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Rotation of 2D orthogonal polynomials

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ABSTRACT

Orientation-independent object recognition mostly relies on rotation invariants. Invariants from moments orthogonal on a square have favorable numerical properties but they are difficult to construct. The paper presents sufficient and necessary conditions, that must be fulfilled by 2D separable orthogonal polynomials, for being transformed under rotation in the same way as are the monomials. If these conditions have been met, the rotation property propagates from polynomials to moments and allows a straightforward derivation of rotation invariants. We show that only orthogonal polynomials belonging to a specific class exhibit this property. We call them Hermite-like polynomials.

1. Introduction

Rotation invariants play a key role in an orientation-invariant object description and recognition. Being a part of rigid-body transformation, object rotation is present almost in all applications, even if the imaging system has been well set up and the experiment has been prepared in a laboratory. Rotation is not trivial to handle mathematically, unlike for instance translation and scaling. For these two reasons, invariants to rotation have been in focus of researchers since the beginning. Invariants composed of image moments, *moment invariants*, belong to the most popular ones [1].

Moment invariants have been mostly constructed from *geometric moments*, which are projections of an image onto a standard monomial basis $x^p y^q$ [1]. A theory which allows to construct complete and independent set of rotation invariants of arbitrary order was proposed by Flusser [2,3]. However, geometric moments are not very suitable for practical applications since they suffer with a numerical instability and precision loss, which decreases the performance of moments of high orders [1]. To overcome that, several authors proposed to employ various *orthogonal (OG) moments* (i.e. moments with respect to certain orthogonal polynomial basis) instead.

In 2D, there exist two families of OG polynomials, which differ from one another by the area of orthogonality – polynomials orthogonal on a disc and polynomials orthogonal on a square/rectangle. The former group is inherently suitable for constructing rotation invariants, because these moments change under rotation in a simple way and the rotation parameter can be eliminated easily. This was noted for example by Teague [4], Khotanzad and Hong [5], and Wallin and Kubler [6] who used Zernike moments, and by other authors who employed pseudo-Zernike moments [7], Fourier–Mellin moments [8,9], Jacobi-Fourier moments [10], and Chebyshev-Fourier moments [11]. The negative aspect of using moments OG on a disc is that they require a mapping of the image into the disc, which is equivalent to image scaling and polar transformation. This operation leads to a precision loss due to the image resampling and also increases the computation time (both can be partially compensated by dedicated algorithms for moment computation, see [12] for instance). That is why some authors turned back to the moments OG on a square/rectangle.

Moments OG on a square can be calculated efficiently and precisely because the grid of the area of orthogonality is the same as the pixel grid of the image. In image processing literature, we can find many representatives of this group of moments. Legendre moments [13–15], Chebyshev moments [16–18], Hermite and Gaussian-Hermite moments [19,20], Krawtchouk moments [21], and Gegenbauer moments [22] are the most popular examples. However, construction of rotation invariants from these moments is generally very difficult. Even for low orders it leads to complicated clumsy formulas. This is why only few papers have followed this tedious approach. Yap et al. [21] did it for Krawtchouk moments, Hosny [15] and Deepika et al. [23] for Legendre moments. For higher orders, general forms of the invariants have not been published yet. Before 2011, the situation looked like a deadlock. We could use either OG moments on a disc on the expense of compu-

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tational precision or geometric moments, which are defined on a rectangular grid but which are not OG and hence unstable.

A significant breakthrough on this field was achieved by Yang et al. [24,25]. They discovered that 2D Hermite moments, which are OG on a square grid, offer a possibility of an easy and efficient design of rotation invariants and demonstrated this on low-order moments. Their elegant technique was later used to construct complete and independent set of rotation invariants of arbitrary orders [26] and was generalized even to 3D Hermite and Gaussian-Hermite moments [27]. Their method is based on the *Yang's Theorem*, which essentially says that Hermite polynomials change under an in-plane rotation exactly in the same way as do the monomials $x^p y^q$. Hence, if we have explicit formulas of rotation invariants from geometric moments (these formulas have been actually known thanks to [2]), it is sufficient just to replace the geometric moments with Hermite moments of the same degree and we end up with rotation invariants from OG moments. Since Hermite polynomials can be evaluated by recurrent relations, we can work with these invariants with an acceptable precision up to very high moment orders [26].

However, an important question has still remained open – are Hermite polynomials *the only OG polynomials* satisfying the rotation property described in the Yang's Theorem, or are there other 2D separable OG polynomials which provide the same possibility of construction of the invariants? In this paper, we answer this question completely. We present the proof that there exists a specific class of polynomials (we call them Hermite-like polynomials because they are in certain sense similar to Hermite polynomials), which are actually the only OG polynomials with this property.

2. Orthogonal polynomials under rotation

Let us first investigate how the monomials $\pi_{pq}(x, y) = x^p y^q$ are transformed under a coordinate rotation by angle θ. Rotation $(x, y) \to (\hat{x}, \hat{y})$ is given as

$$\begin{aligned} \hat{x} &= x\cos\theta - y\sin\theta \\ \hat{y} &= x\sin\theta + y\cos\theta. \end{aligned} \tag{1}$$

After a substitution and application of binomial formula, we obtain the monomial in the rotated coordinates

$$\pi_{pq}(\hat{x}, \hat{y}) = \\ = \sum_{n=0}^{p}\sum_{j=0}^{q}(-1)^n \binom{p}{n}\binom{q}{j}(\cos\theta)^{p-n+j}(\sin\theta)^{q-j+n}x^{p+q-n-j}y^{n+j}. \tag{2}$$

Grouping the variables of the same power together, Eq. (2) can be rewritten into the form

$$\pi_{pq}(\hat{x}, \hat{y}) = \sum_{r=0}^{p+q} k(r, p, q, \theta)x^{p+q-r}y^r, \tag{3}$$

where $k(r, p, q, \theta)$ is a coefficient given as a linear combination of certain powers of $\sin\theta$ and $\cos\theta$ (see [24] for detailed formulas and basic properties of $k(r, p, q, \theta)$).

Now let us move from the monomials $x^p y^q$ to bivariate polynomials $G_{pq}(x, y)$. In this paper, we consider solely 2D *separable* polynomials[1] We assume $G_{pq}(x, y)$ can be expressed as a product

$$G_{pq}(x, y) = G_p(x)G_q(y), \tag{4}$$

where $G_n(x)$ is a univariate polynomial of degree n.

We are particularly interested in the case when polynomials $G_p(x)$ form an orthogonal (possibly weighted orthogonal) system. Obviously, in such a case also the corresponding bivariate polynomials (4) are orthogonal. Due to Favard's Theorem [28], any symmetric OG polynomials[2] can be expressed by a *three-term recurrent relation* of the form

$$G_{p+1}(x) = a_p x G_p(x) - b_p G_{p-1}(x), \text{ for } p \geqslant 1 \tag{5}$$

with an initialization

$$\begin{aligned} G_0(x) &= c_0, \\ G_1(x) &= c_1 x, \end{aligned} \tag{6}$$

where all coefficients are real-valued, $c_0 \neq 0, c_1 \neq 0, a_p \neq 0$ and $b_p > 0$ for every $p \geq 1$. Conversely, any recurrent relation of this form generates symmetric OG polynomials.

The Favard's Theorem allows to work directly with recurrent relations (5) without loss of generality. All properties of the polynomials are determined by the coefficients. For example, the setting $c_0 = c_1 = 1, a_p = (2p+1)/(p+1), b_p = p/(p+1)$ yields Legendre polynomials; $c_0 = c_1 = 1, a_p = 2, b_p = 1$ leads to Chebyshev polynomials of the first kind; $c_0 = 1, c_1 = 2, a_p = 2, b_p = 1$ leads to Chebyshev polynomials of the second kind; and $c_0 = 1, c_1 = 2, a_p = 2, b_p = 2p$ yields Hermite polynomials (see [1] or [29] for more details and other examples).

Now we can proceed to formulate the central theorem of this paper, which introduces necessary and sufficient conditions for a "simple" (i.e. similar to monomials) transformation of OG polynomials under rotation.

Theorem 1. *Let a family of polynomials $G_p(x)$ be defined by recurrence (5) with initialization (6). Then bivariate polynomials $G_{pq}(x, y)$ (4) are transformed under rotation of the coordinates (1) as*

$$G_{pq}(\hat{x}, \hat{y}) = \sum_{r=0}^{p+q} k(r, p, q, \theta)G_{p+q-r}(x)G_r(y), \tag{7}$$

where $k(r, p, q, \theta)$ are from (3), if and only if it holds, for the recurrence coefficients, the following:

$$\begin{aligned} a_p &= \frac{c_1}{c_0} \equiv a, \\ b_p &= b \cdot p, \ b > 0, \text{ for any } p \geqslant 1. \end{aligned} \tag{8}$$

In other words, Theorem 1 says that if the recurrence has certain specific form, then the corresponding 2D OG polynomials are transformed under rotation exactly in the same way as do the monomials, and vice versa. For the proof of Theorem 1 see Appendix A.

Let us show what actually the constraints (8), imposed on the recurrence coefficients, mean. For $c_0 = 1, \ c_1 = a = b = 2$ we obtain exactly Hermite polynomials. Other choices of parameters c_0, a (resp. c_1) and b make a scaling of the variable x, which is the same for all degrees, and scaling of the values of $G_p(x)$, which, however, depends on p. Since this does not change the character of the polynomials, we call the polynomials satisfying (8) *Hermite-like polynomials*. Theorem 2 specifies these polynomials exactly.

Theorem 2. *Let a family of polynomials $P_n(x)$ satisfy (8) with $c_0 = a = b = 1$ and let a family of polynomials $G_n(x)$ satisfy (8) with an arbitrary setting of c_0, a, b; $b > 0$. Then these two polynomial families are linked with each other as*

$$G_n(x) = c_0\sqrt{b^n}P_n(ax/\sqrt{b}). \tag{9}$$

[1] For a short discussion on non-separable polynomials see Section 4.2.

[2] Favard's Theorem holds for general OG polynomials as well; in that case the first factor in (5) has the form $(a_p x + s_p)$ instead of just $a_p x$. Since we look for polynomials with the same rotation properties as $x^p y^q$, it is reasonable to limit ourselves to symmetric OG polynomials (i.e. $s_p = 0$ for any p) which have the same symmetry/antisymmetry as the monomials. Non-zero s_p's yield shifted OG polynomials which do not exhibit this property.

Polynomials $P_n(x)$ are sometimes called *probabilists' Hermite polynomials*. Polynomials $G_n(x)$ are orthogonal on $(-\infty, \infty)$ with respect to weighting function

$$w(x) = e^{-\frac{(ax)^2}{2b}}. \tag{10}$$

Applying Theorem 2 in a transitive manner, we may establish the link between any two polynomial families of this kind. For the proof of Theorem 2 see Appendix B.

3. Rotation invariants from OG moments

In this Section, we show how Theorem 1 can be used for an easy derivation of rotation invariants from OG moments. First, consider geometric moments of image *f*(*x*, *y*)

$$m_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y) dxdy. \tag{11}$$

Under rotation, geometric moments are transformed as

$$\hat{m}_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{x}^p \hat{y}^q f(x,y) dxdy. \tag{12}$$

Substituting from Eq. (3) we obtain

$$\begin{aligned}\hat{m}_{pq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{r=0}^{p+q} k(r,p,q,\theta) x^{p+q-r} y^r f(x,y) dxdy \\ &= \sum_{r=0}^{p+q} k(r,p,q,\theta) m_{p+q-r,r}.\end{aligned} \tag{13}$$

Rotation invariants are such functions of moments that eliminate rotation parameter θ. A consistent theory how to construct them was first proposed in [2], for a deeper insight and links to other approaches see [1]. The main conclusion is that an independent and complete set of rotation invariants from geometric moments can be designed as

$$\begin{aligned}\Phi_{pq} = &\left(\sum_{k=0}^{q_0}\sum_{j=0}^{p_0}\binom{q_0}{k}\binom{p_0}{j}(-1)^{p_0-j} i^{p_0+q_0-k-j} m_{k+j,p_0+q_0-k-j}\right)^{p-q} \\ &\cdot \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}(-1)^{q-j} i^{p+q-k-j} m_{k+j,p+q-k-j}\end{aligned} \tag{14}$$

where $p \geq q$ and p_0, q_0 are fixed user-defined indices (usually very low) such that $p_0 - q_0 = 1$.

If we have OG polynomials $G_{pq}(x, y)$ satisfying conditions (8), then, thanks to Theorem 1, we can only replace geometric moments in (14) with the corresponding OG moments

$$\eta_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_p(x) G_q(y) f(x,y) dxdy \tag{15}$$

and the invariance property of each Φ_{kj} is preserved.[3] On the other hand, Theorem 1 says that Hermite-like moments are the only moments[4] which offer this possibility. That underlines the prominent position of Hermite-like moments in image analysis.

4. Possible extensions

4.1. Extension to 3D

Due to a recent development of 3D imaging devices and technologies, which have become widely accessible, 3D rotation moment invariants started to attract an increasing attention of the researchers [30–36]. The problem of numerical instability of non-orthogonal moments appears in 3D even more seriously because it influences lower moment orders than in 2D. To overcome this, Yang et al. [27] proposed 3D rotation invariants from Gaussian-Hermite moments. They proved that the Yang's Theorem holds in 3D as well and is fully analogous to its 2D ancestor. Thanks to this, we can easily generalize Theorem 1 for the 3D case.

Theorem 3. *Let a family of polynomials $G_p(x)$ be defined by recurrence (5) with initialization (6). Then trivariate polynomials $G_{pqr}(x,y,z) = G_p(x)G_q(y)G_r(z)$ are transformed under rotation of the coordinates by the same coefficients as monomials $x^p y^q z^r$ if and only if the conditions (8) hold for the recurrence coefficients.*

The proof is via the same induction as in 2D, only more laborious. We do not repeat it in the paper. Theorem 2 of course holds regardless of the space dimension.

4.2. The case of non-separable polynomials

The question whether or not Theorem 1 can be extended and reformulated also for non-separable OG polynomials $G_{pq}(x, y)$ (i.e. those that cannot be expressed as a product of two univariate polynomials) is very difficult to answer. We should distinguish between *weakly* and *strongly* non-separable polynomials. Weakly non-separable polynomials can be made separable after the coordinates have been rotated by an appropriate angle. For example, the polynomials $x + y$ and $x - y$ are both weakly non-separable, because when rotating them by $\pi/4$ they become $\sqrt{2}x$ and $\sqrt{2}y$. For weakly non-separable polynomials Theorem 1 holds well, since we can transform them to a separable case by means of rotation.

For strongly non-separable polynomials the answer is unknown. We cannot modify Theorem 1 and follow its original proof, because it is based on the recurrent relations of 1D polynomials. The equivalence between recurrent relations and polynomials in 1D follows from Favard's theorem. However, no such theorem exists in 2D, to our best knowledge. To modify Theorem 1 for strongly non-separable polynomials, we would have to derive a 2D analogue of Favard's theorem, which is a quite challenging open problem. Our conjecture is that no strongly non-separable polynomials change as the monomials but we do not have a proof of this statement.

We hope that from practical point of view this is not a significant restriction. The use of strongly non-separable polynomials and their moments would increase the computing complexity while (probably) not bringing any advantages. Almost nobody has used non-separable polynomials for image analysis purposes; [37] is one of very few exceptions.

5. Conclusion

The paper presents sufficient and necessary conditions, that must be fulfilled by 2D OG polynomials, for being transformed under rotation in the same way as are the monomials. These conditions are given by Theorem 1, which is the main novel result of the paper. If these conditions have been met, the rotation property propagates from polynomials to moments and allows an effortless derivation of rotation invariants. We showed that only Hermite-like polynomials and moments exhibit this property.

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[3] For Gaussian-Hermite moments, this process is described in detail in [26].

[4] For practical purposes, Hermite-like moments may be weighted and normalized to ensure reasonable dynamic range of the invariants but this does not violate their rotation properties.

Appendix A. Proof of Theorem 1

Theorem 1 has a form of an equivalence. To prove it in full, we first prove that the constraints (8) imposed on the parameters are sufficient and then that they are also necessary. The proof is via mathematical induction.

Proof of sufficiency

Let us prove that the validity of (8) implies the validity of (7).

- **Initial step**
Eq. (7) holds trivially for $(p, q) = (0, 0)$. For $(p, q) = (1, 0)$ we have

$$\pi_{10}(\hat{x}, \hat{y}) = \hat{x} = x\cos\theta - y\sin\theta,$$

from where we can see the coefficients $k(r, 1, 0, \theta), r = 0, 1,$ and the validity of (7) can be verified immediately. For $(p, q) = (1, 1)$ we have on one hand

$$\begin{aligned} G_{11}(\hat{x}, \hat{y}) &= G_1(\hat{x})G_1(\hat{y}) \\ &= c_1^2\sin\theta\cos\theta x^2 + c_1^2\cos^2\theta xy \\ &\quad - c_1^2\sin^2\theta xy - c_1^2\sin\theta\cos\theta y^2. \end{aligned} \tag{16}$$

On the other hand,

$$\begin{aligned} &\sum_{r=0}^{2} k(r, 1, 1, \theta)G_{2-r}(x)G_r(y) \\ &= c_1^2\sin\theta\cos\theta x^2 + c_1^2\cos^2\theta xy \\ &- c_1^2\sin^2\theta xy - c_1^2\sin\theta\cos\theta y^2. \end{aligned} \tag{17}$$

Hence, Eq. (7) holds for the initial conditions.
- **Induction step**
Assuming Eq. (7) holds for some positive integers (p, q), let us prove the validity for $(p, q+1)$. We can rewrite the left-hand side of Eq. (7) to the following form:

$$\begin{aligned} \Phi_{left} &= G_p(\hat{x})G_{q+1}(\hat{y}) \\ &= G_p(\hat{x})\left(\frac{c_1}{c_0}\hat{y}G_q(\hat{y}) - bqG_{q-1}(\hat{y})\right) \\ &= \frac{c_1}{c_0}(x\sin\theta + y\cos\theta)\sum_{r=0}^{p+q} k(r, p, q, \theta)G_{p+q-r}(x)G_r(y) \\ &\quad - bq\sum_{r=0}^{p+q-1} k(r, p, q-1, \theta)G_{p+q-r-1}(x)G_r(y). \end{aligned} \tag{18}$$

The right-hand side Φ_{right} can be expressed by means of *Lemma 1* from [24] (the Lemma shows the properties of coefficients $k(r, p, q, \theta)$) as

$$\begin{aligned} \Phi_{right} &= \sum_{r=0}^{p+q+1} k(r, p, q+1, \theta)G_{p+q+1-r}(x)G_r(y) \\ &= \sin\theta\sum_{r=0}^{p+q} k(r, p, q, \theta)G_{p+q+1-r}(x)G_r(y) \\ &\quad + \cos\theta\sum_{r=1}^{p+q+1} k(r-1, p, q, \theta)G_{p+q+1-r}(x)G_r(y). \end{aligned} \tag{19}$$

Since

$$\begin{aligned} &\sum_{r=1}^{p+q+1} k(r-1, p, q, \theta)G_{p+q+1-r}(x)G_r(y) \\ &= \sum_{r=0}^{p+q} k(r, p, q, \theta)G_{p+q-r}(x)G_{r+1}(y) \end{aligned} \tag{20}$$

due to the index shift, we have

$$\begin{aligned} \Phi_{right} &= \sin\theta\sum_{r=0}^{p+q} k(r, p, q, \theta)G_{p+q+1-r}(x)G_r(y) \\ &\quad + \cos\theta\sum_{r=0}^{p+q} k(r, p, q, \theta)G_{p+q-r}(x)G_{r+1}(y). \end{aligned} \tag{21}$$

Substituting the recurrence relations for $G_{p+q+1-r}(x)$ and $G_{r+1}(y)$ in Eq. (21) yields

$$\begin{aligned} \Phi_{right} &= \frac{c_1}{c_0}(x\sin\theta + y\cos\theta)\sum_{r=0}^{p+q} k(r, p, q, \theta)G_{p+q-r}(x)G_r(y) \\ &\quad - b\sin\theta\sum_{r=0}^{p+q}(p+q-r)k(r, p, q, \theta)G_{p+q-1-r}(x)G_r(y) \\ &\quad - b\cos\theta\sum_{r=0}^{p+q} rk(r, p, q, \theta)G_{p+q-r}(x)G_{r-1}(y). \end{aligned} \tag{22}$$

The second and third terms of (22) vanish for $r = p + q$ and $r = 0$, respectively. So, the summation in the second term goes only to $r = p + q - 1$ and the summation of the third term goes from $r = 1$. Incorporating this into (22) and using *Lemma 2* from [24] to simplify the sums, we obtain

$$\Phi_{left} = \Phi_{right},$$

which completes the induction step.
To complete the proof of sufficiency, we should repeat the induction also over p. That is, however, the same as for q due to the symmetry of the problem. The only change is that we employ Lemmas 3 and 4 from [24] instead of Lemmas 1 and 2 which have been used above. We do not repeat the proof for p here.

Proof of necessity

Assuming Eq. (7) holds for any p and q, we derive the constraints (8) on parameters a_p and b_p via induction.

- **Initial step**
Let us calculate $G_{11}(\hat{x}, \hat{y})$. On one hand, via direct calculation, it equals $c_1^2\hat{x}\hat{y}$, which can be further expanded using (1). On the other hand, using the assumption of Theorem 1, we have

$$G_{11}(\hat{x}, \hat{y}) = \sum_{r=0}^{2} k(r, 1, 1, \theta)G_{2-r}(x)G_r(y). \tag{23}$$

Comparing the coefficients of x^2 leads to the constraint

$$a_1 = \frac{c_1}{c_0} \tag{24}$$

while b_1 may be an arbitrary positive real number.
- **Induction step**
We assume Theorem 1 valid for certain $p \geq 1$, i.e. we assume

$$a_p = \frac{c_1}{c_0} \text{ and } b_p = pb,\ b > 0. \tag{25}$$

Let us again express the left-hand side of (7) in the form of (18). The second term of (18) can be expanded by means of

Lemma 2 from [24] (the Lemma is about the properties of the coefficient $k(r, p, q, \theta)$) as

$$qb\sum_{r=0}^{p+q-1} k(r,p,q-1,\theta)G_{p+q-1-r}(x)G_r(y)$$
$$= b\sin\theta\sum_{r=0}^{p+q-1}(p+q-r)k(r,p,q,\theta)G_{p+q-1-r}(x)G_r(y)$$
$$+ b\cos\theta\sum_{r=0}^{p+q-1}(r+1)k(r+1,p,q,\theta)G_{p+q-1-r}(x)G_r(y)$$
$$= b\sin\theta\sum_{r=0}^{p+q}(p+q-r)k(r,p,q,\theta)G_{p+q-1-r}(x)G_r(y)$$
$$+ b\cos\theta\sum_{r=0}^{p+q} rk(r,p,q,\theta)G_{p+q-r}(x)G_{r-1}(y). \quad (26)$$

Hence,

$$G_{p,q+1}(\hat{x},\hat{y}) = \frac{c_1}{c_0}(x\sin\theta + y\cos\theta)\sum_{r=0}^{p+q}k(r,p,q,\theta)G_{p+q-r}(x)G_r(y)$$
$$- b\sum_{r=0}^{p+q}\sin\theta(p+q-r)k(r,p,q,\theta)G_{p+q-1-r}(x)G_r(y)$$
$$- b\sum_{r=0}^{p+q}\cos\theta rk(r,p,q,\theta)G_{p+q-r}(x)G_{r-1}(y). \quad (27)$$

The right-hand side Φ_{right} of Eq. (7) can be rewritten to Eq. (21). Substituting the recurrence relations for $G_{p+q+1-r}(x)$ and $G_{r+1}(y)$ in Eq. (21) yields

$$\Phi_{right} = x\sin\theta\sum_{r=0}^{p+q}a_{p+q-r}k(r,p,q,\theta)G_{p+q-r}(x)G_r(y)$$
$$+ y\cos\theta\sum_{r=0}^{p+q}a_r k(r,p,q,\theta)G_{p+q-r}(x)G_r(y)$$
$$- \sin\theta\sum_{r=0}^{p+q}b_{p+q-r}k(r,p,q,\theta)G_{p+q-1-r}(x)G_r(y)$$
$$- \cos\theta\sum_{r=0}^{p+q}b_r k(r,p,q,\theta)G_{p+q-r}(x)G_{r-1}(y). \quad (28)$$

Comparing the coefficients of terms $x\sin\theta$, $y\cos\theta$, $\sin\theta$ and $\cos\theta$ between (27) and (28) leads to the constraints

$$a_r = \frac{c_1}{c_0}$$
$$b_r = rb, \quad (29)$$

for any $r \leqslant p+q$.

Theorem 1 has been proven completely.

Appendix B. Proof of Theorem 2

The proof is via mathematical induction over degree n.

- **Initial step**
  For $n = 0, 1, 2$ we have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - 1$ and
  $G_0(x) = c_0$, $G_1(x) = c_0 ax$, $G_2(x) = c_0(ax)^2 - c_0 b$.
  The validity of Theorem 2 is evident.
- **Induction step**
  Assuming Theorem 2 is valid up to certain degree n, we prove it for $n+1$. We have to prove that

$$G_{n+1}(x) = c_0\sqrt{b^{n+1}}P_{n+1}(ax/\sqrt{b}). \quad (30)$$

The left-hand side of (30) can be expanded using the recurrence as

$$G_{n+1}(x) = axG_n(x) - bnG_{n-1}(x), \quad (31)$$

which can be further rewritten, by means of the induction assumption, into the form

$$G_{n+1}(x) = c_0ax\sqrt{b^n}P_n(ax/\sqrt{b}) - c_0bn\sqrt{b^{n-1}}P_{n-1}(ax/\sqrt{b})$$
$$= c_0\sqrt{b^{n+1}}[(ax/\sqrt{b})P_n(ax/\sqrt{b}) - nP_{n-1}(ax/\sqrt{b})]. \quad (32)$$

On the other hand, the right-hand side of (30) can be expressed by recurrence of P_{n+1} as

$$c_0\sqrt{b^{n+1}}P_{n+1}(ax/\sqrt{b})$$
$$= c_0\sqrt{b^{n+1}}[(ax/\sqrt{b})P_n(ax/\sqrt{b}) - nP_{n-1}(ax/\sqrt{b})], \quad (33)$$

which is the same as (32). The proof of Theorem 2 has been completed.

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