsilon_1$, and $f_{\epsilon_1} := \frac{2\phi}{(\frac{1}{2} - r + \epsilon_1)^2}$. For $\epsilon_1 \in (r - \frac{1}{2}, r)$, it holds

$$\Omega_-^{v_{\epsilon_1}} = (\frac{1}{2} - r_{\epsilon_1}, \frac{1}{2} + r_{\epsilon_1}) \subset \Omega_-^u = (\frac{1}{2} - r, \frac{1}{2} + r).$$

In view of (16), we can await that these approximations the measure $\mu_{\phi\psi}(v_{\epsilon_1})$ is positive. Approximations of the dual variable are taken in the form $y^*_{\epsilon_2} = \mathcal{I}(p^*)(x)$, where $\mathcal{I}$ denotes piecewise linear nodal and continuous interpolation at the nodes $\{0, \frac{1}{2} - r - \epsilon_2, \frac{1}{2} - r + \epsilon_2, \frac{1}{2} + r - \epsilon_2, \frac{1}{2} + r + \epsilon_2, 1\}$. It differs from p^* in

$$(\frac{1}{2} - r - \epsilon_2, \frac{1}{2} - r + \epsilon_2) \cup (\frac{1}{2} + r - \epsilon_2, \frac{1}{2} + r + \epsilon_2)$$

and

$$\Omega_-^u \subset \Omega_-^{y^*_{\epsilon_2}} = (\frac{1}{2} - r - \epsilon_2, \frac{1}{2} + r + \epsilon_2) \quad \text{for} \quad \epsilon_2 \in (0, r).$$

Hence the condition (17) is not valid allowing $\mu^*_{\phi\psi}(y^*_{\epsilon_2}) > 0$. Notice that in this example $\psi = +\infty$. Therefore, condition

$$\mathrm{div} y^* + f \le 0 \qquad \text{in } \Omega \tag{41}$$

must be satisfied. The function $y^*_{\epsilon_2}$ satisfies this requirement.

In the tests, we set

$$\phi = -1, \qquad f = -14.$$

Then, $r \approx 0.1220$ and $J(u) \approx -6.9446$. The approximations v_{ϵ_1}, $y^*_{\epsilon_2}$ are computed for $\epsilon_1, \epsilon_2 = 2^{-i}/10$, $i = 0, 1, ..., 4$.

Table 1 contains results related to the primal error identity (18) for the approximations v_{ϵ_1}. It shows that both (quadratic and nonlinear) parts of the primal error measure tend to zero as $\epsilon_1 \to 0$. Relative contribution of the nonlinear part is expressed by the quantity

$$\kappa(v_{\epsilon_1}) := 100\, \frac{\mu_{\phi\psi}(v_{\epsilon_1})}{\mathbb{M}(u, v_{\epsilon_1})} \quad [\%]$$

presented in the last right column. For smaller values of ϵ_1 the quadratic part dominates. This is quite natural because the quadratic part is globally distributed over Ω and the nonlinear part $\mu_{\phi\psi}(v_{\epsilon_1})$ has a local support in the set

$$\Omega_-^u \setminus \Omega_-^{v_{\epsilon_1}} \approx (0.3779, 0.3779 + \epsilon_1) \cup (0.6220 - \epsilon_1, 0.6220),$$

whose measure tends to zero (for the considered sequence of approximations).

Table 2 reports on terms in the dual error identity (19) for the approximations $y^*_{\epsilon_2}$. Again, both quadratic and nonlinear parts of the error converge to zero as ϵ_2 converges to zero. The contribution of the nonlinear measure to the dual energy identity is measured by the quantity

$$\kappa(y^*_{\epsilon_2}) := 100\, \frac{\mu^*_{\phi\psi}(y^*_{\epsilon_2})}{\mathbb{M}(p^*, y^*_{\epsilon_2})} \quad [\%].$$

In this example, the functions $y^*_{\epsilon_2}$ and $p^*_{\epsilon_2}$ differ only locally. Hence none of the error parts dominates. The nonlinear part $\mu^*_{\phi\psi}(y^*_{\epsilon_2})$ is supported in the set

$$\Omega_-^{y^*_{\epsilon_2}} \setminus \Omega_-^u \approx (0.3779 - \epsilon_2, 0.3779) \cup (0.6220, 0.6220 + \epsilon_2)$$

and tends to zero as $\epsilon_2 \to 0$.

Table 3 reports on the terms in the primal-dual error identity (21), where the computable nonlinear part Υ is given by (22). Relative contribution of the nonlinear part (in the primal-dual measure) is presented by the quantity $\kappa(v_{\epsilon_1})$.

Table 1 Terms in the primal error identity computed for approximations v_{ϵ_1} of Example 1.

ϵ_1	$\frac{1}{2}\|\nabla(u-v_{\epsilon_1})\|^2$	$\mu_{\phi\psi}(v_{\epsilon_1})$	$\mathbb{M}(u, v_{\epsilon_1})$	$J(v_{\epsilon_1}) - J(u)$	$J(v_{\epsilon_1})$	$\kappa(v_\epsilon)$ [%]
1/10	1.54e-01	4.09e-02	1.95e-01	1.95e-01	–6.749391	20.93
1/20	4.82e-02	6.38e-03	5.45e-02	5.45e-02	–6.890142	11.69
1/40	1.36e-02	9.00e-04	1.45e-02	1.45e-02	–6.930187	6.21
1/80	3.62e-03	1.20e-04	3.74e-03	3.73e-03	–6.940928	3.21
1/160	9.33e-04	1.55e-05	9.49e-04	9.49e-04	–6.943714	1.64

Table 2 Terms in the dual error identity computed for approximations $y^*_{\epsilon_2}$ of Example 1.

ϵ_2	$\frac{1}{2}\|p^* - y^*_{\epsilon_2}\|^2$	$\mu^*_{\phi\psi}(y^*_{\epsilon_2})$	$\mathbb{M}(p^*, y^*_{\epsilon_2})$	$I^*(p^*) - I^*(y^*_{\epsilon_2})$	$I^*(p^*)$	$\kappa(y^*_{\epsilon_2})$ [%]
1/10	3.27e-02	3.27e-02	6.53e-02	6.53e-02	–7.009997	50.00
1/20	4.08e-03	4.08e-03	8.17e-03	8.17e-03	–6.952830	50.00
1/40	5.10e-04	5.10e-04	1.02e-03	1.02e-03	–6.945684	50.00
1/80	6.38e-05	6.38e-05	1.28e-04	1.28e-04	–6.944791	50.00
1/160	7.97e-06	7.98e-06	1.60e-05	1.61e-05	–6.944679	50.01

Table 3 Terms in the primal-dual error identity computed for approximations v_{ϵ_1} and $y^*_{\epsilon_2}$ of Example 1. Here, $\mathbb{M}(v_{\epsilon_1}, y^*_{\epsilon_2})$ denotes $\mathbb{M}(\{u, p^*\}, \{v_{\epsilon_1}, y^*_{\epsilon_2}\})$.

ϵ_1	ϵ_2	$\frac{1}{2}\|\|\nabla v_{\epsilon_1} - y^*_{\epsilon_2}\|\|^2$	$\Upsilon(v_{\epsilon_1}, y^*_{\epsilon_2})$	$\mathbb{M}(v_{\epsilon_1}, y^*_{\epsilon_2})$	$J(v_{\epsilon_1}) - I^*(y^*_{\epsilon_2})$	$\kappa(v_{\epsilon_1}, y^*_{\epsilon_2})$[%]
1/10	1/10	9.72e-02	1.63e-01	2.61e-01	2.61e-01	62.71
1/20	1/10	3.39e-02	8.60e-02	1.20e-01	1.20e-01	71.75
1/20	1/20	3.72e-02	2.55e-02	6.27e-02	6.27e-02	40.64
1/40	1/20	1.05e-02	1.21e-02	2.26e-02	2.26e-02	53.55
1/40	1/40	1.19e-02	3.59e-03	1.55e-02	1.55e-02	23.18
1/80	1/40	3.14e-03	1.61e-03	4.76e-03	4.76e-03	33.94
1/80	1/80	3.38e-03	4.78e-04	3.86e-03	3.86e-03	12.38
1/160	1/80	8.68e-04	2.08e-04	1.08e-03	1.08e-03	19.36
1/160	1/160	9.03e-04	6.17e-05	9.65e-04	9.65e-04	6.40

For illustration, some terms are visualized in Fig. 2 for in the case of approximations v_{ϵ_1} and $y^*_{\epsilon_2}$ generated by $\epsilon_1 = \epsilon_2 = 0.1$.

5.2 Example 2: the two-phase obstacle problem in 1D

Let $\Omega = (-1, 1)$, $f = 0$, $A = 1$, $\alpha_+, \alpha_- > 0$, and u satisfy the boundary conditions $u(-1) = -1$ and $u(1) = 1$. The exact solution is given by the formula

$$u(x) = \begin{cases} -(\frac{\alpha_-}{2})\, x^2 + (\sqrt{2\alpha_-} - \alpha_-)\, x + \sqrt{2\alpha_-} - \frac{\alpha_-}{2} - 1, & x \in [-1, r_-]\,, \\ 0, & x \in (r_-, r_+), \\ (\frac{\alpha_+}{2})\, x^2 + (\sqrt{2\alpha_+} - \alpha_+)\, x - \sqrt{2\alpha_+} + \frac{\alpha_+}{2} + 1, & x \in [r_+, 1]\,, \end{cases}$$

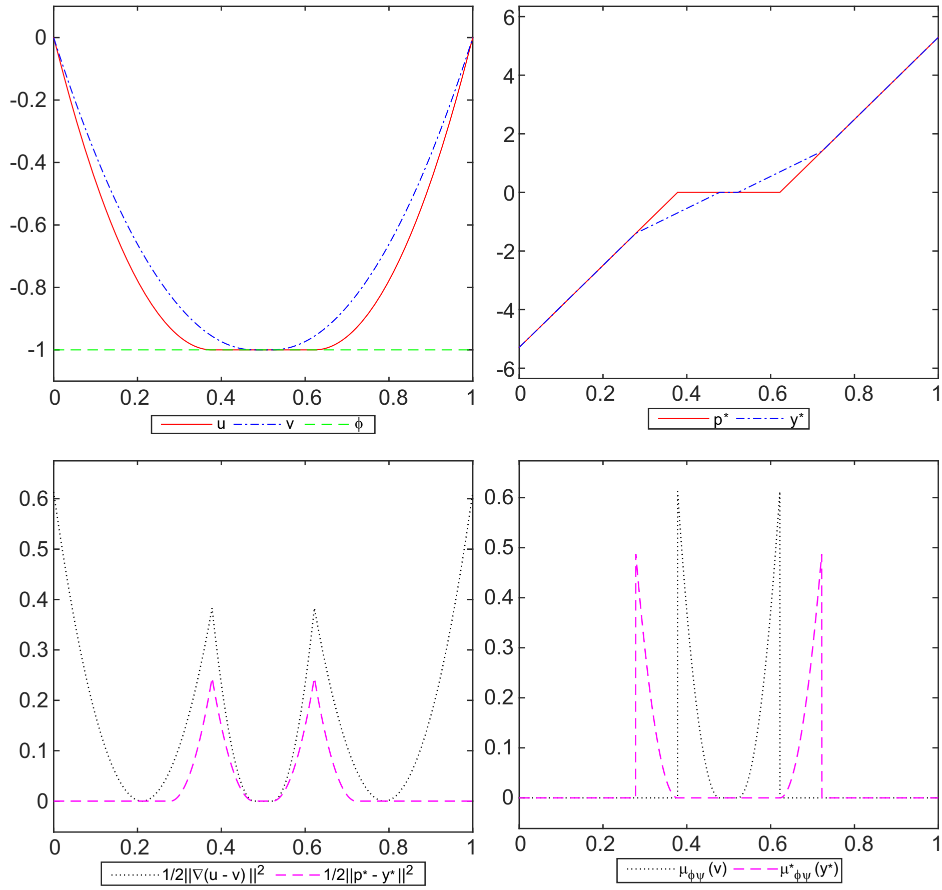

Fig. 2 Example 1 for $\phi = -1$, $f = -14$ and approximations v_{ϵ_1} and $y^*_{\epsilon_2}$ generated by $\epsilon_1 = \epsilon_2 = 0.1$: the exact solution u and its approximation v_{ϵ_1} (top left), the exact flux p^* and its approximation $y^*_{\epsilon_2}$ (top right), quadratic measures $\frac{1}{2}||\nabla(u - v_{\epsilon_1})||^2$ and $\frac{1}{2}||p^* - y^*_{\epsilon_2}||^2$ (bottom left), nonlinear measures $\mu_{\phi\psi}(v_{\epsilon_1})$ and $\mu^*_{\phi\psi}(y^*_{\epsilon_2})$ (bottom right).

where $r_- := \sqrt{\frac{2}{\alpha_-}} - 1$, $r_+ := 1 - \sqrt{\frac{2}{\alpha_+}}$. It is not difficult to show that $J(u) = \frac{2\sqrt{2}}{3}(\sqrt{\alpha_+} + \sqrt{\alpha_-})$. Consider the approximations

$$v_{\epsilon_1}(x) = \begin{cases} -\left(\frac{\alpha_{-\epsilon_1}}{2}\right)x^2 + \left(\sqrt{2\alpha_{-\epsilon_1}} - \alpha_{-\epsilon_1}\right)x + \sqrt{2\alpha_{-\epsilon_1}} - \frac{\alpha_{-\epsilon_1}}{2} - 1, & x \in \left[-1, r_{-\epsilon_1}\right], \\ 0, & x \in (r_{-\epsilon_1}, r_{+\epsilon_1}), \\ \left(\frac{\alpha_{+\epsilon_1}}{2}\right)x^2 + \left(\sqrt{2\alpha_{+\epsilon_1}} - \alpha_{+\epsilon_1}\right)x - \sqrt{2\alpha_{+\epsilon_1}} + \frac{\alpha_{+\epsilon_1}}{2} + 1, & x \in \left[r_{+\epsilon_1}, 1\right], \end{cases}$$

for $\epsilon_1 > 0$, where

$$r_{-\epsilon_1} := r_- + \epsilon_1, \quad r_{+\epsilon_1} := r_+ - \epsilon_1, \quad \alpha_{-\epsilon_1} := \frac{2}{(1 + r_{-\epsilon_1})^2}, \quad \alpha_{+\epsilon_1} := \frac{2}{(1 - r_{+\epsilon_1})^2}.$$

It holds $\Omega_0^{v_{\epsilon_1}} = (r_{-\epsilon_1}, r_{+\epsilon_1}) \subset \Omega_0^u = (r_-, r_+)$ for $\epsilon_1 \in (0, \min\{r_+, -r_-\})$ and the condition (35) is not satisfied (hence we await that $\mu_\omega(v_{\epsilon_1}) > 0$). As previously, $y^*_{\epsilon_2}$ is constructed as $y^*_{\epsilon_2}(x) = \mathcal{I}(p^*)(x)$, $x \in \Omega$, where $\mathcal{I}$ is a piecewise linear nodal and continuous interpolation operator at the nodes $\{-1, r_- - \epsilon_2, r_- + \epsilon_2, r_+ - \epsilon_2, r_+ + \epsilon_2, 1\}$ for small $\epsilon_2 > 0$. These approximations $y^*_{\epsilon_2}$ differ from p^* only locally in the set $(r_- - \epsilon_2, r_- + \epsilon_2) \cup (r_+ - \epsilon_2, r_+ + \epsilon_2)$. We have

$$\begin{aligned} \Omega_-^{y^*_{\epsilon_2}} &= (-1, r_- - \epsilon_2) \subset \Omega_-^u = (-1, r_-) \\ \Omega_+^{y^*_{\epsilon_2}} &= (r_+ + \epsilon_2, 1) \quad \subset \Omega_+^u = (r_+, 1) \end{aligned} \qquad \text{for} \quad \epsilon_2 \in (0, \min\{-r_-, r_+\})$$

and the condition (36) is not satisfied (hence $\mu^*_\omega(y^*_{\epsilon_2})$ may be positive).

In the tests, we set (cf. [4])

$$\alpha_- = \alpha_+ = 8.$$

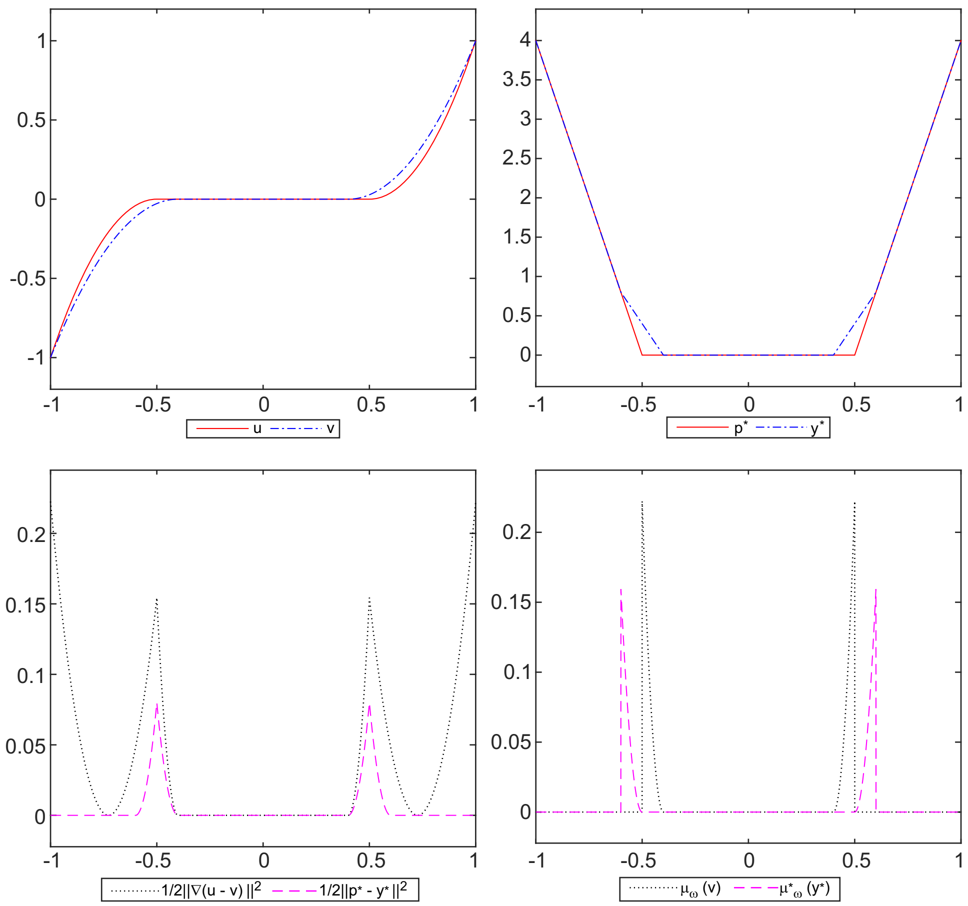

Fig. 3 Example 2 for $\alpha_- = \alpha_+ = 8$ and approximations v_{ϵ_1} and $y^*_{\epsilon_2}$ generated by $\epsilon_1 = \epsilon_2 = 0.1$: the exact solution u and its approximation v_{ϵ_1} (top left), the exact flux p^* and its approximation $y^*_{\epsilon_2}$ (top right), quadratic measures $\frac{1}{2}||\nabla(u - v_{\epsilon_1})||^2$ and $\frac{1}{2}||p^* - y^*_{\epsilon_2}||^2$ (bottom left), nonlinear measures $\mu_\omega(v_{\epsilon_1})$ and $\mu^*_\omega(y^*_{\epsilon_2})$ (bottom right).

and this choice implies $r_- = -0,5, r_+ = 0.5$, and $J(u) = 5\frac{1}{3}$. The approximations $v_{\epsilon_1}, y^*_{\epsilon_2}$ are computed for $\epsilon_1, \epsilon_2 = 2^{-i}/10, i = 0, 1, ..., 4$. Table 4 presents results related to the primal error identity (37). The quadratic part of the primal error measure dominates over the nonlinear part for small values of ϵ_1. This follows from the fact that the quadratic part of error is globally distributed over Ω and the nonlinear part $\mu_\omega(v_{\epsilon_1})$ is distributed in a relatively small subdomain

$$\Omega^u_0 \setminus \Omega^{v_{\epsilon_1}}_0 = (-0.5, -0.5 + \epsilon_1) \cup (0.5 - \epsilon_1, 0.5).$$

Table 5 reports on terms in the dual error identity (38) for all approximations $y^*_{\epsilon_2}$. Again, both quadratic and nonlinear parts converge. Since $y^*_{\epsilon_2}$ and $p^*_{\epsilon_2}$ differ only locally, none of the error parts dominates. The nonlinear part $\mu^*_\omega(y^*_{\epsilon_2})$ is supported in the set

$$(\Omega^u_- \setminus \Omega^{y^*_{\epsilon_2}}_-) \cup (\Omega^u_+ \setminus \Omega^{y^*_{\epsilon_2}}_+) = (0.5 - \epsilon_2, 0.5) \cup (0.5, 0.5 + \epsilon_2)$$

and tends to zero if $\epsilon_2 \to 0$.

Table 6 reports on terms in the primal-dual identity (39), where the computable part Υ is given by (40).

For illustration, some terms are visualized in Fig. 3 for in the case of approximations v_{ϵ_1} and $y^*_{\epsilon_2}$ generated by $\epsilon_1 = \epsilon_2 = 0.1$.

5.3 Example 3: the classical obstacle problem in 2D

We consider a 2D example from [17]. Here, $\Omega = (-1, 1)^2$, $A = \mathbb{I}$, $\phi = 0$, $\psi = +\infty$. For

$$f(\rho) = \begin{cases} -16\rho^2 + 8R^2 & \text{if } \rho > R \in [0, 1), \\ -8(R^4 + R^2) + 8R^2\rho^2 & \text{if } \rho \le R, \end{cases}$$

Table 4 Terms in the primal error identity computed for approximations v_{ϵ_1} of Example 2.

ϵ_1	$\frac{1}{2}\|\nabla(u - v_{\epsilon_1})\|^2$	$\mu_\omega(v_{\epsilon_1})$	$\mathbb{M}(u, v_{\epsilon_1})$	$J(v_{\epsilon_1}) - J(u)$	$J(v_{\epsilon_1})$	$\kappa(v_\epsilon)$ [%]
1/5	2.18e-01	8.72e-02	3.05e-01	3.05e-01	5.638095	28.60
1/10	7.41e-02	1.49e-02	8.89e-02	8.89e-02	5.422222	16.71
1/20	2.20e-02	2.22e-03	2.43e-02	2.42e-02	5.357576	9.14
1/40	6.05e-03	3.06e-04	6.35e-03	6.35e-03	5.339682	4.82
1/80	1.59e-03	4.06e-05	1.63e-03	1.63e-03	5.334959	2.50

Table 5 Terms in the dual error identity computed for approximations $y^*_{\epsilon_2}$ of Example 2.

ϵ_2	$\frac{1}{2}\|p^* - y^*_{\epsilon_2}\|^2$	$\mu^*_\omega(y^*_{\epsilon_2})$	$\mathbb{M}(p^*, y^*_{\epsilon_2})$	$I^*(p^*) - I^*(y^*_{\epsilon_2})$	$I^*(p^*)$	$\kappa(y^*_{\epsilon_2})$ [%]
1/5	8.53e-02	8.53e-02	1.71e-01	1.71e-01	5.162666	50.00
1/10	1.07e-02	1.07e-02	2.13e-02	2.13e-02	5.312000	50.00
1/20	1.33e-03	1.33e-03	2.67e-03	2.67e-03	5.330666	50.00
1/40	1.67e-04	1.67e-04	3.33e-04	3.34e-04	5.333000	50.00
1/80	2.08e-05	2.08e-05	4.17e-05	4.19e-05	5.333291	50.00

Table 6 Terms in the primal-dual error identity computed for approximations v_{ϵ_1} and $y^*_{\epsilon_2}$ of Example 2. Here, $\mathbb{M}(v_{\epsilon_1}, y^*_{\epsilon_2})$ denotes $\mathbb{M}(\{u, p^*\}, \{v_{\epsilon_1}, y^*_{\epsilon_2}\})$.

ϵ_1	ϵ_2	$\frac{1}{2}\|\|\nabla v_{\epsilon_1} - y^*_{\epsilon_2}\|\|^2$	$\Upsilon(v_{\epsilon_1}, y^*_{\epsilon_2})$	$\mathbb{M}(v_{\epsilon_1}, y^*_{\epsilon_2})$	$J(v_{\epsilon_1}) - I^*(y^*_{\epsilon_2})$	$\kappa(v_{\epsilon_1}, y^*_{\epsilon_2})$[%]
1/5	1/5	1.27e-01	3.48e-01	4.75e-01	4.76e-01	73.26
1/10	1/5	5.96e-02	2.00e-01	2.60e-01	2.60e-01	77.05
1/10	1/10	5.10e-02	5.93e-02	1.10e-01	1.10e-01	53.76
1/20	1/10	1.58e-02	2.98e-02	4.56e-02	4.56e-02	65.28
1/20	1/20	1.81e-02	8.82e-03	2.69e-02	2.69e-02	32.76
1/40	1/20	4.93e-03	4.08e-03	9.02e-03	9.02e-03	45.27
1/40	1/40	5.47e-03	1.21e-03	6.68e-03	6.69e-03	18.10
1/80	1/40	1.42e-03	5.35e-04	1.96e-03	1.96e-03	27.33
1/80	1/80	1.51e-03	1.59e-04	1.67e-03	1.67e-03	9.51

where (ρ, θ) are the polar coordinates, the exact solution is

$$u(\rho) = \max\{\rho^2 - R^2, 0\}.$$

The lower coincidence set reads $\Omega^u_- = \{\rho : \rho < R\}$ and the corresponding energy is given by the relation (see [14])

$$J(u) = 192\left(\frac{12}{35} - \frac{28R^2}{45} + \frac{R^4}{3}\right) - 32R^2\left(\frac{28}{45} - \frac{4R^2}{3} + R^4\right) + \frac{2}{3}\pi R^8.$$

In all tests, we take the exact solution u with $R = 0.6$ (then $J(u) \approx 28.019193$). We consider three kinds of approximations $v \in K$ converging to the exact minimizer u.

5.3.1

The first kind approximations are defined by the relation

$$v_{\epsilon_1} := \tilde{v}_{R+\epsilon_1, R+2\epsilon_1}(\rho),$$

where $\epsilon_1 > 0$ is a small positive parameter,

$$\tilde{v}_{r_1,r_2}(\rho) := \begin{cases} 0, & \text{if } \rho \le r_1 \\ u - u(r_1)\frac{r_2-\rho}{r_2-r_1}, & \text{if } r_1 \le \rho \le r_2 \\ u, & \text{if } \rho \ge r_2 \end{cases},$$

and $R \le r_1 \le r_2 < 1$. In this case, $\Omega_-^u \subset \Omega_-^{v_{\epsilon_1}}$ and the condition (16) implies that the nonlinear measure $\mu_{\phi\psi}(v_{\epsilon_1})$ is zero. Table 7 reports on terms in the primal error identity (18). Here, the nonlinear measure $\mu_{\phi\psi}(v_{\epsilon_1})$ is indeed zero and only the quadratic measure $\frac{1}{2}\|\nabla(u - v_{\epsilon_1})\|^2$ contributes to the error measure. Fig. 4 displays the approximation v_{ϵ_1} for $\epsilon_1 = 1/8$ and the densities of the nonlinear and the quadratic measures. The positive density of the quadratic measure is localized in the annulus of radii $R + \epsilon_1$ and $R + 2\epsilon_1$.

5.3.2

The second kind approximations are defined by the relation

$$v_{\epsilon_1} := u + \epsilon_1\, w,$$

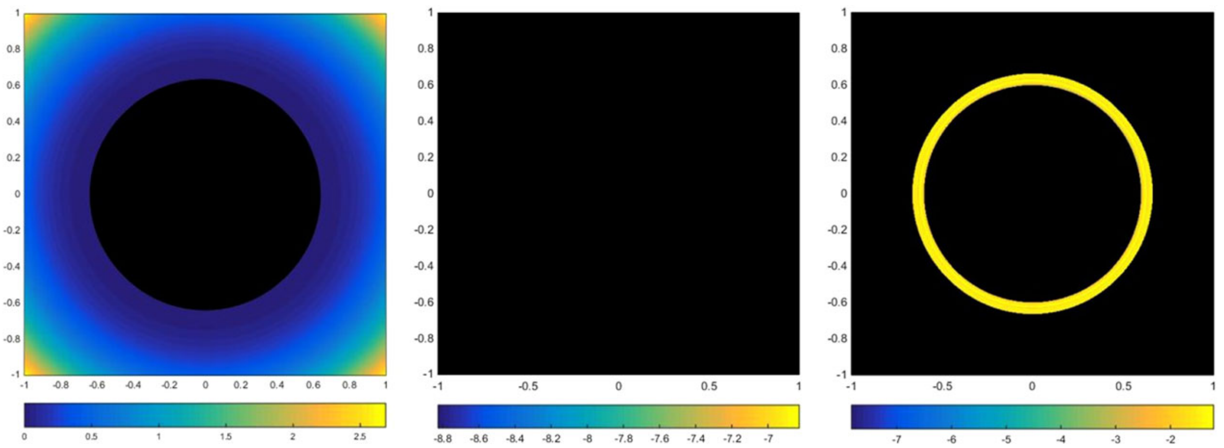

Fig. 4 Example 3: the first kind approximation v_{ϵ_1} for $\epsilon_1 = 1/8$ (left) and densities of the nonlinear measure $\mu_{\phi\psi}(v_{\epsilon_1})$ (middle) and of the quadratic measure $\frac{1}{2}\|\nabla(u - v_{\epsilon_1})\|^2$ (right). Densities are displayed in a log-100 scale.

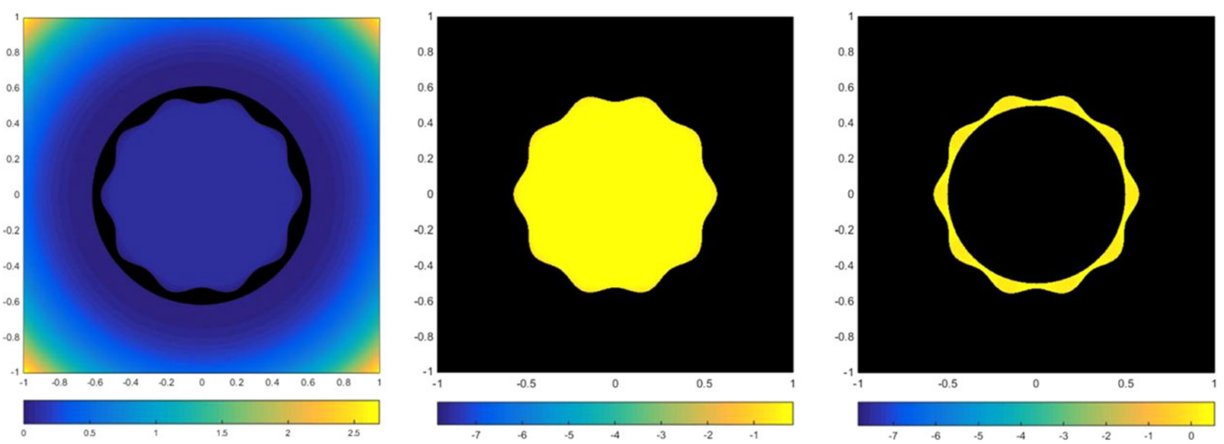

Fig. 5 Example 3: the second kind approximation v_{ϵ_1} for $\epsilon_1 = 1/8$ (left) and densities of the nonlinear measures $\mu_{\phi\psi}(v_{\epsilon_1})$ (middle) and of the quadratic measures $\frac{1}{2}\|\nabla(u - v_{\epsilon_1})\|^2$ (right). Densities are displayed in a log-100 scale.

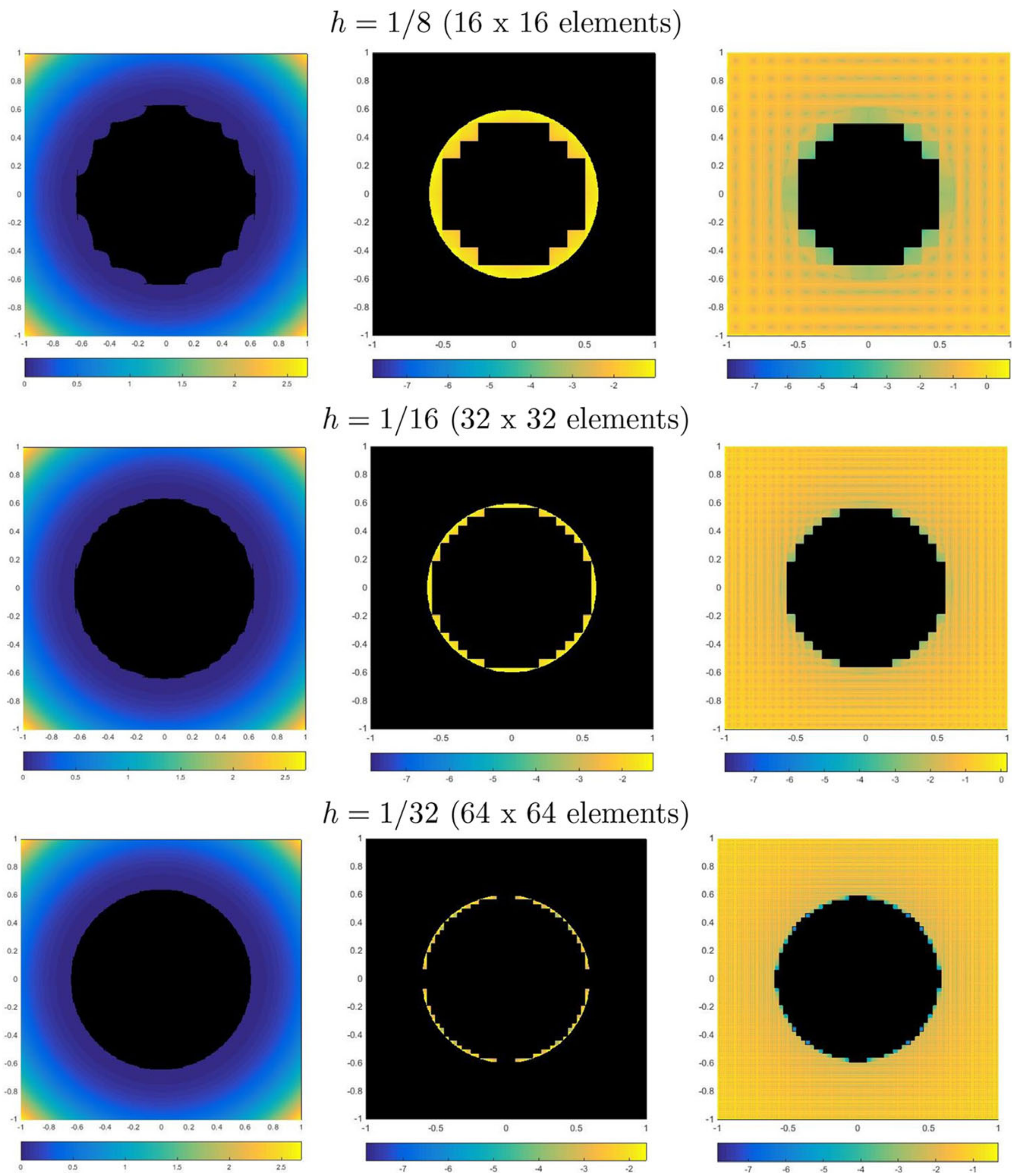

Fig. 6 Example 3: the third kind approximations u_h (left column) for various h and their corresponding nonlinear measures $\mu_{\phi\psi}(u_h)$ (middle column) and quadratic measures $\frac{1}{2}\|\nabla(u-u_h)\|^2$ (right column). Densities are displayed in a log-100 scale.

Table 7 Terms in the primal error identity computed for the first kind approximations v_{ϵ_1} of Example 3.

ϵ_1	$\frac{1}{2}\|\nabla(u - v_{\epsilon_1})\|^2$	$\mu_{\phi\psi}(v_{\epsilon_1})$	$\mathbb{M}(u, v_{\epsilon_1})$	$J(v_{\epsilon_1}) - J(u)$	$J(v_{\epsilon_1})$	$\kappa(v_\epsilon)$ [%]
1/4	4.59e+00	0	4.59e+00	4.59e+00	32.604407	0.00
1/8	3.32e-02	0	3.32e-02	3.32e-02	28.051920	0.00
1/16	3.05e-03	0	3.05e-03	3.06e-03	28.021814	0.00

Table 8 Terms in the primal error identity computed for the second kind approximations v_{ϵ_1} of Example 3.

ϵ_1	$\frac{1}{2}\|\nabla(u - v_{\epsilon_1})\|^2$	$\mu_{\phi\psi}(v_{\epsilon_1})$	$\mathbb{M}(u, v_{\epsilon_1})$	$J(v_{\epsilon_1}) - J(u)$	$J(v_{\epsilon_1})$	$\kappa(v_\epsilon)$ [%]
1/4	2.43e+00	7.62e-01	3.19e+00	3.19e+00	31.210070	23.89
1/8	6.07e-01	3.81e-01	9.88e-01	9.88e-01	29.007178	38.56
1/16	1.52e-01	1.91e-01	3.42e-01	3.42e-01	28.361163	55.66
1/32	3.80e-02	9.53e-02	1.33e-01	1.33e-01	28.152013	71.52

Table 9 Terms in the dual error identity computed for approximations $y^*_{\epsilon_2}$ of Example 3.

ϵ_2	$\frac{1}{2}\|p^* - y^*_{\epsilon_2}\|^2$	$\mu^*_{\phi\psi}(y^*_{\epsilon_2})$	$\mathbb{M}(p^*, y^*_{\epsilon_2})$	$I^*(p^*) - I^*(y^*_{\epsilon_2})$	$I^*(p^*)$	$\kappa(y^*_{\epsilon_2})$ [%]
1/4	2.08e-02	2.60e-01	2.80e-01	2.80e-01	27.738317	92.57
1/8	5.21e-03	1.30e-01	1.35e-01	1.35e-01	27.883746	96.14
1/16	1.30e-03	6.49e-02	6.62e-02	6.62e-02	27.952554	98.03

Table 10 Terms in the primal-dual error identity computed for the second kind approximations v_{ϵ_1} and approximations $y^*_{\epsilon_2}$ of Example 3. Here, $\mathbb{M}(v_{\epsilon_1}, y^*_{\epsilon_2})$ denotes $\mathbb{M}(\{u, p^*\}, \{v_{\epsilon_1}, y^*_{\epsilon_2}\})$.

ϵ_1	ϵ_2	$\frac{1}{2}\|\|\nabla v_{\epsilon_1} - y^*_{\epsilon_2}\|\|^2$	$\Upsilon(v_{\epsilon_1}, y^*_{\epsilon_2})$	$\mathbb{M}(v_{\epsilon_1}, y^*_{\epsilon_2})$	$J(v_{\epsilon_1}) - I^*(y^*_{\epsilon_2})$	$\kappa(v_{\epsilon_1}, y^*_{\epsilon_2})$[%]
1/4	1/4	2.40e+00	1.08e+00	3.47e+00	3.47e+00	31.00
1/8	1/4	6.01e-01	6.68e-01	1.27e+00	1.27e+00	52.64
1/8	1/8	5.99e-01	5.25e-01	1.12e+00	1.12e+00	46.69
1/16	1/8	1.50e-01	3.27e-01	4.77e-01	4.77e-01	68.53
1/16	1/16	1.50e-01	2.59e-01	4.09e-01	4.09e-01	63.36

Table 11 Terms in the primal error identity computed for the third kind (fem) approximations u_h of Example 3.

h	$\frac{1}{2}\|\nabla(u - u_h)\|^2$	$\mu_{\phi\psi}(u_h)$	$\mathbb{M}(u, u_h)$	$J(u_h) - J(u)$	$J(u_h)$	$\kappa(u_h)$ [%]
1/4	6.38e+00	6.66e-03	6.39e+00	6.39e+00	34.410421	0.10
1/8	5.43e-01	6.71e-04	5.43e-01	5.43e-01	28.561970	0.12
1/16	6.43e-02	8.60e-05	6.44e-02	6.44e-02	28.083185	0.13
1/32	1.08e-02	7.30e-06	1.08e-02	1.09e-02	28.029611	0.07

where $\epsilon_1 > 0$ is a small positive parameter,

$$w(\rho,\theta) := \begin{cases} 1, & \text{if } \rho \le r \in (0,R) \\ 1 - \frac{\rho - r}{\tilde{r}(\theta) - r}, & \text{if } r \le \rho \le \tilde{r}(\theta) \\ 0, & \text{if } \rho \ge \tilde{r}(\theta) \end{cases},$$

and $\tilde{r}(\theta) := r + (R-r)(\frac{2+\cos(k\theta)}{4})$ for some $k \in \mathbb{Z}$. This construction ensures that $v_{\epsilon_1} \ge u$ in Ω and $\Omega_-^{v_{\epsilon_1}} \subset \Omega_-^u$ for any choice of $\epsilon_1 > 0$. For the testing, we consider $r = 0.5$ and $k = 10$. Table 3 reports on terms in the primal error identity (18). Here, the nonlinear measure dominates over the quadratic measure for smaller values of ϵ_1. Fig. 5 displays the approximation v_{ϵ_1} for $\epsilon_1 = 1/8$ and the densities of the nonlinear and the quadratic measures.

Remark 7. Table 8 leads to a conclusion important for numerical analysis. The corresponding sequence of approximations converges to the exact minimizer in the energy norm: $v_{\epsilon_1} \to u \in K$ as $\epsilon_1 \to 0$. In view of this fact, the nonlinear part of the primal error measure must converge to zero

$$\mu_{\phi\psi}(v_{\epsilon_1}) \to \mu_{\phi\psi}(u) = 0 \qquad \text{as } \epsilon_1 \to 0.$$

At the same time, we see that the shape of the approximate lower coincidence set $\Omega_-^{v_{\epsilon_1}}$ depends on R, r, k, but it does not depend on ϵ_1. Therefore, $\Omega_-^{v_{\epsilon_1}}$ never approximates the exact lower coincidence set Ω_-^u for any choice of ϵ_1! In other words, approximations of the free boundary produced by a minimizing sequence may be arbitrary coarse.

In addition, we define certain dual approximations $y^*_{\epsilon_2}$. Then, the dual and primal-dual estimates can be also evaluated. In the simplest case, we set

$$y^*_{\epsilon_2} = p^* + \epsilon_2 z, \qquad z(\rho) := -(1/4)\nabla\rho^2, \qquad \epsilon_2 > 0.$$

Since $\operatorname{div} z = -1$ in Ω, the function $y^*_{\epsilon_2}$ satisfies the equilibrium constrain (41) for any $\epsilon_2 > 0$. Tables 9 and 10 report on terms in the dual error (19) and the primal-dual identities (21) for some selected approximations.

5.3.3

The third kind approximations u_h are provided by the finite element method (FEM) as piecewise bilinear and continuous functions (known as Q_1 elements). Here, the square domain Ω is subdivided in $n \times n$ square elements with the length h (therefore, $h = 2/n$). For a given h, the approximation u_h is computed by the method discussed in [13]. Table 11 reports on terms in the primal error identity (18). Here, the nonlinear measure is nonzero but significantly smaller than the quadratic measure. Fig. 6 displays approximation u_h (left column) and their corresponding nonlinear measures (middle column) and quadratic measures (right column).

Remark 8. The Matlab code generating all computational results is available at `http://www.mathworks.com/matlabcentral/fileexchange/63817.`

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