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On the distinguishability and observer design for single-input single-output continuous-time switched affine systems under bounded disturbances with application to chaos-based modulation

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ABSTRACT

Switched Affine Systems (*SAS's*) is a class of Hybrid Systems composed of a collection of Affine Systems (*AS's*) and a switching signal that determines, at each time instant, the evolving affine subsystem. This paper is concerned with the observability and observer design for single-input single-output (*SISO*) *SAS's* under unknown perturbation, for the case that no information about the switching signal is available. It is firstly demonstrated that in the presence of disturbances every pair of *AS's* is always indistinguishable from the continuous output, meaning that it is not possible to infer the evolving *AS* by using only the information provided by the output of the *SAS*. Nevertheless, by taking advantage of the knowledge on the disturbance bound, new distinguishability conditions are derived, making possible to distinguish the evolving *AS*. By using these new distinguishability conditions, an observer scheme for *SISO SAS's*, subject to unknown switching signal and unknown perturbations, is presented. Such an observer scheme determines in finite-time the evolving *AS*. Furthermore, it estimates both the state of the system and the disturbance. Finally, the proposed observer scheme is effectively applied for a non-autonomous chaotic modulation application, which is an attractive method for spread-spectrum secure communication in which the message is fed as a disturbance to a chaotic *SAS* and the output is then transmitted through an open channel to a receiver, which is an observer algorithm that recovers the message (the disturbance) from the output signal.

1. Introduction

Switched Affine Systems (*SAS's*) are composed of a collection of Affine Systems (*AS's*) and a switching signal determining, at each time instant, the evolving affine subsystem. Although *SAS's* are formed of simple *AS's*, this class of systems may exhibit highly non-linear behaviors, such as chaos [34,26,48,52,27,32,38,13], under a suitable selection of the affine subsystems and the switching rule.

SAS's are interesting models for applications in different engineering areas. For instance, process systems frequently include the operation of discrete actuators, each combination of the actuator states leads to an operation mode in which the behavior of the system is ruled by a continuous model [1,25,8]. In the same area, nonlinear continuous dynamics are frequently approximated by *AS's* operating at different operation points, thus the active *AS* depends on the continuous state, leading to autonomous switching as a piecewise affine system [39,43]. In power electronic systems, the presence of semiconductors that may be either controlled or autonomously driven together with linear components leads to switched linear models [17,44]. The analysis and control of these kinds of systems, based on *SAS's*, are frequently affected by disturbances and parametric variations, particularly when a *SAS* model is used to approximate a nonlinear continuous behavior like in process control. For this reason, such analysis should be performed by taking into account the disturbances affecting the system.

This paper is concerned with the observability and the observer design for single-input single-output (*SISO*) perturbed *SAS's*, under the assumption that the switching signal in unknown. Mainly considering

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the unperturbed case, the observability has been extensively studied for Switched Linear Systems (*SLS*) in [46,15,7], whose results can be straightforwardly extended to *SAS's*, by means of the so-called distinguishability property, which allows to infer the currently evolving subsystem based on the input–output information only. Regarding the perturbed case, in [18] the distinguishability property using the input–output information of the perturbed *SLS* has been characterized, based on invariant subspaces analysis. Nevertheless, as we show hereinafter, when disturbances (unknown inputs) are considered for single-input single-output *SAS's*, every pair of affine systems are indistinguishable from the output, i.e. the output information is not sufficient to determine the evolving *AS*. Thus, for this frequent case, new distinguishability analysis that take into account additional information, such as the knowledge on the disturbance bound, are required.

Regarding the observer design in *SLS*, many contributions have been presented in the literature in the last years [47,12,20,31,49,33,9,11,19]. However, to the best of our knowledge the available results are not applicable to the problem under consideration in this paper. For instance, in [47], in order to infer the evolving subsystem and the unknown input, it is required to know not only the output but also the initial condition. In [12,20,31,49,33], in order to recover the state and the unknown input, it is required to know the switching signal (i.e. the evolving system is always known). In our setting, the switching signal is unknown. Regarding observers for *SLS* subjected to an unknown switching signal [9,11,35,19], most of the methods consider the unperturbed case and are valid for observable *SLS* that require that each pair of subsystems is distinguishable [19]. Nevertheless, such results cannot be straightforwardly applied to the perturbed case. In addition, we show that, unlike [35], the continuous state and the switching signal can be estimated in the perturbed case even if there is not a common transformation that transforms every *AS* to the observability form.

For unperturbed systems, multi-observer structures have been proposed in the framework of supervisor [22] and adaptive control [3,40], where they are used to determine a suitable controller (from a bank of controllers) for the evolving process based on the smallness of the output estimation error. Unfortunately, as shown hereinafter, when considering perturbed *SISO AS's*, each observer in the multi-observer structure may give a zero output estimation error. Thus, a different decision method for inferring the evolving *AS* and for the observer design is required.

1.1. Contribution

We first show that every pair of observable *SISO SAS's* become indistinguishable from the output when disturbances are present, i.e., it is not possible to infer which *AS* of the *SAS* is evolving using only the output trajectory. For this reason, we derive new distinguishability results, according to which, by taking advantage on the knowledge of the disturbance bound, a pair of perturbed *SISO AS's* may become distinguishable.

Furthermore, in this paper we present an observer scheme for perturbed *SISO SAS's* subject to an unknown switching signal, where the continuous state, the evolving subsystem and the unknown disturbance are inferred from the output information and the knowledge of the disturbance bound. Although our approach uses a multi-observer structure to infer the evolving subsystem, this task would be impossible to achieve by using other multi-observer structures already proposed, as [22,3,40], which are mainly applicable in the framework of supervisor and adaptive control for unperturbed systems.

Finally, it is shown that the proposed observer can be effectively applied for chaos-based modulation, in particular, to the non-autonomous chaotic modulation. In particular, to the non-autono mous chaotic modulation [50] (also known as message-embedded modulation [14,2,29,37]) using chaotic attractors generated by *SAS's*, where a message is embedded by means of a non-linear function that affects the phase of the chaotic attractor, thus acting as a disturbance. The modulation signal is obtained as the output of the chaotic system which can be transmitted through an open channel. At the receiver, the proposed unknown input observer recovers the message using the knowledge of the nominal system. Thus, the proposed observer enables a wide class of chaotic attractors generated by *SAS's* (see e.g. [34,26,48,52,27,32,38,13]) to be used in non-autonomous chaotic modulation. To the best of our knowledge, the non-autonomous modulation using general classes of *SAS's* with chaotic behavior has not been presented in the literature.

The manuscript is organized as follows. Section 2 recalls basic concepts on *SAS's* and the distinguishability property on these systems. Section 3 introduces a new distinguishability condition that will be used in the observer design. Section 4 introduces the observer scheme. Section 5 presents the application of the proposed methodology to chaos-based non-autonomous modulation. Finally, some conclusions are presented in Section 6.

2. Preliminaries

Definition 1. A *SAS* $\Sigma_{\sigma(t)}$ is composed of a collection of *AS's* $\mathcal{F}=\{\Sigma_1,\ldots,\Sigma_m\}$, each one evolving in the state space $\mathcal{X}=\mathbb{R}^n$, and a switching rule $\sigma:\mathbb{R}_{\geq 0}\to\{1,\ldots,m\}$ determining the evolving *AS* at each time. The state equation of a *SISO* $\Sigma_{\sigma(t)}$ is

$$\begin{aligned}\dot{x}(t)&=A_{\sigma(t)}x(t)+b_{\sigma(t)}+s_{\sigma(t)}d(t),\quad x(t_0)=x_0\in\mathcal{X}_0\subsetneq\mathcal{X},\\ y(t)&=c_{\sigma(t)}x(t),\\ \sigma&:\mathbb{R}_{\geq 0}\to\{1,\ldots,m\}\end{aligned}\tag{1}$$

where $y\in\mathbb{R}$ is the output signal and $d\in\mathbb{R}$ is an unknown input signal (e.g., disturbance) assumed to be continuous and piecewise differentiable. The evolving *AS*, when $\sigma(t)=i$, is represented by Σ_i (A_i,b_i,c_i,s_i) or simply by Σ_i, where A_i, b_i, c_i and s_i are constant matrices and vectors of appropriate dimensions. In our setting, the switching signal $\sigma(t)$ is assumed to be unknown (i.e. the currently evolving *AS* is not known).

In the sequel, expressions $y^i(t,x_0,d(t))$ and $x^i(t,x_0,d(t))$ will be used to denote the output and the state trajectories, respectively, obtained when the system Σ_i is evolving from the initial state x_0 under the disturbance $d(t)$.

When a *SAS* (1) is composed of *AS's*, it may occur that fundamental properties of *AS's* may not be preserved in the occurrence of switching among them. Furthermore, highly nonlinear dynamics such as chaotic behavior can be generated by a simple *SAS*.

Throughout this paper, it will be assumed that the considered *SAS's* fulfill the following assumptions.

Assumptions.

1. Each *AS* $\Sigma_i(A_i,b_i,c_i,s_i)$ composing the *SAS* is assumed to be observable with unknown input $d(t)$ (also known as strongly observable), i.e. the pair (A_i,c_i) is observable, the pair (A_i,s_i) is controllable and the triple (A_i,s_i,c_i) has no transmission zeros.
2. The set $\mathcal{X}_0$ of all possible initial conditions is bounded, i.e., $\|x_0\|<\delta$ for all $x_0\in\mathcal{X}_0$, with known δ.
3. Zeno behavior is excluded and there is a minimum dwell-time between any two switching instants. However, only the minimum dwell time for the first switching τ_d is assumed to be known.
4. The disturbance $d(t)$ is assumed to satisfy for all $t\geq 0$, $|d(t)|<D$ and $|\dot{d}(t)|<L$ with known constants D and L.

Since Zeno behavior is excluded, solutions to (1) are understood in the sense of Carathéodory [30, Section 1.2], which are absolutely continuous and piecewise differentiable functions [30].

For the case of state dependent switching, the minimum dwell-time condition for the first switching can be enforced by imposing a suitable region for the system's initial condition, which can be computed by using the bound of the disturbance and minimum-time control methods that ensure that the switching condition will not be satisfied before a predefined time regardless of the disturbance affecting the *SAS*, see e.g. [10, Algorithm 3.5.1 and Problem 3.5.1].

2.1. Distinguishability in perturbed SISO SAS's

An important concept related to the observability and the observer design problems in *SAS's* is the distinguishability property, which deals with the possibility of inferring, from the continuous output of the *SAS*, the evolving *AS* even in the presence of disturbance [18]. Here we call such property as output-distinguishability to emphasize that it is based on the output information only. Let us formally introduce such property.

Definition 2. The *AS* Σ_i is said to be output-indistinguishable from the *AS* Σ_j if there exists a pair $(x_0, d(t))$ applied to Σ_i and $(x_0', d'(t))$ applied to Σ_j such that Σ_i and Σ_j produce the same output, i.e.

$$\exists x_0, x_0', d(t), d'(t) \text{ such that } y^i(t, x_0, d(t)) = y^j(t, x_0', d'(t)), \quad \forall t \geq 0 \tag{2}$$

otherwise Σ_i is said to be output-distinguishable from Σ_j.

Thus, if Σ_i is output-indistinguishable from Σ_j it is impossible to determine from the output of the *SAS* whether the evolving *AS* is Σ_i or Σ_j, the initial condition is x_0 or x_0' and the affecting disturbance is $d(t)$ or $d'(t)$.

Notice that, in general, the pairs $(x_0, d(t))$ and $(x_0', d'(t))$ such that (2) holds are not required to be equal. For the case when they are equal the continuous initial condition and the affecting disturbance can be uniquely determined even if it is impossible to assert which *AS* is the evolving one.

3. New distinguishability conditions for perturbed *SISO SAS's*

In this section, it is shown that any pair of *SISO* perturbed *AS's* is output-indistinguishable. For this reason, a new distinguishability condition is introduced in order to support the design of the observer algorithm to be introduced in Section 4. This new condition takes advantage of the known disturbance bound (Assumption 4) in order to gain distinguishability.

Proposition 1. *Any two SISO AS's Σ_i and Σ_j are output-indistinguishable under disturbance. Otherwise stated, for every initial condition x_0 and disturbance $d(t)$ applied to Σ_i there exist an initial condition x_0' and a disturbance $d'(t)$ (not necessarily equal to x_0 and $d(t)$) applied to Σ_j such that the corresponding output trajectories are equal, i.e. $y^i(t, x_0, d(t)) = y^j(t, x_0', d'(t))$ for all time $t \geq 0$, thus making impossible to infer from the output which is the evolving AS.*

Proof. Consider a similarity transformation $x = T_i \check{x}$ such that $T_i = \mathcal{O}_{\Sigma_i}^{-1}$ and $\mathcal{O}_{\Sigma_i}$ is the observability matrix of the *AS* Σ_i (recall that a nonsingular coordinate transformation does not change the input–output behavior of the *AS*). By Assumption 1, such similarity transformation is well defined and the transformed system is in the observability canonical form with the unknown disturbance $d(t)$ affecting only the last state variable [24].

In the new coordinates the *AS* is represented by

$$\begin{aligned}
\dot{\check{x}}_1 &= \check{x}_2 + \check{b}_1^i \\
\dot{\check{x}}_2 &= \check{x}_3 + \check{b}_2^i \\
&\vdots \\
\dot{\check{x}}_n &= -a_n^i \check{x}_1 - a_{n-1}^i \check{x}_2 - \cdots - a_1^i \check{x}_n + \check{b}_n^i + \beta^i d(t) \\
y(t) &= \check{x}_1
\end{aligned}$$

where

$$s^n + a_1^i s^{n-1} + \cdots + a_{n-1}^i s + a_n^i \tag{3}$$

is the characteristic polynomial of A_i. Now, let us introduce the variable transformation $\bar{x}_1 = \check{x}_1$ and $\bar{x}_k = \check{x}_k + \check{b}_{k-1}^i$, $k \in \{2, \ldots, n\}$, thus $\bar{x} = T_i^{-1}(x + b_i)$. Then, the *AS* Σ_i can be represented as

$$\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= \bar{x}_3 \\
&\vdots \\
\dot{\bar{x}}_n &= \alpha^i(\bar{x}) + \beta^i d(t) \\
y(t) &= \bar{x}_1
\end{aligned} \tag{4}$$

where

$$\alpha^i(\bar{x}) = -a_n^i \bar{x}_1 - \sum_{k=2}^{n} a_{n-k+1}^i \left(\bar{x}_k - \check{b}_{k-1}^i \right) + \check{b}_n^i. \tag{5}$$

Notice that the new state variables are the output and their derivatives, i.e.

$$\bar{x}_k = \frac{d^{k-1} y(t)}{dt^{k-1}}, \quad \forall k \in \{1, \ldots, n\} \tag{6}$$

Now, applying the analogous transformation procedure to the *AS* $\Sigma_j(A_j, b_j, c_j, s_j)$, with $\bar{x} = T_j^{-1}(x + b_j)$, Σ_j can be represented as

$$\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= \bar{x}_3 \\
&\vdots \\
\dot{\bar{x}}_n &= \alpha^j(\bar{x}) + \beta^j d(t) \\
y(t) &= \bar{x}_1
\end{aligned} \tag{7}$$

Notice that if the disturbance $d(t)$ is applied to Σ_i and the signal

$$d'(t) = \frac{1}{\beta^j} \left(\alpha^i(\bar{x}) - \alpha^j(\bar{x}) + \beta^i d(t) \right) \tag{8}$$

is applied to Σ_j as a disturbance then (4) and (7) have the same output behavior. Therefore, the output trajectory obtained when the disturbance $d(t)$ is applied to the system Σ_i with the initial condition x_0 is equal to that obtained when the disturbance $d'(t)$ in (8) is applied to Σ_j with the initial condition $x_0' = T_j T_i^{-1}(x_0 + b_i) - b_j$. Notice that it is possible to compute such x_0' and $d(t)'$ for any pair x_0 and $d(t)$. Thus, it is impossible to determine from the output whether the evolving subsystem is Σ_i or Σ_j. Similarly, it is impossible to determine from the output whether the initial condition is x_0 or x_0'. Therefore, the *AS* Σ_i is output-indistinguishable from Σ_j. □

The previous proposition establishes that any pair of perturbed *SISO AS's* are always output-indistinguishable. The next proposition additionally establishes that Σ_i and Σ_j become output-indistinguishable only with disturbances of the form (8).

Proposition 2. *Let Σ_i and Σ_j be perturbed AS's satisfying Assumption 1. Suppose the disturbance $d(t)$ is applied to the system Σ_i with the initial condition x_0 and the signal $d'(t)$ is applied to Σ_j as a disturbance with the initial condition x_0'. Both AS's produce the same output trajectories iff $d'(t)$ fulfills (8) and $x_0' = T_j T_i^{-1}(x_0 + b_i) - b_j$, where $T_k^{-1} = \mathcal{O}_{\Sigma_k}$ with $\mathcal{O}_{\Sigma_k}$ being the observability matrix of Σ_k, $k = i, j$.*

Furthermore, the generated state trajectories $x^i(t, x_0, d(t))$ and $x^j(t, x_0', d'(t))$ fulfill with

$$x^j(t, x_0, d(t)) = T_j T_i^{-1}(x^i(t, x_0', d'(t)) + b_i) - b_j.$$

Proof. The sufficiency has been demonstrated above. To prove the necessity, assume that $\exists x_0, x_0', d(t), d'(t)$ such that $y^i(t, x_0, d(t)) = y^j(t, x_0', d'(t))$. Let us consider the coordinate transformations $\bar{x}^i = T_i^{-1}(x^i + b_i)$ and $\bar{x}^j = T_j^{-1}(x^j + b_j)$ for Σ_i and Σ_j, respectively (which do not affect the input–output behavior of the *AS*'s). Since $y^i(t, x_0, d(t)) = y^j(t, x_0', d'(t))$, $\forall t \geq t_0$, then $\frac{d^k}{dt^k} y^i(t, x_0, d(t)) = \frac{d^k}{dt^k} y^j(t, x_0', d'(t))$, for all $k \geq 0$, which implies that $\bar{x}^i = \bar{x}^j$ and $\alpha^i(\bar{x}^i) + \beta^i d(t) = \alpha^j(\bar{x}^j) + \beta^j d'(t)$. Based on these equations, it is easy to see that $d'(t)$ must be equal to (8) in order to make Σ_i output-indistinguishable from Σ_j and that, in such case, $x^j(t) = T_j T_i^{-1} (x^i(t) + b_i) - b_j$, which particularly holds for $x^j(0) = x_0' = T_j T_i^{-1} (x_0 + b_i) - b_j$. □

Lemma 1. *Consider a SAS and suppose it is evolving in either Σ_i. By using the knowledge of the disturbance bound D imposed by Assumption 4, it can be inferred whether the system evolves in Σ_i or Σ_j if the signal $d'(t)$ that, if applied to Σ_j as a disturbance, would make Σ_j output-indistinguishable from Σ_i does not satisfy the disturbance bound condition for a proper time interval, i.e. if for a proper time interval it holds that*

$$|d'(t)| = \left| \frac{1}{\beta^j} \left(\alpha^i(\bar{x}) - \alpha^j(\bar{x}) + \beta^i d(t) \right) \right| > D \tag{9}$$

Proof. The proof follows directly from the previous propositions. □

Notice that, in some cases, this bound condition does not provide additional information for instance when $\beta^i/\beta^j < 1$ and the system is evolving in a certain state region where $\alpha^i(\bar{x}) - \alpha^j(\bar{x})$ is relatively small with respect to $\beta^i d(t)$. The following theorem formalizes this new distinguishability condition.

Lemma 2. *Consider a SAS and two AS's, Σ_i and Σ_j, of the collection. Consider the disturbance bound D imposed by Assumption 4. Suppose that the system is evolving in either Σ_i or Σ_j. Let $\mathcal{B}_{\bar{x}}^{ij} \subseteq \mathbb{R}^n$ be the set of vectors $x' \in \mathbb{R}^n$ that fulfills*

$$\left| \frac{1}{\beta_j} \left(\alpha^i(x') - \alpha^j(x') \right) \right| > D \left(1 + \left| \frac{\beta_i}{\beta_j} \right| \right) \tag{10}$$

where the polynomial functions $\alpha^i(x')$ and $\alpha^j(x')$ are given by (5). During the evolution of the SAS, if for a proper time interval $\bar{x}(t) \in \mathcal{B}_{\bar{x}}^{ij}$, where $\bar{x}_k(t) = \frac{d^{k-1} y(t)}{dt^k}$ for $k \in [1,..,n]$, then it can be inferred whether the system evolves in Σ_i or Σ_j. In such case, it is said that the pair Σ_i and Σ_j is bound-distinguishable.

Proof. Considering $|d(t)| \leq D$, the condition (10) implies the condition (9). Thus, whenever the system is evolving in a state region such that $\bar{x}(t) \in \mathcal{B}_{\bar{x}}^{ij}$, the conditions of Lemma 1 hold. Then it is possible to determine whether Σ_i or Σ_j is evolving. □

Additional information can be exploited for the case when each *LS* in known to only evolve in a sub-region of the state space $\mathcal{X}$. Such is the case in chaotic attractors where there exists a basin of attraction or switching *AS's*, where the switching among the *AS's* is state dependent.

The sub-region in which a *LS* Σ_i is known to only evolve in is defined as the containing set $\mathcal{C}_{\Sigma_i}$. The following result shows how the information on containing sets can be used to distinguish between *AS's*.

Lemma 3. *Consider two AS's Σ_i and Σ_j in the SAS. Let $\mathcal{C}_{\Sigma_i}$ and $\mathcal{C}_{\Sigma_j}$ be two containing sets for the evolution of $x(t)$ on Σ_i and Σ_j, respectively. Suppose that the evolving AS is either Σ_i or Σ_j. If the output trajectory is such that in a proper time interval*

$$T_i \bar{x} - b_j \in \mathcal{C}_{\Sigma_i} \quad \textit{and} \quad T_j \bar{x} - b_j \notin \mathcal{C}_{\Sigma_j}, \tag{11}$$

where $\bar{x}_k(t) = \frac{d^{k-1} y(t)}{dt^k}$ for $k \in [1, \ldots, n]$, then it can be inferred that the evolving system is Σ_i, not Σ_j. In such case Σ_i is said to be containing set-distinguishable from Σ_j.

Proof. The proof follows from Proposition 2. If Σ_i is evolving and $\bar{x}$ is such that $\bar{x}_k(t) = \frac{d^{k-1} y(t)}{dt^k}$ for $k \in [1, \ldots, n]$, the evolving state trajectory is $x(t) = T_i \bar{x}(t) - b_j$, which according to the knowledge on the containing set should satisfy $T_i \bar{x}(t) - b_j \in \mathcal{C}_{\Sigma_i}$. On the contrary, if we suppose that the evolving *AS* is Σ_j then the state trajectory would be $T_j \bar{x}(t) - b_j$, but since $T_j \bar{x}(t) - b_j \notin \mathcal{C}_{\Sigma_j}$ then Σ_j is not evolving. Therefore, it can be asserted that Σ_i is evolving. □

The information of containing sets can be used together with the disturbance bound to explore distinguishability. The following theorem involves the previous results.

Theorem 1. *Let $\mathcal{C}_{\Sigma_i}$ be the containing set for the evolution of Σ_i, $i = 1, \ldots, m$ (if such information is unavailable or the evolution of Σ_i is unconstrained then consider $\mathcal{C}_{\Sigma_i} = \mathbb{R}^n$), and let D be the bound on the disturbance imposed in Assumption 4. Then the continuous and discrete states of the SAS are observable for every $\bar{x}$ (where $\bar{x}_k(t) = \frac{d^{k-1} y(t)}{dt^k}$ for $k \in [1, \ldots, n]$) such that pairwise Σ_i and Σ_j are either bound-distinguishable or containing set-distinguishable.*

Proof. The proof follows from Lemmas 2, 3 and Assumption 1. □

4. Observers synthesis

In this section, an observer structure for *SISO SAS's*, fulfilling the aforementioned assumptions, is proposed. In the proposed structure, an observer is designed for each *AS* in the collection.

To illustrate our approach we use the High Order Sliding mode (*HOSM*) differentiator proposed in [28]. Nevertheless, any exact differentiator algorithm can be used. An alternative for the observers design is the uniform robust exact differentiator proposed in [4] (in such case, the convergence-time is bounded, independently on the initial condition).

Another alternative is a step-by-step observer based on the super-twisting algorithm, for which the observer gains can be easily chosen with a priori time-convergence bound by using the results from [45]; the drawback of this approach however is that the observer is not global.

The idea is to design the observer for each *AS* in the *SAS* in such a way that the error dynamic coincide with the differentiation error of an exact differentiator, illustrated hereinafter for the *HOSM* differentiator proposed in [28].

4.1. Estimation of the currently evolving AS

In this subsection an observer algorithm based on the HOSM differentiator described in [28] is proposed for the detection of the evolving *AS* and the estimation of the current continuous state and the affecting disturbance.

Proposition 3. *The AS $\Sigma_i(A_i, b_i, c_i, s_i)$ admits the following global finite-time observer:*

$$\begin{aligned}
\dot{\tilde{x}}_1 &= -a_1^i y + \tilde{x}_2 + \check{b}_1^i + l_1 \rho |y - \tilde{x}_1|^{n/(n+1)} \text{sign}(y - \tilde{x}_1) \\
&\vdots \\
\dot{\tilde{x}}_j &= -a_j^i y + \tilde{x}_{j+1} + \check{b}_j^i + l_j \rho^j |y - \tilde{x}_1|^{(n-j+1)/(n+1)} \text{sign}(y - \tilde{x}_1) \\
&\vdots \\
\dot{\tilde{x}}_n &= -a_n^i y + \check{b}_n^i + \tilde{d}(t) + l_n \rho^n |y - \tilde{x}_1|^{1/(n+1)} \text{sign}(y - \tilde{x}_1) \\
\dot{\tilde{d}} &= l_{n+1} \rho^{n+1} \text{sign}(y - \tilde{x}_1) \\
\tilde{y} &= \tilde{x}_1
\end{aligned} \tag{12}$$

where $l_1, \ldots, l_n$ and ρ are observer parameters to be adjusted, $a_1^i, \ldots, a_n^i$ are the coefficients of the characteristic polynomial (3) of A_i, and $\check{b}_j^i$ is the j-th element of the vector $\check{b}_i = \mathcal{T}_i^{-1} b_i$ with

$$\mathcal{T}_i^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1^i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n-1}^i & a_{n-2}^i & \cdots & 1 \end{bmatrix} \begin{bmatrix} c_i \\ c_i A_i \\ \vdots \\ c_i A_i^{n-1} \end{bmatrix}. \tag{13}$$

Solutions to (12) are understood in the sense of Filippov [30,28].

The state and the disturbance estimates are given by $\hat{x}(t) = \mathcal{T}_i \tilde{x}$ and $\hat{d}(t) = \frac{1}{\beta_i} \tilde{d}(t)$, respectively. In other words, if the AS Σ_i is evolving with the state trajectory $x(t)$ and the affecting disturbance $d(t)$ then, after the observer converges in finite-time, the estimates $\hat{x}(t) = x(t)$ and $\hat{d}(t) = d(t)$ will be produced.

Proof. Notice that $\mathcal{T}_i$ as in (13) is the similarity transformation taking the i-th *AS* into the observer canonical form [24]. Then, by substitution, the estimation error $e(t) = \mathcal{T}_i^{-1} x(t) - \tilde{x}(t)$ evolves as

$$\begin{aligned}
&\dot{e}_1 = e_2 - l_1 \rho |e_1|^{n/(n+1)} \mathrm{sign}(e_1) \\
&\vdots \\
&\dot{e}_j = e_{j+1} - l_j \rho^j |e_1|^{(n-j+1)/(n+1)} \mathrm{sign}(e_1) \quad \text{for } j = 1, \ldots, k-1. \\
&\vdots \\
&\dot{e}_n = e_d - l_n \rho^n |e_1|^{1/(n+1)} \mathrm{sign}(e_1) \\
&\dot{e}_d = \beta^i \dot{d}(t) - l_{n+1} \rho^{n+1} \mathrm{sign}(e_1) \\
&\quad \text{where } e_d = \beta^i d(t) - \tilde{d}(t).
\end{aligned} \tag{14}$$

This error evolution coincides with that of the differentiation error of the *HOSM* differentiator [28]. Thus, with an appropriate gain selection ($l_1, \ldots, l_n$ and ρ), the estimation error $e(t)$ will converge to zero after a finite-time $\bar{t}$ (it is well-known that such $\bar{t}$ can be made arbitrarily small with a suitable selection of the gains), i.e. for all $t \geq \bar{t}$, $\hat{x}(t) = \mathcal{T}_i \tilde{x}(t) = x(t)$, and $\hat{d}(t)$ converges to $d(t)$. □

Let us provide a couple of comments regarding practical issues during the observers synthesis:

- Previous proposition establishes that, if the gains are appropriated adjusted, the observer (12) corresponding to the evolving *AS* will accurately estimate the state and the disturbance in finite-time. See [28] for a selection of the observer gains for the error dynamics (14) for up to 5th order. See [45] for the gain selection of a second order error dynamic (14) together with a tight estimation of the convergence time bound. For instance, for a third order error dynamic (14), a proper gain selection is $l_1 = 9.5608$, $l_2 = 6.8681$, $l_3 = 0.0219$ and $\rho^{n-1} > |\beta_i| L/0.0081$ [42], for a convergence time bound see [42].
- In our framework, it is expected that the observer corresponding to the evolving *AS* converges before the first switching. Thus, the time convergence must be lower bounded by the dwell time τ_d imposed in Assumption 3. It is known that such convergence bound can be obtained with an appropriate selection of the observer gains, as the initial condition lies in a known bounded set according to Assumption 2. It is shown in Appendix A that such convergence bound can be obtained with an appropriate selection of the observer gains, provided that the initial condition is bounded as required by Assumption 2. In particular, assume that using the gain selection $\bar{l}_1, \bar{l}_2, \ldots, \bar{l}_{n+1}$, $\bar{\rho}$, a time convergence bound $\overline{T}_f$ is obtained. It is shown in Appendix A that the convergence time bound for a gain selection $l_1 = \bar{\rho} \bar{l}_1$, $l_2 = \bar{\rho}^2 \bar{l}_2$, …, $l_{n+1} = \bar{\rho}^{n+1} \bar{l}_{n+1}$ and $\rho \geq 1$ is given by $T_f = \overline{T}_f / \rho$.

The next proposition demonstrates that if the observer of another *AS* converges, then the estimate of the disturbance will be equal to that of (8). Thus, by means of such estimated disturbance and Lemma 1, the estimations provided by such observer can be discarded.

Proposition 4. *Let Σ_i be the evolving AS with $x(t)$ as the state trajectory and $d(t)$ as the affecting disturbance. Let the observer associated to Σ_j be designed as illustrated in Proposition 3. If the output estimation error of the observer associated to Σ_j, denoted as $e_y^j = y - \tilde{y}^j$, becomes zero then the observer will produce an estimate of the state $\hat{x}(t)$ and the disturbance $\hat{d}(t)$ making Σ_i output-indistinguishable from Σ_j, where $\hat{d}(t)$ has the form of (8).*

Proof. For the sake of simplicity, assume that the *AS* Σ_i and Σ_j are represented in the observer canonical form [24]. Let Σ_i be the evolving *AS* with $x(t)$ being the state trajectory and let $\tilde{x}^j(t)$ be the state of the observer associated to Σ_j. Denote the entries of the error vector as $\hat{e}_k^j = x_k - \tilde{x}_k^j$, $k = 1, \ldots, n$. Thus, if $y - \tilde{x}_1^j = \hat{e}_1^j = 0$ then the dynamic behavior of $\hat{e}^j = \left[\hat{e}_1^j \cdots \hat{e}_n^j\right]^T$ becomes

$$\begin{aligned}
&0 = \hat{e}_2^j + \left(-a_1^i y + \check{b}_1^i\right) - \left(-a_1^j y + \check{b}_1^j\right) \\
&\dot{\hat{e}}_2^j = \hat{e}_3^j + \left(-a_2^i y + \check{b}_2^i\right) - \left(-a_2^j y + \check{b}_2^j\right) \\
&\vdots \\
&\dot{\hat{e}}_n^j = \left(-a_n^i y + \check{b}_n^i + \beta^i d(t)\right) - \left(-a_n^j y + \check{b}_n^j + \tilde{d}^j(t)\right)
\end{aligned} \tag{15}$$

Differentiating the first equation and combining it with the second equation in (15) we get $\hat{e}_3^j = -\left(-a_1^i \dot{y} - a_2^i y + \check{b}_2^i\right) + \left(-a_1^k \dot{y} - a_2^k y + \check{b}_2^i\right)$. Differentiating $\hat{e}_3^j$ and combining it with (15), we get $\hat{e}_4^j = -\left(-a_1^i \ddot{y} - a_2^i \dot{y} + a_3^i y + \check{b}_2^i\right) + \left(-a_1^k \ddot{y} - a_2^k \dot{y} + a_3^j y + \check{b}_2^i\right)$. Following this procedure we get that $\hat{e}_n^j = -\left(-a_1^i \overset{(n-2)}{y} - \cdots - a_{n-1}^i y + \check{b}_2^i\right) + \left(-a_1^k \overset{(n-1)}{y} - \cdots - a_{n-1}^j y + \check{b}_2^i\right)$. Differentiating $\hat{e}_n^j$ and combining it with the last equation of (15) we get that $\hat{d}^j(t) = \frac{1}{\beta^j} \tilde{d}^j(t)$, with $\tilde{d}^j(t) = \left(\alpha(i, \bar{x}) - \alpha(j, \bar{x}) + \beta^i d(t)\right)$, thus $\hat{d}^j(t)$ is equal to (8). Since with $e_y^j = y - \tilde{y}^j = 0$ the observer (12) becomes a copy of Σ_j represented in the canonical observer form producing the same output information as Σ_i then the state estimate obtained by the observer associated to Σ_j is the one given in Proposition 2. □

Remark 1. In [9], a super-twisting based step-by-step observer was proposed for autonomous switched nonlinear systems, i.e. where no inputs are present. Such approach requires, for each step, a super-twisting algorithm, which is designed by using the known bounds for the first n derivatives of the output. In our approach, which allows to cope with unknown inputs, such bound knowledge is not required for the convergence of the observer (12), instead only the knowledge on the bound for $|\dot{d}(t)|$ is needed (Assumption 4).

4.2. Observer scheme

The complete observer scheme (Fig. 1) includes a collection of finite-time observers, one for each *AS*, determining thus the evolving *AS* by detecting the only observer that satisfies $|\tilde{d}(t)| \leq D$ and $e(t) = y(t) - \tilde{y}(t) = 0$ for a proper time interval. The state estimate is given by the observer of the evolving *AS*, once it is

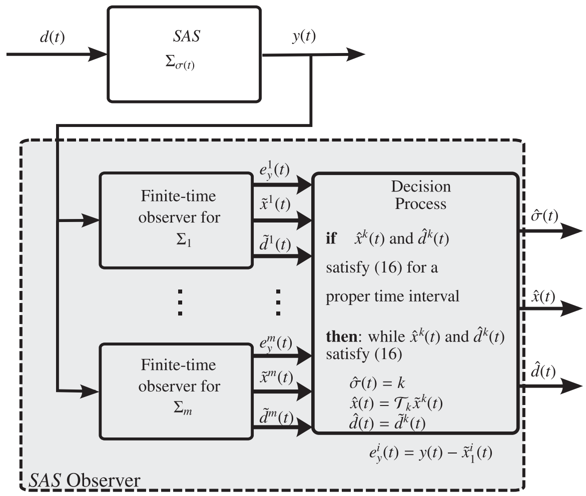

Fig. 1. *SAS* observer.

determined. This is formally stated in the following proposition. In the sequel, let us denote as $\hat{x}^i(t)$, $\hat{y}^i(t)$ and $\hat{d}^i(t)$ the estimates of the state, the output and the disturbance provided by the observer of the i-th *AS*, respectively. Similarly, let us denote as $e^i(t) = y(t) - \hat{y}^i(t)$ the estimation output error provided by such observer.

Proposition 5. *Let $\Sigma_{\sigma(t)}$ be a SAS and consider a collection of observers of the form (12), one for each AS in the SAS, evolving in parallel, as depicted in Fig. 1. Suppose that the system remains in the initial AS a time longer than τ_d, and suppose that each observer has a time convergence bound $\tau \ll \tau_d$. Suppose that the system evolves inside a region in which the output and their derivatives fulfill $\overline{x} \in \mathcal{B}^{ij}_{\overline{x}}$, for every pair of AS's Σ_i and Σ_j, as defined in Theorem 1. Then, the state of the switching signal $\sigma(t)$ can be detected by the index k of the only k-th observer satisfying*

$$\hat{x}^k(t) \in \mathcal{C}_{\Sigma_k} \quad \text{and} \ \left|\hat{d}^k(t)\right| \le D \ \text{and} \ e_y^k(t) = 0 \quad \forall t \ \text{in a proper interval}[\tau, \tau + \Delta t], \quad \Delta t > 0 \tag{16}$$

Once it is inferred that the evolving AS is Σ_k, an exact estimate of the continuous state of the SAS and the affecting disturbance is provided by the observer associated to Σ_k.

Proof. If Σ_i is evolving with $x(t)$ as the state trajectory and $d(t)$ as the affecting disturbance, then the observer associated to the *AS* Σ_j either gives $e_y \neq 0 \ \forall t > \tau$, from which it can be asserted that Σ_k is not the evolving *AS*, or it precisely produces an estimate of the state $\hat{x}^k(t)$ and the disturbance $\hat{d}^k(t)$ when Σ_i is output-indistinguishable from Σ_k, according to Proposition 4. However, the conditions of Theorem 1 are satisfied, thus if $i \neq k$ (i.e. for an observer not associated to the evolving *AS*) then the condition (16) cannot hold for a proper time interval. Consequently, $|\hat{d}^k(t)| > D$ and thus it can be asserted that Σ_k is not the evolving *AS*. On the contrary, if $i = k$ (i.e. for the observer associated to the evolving *AS*) then, according to Proposition 3, exact estimates of the evolving state trajectory $x(t)$ and the affecting disturbance $d(t)$ are obtained by the observer, which clearly is consistent with the knowledge on the disturbance bound, i.e. (16) is satisfied. □

According to the previous proposition, the evolving *AS* can be detected as the index k of the only observer satisfying (16). After a switching occurrence, the same observer can no longer maintain the condition (16). Thus, the switching occurrence is detected when such condition no longer holds.

Based on the previous propositions, the observer structure is presented in the next algorithm. The observer structure estimates the switching signal, the continuous state and the affecting disturbance.

Algorithm 1. State and disturbance observer implementation.

1: **Input** The collection of *AS*'s $\mathcal{F}$. The disturbance bound D. The first switching dwell time τ_d. The bound for the possible initial state δ. The output evolution $y(t)$ of the *SAS*.

2: **Output** The estimates of the state and the disturbance are given by $\hat{x}(t)$ and $\hat{d}(t)$, respectively.

3: *Synthesis:*

4: • Design an observer (12) with time convergence bound $\tau \ll \tau_d$ for each $\Sigma_i \in \mathcal{F}$.

5: • Initialize each observer state and the disturbance as $\tilde{x}^i(0) = 0$ and $\tilde{d}^i(0) = 0$. Initialize the switching signal estimate as $\hat{\sigma}(0) = 1$. Initialize the state estimate and disturbance estimate as $\hat{x}(0) = 0$ and $\hat{d}(0) = 0$, respectively.

6: *Operation:*

7: • All the observers run in parallel.

8: • If there is an observer of an *AS* Σ_k satisfying (16), i.e. such that $e_y^k(t) = 0$ and $|\hat{d}^k(t)| \le D$ for a proper time interval, then set $\hat{\sigma}(t) = k$, $\hat{x}(t) = \mathcal{T}_k \hat{x}^k(t)$ and $\hat{d}(t) = \hat{d}^k(t)$.

9: • After the evolving *AS* Σ_k has been detected, a switching occurring at time t_s can be detected as the time instant when the observer associated to Σ_k no longer satisfies (16). In that case, the observer associated to each *AS* Σ_i has to be reinitialized as $\tilde{x}^i(t_s) = \mathcal{T}_i^{-1} \hat{x}^k(t_s)$ and $\tilde{d}^i(t_s) = \tilde{d}^k(t_s)$.

Proposition 6. *Consider a SAS $\Sigma_{\sigma(x)}$ fulfilling Assumptions 1–4. Suppose that the state of $\Sigma_{\sigma(x)}$ evolves in a state region such that the output and their derivatives fulfill the condition in Theorem 1. Then the state $x(t)$ of $\Sigma_{\sigma(x)}$, the disturbance $d(t)$ and the switching signal $\sigma(t)$ are estimated by the observer structure of Fig. 1 following Algorithm 1.*

Proof. Proposition 5 guarantees that if the *AS* Σ_k is evolving then only its associated observer will satisfy condition (16) for a proper time interval, thus the evolving system is detected and exact estimates of the continuous state $x(t)$ and the affecting disturbance $d(t)$ are given by $\hat{x}^k(t)$ and $\hat{d}(t)$, respectively. Next, after a switching occurrence the condition of Proposition 5 can no longer hold, thus the switching time is detected. Since no jumps occur in the continuous state of $\Sigma_{\sigma(x)}$, by reinitializing each observer with the estimated value at the switching time, all the observers have accurate estimates of the state and the disturbance, but only the observer associated with the new evolving system will satisfy condition (16). Consequently, no additional time is required for the convergence of the observer scheme. Following in this way the switching signal $\sigma(t)$, the continuous state $x(t)$ and the affecting disturbance $d(t)$ are continuously estimated. □

5. Application to chaos-based non-autonomous modulation

A *SAS* may exhibit a complex nonlinear behavior, such as chaos, under a suitable selection of the affine subsystems and the switching rule. Chaotic *SAS's* exhibit properties like wide spread spectrum, dense periodic orbits and strong dependence on the initial conditions. These features make them suitable for communication applications, mainly because broadband information carriers enhance the robustness of communication channels against interferences with narrow-band disturbances, which is the

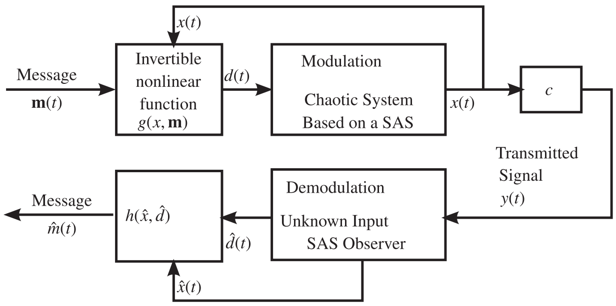

Fig. 2. Chaotic modulation/demodulation process.

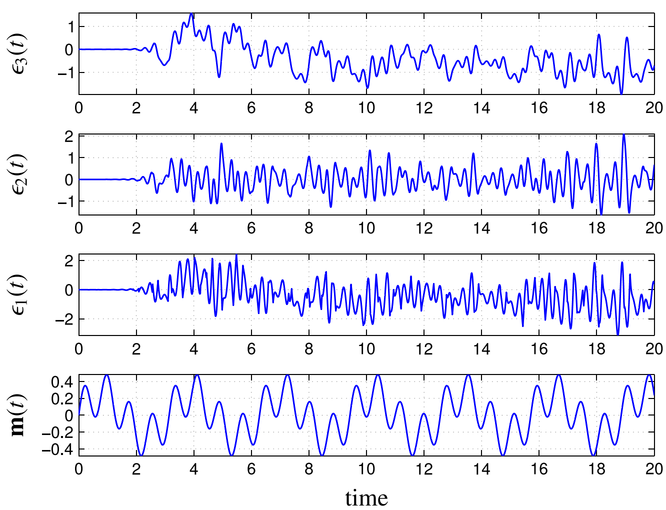

Fig. 3. High sensitivity to the initial condition. Let $x(t)$ and $x'(t)$ be trajectories of $\Sigma_{\sigma(x)}$ starting at the initial condition $x_0 = [1\ 0\ 0.5]^T$ and $x'_0 = [0.9999\ 0\ 0.5]^T$, respectively. The difference $\epsilon(t) = x(t) - x'(t)$ is shown above.

basis of spread-spectrum communication techniques. In chaos-based communications, the broadband coding signal is generated at the physical layer rather than algorithmically, as in code division multiple access [5]. Additionally, the irregular signals and seemingly randomness of chaotic systems make them useful to hide information, enhancing software-based encryption, to achieve privacy in the communication [50,14].

One of the chaos-based modulation methods is the non-autonomous chaotic modulation [50] (also known as message-embedded modulation [14,2,29,37]), which has been previously considered using the Lorenz system [29] and the Generalized Lorenz system [14] as the chaotic attractors. In this modulation method, a message is embedded by means of a non-linear function that is then fed to the chaotic system as an input. The modulated signal (which is an analog signal) to be transmitted is obtained as the output of the chaotic system. The receiver recovers the message from the transmitted signal by synchronization with the emitter. From a control theory perspective, the original message is estimated from the modulated signal by means of a disturbance observer that takes advantage from the knowledge of the nominal system.

In our approach, the chaotic attractor for chaos-based non-autonomous modulation is assumed to be generated by a *SAS*. One of the main advantages of using *SAS's* is the simple circuitry required for the implementation of the chaotic system for instance using Chua's circuit [34], DC to DC converters [16], or the general jerk circuit [51,38].

In this section we show that the proposed observer design for perturbed *SISO SAS's* can be applied for the non-autonomous chaotic modulation using general chaotic attractors generated by *SAS's* (see e.g. [34,26,48,52,27,32,13]). This modulation/demodulation process is depicted in Fig. 2.

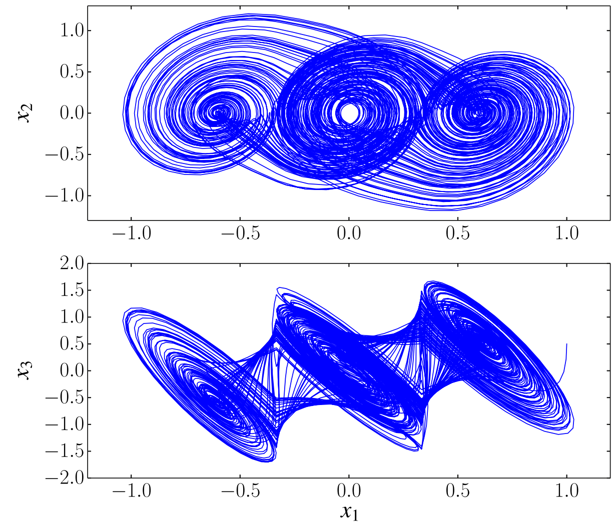

Fig. 4. Preservation of the multiscroll attractors under small disturbances.

5.1. Multiscroll attractors by switched affine systems

Let us consider for instance the multiscroll attractor $\Sigma_{\sigma(t)}$ proposed in [13] given by the following collection of *AS's*

	A	b	s
Σ_1	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & -0.2461 & 1 \\ -8.0521 & -2.0060 & -1.1102 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 5.5355 \end{bmatrix}$	$\begin{bmatrix} 0.0309 \\ -0.1241 \\ 0 \end{bmatrix}$
Σ_2	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & -0.2461 & 1 \\ -6.8438 & -2.0060 & -1.1102 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.04225 \\ -0.1441 \\ 0 \end{bmatrix}$
Σ_3	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & -0.2461 & 1 \\ -8.0521 & -2.0060 & -1.1102 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ -5.5355 \end{bmatrix}$	$\begin{bmatrix} 0.0309 \\ -0.1241 \\ 0 \end{bmatrix}$

together with a state dependent switching rule:

$$\sigma(t) = \begin{cases} 1 & \text{if } x_1(t) \geq 1/3 \\ 2 & \text{if } -1/3 < x_1(t) < 1/3 \\ 3 & \text{if } x_1(t) \leq -1/3 \end{cases}$$

Notice that the state x is unknown at the receiver, thus if the switching depends on an unmeasured state then the switching signal $\sigma(t)$ is unknown at the receiver.

Because of the message-embedding method of Fig. 2 using the nonlinear function $g(x, \mathbf{m})$, the estimates of both the continuous state and the disturbance are required for recovering the hidden message $\mathbf{m}(t)$. Moreover, once $x(t)$ and $d(t)$ are estimated the recovery of the hidden message by means of the function $h(\hat{x}, \hat{d})$ is straightforward. For this reason, in this example we focus on the estimation of both $x(t)$ and $d(t)$ and for simplicity the function $g(x, \mathbf{m})$ of Fig. 2 is assumed to be such that $g(x, \mathbf{m}) = m$, hence $d(t) = \mathbf{m}(t)$ (such method is known as non-autonomous chaotic modulation), however different $g(x, \mathbf{m})$ functions enhancing the security of the communication can be straightforwardly addressed.

In [13], it was shown that $\Sigma_{\sigma(t)}$ as defined above is a chaotic attractor. Fig. 3 shows the high sensitivity to the initial condition in the presence of the disturbance and Fig. 4 displays the multi-scroll behavior of $\Sigma_{\sigma(t)}$.

Although measurement (or channel) noise is not affecting the system, this example has practical applications in fiber–optic and visible–light communications, for instance, visible light communication systems in indoor have very high signal-to-noise-ratio

(*SNR*) in the range of 40–70 dB [41,6]. Under such *SNR* the effect of noise is negligible.

Let us analyze the fulfilling of Assumptions 1–4 and the condition of Theorem 1 for this application example.

- First, it is always possible to impose a suitable bound for the message and hence on the disturbance.
- The evolution inside the basin of attraction of a chaotic attractor is confined inside an invariant bounded set [21]. For instance, such bounded set can be obtained by using the methodology proposed in [23], where piecewise quadratic Lyapunov functions are used to derive tight bounds for the chaotic oscillations and for the evolution of each individual *AS*.
- Zeno behavior does not occur in this system. This can be seen from the vector field and the definition of the switching signal, since the vector fields of the *AS*'s point in the same direction relative to the switching surface [30]. In detail, let us denote as $\Omega_1 = \{x | x_1 = 1/3\}$ the hyperplane between Σ_1 and Σ_2. Then, it is easy to verify that the vector fields of Σ_1 and Σ_2 are in the same direction (in the direction of x_2) in the neighborhood of Ω_1. This also occurs for the switching hyperplane $\Omega_2 = \{x | x_1 = -1/3\}$ between Σ_2 and Σ_3.
- Regarding the assumption on the minimum dwell time for the first switching, regions for suitable initial conditions guaranteeing that no switching may occur before τ_d can be found by using minimum-time control methods, which allow to obtain, by computational methods, the set of states that can be reached from x_0 by a bounded control with time $t \le \tau_d$ (in our case a bounded disturbance), see [10], Algorithm 3.5.1 and Problem 3.5.1. Thus, appropriate bounds for the initial conditions can be provided by the system designer.
- A suitable output for each *AS* can be selected by the designer to fulfill Assumption 1 together with the bound-distinguishability condition in Theorem 1. For instance,if $y = x_2$ then Σ_1 and Σ_3 can be proved to be indistinguishable by showing that both AS's have the same set of equations when written in the form of (4), with $\alpha^1(\bar{x}) = \alpha^3(\bar{x}) = -9.1623\bar{x}_1 - 3.2792\bar{x}_2 - 1.3563\bar{x}_3$. Moreover, $\alpha^2(\bar{x}) - \alpha^1(\bar{x}) = 1.2083\bar{x}_1$ and the condition of Theorem 1 is not satisfied for $\bar{x}_1 = x_2 \in (-1, 1)$. On the contrary,if $y = x_3$ and the disturbance $d(t) = \mathbf{m}(t)$ is bounded by $D=1$ then the condition of Theorem 1 is satisfied. In detail, $\beta^1 = \beta^2 = 1$ and

$$\alpha^1(\bar{x}) = -9.1623\bar{x}_1 - 3.2792\bar{x}_2 - 1.3563\bar{x}_3 + 5.5355$$
$$\alpha^2(\bar{x}) = -7.9540\bar{x}_1 - 3.2792\bar{x}_2 - 1.3563\bar{x}_3$$
$$\alpha^3(\bar{x}) = -9.1623\bar{x}_1 - 3.2792\bar{x}_2 - 1.3563\bar{x}_3 - 5.5355.$$

Thus, $\left|\alpha^1(\bar{x}) - \alpha^3(\bar{x})\right| = 11.071 > 2D$ and for $\bar{x}_1 = x_3 \in (-2.5, 2.5)$

$$\left|\alpha^2(\bar{x}) - \alpha^1(\bar{x})\right| = \left|-5.5355 + 1.2083\bar{x}_1\right| > 2D \quad \text{and} \quad \left|\alpha^2(\bar{x}) - \alpha^3(\bar{x})\right| = \left|5.5355 + 1.2083\bar{x}_1\right| > 2D.$$

It can be seen by numerical simulation that the evolution inside the basin of attraction satisfies $x_3 \in (-2.5, 2.5)$ as shown in Fig. 4.

Now, let us report the results. The demodulation process, i.e. the chaotic synchronization and the estimation of the signal $d(t)$, for the chaotic system $\Sigma_{\sigma(t)}$ described above is shown in Figs. 5–7.

In Fig. 5, in time point ❶ it is shown that only the observer associated with the evolving *AS* Σ_3 is able to satisfy the conditions of Proposition 5 for a proper time interval before the first switching. Thus, it is asserted that Σ_3 is evolving. Next, the switching occurrence ❷ is detected when the observer associated to Σ_3 no longer maintains $\left|\tilde{d}^k(t)\right| < D$ with $e_y^k(t) = 0$, as indicated by ❸. Once the switching occurrence is detected then each observer is reinitialized, as indicated in ❹, and after the reinitialization only the observer associated to Σ_2 is able to maintain the

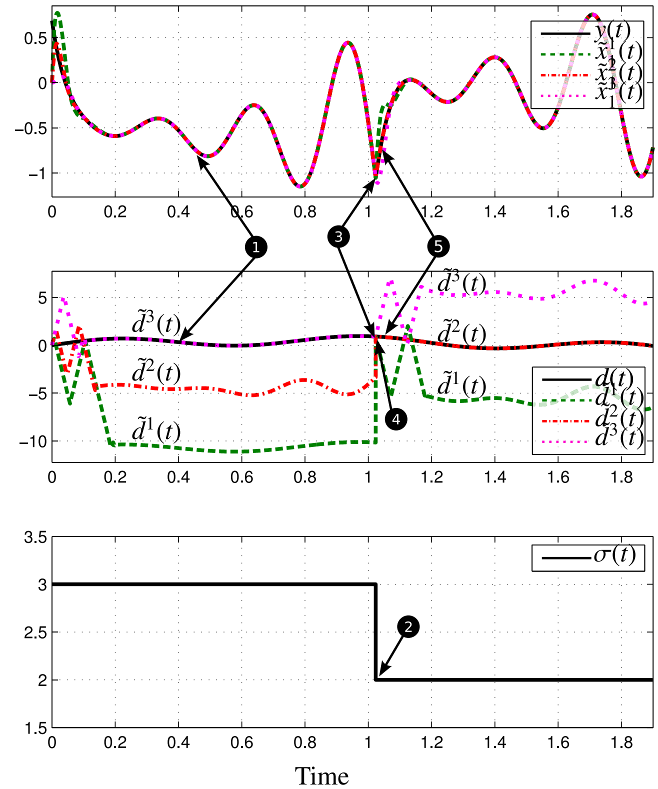

Fig. 5. From top to bottom: *SAS* output vs output estimation of each observer, disturbance vs estimation of the disturbance of each observer, switching signal. **Estimation Process:** ❶ Estimation of the evolving *AS*. ❷ Switching occurrence. ❸ Switching Detection. ❹ Reinitialization of the observer. ❺ Detection of the subsequent evolving *AS*.

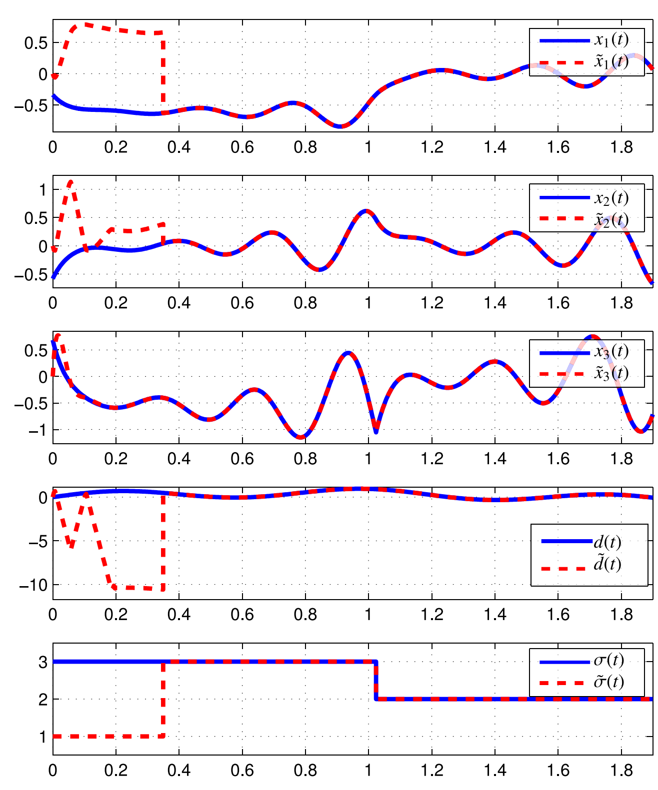

Fig. 6. Chaotic synchronization; estimation of the continuous state $x(t)$, the switching signal $\sigma(t)$ and the information signal $d(t)$.

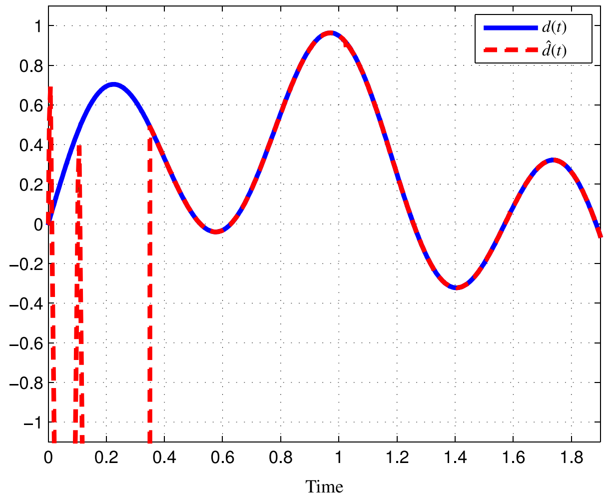

Fig. 7. Estimation of the signal $d(t)$ by the *SAS* observer.

condition $\left|\hat{d}^k(t)\right| < D$ with $e_y^k(t) = 0$ as shown by ❺. Consequently, the switching signal $\sigma(t)$, the continuous state $x(t)$ and the signal $d(t)$ can be estimated, as shown in Fig. 6.

The discontinuities in the estimated variables that appear in Figs. 6 and 7 occur because the initial condition of the estimated switching signal was $\hat{\sigma}(t_0) = 1$, thus the continuous state and the disturbance were estimated as $\hat{x}(t) = \hat{x}^1(t)$ and $\hat{d}(t) = \hat{d}^1(t)$, respectively. Once the evolving *AS* is detected this value was updated to $\hat{\sigma}(t_0) = 3$ and the estimates of the continuous state and the affecting disturbance were updated to $\hat{x}(t) = \hat{x}^3(t)$ and $\hat{d}(t) = \hat{d}^3(t)$. The estimate of the signal $d(t)$ by the *SAS* observer is also shown in Fig. 7.

6. Conclusions

Regarding observability of *SISO SAS's*, in this paper it has been shown that in the presence of disturbances every pair of *AS's* are always indistinguishable from the continuous output. Nevertheless, it has been demonstrated that by taking advantage of the knowledge on the disturbance bound, it would be possible to distinguish which is the evolving *AS*. By using such information, new distinguishability conditions have been introduced.

An observer scheme for *SISO SAS's* subject to unknown switching signals and unknown perturbations has been presented. It has been shown that the proposed observer can be effectively applied in the non-autonomous chaotic modulation, which is an attractive method for spread-spectrum secure communications [36], using *SAS's* with chaotic behavior.

Acknowledgments

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Appendix A

Proposition 7. *Let the initial conditions of the observer (12) be taken as zero and let the continuous initial condition of the SAS (1) be bounded by δ, i.e. $\|x_0\| < \delta$ with a known constant δ, as in Assumption 2. Then for every constant τ_k, the gains of (12) can be designed such that the estimation error (14) converges to the origin in a finite time lower than τ_k.*

Proof. Consider the error dynamics given in (14) which is finite-time stable. Take $\rho \geq 1$ and consider the time-scaling $\check{t} = t\rho$ together with the coordinate change $\epsilon = P\check{\epsilon}$ with $P = diag(1, \rho, \ldots, \rho^n)$ and $\check{\epsilon} = [\check{e}_1 \cdots \check{e}_n\ \check{e}_d]$ and $d = \rho^{n+1}\check{d}$. These transformations and time scaling take (14) into the following form:

$$\frac{d(\check{e}_1)}{d\check{t}} = \check{e}_2 - l_1|\check{e}_1|^{n/(n+1)}\text{sign}(\check{e}_1)$$
$$\vdots$$
$$\frac{d(\check{e}_n)}{d\check{t}} = \check{e}_d - l_n|\check{e}_1|^{1/(n+1)}\text{sign}(\check{e}_1)$$
$$\frac{d(\check{e}_d)}{d\check{t}} = \dot{\check{d}}(t) - l_{n+1}\text{sign}(\check{e}_1) \tag{A.1}$$

which does not depend on ρ. Therefore, (A.1) is finite-time stable and $\forall \delta > 0$, exists $\check{\tau}_d$ such that it holds

$$\check{\epsilon}(\check{t}, \check{\epsilon}_0) = 0, \quad \forall \check{\epsilon}_0 \text{ such that } \|\check{\epsilon}_0\| < \delta \text{ and } \forall \check{t} \geq \check{\tau}_d$$

where $\check{\epsilon}_0 = \check{\epsilon}(t_0)$. Going back to the original coordinates ϵ and the real time t, the above implies that

$$\epsilon(t, \epsilon_0) = 0, \quad \forall \epsilon_0 \text{ such that } \|\epsilon_0\| < \delta \text{ and } \forall t \geq \check{\tau}_d/\rho.$$

Indeed, the above implication is correct as the inequality $\|\epsilon_0\| < \delta$ clearly implies that $\|\check{\epsilon}_0\| < \delta$ due to the straightforward inequality

$$\|\check{\epsilon}\| \leq \|\epsilon\|, \quad \forall \rho \geq 1.$$

Therefore $\forall \delta, \tau_d > 0$ there exists $\rho(\delta, \tau_d)$ such that $\epsilon(t, \epsilon_0) = 0$, $\forall \epsilon_0$ such that $\|\epsilon_0\| < \delta$ and $\forall t \geq \tau_d$. □

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