Markov Processes Relat. Fields **23**, 87–101 (2017)

Markov Processes and **Related Fields**

A Simple Rank-Based Markov Chain with Self-Organized Criticality*

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Received October 12, 2015, revised May 26, 2016

Abstract. We introduce a self-reinforced point processes on the unit interval that appears to exhibit self-organized criticality, somewhat reminiscent of the well-known Bak – Sneppen model. The process takes values in the finite subsets of the unit interval and evolves according to the following rules. In each time step, a particle is added at a uniformly chosen position, independent of the particles that are already present. If there are any particles to the left of the newly arrived particle, then the left-most of these is removed. We show that all particles arriving to the left of $p_c = 1 - e^{-1}$ are a.s. eventually removed, while for large enough time, particles arriving to the right of p_c stay in the system forever.

KEYWORDS: self-reinforcement, self-organized criticality, canyon

AMS SUBJECT CLASSIFICATION: Primary 82C27; secondary 60K35, 82C26, 60J05

1. Introduction and results

1.1. Main results

Let $(U_k)_{k\geq 1}$ be an i.i.d. collection of uniformly distributed $[0,1]$-valued random variables. For each finite subset x of $[0,1]$, we inductively define a sequence $X^x = (X^x_k)_{k\geq 0}$ of random finite subsets of $[0,1]$ by $X^x_0 := x$, $M^x_{k-1} := \min(X^x_{k-1} \cup \{1\})$ and

$$X^x_k := \begin{cases} X^x_{k-1} \cup \{U_k\} & \text{if } U_k < M^x_{k-1}, \\ (X^x_{k-1} \cup \{U_k\}) \backslash \{M^x_{k-1}\} & \text{if } U_k > M^x_{k-1}. \end{cases} \qquad (k \geq 1). \tag{1.1}$$

*Work sponsored by GAČR grants: P201/12/2613 and 15-08819S.

In words, this says that M^x_{k-1} is the minimal element of X^x_{k-1} and that the set X^x_k is constructed from X^x_{k-1} by adding U_k, and in case that $M^x_{k-1} < U_k$, removing M^x_{k-1} from X^x_{k-1}. Since the $(U_k)_{k\geq 1}$ are i.i.d. and X^x_k is a function of X^x_{k-1} and U_k, it is clear that X^x is a Markov chain. (In fact, we have just given a *random mapping representation* for it.) The state space of X^x is the set $\mathcal{P}_{\rm fin}[0,1]$ of all finite subsets of $[0,1]$, which is naturally isomorphic to the space of all simple counting measures on $[0,1]$ (i.e., $\mathbb{N}$-valued measures ν such that $\nu(\{u\}) \leq 1$ for all $u \in [0,1]$). We equip this space with the topology of weak convergence and the associated Borel-σ-algebra.

The process X^x is an example of a Markov process with self-reinforcement (compare [15]), since the number of particles in the system can grow without bound and influences the fate of newly arrived particles. As we will see in a moment, it also appears to exhibit self-organized criticality in a way that is reminiscent of the well-known Bak − Sneppen model. The empirical distribution function $F^x_k(q) := \big|X^x_k \cap [0,q]\big|$ can loosely be interpreted as the profile of a canyon being cut out by a river. If $U_k < M^x_{k-1}$, then the river cuts deeper into the rock. If $U_k > M^x_{k-1}$, then the slope of the canyon between U_k and the river is eroded one step down.

Our first result says that particles arriving on the left of the critical point $p_{\rm c} := 1 - e^{-1}$ are eventually removed from the system, but for large enough time, particles arriving on the right of $p_{\rm c}$ stay in the system forever.

Theorem 1.1 (A.s. behavior of the minimum). *For any finite $x \subset [0,1]$, one has*

$$\limsup_{k\to\infty} M^x_k = 1 - e^{-1} \quad \text{a.s.} \tag{1.2}$$

To understand Theorem 1.1 better, note that for each $0 \leq q \leq 1$, the restriction $X^x_k \cap [0,q]$ of X^x_k to $[0,q]$ is a Markov chain. Indeed, particles arriving on the right of q just have the effect that in each time step, with probability $1-q$, the minimal element of $X^x_k \cap [0,q]$, if there is one, is removed, while no new particles are added inside $[0,q]$. Theorem 1.1 says that this Markov chain is recurrent for $q < p_{\rm c}$ and transient for $q > p_{\rm c}$. For any $q \in [0,1]$, let

$$\tau^q_\emptyset := \inf\{k > 0 : X^\emptyset_k \cap [0,q] = \emptyset\} \tag{1.3}$$

be the first time the restricted process $X^\emptyset_k \cap [0,q]$ returns to the empty set. Letting $\mathbb{P}^x$ denote the law of X^x, we have the following result.

Theorem 1.2 (Recurrence versus transience). *Let $p_{\rm c} := 1 - e^{-1}$. Then*

$$\begin{aligned}
&\mathbb{E}^\emptyset[\tau^q_\emptyset] = \big(1 + \log(1-q)\big)^{-1} && (q < p_{\rm c}),\\
&\mathbb{E}^\emptyset[\tau^q_\emptyset] = \infty \quad \textit{and} \quad \mathbb{P}^\emptyset[\tau^q_\emptyset < \infty] = 1 && (q = p_{\rm c}),\\
&\mathbb{P}^\emptyset[\tau^q_\emptyset = \infty] > 0 && (q > p_{\rm c}).
\end{aligned} \tag{1.4}$$

Numerical simulations strongly suggest that at $q = p_c$, the probability $\mathbb{P}[\tau^q_\emptyset \geq k]$ decays as $k^{-1/2}$. We briefly comment on this in Section 2.5. In particular, it seems this can be proved provided it can be shown that a certain martingale occurring in our proofs behaves on long time scales like Brownian motion. Such a proof would establish self-organized criticality for our process. Our process is self-organized in the sense that it finds the transition point p_c by itself. In particular, one does not have to tune a parameter of the model to exactly the right value to see the (presumed) power-law critical behavior at p_c.

In the positive recurrent regime $q < p_c$, it is not hard to show that the process is ergodic, and as a result we also obtain the following result. Below, we call a subset of $[0, p_c)$ *locally finite* if its intersection with any compact subset of $[0, p_c)$ is finite.

Theorem 1.3 (Ergodicity of restricted process). *There exists a random, locally finite subset $X_\infty \subset [0, p_c)$ such that, regardless of the initial state x,*

$$\mathbb{P}\big[X^x_k \cap [0, q] \in \cdot\,\big] \underset{k\to\infty}{\longrightarrow} \mathbb{P}\big[X_\infty \cap [0, q] \in \cdot\,\big] \qquad (0 < q < p_c), \tag{1.5}$$

where $\to$ denotes convergence of probability measures in total variation norm distance. The random point set X_∞ a.s. consists of infinitely many points.

1.2. Relation to known models

Models for email communication

Our model appears to be new, but it is similar to a number of other models that have been studied in the literature. To start with the simplest one, consider the following model for email communication. (The model described here is from [5], but very similar to a model introduced in [7]; see also [21] for a similar model with an interpretation coming from mathematical finance.) A person receives emails at times of a Poisson process with intensity λ_{in}, and assigns to each email a priority. Priorities are i.i.d. and uniformly distributed on $[-\lambda_{\text{in}}, 0]$ (with 0 the highest priority). At times of a Poisson process with intensity $\lambda_{\text{out}} := 1$, the person answers the email in the inbox with the highest priority, if there is one, and does nothing otherwise. Letting $X_t \subset [-\lambda_{\text{in}}, 0]$ denote the set of priorities of unanswered emails that are at time t in the inbox, we observe that the process is consistent in the sense that for each $0 \leq \lambda \leq \lambda_{\text{in}}$, the process $(X_t \cap [-\lambda, 0])_{t\geq 0}$ is a Markov process, that has the same dynamics as the original model but with λ_{in} replaced by λ. In this sense, the model is similar to our canyon model, but the email model has the stronger property that if we throw away even more information and only consider the number $N^\lambda_t := |X_t \cap [-\lambda, 0]|$ of emails with priority above $-\lambda$, then even this is a Markov process. Indeed, it is easy to see that N^λ_t jumps $n \mapsto n + 1$ with rate λ and $n \mapsto n - 1$ with rate $1_{\{n>0\}}$. Standard results say that this Markov chain is positive recurrent for $\lambda < 1$, null recurrent for $\lambda = 1$, and transient for $\lambda > 1$.

For the email process we have just described, in [5], a limit theorem is proved for the equilibrium distribution of unanswered emails with priorities just above the critical point $\lambda_c = 1$, linking it to the convex hull of Brownian motion. In [7], a very similar model in discrete time is studied where priorities are uniformly distributed on $[0,1]$ and in each step, one email is answered and $m \geq 2$ new emails arrive. It is shown that the critical priority for this model is $p_c = 1 - 1/m$ and the probability that an incoming email has to wait time t before being answered, given that it is answered at all, decays as $t^{-3/2}$. The proof uses a mapping to invasion percolation on a regular tree.

The model in [7] was inspired by a similar model in [3] where the number N of emails in the inbox is fixed. In this model, in each step, with probability p the email with the highest priority is answered, and with the remaining probability, a random email from the inbox is answered. After this, a new email with a new random priority arrives. If one sends first $p \to 1$ and then $t \to \infty$, then the probability that an incoming email is answered after t time steps is asymptotically of order t^{-1}. This was first shown for an inbox containing $N = 2$ emails in [22] and then for general N in [2]. In the latter paper, it is also argued that keeping $0 < p < 1$ fixed and sending first $N \to \infty$ and then $t \to \infty$, the probability of waiting a time t for an answer decays like $t^{-3/2}$. In [6], a mapping of the model in [3] to invasion percolation is described.

The Bak – Sneppen model

The Bak – Sneppen model, introduced in [4], is one of the best-known models believed to exhibit self-organized criticality. It is a Markov chain $(X_k)_{k\geq 0}$ with state space $[0,1]^N$. Writing $X_k = \big(X_k(0), \ldots, X_k(N-1)\big)$, one interprets $X_k(i) \in [0,1]$ as the fitness of species i at time k. One thinks of the species as being situated on a ring, where $i-1$ and $i+1$ (calculated modulo N) are the neighbors of i. Initially, all fitnesses are independent and uniformly distributed. In each step, the species i with the lowest fitness is selected, and X_{i-1}, X_i, and X_{i+1} are all replaced by new, independent and uniformly chosen fitnesses. It is believed that there exists a critical value $f_c \approx 0.6672(2)$ such that for large N, in equilibrium, the fitness $X(0)$ is approximately uniformly distributed on $[f_c, 0]$. For $0 < f < 1$, excursions away from the set $[f,1]^N$ are called *avalanches*. During an avalanche, to decide what the next move is, one only needs information about fitnesses below f, and in view of this consecutive avalanches are i.i.d. It is believed that for $f < f_c$, the length of an avalanche and the number of species affected have a limit law as $N \to \infty$ with exponential tail, but at $f = f_c$ one should find power-law decay signifying self-organized criticality.

The best available rigorous results for the Bak – Sneppen model can be found in [13, 14]. A key tool in the latter paper is the *locking thresholds representa-*

tion, which shows that at each time, the fitnesses $X_k(0), \dots, X_k(N-1)$ can be thought of as being independent and uniformly distributed in random intervals $[Y_k(0), 1], \dots, [Y_k(N-1), 1]$. The paper defines two critical fitnesses $0 < f_c \le f'_c < 1$, where f_c resp. f'_c is the first point where avalanches (in the limit $N \to \infty$) have infinite expected size, resp. are with positive probability infinite. Assuming that $f_c = f'_c$, it is shown that in the limit $N \to \infty$, in equilibrium fitnesses are approximately uniformly distributed on $[f_c, 1]$.

In [12], a modified Bak – Sneppen model is introduced. Here, if i if the species with the lowest fitness, then instead of redrawing the fitnesses of $i-1, i$ and $i+1$, one redraws the fitnesses of i and one random other species j, chosen uniformly from the population. In a sense, this is the mean-field version of the original Bak – Sneppen model. For this model, it has been proved in [12] that $f_c = 1/2$. Moreover, the probability that an avalanche below some chosen fitness level f has a duration longer than t decays exponentially in t for $f < f_c$, but decays as $t^{-1/2}$ at $f = f_c$. This is proved by setting up a coupling with a branching process, which is subcritical for $f < f_c$ and critical for $f = f_c$.

The Stigler – Luckock model

In the Stigler – Luckock model, first introduced in [18] and reinvented and generalized by Luckock in [10] and again independently reinvented in [16] and [23, 24], traders place buy and sell orders according to independent Poisson processes. Letting $I = (I_-, I_+)$ denote the interval of possible prices, buy (resp. sell) orders arrive in $A \subset I$ with intensity $\mu_-(A)$ (resp. $\mu_+(A)$), where $\mu_\pm$ are finite measures on I. If a buy order arrives at a price x while the order book already contains a sell order at a price $y < x$, then the buy order together with the lowest sell order are immediately removed. Similarly, newly arriving sell orders at a price x are immediately matched against the best available buy order at a price $x' > x$, if such an order exists, and stay in the order book otherwise. Assuming stationarity, Luckock derived a differential equation from which he was able to calculate the equilibrium distribution of the best buy and sell offer in the order book. In particular, he was able to calculate two prices $x_- < x_+$ so that the best sell offer never drops below x_- and the best buy offer never climbs above x_+, with the result that buy orders below x_- and sell orders above x_+ are never matched.

Mathematically, proving existence of the type of stationary distributions that Luckock postulates remains an open problem. Recent progress was made in [20], where for each subinterval $J = (J_-, J_+) \subset I$, a *restricted model on* J is defined in such a way that sell orders that arrive on the left of J_- can still be matched to the best available buy order, but if no such order exist, are not written into the order book; similar rules apply to buy orders that arrive on the right of J_+. For such a restricted model, [20] gives necessary and sufficient conditions for positive recurrence. In particular, if x_-, x_+ are the prices of Luckock, then it is shown that for any $x_- < J_- < J_+ < x_+$, the restricted model is positive recurrent, while for $J_- < x_- < x_+ < J_+$ it is not. Note, however, that the restricted model on J (as defined above) and the original model restricted to J evolve in the same way only as long as the best buy and sell offers remain inside J. In view of this, it has so far not been possible to draw rigorous conclusions about the original model from the behavior of the restricted model.

The main tool in [20], which was inspired by Lemma 2.1 below which was discovered before [20] was written, are weight functions that give a weight to buy and sell orders that depends on their price. By choosing these weight functions in the right way, as solutions to a suitable differential equation, a Lyapunov function can be constructed. In [9], a result in the spirit of our Theorem 1.1 is proved for the Stigler – Luckock model, under certain technical restrictions. An important tool in their work are comparison lemmas in the spirit of our Lemmas 2.3 and 2.4.

In [17], an extension of the Stigler – Luckock model is introduced with richer behavior (in particular, a transition between self-organized criticality and different behavior). See also [20] for references to a number of related models for the evolution of an order book, which however do not exhibit self-organized criticality.

In [16, 19], a variation of the Stigler – Luckock model is introduced where in each step, two traders arrive who want to buy and sell at prices that are infinitesimally close, but ordered so that they just don't match. This model can alternatively be interpreted as a model for canyon formation and motivated the model of the present paper. Numerically, its behavior seems to be governed by the same prices x_-, x_+ as for the Stigler – Luckock model.

Other models

Somewhat similar in spirit to the previous models is also the model [8], which is basically a supercitical branching process in which fitnesses are assigned to the particles, and those killed have the lowest fitness. We should also mention the branching Brownian motions with strong selection treated in [11], which also use a rule of the type "kill the lowest particle" and exhibit self-organized criticality.

We note that in the construction of many of the models discussed so far and in particular also our model, only the relative order of the points (i.e., their rank or priority) matters, so replacing the uniform distribution on $[0, 1]$ by any other atomless law on $\mathbb{R}$ yields the same model up to a continuous transformation of space. Starting from the empty initial state, adding points one by one, and taking notice only of their relative order, one in effect constructs after k steps a random permutation of k elements. In view of this, our quantities of interest may be described as functions of such a random permutation. This is somewhat reminiscent of the way the authors of [1] use what they call Hammersley's process to study the longest increasing subsequence of a random permutation. There is an extensive literature on functions of random permutations, but none of those studied so far seem relevant for our process.

Also, although our model is an example of a Markov process with self-reinforcement, it does not seem to have close connections to any of the classical models with self-reinforcement studied so far, as reviewed in [15].

2. Proofs

2.1. Weight functions

Our main tool for proving Theorems 1.1–1.3 are linear functionals that count the number of points in the interval $[0,q]$ weighted with a function that depends on their position. This method significantly simplifies methods used in a number of preprints predating the present paper, which for the interest of the reader are available as [19], versions 1–3. The method of using weight functions was first discovered for the present model but has in the mean time already successfully been applied in [20] to the Stigler−Luckock model.

As already mentioned in Section 1.2, by a simple transformation of space, we may replace the uniformly distributed random variables $(U_k)_{k\geq 1}$ by real random variables having any non-atomic distribution. At present, it will be more convenient to work with exponentially distributed random variables with mean one, so we transform the unit interval $[0,1]$ into the closed halfline $[0,\infty]$ with the transformation

$$q \mapsto f(q) := -\log(1-q)$$

and set $\sigma_k := f(U_k)$ $(k \geq 1)$. Letting $Y_k := f(X_k)$ denote the image of X_k under f, we then have

$$Y_k := \begin{cases} Y_{k-1} \cup \{\sigma_k\} & \text{if } \sigma_k < N_{k-1}, \\ (Y_{k-1} \cup \{\sigma_k\}) \backslash \{N_{k-1}\} & \text{if } \sigma_k > N_{k-1}. \end{cases} \quad (k \geq 1), \tag{2.1}$$

where $N_k := \min(Y_k \cup \{\infty\})$. We let $\mathcal{P}_{\rm fin}[0,\infty]$ denote the set of all finite subsets of $[0,\infty]$.

Lemma 2.1 (Linear functions of the process). *Define* $L_t : \mathcal{P}_{\rm fin}[0,\infty] \to \mathbb{R}$ *as*

$$L_t(Y) := \sum_{s \in Y} e^s 1_{[0,t]}(s), \quad t \geq 0. \tag{2.2}$$

Then, for the process started in any deterministic initial state Y_0*, one has*

$$\mathbb{E}\big[L_t(Y_1)\big] - L_t(Y_0) = t - 1_{[0,t]}(N_0). \tag{2.3}$$

Proof. For any bounded measurable function $w : [0, \infty] \to \mathbb{R}$, we calculate

$$\begin{aligned}\mathbb{E}\big[L_t(Y_1)\big] - F(Y_0) &= \int_0^{N_0} w(s)e^{-s}\mathrm{d}s + \int_{N_0}^{\infty} \big(w(s) - w(N_0)\big)e^{-s}\mathrm{d}s \\ &= \int_0^{\infty} w(s)e^{-s}\mathrm{d}s - w(N_0)e^{-N_0}.\end{aligned} \tag{2.4}$$

Setting $w(s) := e^s 1_{[0,t]}(s)$, we arrive at (2.3). □

2.2. The positive recurrent regime

For each $t \geq 0$, we let

$$Y_k^{(t)} := Y_k \cap [0, t] \qquad (k \geq 0) \tag{2.5}$$

denote the restriction of the process Y_k to the interval $[0, t]$, which is itself a Markov chain. Using Lemma 2.1, it is easy to show that for $t < 1$, this Markov chain is positive recurrent and ergodic. We let $\mathbb{P}^{\emptyset}$ denote the law of the process started in the empty configuration $Y_0 = \emptyset$ and we let

$$\tau_{\emptyset}^{(t)} := \inf\{k > 0 : Y_k^{(t)} = \emptyset\} \tag{2.6}$$

denote the first return time of $Y^{(t)}$ to the empty configuration.

Proposition 2.1 (Positive recurrent regime). *For $0 \leq t < 1$, one has*

$$\mathbb{E}^{\emptyset}[\tau_{\emptyset}^{(t)}] = (1 - t)^{-1}. \tag{2.7}$$

Moreover, the process $Y^{(t)}$ has an invariant law ν on $\mathcal{P}_{\mathrm{fin}}[0, t]$ such that $\nu(\{\emptyset\}) = 1 - t$ and the process started in an arbitrary initial law satisfies

$$\big\|\mathbb{P}[Y_k^{(t)} \in \cdot\,] - \nu\big\| \underset{k\to\infty}{\longrightarrow} 0, \tag{2.8}$$

where $\|\cdot\|$ denotes the total variation norm.

Proof. For $t < 1$, the function L_t is a Lyapunov function. Using this and the fact that $\mathbb{P}^{\emptyset}[Y_1^{(t)} = \emptyset] > 0$, which shows that our process is aperiodic in an appropriate sense, standard results (see, e.g., [20, Prop. 19]) show that $\mathbb{E}^{Y}[\tau_{\emptyset}^{(t)}] < \infty$ for any deterministic $Y \in \mathcal{P}_{\mathrm{fin}}[0, t]$, and there exists an invariant law ν such that (2.8) holds.

Let $(Y_k^{(t)})_{k\in\mathbb{Z}}$ be the corresponding stationary process. Then Lemma 2.1 tells us that

$$\mathbb{E}[L_t(Y_1^{(t)}) - L_t(Y_0^{(t)})] = t - \mathbb{P}[Y_0^{(t)} \neq \emptyset]. \tag{2.9}$$

By stationarity and Lemma 2.2 below, the left-hand side of this equation is zero, so the invariant law satisfies $\nu(\{\emptyset\}) = 1 - t$. By standard renewal arguments, $\mathbb{E}^{\emptyset}[\tau_{\emptyset}^{(t)}] = \nu(\{\emptyset\})^{-1}$, so (2.7) follows. □

Lemma 2.2 (Stationary increments). *Let $(F(k))_{k\in\mathbb{Z}}$ be a stationary process, and assume that $\mathbb{E}\big[|F(1) - F(0)|\big] < \infty$. Then $\mathbb{E}\big[F(1) - F(0)\big] = 0$.*

Proof. For $M > 0$, let $F^M(k) := F(k)$ if $-M \leq F(k) \leq M$ and $F^M(k) := M$ or $-M$ if $F(k) \geq M$ or $F(k) \leq -M$, respectively. By stationarity, $\mathbb{E}[F^M(1)] = \mathbb{E}[F^M(0)]$ and hence $\mathbb{E}[F^M(1) - F^M(0)] = 0$. Letting $M \uparrow \infty$, using the fact that $|F^M(1) - F^M(0)| \leq |F(1) - F(0)|$ and dominated convergence, we conclude that $\mathbb{E}[F(1) - F(0)] = 0$. □

2.3. The lower invariant process

It turns out to be possible to find stationary solutions of the inductive formula (2.1) that are defined for all $k \in \mathbb{Z}$, and this will very helpful in proving our main theorems. A crucial observation is that solutions to the inductive formula (2.1) are monotone in the starting configuration.

Lemma 2.3 (First comparison lemma). *Let y and $\tilde{y}$ be finite subsets of $[0,1]$ and let $(Y_k)_{k\geq 0}$ and $(\tilde{Y}_k)_{k\geq 0}$ be defined by the inductive relation (2.1) with $Y_0 = y$ and $\tilde{Y}_0 = \tilde{y}$. Then $y \subset \tilde{y}$ implies that $Y_k \subset \tilde{Y}_k$ for all $k \geq 0$.*

Proof. It suffices to show that $Y_{k-1} \subset \tilde{Y}_{k-1}$ implies $Y_k \subset \tilde{Y}_k$. Adding the point σ_k to both Y_{k-1} and $\tilde{Y}_{k-1}$ obviously preserves the order of inclusion, as does simultaneously removing the minimal elements N_{k-1} from Y_{k-1} and $\tilde{N}_{k-1}$ from $\tilde{Y}_{k-1}$. Since $Y_k \subset \tilde{Y}_k$ we have $N_{k-1} \geq \tilde{N}_{k-1}$ and it may happen that $\tilde{N}_{k-1} < \sigma_k \leq N_{k-1}$, in which case we remove $\tilde{N}_{k-1}$ from $\tilde{Y}_{k-1}$ but not N_{k-1} from Y_{k-1}, but in this case $\tilde{N}_{k-1}$ is not an element of Y_{k-1} so again the order is preserved. □

We will be interested in stationary solutions to the inductive relation (2.1). To this aim, we consider a two-way infinite sequence $(\sigma_k)_{k\in\mathbb{Z}}$ of i.i.d. exponentially distributed random variables with mean one. For each $m \in \mathbb{Z}$, we let $(Y_{m,k})_{k\geq m}$ denote the solution to the inductive relation (2.1) started in $Y_{m,m} := \emptyset$. Since $Y_{m-1,m} \supset \emptyset = Y_{m,m}$, we see by Lemma 2.3, that $Y_{m-1,k} \supset Y_{m,k}$ for all $k \geq m$, so there exists a collection $(\overline{Y}_k)_{k\in\mathbb{Z}}$ of countable subsets of $[0,\infty)$ such that

$$Y_{m,k} \uparrow \overline{Y}_k \quad \text{as } m \downarrow -\infty. \tag{2.10}$$

We call the limit process $(\overline{Y}_k)_{k\in\mathbb{Z}}$ from (2.10) the *lower invariant process*. The following proposition is the main result of the present subsection.

Proposition 2.2 (Lower invariant process). *For all $k \in \mathbb{Z}$, one has*

$$\big|\overline{Y}_k \cap [0,t]\big| \begin{cases} < \infty & \text{a.s.} \quad \text{if } t \in [0,1), \\ = \infty & \text{a.s.} \quad \text{if } t \in [1,\infty). \end{cases} \tag{2.11}$$

The set $\overline{Y}_k$ a.s. has a minimal element $\overline{N}_k := \min(\overline{Y}_k)$, whose distribution is given by

$$\mathbb{P}[\overline{N}_k < t] = t \wedge 1 \qquad (k \in \mathbb{Z},\ t \geq 0). \tag{2.12}$$

Moreover, the process $(\overline{Y}_k)_{k\in\mathbb{Z}}$ solves the inductive relation (2.1) for all $k \in \mathbb{Z}$.

Proof. Proposition 2.1 tells us that for each $t \in [0,1)$ and fixed $k \in \mathbb{Z}$, the random variables $Y_{m,k}$ converge in distribution as $m \downarrow -\infty$ to a limit with law ν satisfying $\nu(\{\emptyset\}) = 1 - t$. This shows that $\overline{Y}_k \cap [0,t]$ is a.s. finite for each $t < 1$ and $\mathbb{P}\big[\overline{Y}_k \cap [0,t] = \emptyset\big] = 1 - t$. It follows that $\overline{Y}_k \cap [0,1)$ is a locally finite subset of $[0,1)$ and

$$\mathbb{P}\big[\overline{Y}_k \cap [0,1) = \emptyset\big] = \lim_{t\uparrow 1} \mathbb{P}\big[\overline{Y}_k \cap [0,t] = \emptyset\big] = 0. \tag{2.13}$$

If $\mathbb{P}\big[\big|\overline{Y}_k \cap [0,1)\big| \leq N\big]$ were positive for some $N < \infty$, then it is easy to see that $\mathbb{P}[\overline{Y}_{k+N} = \emptyset]$ would also be positive, so by stationarity we conclude that $\overline{Y}_k \cap [0,1)$ is a.s. infinite. In particular, this implies that $\overline{Y}_k \cap [0,t] = \infty$ a.s. for each $t \geq 1$. Moreover, (2.12) now follows from the fact that $\mathbb{P}\big[\overline{Y}_k \cap [0,t] = \emptyset\big] = 1 - t$.

Our arguments so far show that almost surely, $\overline{N}_{k-1} < 1$ and $\overline{Y}_{k-1} \cap [0,t]$ is a finite set for all $t < 1$. Choose $\overline{N}_{k-1} < t < 1$. Then a.s. there exists an $M > -\infty$ such that $Y_{m,k-1} \cap [0,t] = \overline{Y}_{k-1} \cap [0,t]$ for all $m \leq M$. Using this and the fact that $Y_{m,k-1}$ and $Y_{m,k}$ are related by the inductive relation (2.1), we see that also $\overline{Y}_{k-1}$ and $\overline{Y}_k$ are related as in (2.1). □

2.4. Proof of the theorems

Using the transformation $\sigma_k = -\log(1 - U_k)$, Theorem 1.2 can equivalently be formulated for the process Y from (2.1) as follows.

Theorem 2.1 (Recurrence versus transience). *Let $Y^{(t)}$ be the restricted process from (2.5) and let $\tau^{(t)}_\emptyset$ as in (2.6) denote the first return time of $Y^{(t)}$ to the empty configuration. Then*

$$\begin{aligned} &\mathbb{E}^\emptyset[\tau^{(t)}_\emptyset] = (1-t)^{-1} && (t < 1), \\ &\mathbb{E}^\emptyset[\tau^{(t)}_\emptyset] = \infty \quad \text{and} \quad \mathbb{P}^\emptyset[\tau^{(t)}_\emptyset < \infty] = 1 && (t = 1), \\ &\mathbb{P}^\emptyset[\tau^{(t)}_\emptyset = \infty] > 0 && (t > 1). \end{aligned} \tag{2.14}$$

Proof. The case $t < 1$ has already been proved in Proposition 2.1. Clearly $t \leq t'$ implies $\tau_\emptyset^{(t)} \leq \tau_\emptyset^{(t')}$ a.s., so the formula for $t < 1$ implies that $\mathbb{E}^\emptyset[\tau_\emptyset^{(t)}] = \infty$ for all $t \geq 1$.

For $t = 1$, Lemma 2.1 implies that

$$M_k := L_1(Y_{k\wedge\tau_\emptyset^{(1)}}) \qquad (k \geq 0) \tag{2.15}$$

is a nonnegative martingale, so the a.s. limit $M_\infty := \lim_{k\to\infty} M_k$ exists. It is easy to see that M_k cannot converge to a positive limit, so we conclude that $M_\infty = 0$ a.s. and hence $\tau_\emptyset^{(1)} < \infty$ a.s.

For $t > 1$, we observe that for any $0 \leq s < t$,

$$\big|Y_n \cap (s,t]\big| \geq \sum_{k=1}^{n} 1_{\{s < \sigma_k < t\}} - \sum_{k=1}^{n} 1_{\{Y_{k-1} \cap [0,s] = \emptyset\}}. \tag{2.16}$$

By the strong law of large numbers and Proposition 2.1

$$n^{-1} \sum_{k=1}^{n} 1_{\{s < \sigma_k < t\}} \underset{n\to\infty}{\longrightarrow} (e^{-s} - e^{-t}) \tag{2.17}$$

and

$$n^{-1} \sum_{k=1}^{n} 1_{\{Y_{k-1} \cap [0,s] = \emptyset\}} \underset{n\to\infty}{\longrightarrow} 1 - s \quad \text{a.s.}$$

Choosing s close enough to 1 such that $1 - s < e^{-s} - e^{-t}$, we see that $\big|Y_n \cap (s,t]\big| \to \infty$ a.s., and hence $\mathbb{P}^\emptyset[\tau_\emptyset^{(t)} = \infty] > 0$. □

Proof of Theorem 1.1. It follows from Proposition 2.1 that regardless of the initial state, $\limsup_{k\to\infty} N_k \geq 1$. On the other hand, in the proof of Theorem 2.1 we have seen that $|Y_n \cap [0,t]| \to \infty$ a.s. for any $t > 1$, proving that $\limsup_{k\to\infty} N_k \leq 1$. Translating these results to the process X yields Theorem 1.1. □

Proof of Theorem 1.3. This follows from results that have already been proved for the process Y from (2.1). The ergodic statement (1.5) follows from Proposition 2.1, and the fact that X_∞ is an infinite, but locally finite subset of $[0, p_\mathrm{c})$ follows from Proposition 2.2. □

2.5. Some concluding remarks

For the process that is the subject of the present paper, two open problems seem worth investigating. For the transformed process Y from (2.1), one would like to know if it is true, as numerical simulations suggest, that at the critical point $t_c := 1$, the tail of the return probability decays as $\mathbb{P}^\emptyset[\tau^{(t_c)} \geq k] \sim k^{-1/2}$. Second, one would like to know the asymptotic shape of the locally finite point set $\overline{Y}_k \cap [0,1)$ of Proposition 2.2 near the critical point $t_c = 1$.

Both these problems have been resolved for the email model described in Section 1.2. In particular, in [5], for the email model, it has been shown that the (random) distribution function of the lower invariant process, properly rescaled, converges to a limit as one approaches the critical point, and the limit can be expressed in terms of the convex hull of Brownian motion. The proof of this fact essentially uses that for the email process, the number of emails in the inbox above a certain priority is a random walk with reflection at the origin, which properly rescaled converges to a (drifted) Brownian motion.

To prove analogue results for the present, more complicated canyon model, it seems essential to understand the point-counting process

$$F_t(k) := \big|Y_k \cap [0,t]\big| \qquad \big(k \in \mathbb{N},\ t \in [0,\infty)\big), \tag{2.18}$$

and similarly $\overline{F}_t(k)$, which is defined in terms of the lower invariant process $\overline{Y}$. Although these are not Markov processes, one possible guess is that for $t < 1$ close to one, suitable rescaled, they converge to drifted Brownian motions with reflection at the origin. Alternatively, defining L_t as in (2.2), one may also look at the processes $L_t(Y_k)$ and $L_t(\overline{Y}_k)$, which count particles in $[0,t]$ weighted with the function e^x. In view of Lemma 2.1, one has good control over the compensator of these processes and hence convergence to reflected Brownian motion may be easier to prove.

We mention two technical facts that were used in preprints preceding the present paper ([19], versions 1–3) and that may still be of interest.

First, due to the memoryless property of the exponential distribution of the random variables σ_k, it is not hard to check that the function valued process $(F_t)_{t\geq 0}$, with $F_t = (F_t(k))_{k\geq 0}$ as in (2.18), is a continuous-time Markov process, where the parameter t plays the role of time. Indeed, at the time $t = \sigma_k$, the function F_t changes as

$$F_t(k') = \begin{cases} F_{t-}(k') + 1 & \text{if } k \leq k' < \kappa_t(k), \\ F_{t-}(k') & \text{otherwise} \end{cases} \qquad (k' \geq 0), \tag{2.19}$$

where F_{t-} denotes the state immediately prior to time t and

$$\kappa_t(k) := \inf\{k' > k : F_{t-}(k'-1) = 0 = F_{t-}(k')\}, \tag{2.20}$$

with the convention that $\inf\emptyset := \infty$. Similarly, $(\overline{F}_t)_{0\leq t<1}$, with $\overline{F}_t = (\overline{F}_t(k))_{k\in\mathbb{Z}}$ is also a Markov process, which, however, is only well-defined until time one. Also $L_t(Y_k)$ and $L_t(\overline{Y}_k)$ evolve as a function of t in a Markovian way, but in this case, in (2.19), the term $+1$ needs to be replaced by $+e^t$.

A second technical fact that is perhaps of interest is a second comparison lemma, similar to Lemma 2.3, but using a different order. Roughly speaking, it says that for $Y_k \cap [0,t]$ to avoid becoming the empty set, it is good to have many particles that are situated as far as possible to right in the interval $[0,t]$.

Lemma 2.4 (Second comparison lemma). *For each $0 \leq s \leq t$ and finite $y \subset [0,\infty)$, let $F^y_{s,t}(k) := \big|Y^y_k \cap [s,t]\big|$ $(k \geq 0)$. Fix $t > 0$ and let $x, y \subset [0,\infty)$ be finite. Then*

$$F^x_{s,t}(0) \leq F^y_{s,t}(0)\ \forall s \in [0,t] \quad \textit{implies} \quad F^x_{s,t}(k) \leq F^y_{s,t}(k)\ \forall s \in [0,t],\ k \geq 0. \tag{2.21}$$

Proof. It suffices to prove (2.21) for $k = 1$; the general statement follows by induction. Without loss of generality, we may also assume that x and y are subsets of $[0,t]$. Order the elements of x and y as $x = \{x_1,\ldots,x_n\}$ and $y = \{y_1,\ldots,y_m\}$ with $x_n < \cdots < x_1$ (in this order!) and $y_m < \cdots < y_1$. Then the assumption that $F^x_{s,t}(0) \leq F^y_{s,t}(0)$ $\forall s \in [0,t]$ is equivalent to the statement that $m \geq n$ and $x_i \leq y_i$ for all $i = 1,\ldots,n$. We must show that we can order the elements of $\tilde{x} := Y^x_1 \cap [0,t]$ and $\tilde{y} := Y^y_1 \cap [0,t]$ in the same way. We distinguish three different cases.

Case I: $\sigma_1 < x_n$. In this case, no points are removed from x while $\tilde{x}_{n+1} := \sigma_1$ is added as the $(n+1)$-th element. Since $x_n \leq y_n$, the elements $y_1,\ldots,y_n$ remain unchanged while $\tilde{y}_{n+1}$ is the maximal element of $\{\sigma_1\} \cup \{y_m,\ldots,y_{n+1}\}$, which lies on the right of $\tilde{x}_{n+1} = \sigma_1$.

Case II: $x_n < \sigma_1 < t$. In this case, x_n is removed from x and there exist $1 \leq n' \leq n$ and $n' \leq m' \leq m+1$ such that σ_1 is inserted into x between the n'-th and $(n'-1)$-th element and into y between the m'-th and $(m'-1)$-th element, where we allow for the cases that $n' = 1$ (σ_1 is added at the right end of x and possibly also of y) and $m' = m+1$ (σ_1 is added at the left end of y). The elements of the new sets $\tilde{x}$ and $\tilde{y}$, ordered from low to high, are now

$$\begin{aligned} \{x_{n-1},\ldots,x_{m'},x_{m'-1},\ldots,x_{n'},\sigma_1,x_{n'-1},\ldots,x_1\} &= \tilde{x}, \\ \{y_{m-1},\ldots,y_n,y_{n-1},\ldots,y_{m'},\ \sigma_1,y_{m'-1},\ldots,y_{n'},\ y_{n'-1},\ldots,y_1\} &= \tilde{y}. \end{aligned} \tag{2.22}$$

Here $x_{n-1},\ldots,x_{m'}$ lie on the left of $y_{n-1},\ldots,y_{m'}$, and likewise $x_{n'-1},\ldots,x_1$ lie on the left of $y_{n'-1},\ldots,y_1$, respectively. Since moreover

$$x_{m'-1} < \cdots < x_{n'} < \sigma_1 < y_{m'-1} < \cdots < y_{n'}, \tag{2.23}$$

these elements are ordered in the right way too.

Case III: $t < \sigma_1$. In this case, the lowest elements of x and y are removed while no new elements are added, which obviously also preserves the order. □

Acknowledgements

The main method used in the proofs was discovered in a joint discussion with Marco Formentin. I would further like to thank Persi Diaconis, Patrik Ferrari, Ronald Meester, Martin Ondreját, Frank Redig, Jan Seidler, Martin Šmíd, and Rongfeng Sun for useful discussions and in particular Jan van Neerven, Marco Formentin, and Martin Šmíd for drawing my attention to the Bak – Sneppen model, Barabási's queueing system, and the Stigler – Luckock model, respectively.

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