Weak lower semicontinuity by means of anisotropic parametrized measures

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Abstract It is well known that besides oscillations, sequences bounded only in L^1 can also develop concentrations, and if the latter occurs, we can at most hope for weak* convergence in the sense of measures. Here we derive a new tool to handle mutual interferences of an oscillating and concentrating sequence with another weakly converging sequence. We introduce a couple of explicit examples showing a variety of possible kinds of behavior and outline some applications in Sobolev spaces.

1 Introduction

Mutual interactions of oscillations and concentrations appears in many problems of optimal control and calculus of variations. We refer, for example, to [23, 7] for optimal control of dynamical systems with oscillations and concentrations, or to [24] for a model of mechanical debonding. Analytical problems related to these phenomena in the calculus of variations are described in detail in [6]. Moreover, oscillations, concentrations, and discontinuities naturally appear in problems of the variational calculus where one is interested in weak lower semicontinuity in the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$ for a sufficiently regular domain $\Omega\subset\mathbb{R}^n$ and $m,n\geq 1$. Indeed, consider

$$I(u) := \int_\Omega h(x,u(x),\nabla u(x))\,\mathrm{d}x\ , \tag{1}$$

where $h:\bar{\Omega}\times\mathbb{R}^m\times\mathbb{R}^{m\times n}\to\mathbb{R}$ is continuous and such that $|h(x,r,s)|\leq C(1+|r|^q+|s|^p)$ for some $C>0$, $p>1$, and $q\geq 1$ so small that $W^{1,p}(\Omega;\mathbb{R}^m)$ compactly embeds into $L^q(\Omega;\mathbb{R}^m)$. We would like to point out that such integrands also appear in analysis of mechanical problems [27, 28]. If one wants to investigate lower semicontinuity of I with respect to the weak topology in $W^{1,p}(\Omega;\mathbb{R}^m)$, a usual way is to show first that

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$$\lim_{k\to\infty}\int_\Omega h(x,u(x),\nabla u_k(x))\,\mathrm{d}x = \lim_{k\to\infty}\int_\Omega h(x,u_k(x),\nabla u_k(x))\,\mathrm{d}x\ . \tag{2}$$

for a suitable sequence $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, and then to prove that the left-hand side of (2) is bounded from below by $\int_\Omega h(x,u(x),\nabla u(x))\,\mathrm{d}x$. That, however, is not possible without some additional assumptions on h or $\{u_k\}$. We refer to [1] or [5] for such cases. Indeed, if $p\le n$ then u and u_k, $k\in\mathbb{N}$, are not necessarily continuous and if $\{|\nabla u_k|^p\}$ is not uniformly integrable then concentrations can interact with $\{u_k\}_{k\in\mathbb{N}}$. This phenomenon is clearly visible in the following example.

Example 1. Consider $\Omega = B(0,1)$, the unit ball in $\mathbb{R}^n$ centered at the origin, a mapping $w\in W_0^{1,p}(B(0,1);\mathbb{R}^m)$, $p>1$, extended by zero to the whole space and $u_k(x) := k^{n/p-1}w(kx)$. Hence $u_k \rightharpoonup u := 0$ in $W^{1,p}(B(0,1);\mathbb{R}^m)$ as $k\to\infty$. Assume that h as above is positively p-homogeneous in the last variable, i.e., $h(x,r,\alpha s) = \alpha^p h(x,r,s)$, for all (x,r,s) admissible and all $\alpha\ge 0$. Then a simple calculation yields

$$\begin{aligned}
&\liminf_{k\to\infty}\int_{B(0,1)} h(x,u_k(x),\nabla u_k(x))\,\mathrm{d}x = \liminf_{k\to\infty}\int_{B(0,1)} k^n h(x,k^{n/p-1}w(kx),\nabla w(kx))\,\mathrm{d}x\\
&= \liminf_{k\to\infty}\int_{B(0,1)} h(\frac{y}{k},k^{n/p-1}w(y),\nabla w(y))\,\mathrm{d}y\\
&= \begin{cases} \int_{B(0,1)} h(0,w(y),\nabla w(y))\,\mathrm{d}y & \text{if } p=n,\\ \int_{B(0,1)} h(0,0,\nabla w(y))\,\mathrm{d}y & \text{if } p>n,\\ \liminf_{k\to\infty}\int_{B(0,1)} h(y/k,k^{n/p-1}w(y),\nabla w(y))\,\mathrm{d}y & \text{if } p<n. \end{cases}
\end{aligned} \tag{3}$$

We see that if $p>n$ then (2) really holds. On the other hand, if $p=n$ the map u appears in the limit besides its gradient and the most complex case is $p<n$ where the limit cannot be calculated explicitly. Notice that the sequence $\{|\nabla u_k|^p\}_{k\in\mathbb{N}}\subset L^1(\Omega)$ is uniformly bounded in this space and concentrates at $x=0$, i.e., $|\nabla u_k|^p \overset{*}{\rightharpoonup} \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^m)}\delta_0$ in $\mathscr{M}(\overline{B(0,1)})$ as $k\to\infty$. Here δ_0 denotes the Dirac measure supported at the origin and $\mathscr{M}(\overline{B(0,1)})$ denotes the set of Radon measures on $\overline{B(0,1)}$.

If $p=1$, concentrations of the gradient can even interact with jump discontinuities.

Example 2. Consider $\Omega=(0,1)$ and a sequence $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,1}(-1,1)$ such that $u_k\to u$ in $L^q(-1,1)$ for every $1\le q<+\infty$. We are interested in

$$\lim_{k\to\infty}\int_{-1}^1 f(u_k(x))\,\psi(u_k'(x))\,\mathrm{d}x$$

for continuous function ψ such that with $|\psi|\le C(1+|\cdot|)$ with some constant $C>0$ and continuous $f:\mathbb{R}\to\mathbb{R}$. If ψ is the identity map then the calculation is easy, namely the limit equals $\liminf_{k\to\infty}(F(u_k(1))-F(u_k(-1)))$ where F is the primitive of f. In case of more general ψ, the situation is more involved. Let

$$u_k(x) := \begin{cases} 0 & \text{if } -1\le x\le 0,\\ kx & \text{if } 0\le x\le 1/k,\\ 1 & \text{if } 1/k\le x\le 1. \end{cases}$$

Assume further that $\lim_{t\to\infty}\psi(t)/t$ exists. Then it is easy to see that

$$\lim_{k\to\infty}\int_{-1}^1 f(u_k(x))\,\psi(u_k'(x))\,\mathrm{d}x = (f(0)+f(1))\,\psi(0) + \Big(\int_0^1 f(x)\,\mathrm{d}x\Big)\lim_{k\to\infty}\frac{\psi(k)}{k}\ . \tag{4}$$

The sequence of $\{u_k'\}_{k\in\mathbb{N}}$ concentrates at zero which is exactly the point of discontinuity of the pointwise limit of $\{u_k\}_{k\in\mathbb{N}}$ which we denote by u. Also notice that $u_k' \overset{*}{\rightharpoonup} \delta_0$ in

$\mathscr{M}([-1,1])$ for $k \to \infty$. Hence, the second term on the right-hand side of (4) suggests that we should refine the definition of u at zero by saying that $u(0)$ is the Lebesgue measure supported on the interval of the jump of u, i.e., on the interval $(0,1)$.

In this contribution, we introduce a new tool which allows us to describe limits of nonlinear maps along sequences that oscillate, concentrate, and concentrations possibly interfere with discontinuities. While oscillations are successfully treated by Young measures [34] or [4], to handle oscillations and concentrations require finer tools as in, e.g., Young measures and varifolds [2] or DiPerna-Majda measures [9]. We also refer to [22] for an explicit characterization of the DiPerna-Majda measures and to [13, 16] for characterization of those measures which are generated by sequences of gradients, as well as to [19] and [3] for related results in case $p = 1$.

1.1 Basic notation

Let us start with a few definitions and with the explanation of our notation. If not said otherwise, we will assume throughout this article that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary. Furthermore, $C(\Omega;\mathbb{R}^m)$ (respectively $C(\bar{\Omega};\mathbb{R}^m)$) is the space of continuous functions defined on Ω (respectively $bar\Omega$) with values in $\mathbb{R}^m$. Here, as well as in smilar notation for other function spaces, if the dimension of the target space is $m = 1$, then $\mathbb{R}^m$ is omitted and we only write $C(\Omega)$. In what follows $\mathscr{M}(S)$ denotes the set of regular countably additive set functions on the Borel σ-algebra on a metrizable set S (cf. [10]), its subset, $\mathscr{M}_1^+(S)$, denotes regular probability measures on a set S. We write "γ-almost all" or "γ-a.e." if we mean "up to a set with the γ-measure zero". If γ is the n-dimensional Lebesgue measure we omit writing γ in the notation. The support of a measure $\sigma \in \mathscr{M}(\Omega)$ is the smallest closed set S such that $\sigma(A) = 0$ if $S \cap A = \emptyset$. If $\sigma \in \mathscr{M}(\bar{\Omega})$ we write σ_s and d_σ for the singular part and density of σ defined by the Lebesgue decomposition (with respect to the Lebesgue measure), respectively. By $L^p(\Omega;\mathbb{R}^m)$ we denote the usual Lebesgue space of $\mathbb{R}^m$-valued maps. Further, $W^{1,p}(\Omega;\mathbb{R}^m)$ where $1 \le p \le +\infty$ denotes the usual Sobolev space (of $\mathbb{R}^m$-valued functions) and $W_0^{1,p}(\Omega;\mathbb{R}^m)$ denotes the completion of $C_0^\infty(\Omega,\mathbb{R}^m)$ (smooth functions with support in Ω) in $W^{1,p}(\Omega;\mathbb{R}^m)$. We say that Ω has the extension property in $W^{1,p}$ if every function $u \in W^{1,p}(\Omega)$ can be extended outside Ω to $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ and the extension operator is linear and bounded. If Ω is an arbitrary domain and $u, w \in W^{1,p}(\Omega,\mathbb{R}^m)$ we say that $u = w$ on $\partial\Omega$ if $u - w \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. We denote by 'w-lim' or by $\rightharpoonup$ the weak limit. Analogously we indicate weak* limits by $\stackrel{*}{\rightharpoonup}$.

1.2 Quasiconvex functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that a function $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if for any $s_0 \in \mathbb{R}^{m\times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$

$$\psi(s_0)|\Omega| \le \int_\Omega \psi(s_0 + \nabla\varphi(x))\,\mathrm{d}x\,.$$

If $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ is not quasiconvex we define its quasiconvex envelope $Q\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ as

$$Q\psi(s) = \sup\{h(s);\ h \le \psi;\ h : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex }\} \tag{5}$$

and we put $Q\psi = -\infty$ if the set on the right-hand side of (5) is empty. If ψ is locally bounded and Borel measurable then for any $s_0 \in \mathbb{R}^{m\times n}$ (see [8])

$$Q\psi(s_0) = \inf_{\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \frac{1}{|\Omega|}\int_\Omega \psi(s_0+\nabla\varphi(x))\,\mathrm{d}x\,. \tag{6}$$

1.3 Young measures

For $p \geq 0$ we define the following subspace of the space $C(\mathbb{R}^{m\times n})$ of all continuous functions on $\mathbb{R}^{m\times n}$:

$$C_p(\mathbb{R}^{m\times n}) = \{\psi \in C(\mathbb{R}^{m\times n});\psi(s) = o(|s|^p) \text{ for } |s|\to\infty\}\ ,$$

with the obvious modification for any Euclidean space instead of $\mathbb{R}^{m\times n}$. The Young measures on a measurable set $\Lambda \subset \mathbb{R}^l$ are weakly* measurable mappings $x \mapsto \nu_x : \Lambda \to \mathscr{M}(\mathbb{R}^{m\times n})$ with values in probability measures; and the adjective "weakly* measurable" means that, for any $\psi \in C_0(\mathbb{R}^{m\times n})$, the mapping $\Lambda \to \mathbb{R} : x \mapsto \langle \nu_x,\psi\rangle = \int_{\mathbb{R}^{m\times n}} \psi(s)\nu_x(\mathrm{d}s)$ is measurable in the usual sense. Let us remind that, by the Riesz theorem the space $\mathscr{M}(\mathbb{R}^{m\times n})$, normed by the total variation, is a Banach space which is isometrically isomorphic with $C_0(\mathbb{R}^{m\times n})^*$. Let us denote the set of all Young measures by $\mathscr{Y}(\Lambda;\mathbb{R}^{m\times n})$.

Below, we are mostly interested in the case $\Lambda = \Omega$, i.e., a bounded domain. It is known that $\mathscr{Y}(\Omega;\mathbb{R}^{m\times n})$ is a convex subset of $L^\infty_{\mathrm{w}*}(\Omega;\mathscr{M}(\mathbb{R}^{m\times n})) \cong L^1(\Omega;C_0(\mathbb{R}^{m\times n}))^*$, where the index "$w*$" indicates the property "weakly* measurable". A classical result [34] is that, for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^\infty(\Omega;\mathbb{R}^{m\times n})$, there exists its subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathscr{Y}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\forall\psi \in C_0(\mathbb{R}^{m\times n}) : \quad \lim_{k\to\infty} \psi\circ y_k = \psi_\nu \qquad \text{weakly* in } L^\infty(\Omega)\ , \tag{7}$$

where $[\psi\circ y_k](x) = \psi(y_k(x))$ and

$$\psi_\nu(x) = \int_{\mathbb{R}^{m\times n}} \psi(s)\nu_x(\mathrm{d}s)\ . \tag{8}$$

Let us denote by $\mathscr{Y}^\infty(\Omega;\mathbb{R}^{m\times n})$ the set of all Young measures which are created by this way, i.e. by taking all bounded sequences in $L^\infty(\Omega;\mathbb{R}^{m\times n})$. Note that (7) actually holds for any $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ continuous.

A generalization of this result was formulated by Schonbek [32] (cf. also [4]): if $1 \leq p < +\infty$: for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ there exists its subsequence (denoted by the same indices) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathscr{Y}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\forall\psi \in C_p(\mathbb{R}^{m\times n}) : \quad \lim_{k\to\infty} \psi\circ y_k = \psi_\nu \qquad \text{weakly in } L^1(\Omega)\ . \tag{9}$$

We say that $\{y_k\}$ generates ν if (9) holds. Let us denote by $\mathscr{Y}^p(\Omega;\mathbb{R}^{m\times n})$ the set of all Young measures which are created by this way, i.e. by taking all bounded sequences in $L^p(\Omega;\mathbb{R}^{m\times n})$. The subset of $\mathscr{Y}^p(\Omega;\mathbb{R}^{m\times n})$ containing Young measures generated by gradients of $W^{1,p}(\Omega;\mathbb{R}^m)$ maps will be denoted by $\mathscr{G}\mathscr{Y}^p(\Omega;\mathbb{R}^{m\times n})$. An explicit characterization of this set is due to Kinderlehrer and Pedregal [18, 17].

1.4 DiPerna-Majda measures

1.4.1 Definition and basic properties

Let $\mathscr{R}$ be a complete (i.e. containing constants, separating points from closed subsets and closed with respect to the Chebyshev norm) separable ring of continuous bounded functions $\mathbb{R}^{m\times n}\to\mathbb{R}$. It is known [11, Sect. 3.12.21] that there is a one-to-one correspondence $\mathscr{R}\leftrightarrow\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$ between such rings and metrizable compactifications of $\mathbb{R}^{m\times n}$; by a compactification we mean here a compact set, denoted by $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$, into which $\mathbb{R}^{m\times n}$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^{m\times n}$ and its image in $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$. Similarly, we will not distinguish between elements of $\mathscr{R}$ and their unique continuous extensions defined on $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$. This means that if $i:\mathbb{R}^{m\times n}\to\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$ is the homeomorphic embedding and $\psi_0\in\mathscr{R}$ then the same notation is used also for $\psi_0\circ i^{-1}: i(\mathbb{R}^{m\times n})\to\mathbb{R}$ and for its unique continuous extension to $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$.

Let $\sigma\in\mathscr{M}(\bar\Omega)$ be a positive Radon measure on a closure of a bounded domain $\Omega\subset\mathbb{R}^n$. A mapping $\hat\nu: x\mapsto\hat\nu_x$ belongs to the space $L^\infty_{\mathrm{w}*}(\bar\Omega,\sigma;\mathscr{M}(\beta_{\mathscr{R}}\mathbb{R}^{m\times n}))$ if it is weakly* σ-measurable (i.e., for any $\psi_0\in C_0(\mathbb{R}^{m\times n})$, the mapping $\bar\Omega\to\mathbb{R}: x\mapsto\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\psi_0(s)\hat\nu_x(\mathrm{d}s)$ is σ-measurable in the usual sense). If additionally $\hat\nu_x\in\mathscr{M}_1^+(\beta_{\mathscr{R}}\mathbb{R}^{m\times n})$ for σ-a.a. $x\in\bar\Omega$ the collection $\{\hat\nu_x\}_{x\in\bar\Omega}$ is the so-called Young measure on $(\bar\Omega,\sigma)$ ([34], see also [4, 31]).

DiPerna and Majda [9] shown that having a bounded sequence in $L^p(\Omega;\mathbb{R}^{m\times n})$ with $1\le p<+\infty$ defined on an open domain $\Omega\subseteq\mathbb{R}^n$, there exists its subsequence (denoted by the same indices) a positive Radon measure $\sigma\in\mathscr{M}(\bar\Omega)$ and a Young measure $\hat\nu: x\mapsto\hat\nu_x$ on $(\bar\Omega,\sigma)$ such that $(\sigma,\hat\nu)$ is attainable by a sequence $\{y_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{m\times n})$ in the sense that $\forall g\in C(\bar\Omega)$ and $\forall\psi_0\in\mathscr{R}$:

$$\lim_{k\to\infty}\int_\Omega g(x)\psi(y_k(x))\mathrm{d}x=\int_{\bar\Omega}g(x)\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\psi_0(s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x)\ , \tag{10}$$

where

$$\psi\in\Upsilon^p_{\mathscr{R}}(\mathbb{R}^{m\times n}):=\{\psi_0(1+|\cdot|^p);\ \psi_0\in\mathscr{R}\}. \tag{11}$$

In particular, putting $\psi_0\equiv 1\in\mathscr{R}$ in (10) we can see that

$$\lim_{k\to\infty}(1+|y_k|^p)\ =\ \sigma\quad\text{ weakly* in }\ \mathscr{M}(\bar\Omega)\ . \tag{12}$$

If (10) holds, we say that $\{y_k\}_{\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$. Let us denote by $\mathscr{DM}^p_{\mathscr{R}}(\Omega;\mathbb{R}^{m\times n})$ the set of all pairs $(\sigma,\hat\nu)\in\mathscr{M}(\bar\Omega)\times L^\infty_{\mathrm{w}*}(\bar\Omega,\sigma;\mathscr{M}(\beta_{\mathscr{R}}\mathbb{R}^{m\times n}))$ attainable by sequences from $L^p(\Omega;\mathbb{R}^{m\times n})$; note that, taking $\psi_0=1$ in (10), one can see that these sequences must be inevitably bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$.

It is well known [31] that (10) can also be rewritten with the help of classical Young measures as

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega g(x)\psi(y_k(x))\mathrm{d}x=&\int_\Omega\int_{\mathbb{R}^{m\times n}}g(x)\psi(s)\nu_x(\mathrm{d}s)\mathrm{d}x\\&+\int_{\bar\Omega}g(x)\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\psi_0(s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x),\end{aligned} \tag{13}$$

where $\{\nu_x\}_{x\in\Omega}\in\mathscr{Y}^\infty(\Omega,\mathbb{R}^{m\times n})$ and $\{\nu_x\}_{x\in\Omega}$ are as in (10).

There are two prominent examples of compactifications of $\mathbb{R}^{m\times n}$. The simplest example is the so-called one point compactification which corresponds to the ring of continuous bounded functions which have limits if the norm of its argument tends to infinity, i.e., we denote $\psi_0(\infty):=\lim_{|s|\to+\infty}\psi_0(s)$.

A richer compactification is the one by the sphere. In that case, we consider the following ring of continuous bounded functions:

$$\mathscr{S} := \big\{ \psi_0 \in C(\mathbb{R}^{m\times n}) :\ \text{there exist } c \in \mathbb{R}\ ,\ \psi_{0,0} \in C_0(\mathbb{R}^{m\times n}),\ \text{and } \psi_{0,1} \in C(S^{(m\times n)-1}) \text{ s.t.}$$
$$\psi_0(s) = c + \psi_{0,0}(s) + \psi_{0,1}\left(\frac{s}{|s|}\right)\frac{|s|^p}{1+|s|^p} \text{ if } s \neq 0 \text{ and } \psi_0(0) = \psi_{0,0}(0)\big\}\ , \qquad (14)$$

where $S^{m\times n-1}$ denotes the $(mn-1)$-dimensional unit sphere in $\mathbb{R}^{m\times n}$. Then $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$ is homeomorphic to the unit ball $\overline{B(0,1)} \subset \mathbb{R}^{m\times n}$ via the mapping $d : \mathbb{R}^{m\times n} \to B(0,1)$, $d(s) := s/(1+|s|)$ for all $s \in \mathbb{R}^{m\times n}$. Note that $d(\mathbb{R}^{m\times n})$ is dense in $\overline{B(0,1)}$.

The following proposition from [22] explicitly characterizes the set of DiPerna-Majda measures $\mathscr{DM}^p_{\mathscr{R}}(\Omega;\mathbb{R}^{m\times n})$.

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain such that $|\partial\Omega| = 0$, $\mathscr{R}$ be a separable complete subring of the ring of all continuous bounded functions on $\mathbb{R}^{m\times n}$ and $(\sigma,\hat\nu) \in \mathscr{M}(\bar\Omega)\times L^\infty_{\rm w}(\bar\Omega,\sigma;\mathscr{M}(\beta_{\mathscr{R}}\mathbb{R}^{m\times n}))$ and $1 \le p < +\infty$. Then the following two statements are equivalent with each other:*

(i) *the pair $(\sigma,\hat\nu)$ is the DiPerna-Majda measure, i.e. $(\sigma,\hat\nu) \in \mathscr{DM}^p_{\mathscr{R}}(\Omega;\mathbb{R}^{m\times n})$,*

(ii) *The following properties are satisfied simultaneously:*

1. *σ is positive,*
2. *$\sigma_{\hat\nu} \in \mathscr{M}(\bar\Omega)$ defined by $\sigma_{\hat\nu}({\rm d}x) = (\int_{\mathbb{R}^{m\times n}} \hat\nu_x({\rm d}s))\sigma({\rm d}x)$ is absolutely continuous with respect to the Lebesgue measure ($d_{\sigma_{\hat\nu}}$ will denote its density),*
3. *for a.a. $x \in \Omega$ it holds*
$$\int_{\mathbb{R}^{m\times n}} \hat\nu_x({\rm d}s) > 0, \qquad d_{\sigma_{\hat\nu}}(x) = \left(\int_{\mathbb{R}^{m\times n}} \frac{\hat\nu_x({\rm d}s)}{1+|s|^p}\right)^{-1} \int_{\mathbb{R}^{m\times n}} \hat\nu_x({\rm d}s)\ ,$$
4. *for σ-a.a. $x \in \bar\Omega$ it holds*
$$\hat\nu_x \ge 0, \qquad \int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} \hat\nu_x({\rm d}s) = 1\ .$$

Remark 1. Consider a metrizable compactification $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$ of $\mathbb{R}^{m\times n}$ and the corresponding separable complete closed ring $\mathscr{R}$ with its dense subset $\{\psi_k\}_{k\in\mathbb{N}}$. We take a bounded continuous function $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$, $\psi \notin \mathscr{R}$ and take a closure (in the Chebyshev norm) of all the products of elements from $\{\psi\}\cup\{\psi_k\}_{k\in\mathbb{N}}$. The corresponding ring is again separable and the corresponding compactification is metrizable but strictly finer than $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$.

The following result can be found in [16] and its extension in [21]. Here and in the sequel d_σ denotes density of the absolutely continuous part of σ with respect to the Lebesgue measure $\mathscr{L}^n$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the extension property in $W^{1,p}$, $1 < p < +\infty$ and $(\sigma,\hat\nu) \in \mathscr{DM}^p_{\mathscr{R}}(\Omega;\mathbb{R}^{m\times n})$. Then then there is a bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_k = u_j$ on $\partial\Omega$ for any $j,k \in \mathbb{N}$ and $\{\nabla u_k\}_{k\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$ if and only if the following three conditions hold:*

$$\exists u \in W^{1,p}(\Omega;\mathbb{R}^m) :\ \textit{for a.a. } x \in \Omega\textit{:}\ \nabla u(x) = d_\sigma(x)\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p}\hat\nu_x({\rm d}s)\ , \qquad (15)$$

for almost all $x \in \Omega$ and for all $\psi_0 \in \mathbb{R}$ and $\psi(s) := (1+|s|^p)\psi_0(s)$, the

$$Q\psi(\nabla u(x)) \le d_\sigma(x)\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} \psi_0(s)\hat\nu_x({\rm d}s)\ , \qquad (16)$$

for σ-almost all $x \in \bar{\Omega}$ and all $\psi_0 \in \mathscr{R}$ with $Q\psi > -\infty$, where $\psi(s) := (1+|s|^p)\psi_0(s)$,

$$0 \le \int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \psi_0(s)\hat{\nu}_x(\mathrm{d}s) \ . \tag{17}$$

Remark 2. Inequality (16) can be written in terms of $\nu = \{\nu_x\}$, the Young measure generated by $\{u_k\}$, as follows [17]: There exists a zero-measure set $\omega \subset \Omega$ such that for every $x \in \Omega \setminus \omega$

$$\psi(\nabla u(x)) \le \int_{\mathbb{R}^{m\times n}} \psi(s)\nu_x(\mathrm{d}s) \ , \tag{18}$$

for all $\psi : \mathbb{R}^{m\times n} \to \mathbb{R}$ quasiconvex and such that $|\psi| \le C(1+|\cdot|^p)$ for some $C > 0$.

Theorem 1 can be used to obtain weak lower semicontinuity results along sequences with prescribed boundary data [16]. If we do not control boundary conditions the situation is much more subtle. To the best of our knowledge, the first results in this direction are due to Meyers [25] who also deals with higher-order variational problems. However, his condition is stated in terms of sequences. A refinement was proved in [20, Thm. 1.6], showing that even near the boundary, the necessary and sufficient conditions for weak lower semicontinuity in terms of the integrand can be expressed in terms of localized test functions, similar to quasiconvexity:

Theorem 2. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the C^1-boundary. Let $\tilde{h} : \bar{\Omega} \times \mathbb{R}^{m\times n} \to \mathbb{R}$ be continuous and such that $\tilde{h}(\cdot,s)/(1+|s|^p)$ is bounded and continuous in $\bar{\Omega}$, uniformly in s. Then $J(u) := \int_\Omega \tilde{h}(x,\nabla u(x))\,\mathrm{d}x$ is weakly lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if the following two conditions hold simultaneously:*

(i) $\tilde{h}(x,\cdot)$ is quasiconvex for all $x \in \Omega$;

(ii) for every $x_0 \in \partial\Omega$ and for every $\varepsilon > 0$, there exists $C_\varepsilon \ge 0$ such that

$$\int_{D_\rho} \tilde{h}(x_0,\nabla\varphi(x))\,\mathrm{d}x \ge -\varepsilon \int_{D_\rho} |\nabla\varphi(x)|^p\,\mathrm{d}x - C_\varepsilon \ \ \text{for every } \varphi \in C_c^\infty(B(0,1);\mathbb{R}^m). \tag{19}$$

Here, $D_\rho := \{x \in B(0,1);\ x\cdot\rho < 0\}$ where ρ denotes the outer unit normal to $\partial\Omega$ at x_0.

If $\tilde{h}$ satisfies (ii) we say that it has p-quasisubcritical growth from below (p-qscb) at x_0.

2 Anisotropic parametrized measures generated by pairs of sequences

This section is devoted to a new tools which might be seen as a multiscale oscillation/concentration measures. It is a generalization of the approach introduced in [30] where only oscillations were taken into account. We also wish to mention that if $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^m)$ for $1 < p < \infty$ then (at least for a nonrelabeled subsequence) the Young measure generated by the pair $\{(u_k,\nabla u_k)\}$ is $\xi_x(\mathrm{d}(r,s)) = \delta_{u(x)}(\mathrm{d}r)\nu_x(\mathrm{d}s)$ for almost all $x \in \Omega$. Here u is the weak limit of $\{u_k\}_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and $\{\nu_x\}_{x\in\Omega}$ is the Young measure generated by $\{\nabla u_k\}$. We refer to [29] for the proof of this statement. If we are interested also in concentrations of $\{|\nabla u_k|^p\}$ and in their interactions with $\{u_k\}$ the situation is more involved.

As before, let $\mathscr{R}$ be a complete separable ring of continuous bounded functions $\mathbb{R}^{m\times n} \to \mathbb{R}$. Similarly, we take a complete separable ring $\mathscr{U}$ of continuous bounded real-valued functions on $\mathbb{R}^m$, and denote the corresponding metrizable compactification of $\mathbb{R}^m$ by $\beta_{\mathscr{U}}\mathbb{R}^m$. We will consider the ring $C(\bar{\Omega})\otimes\mathscr{U}\otimes\mathscr{R}$, the subset of bounded continuous functions on $\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$ spanned by $\{(x,s,r)\mapsto g(x)f_0(r)\psi_0(s) : g \in C(\bar{\Omega}),\ \ f_0 \in \mathscr{U},\ \ \psi_0 \in \mathscr{R}\}$. Also notice that $\beta_{\mathscr{U}}\mathbb{R}^m \times \beta_{\mathscr{R}}\mathbb{R}^{m\times n} = \beta_{\mathscr{U}\otimes\mathscr{R}}(\mathbb{R}^m\times\mathbb{R}^{m\times n})$. Finally, notice that the linear hull of $\{g \otimes f_0 \otimes \psi_0 : g \in C(\bar{\Omega}), f_0 \in C(\beta_{\mathscr{U}}), \psi_0 \in C(\beta_{\mathscr{R}}\mathbb{R}^{m\times n})\}$ is dense in $C(\bar{\Omega} \times \beta_{\mathscr{U}}\mathbb{R}^m \times \beta_{\mathscr{R}}\mathbb{R}^{m\times n})$ due to the Stone-Weierstrass theorem. Here, $[g \otimes f_0 \otimes \psi_0](x,r,s) := g(x)f_0(r)\psi_0(s)$ for all $x \in \bar{\Omega}$, $r \in \mathbb{R}^m$, and all $s \in \mathbb{R}^{m\times n}$.

Remark 3. There always exists a separable ring into which a given continuous bounded function f_0 belongs. Indeed, consider a ring $\mathscr{U}_0$ of continuous functions which possess limits if the norm of their argument tends to infinity. This ring to the one-point compactification of $\mathbb{R}^m$. If f_0 does not belong to $\mathscr{U}_0$ we construct a larger ring from f_0 and $\mathscr{U}$ by taking the closure (in the maximum norm) of all products of $\{f_0\} \cup \mathscr{U}$.

2.1 Representation of limits using parametrized measures

The following statement is rather standard generalization of the DiPerna-Majda Theorem to the anisotropic case. It can be obtained using a special case of the representation theorem in [15].[1]

Theorem 3. *Let $1 \le q \le +\infty$, $1 \le p < +\infty$ and*

$$Y^{q,p}(\Omega,\mathscr{U},\mathscr{R}) = \{h_0(r,s)(1+|r|^q+|s|^p) : h_0 \in C(\bar{\Omega} \times \beta_{\mathscr{U}}\mathbb{R}^m \times \beta_{\mathscr{R}}\mathbb{R}^{m\times n})\}.$$

Moreover, let $\{u_k\}_{k\in\mathbb{N}}$ be bounded sequence in $L^q(\Omega;\mathbb{R}^m)$ and $\{w_k\}$ a bounded sequence in $L^p(\Omega;\mathbb{R}^{m\times n})$. Then there is a subsequence $\{(u_k,w_k)\}$ (denoted by the same indeces), a measure $\hat{\sigma}(dx)$ such that

$$(1+|u_k|^q+|w_k|^p)dx \stackrel{*}{\rightharpoonup} \hat{\sigma},$$

and a family of probability measures $\{\hat{\gamma}_x\}_{x\in\bar{\Omega}} \in L^\infty_{\mathrm{w}*}(\bar{\Omega},\mathscr{M}(\beta_{\mathscr{U}}\mathbb{R}^m \times \beta_{\mathscr{R}}\mathbb{R}^{m\times n});\hat{\sigma})$ such that for any $h \in Y^{q,p}(\Omega,\mathscr{U},\mathscr{R})$ and any $g \in C(\bar{\Omega})$ we have

$$\lim_{k\to\infty}\int_\Omega g(x)h_0(u_k(x),w_k(x))(1+|u_k(x)|^q+|w_k(x)|^p)dx \to \int_{\bar{\Omega}} g(x)\int_{\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} h_0(r,s)\hat{\gamma}_x(dr,ds)\hat{\sigma}(\mathrm{d}x).$$

Remark 4. In a sense, the pair $(\hat{\sigma},\hat{\gamma})$ is an anisotropic (q,p) DiPerna-Majda measure generated by the sequence $\{(u_k,w_k)\}$, generalizing the isotropic case $p=q$. However, while this approach is a rather intuitive generalization of standard DiPerna-Majda measures, it has a drawback: Several extremely simple and often prototypical choices for the integrands which we would like to use in applications are not admissible. For instance, $h(x,r,s) := |s|^p$ *never* is an element of $Y^{q,p}(\Omega,\mathscr{U},\mathscr{R})$, because the limit of $h_0(x,r,s) := |s|^p(1+|r|^q+|s|^p)^{-1}$ as $|(r,s)| \to \infty$ does not exist: we get 1 as $|s| \to \infty$ for fixed r, and 0 as $|s| \to \infty$ for fixed r. Hence, this function h_0 does not have a continuous extension to the compactification $\beta_{\mathscr{U}} \times \beta_{\mathscr{R}}$ of $\mathbb{R}^m \times \mathbb{R}^{m\times n}$. Similarly, $h(x,r,s) := |r|^q$ is not admissible, either. Note that this problem is completely independent of the choice of compactifications.

In view of the issue pointed out in Remark 4, we will not use Theorem 3 and its class of anisotropic DiPerna-Majda measures below. Instead, our next statement provides an alternative approach which in particular does allow integrands of the form $h(x,r,s) := |s|^p$.

Theorem 4. *Let $1 \le q \le +\infty$ and $1 \le p < +\infty$. Let $\{u_k\}_{k\in\mathbb{N}}$ be bounded sequence in $L^q(\Omega;\mathbb{R}^m)$ and $\{w_k\}$ a bounded sequence in $L^p(\Omega;\mathbb{R}^{m\times n})$. Then there is a (non-relabeled) subsequence $\{(u_k,w_k)\}$, a DiPerna-Majda measure $(\sigma,\hat{\nu}) \in \mathscr{DM}^p_{\mathscr{R}}(\Omega;\mathbb{R}^{m\times n})$ and $\hat{\mu} \in \mathscr{Y}(\bar{\Omega} \times \beta_{\mathscr{R}}\mathbb{R}^{m\times n};\beta_{\mathscr{U}}\mathbb{R}^m)$, such that for every $f_0 \in \mathscr{U}$, every $\psi_0 \in \mathscr{R}$ and every $g \in C(\bar{\Omega})$*

[1] in [15] it is assumed that the compactification of the entire space $\mathbb{R}^m \times \mathbb{R}^{m\times n}$ is a subset in $\mathbb{R}^N$ for some $N \in \mathbb{N}$. This however is not required for the proof in [15] which only uses separability of the compactification.

$$\lim_{k\to\infty}\int_\Omega g(x)f_0(u_k(x))\psi(w_k(x))\,\mathrm{d}x = \int_{\bar\Omega}\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} g(x)f_0(r)\psi_0(s)\hat\mu_{s,x}(\mathrm{d}r)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x)\ , \tag{20}$$

where $\psi(s) := \psi_0(s)(1+|s|^p)$. *Moreover, measure* $(\sigma,\hat\nu)$ *is generated by* $\{w_k\}$.

Proof. Due to separability of $\mathscr{U}$, $\mathscr{R}$ and of $C(\bar\Omega)$ there is a (non-relabeled) subsequence of $\{(u_k,w_k)\}$ such that for all $[g\otimes f_0\otimes\psi_0]\in C(\bar\Omega)\times C(\beta_{\mathscr{U}}\mathbb{R}^m)\times C(\beta_{\mathscr{R}}\mathbb{R}^{m\times n})$ and $\psi(s) := \psi_0(s)(1+|s|^p)$

$$\lim_{k\to\infty}\int_\Omega g(x)f_0(u_k(x))\psi(w_k(x))\,\mathrm{d}x = \langle\Lambda, g\otimes f_0\otimes\psi_0\rangle\ , \tag{21}$$

for some $\Lambda\in\mathscr{M}(\bar\Omega\times\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n})$.

We further define $\hat T_\Lambda : \mathscr{U}\times\mathscr{R}\to C(\bar\Omega)^* = \mathscr{M}(\bar\Omega)$ by $\langle\hat T_\Lambda(f_0,\psi_0),g\rangle := \langle\Lambda, g\otimes f_0\otimes\psi_0\rangle$. Let $\sigma\in\mathscr{M}(\bar\Omega)$ be the weak* limit of $\{1+|w_k|^p\}$. Then we see that due to (21)

$$|\langle\hat T_\Lambda(f_0,\psi_0),g\rangle| = |\langle\Lambda,g\otimes f_0\otimes\psi_0\rangle| \le \|f_0\|_{C(\mathbb{R}^m)}\|\psi_0\|_{C(\mathbb{R}^{m\times n})}\int_{\bar\Omega} g(x)\,\sigma(\mathrm{d}x)\ . \tag{22}$$

This means that $\hat T_\Lambda(f_0,\psi_0)$ is absolutely continuous with respect to σ and by the Radon-Nikodým theorem there is $T_\Lambda : \mathscr{U}\times\mathscr{R}\to L^1(\bar\Omega;\sigma)$ such that for any Borel subset $\omega\subset\bar\Omega$ we get $\hat T_\Lambda(f_0,\psi_0)(\omega) = \int_\omega T_\Lambda(f_0,\psi_0)(x)\sigma(\mathrm{d}x)$. Consequently, the right-hand side of (21) can be written as $\int_{\bar\Omega} T_\Lambda(f_0,\psi_0)(x)g(x)\sigma(\mathrm{d}x)$.

As $\mathscr{U}\times\mathscr{R}$ is separable, $\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$ is metrizable and separable (with $\mathbb{R}^m\times\mathbb{R}^{m\times n}$ a dense subset) and σ is a regular measure, the linear span of $C(\bar\Omega)\otimes C(\beta_{\mathscr{U}}\mathbb{R}^m)\otimes C(\beta_{\mathscr{R}}\mathbb{R}^{m\times n})$ is dense in $L^1(\bar\Omega,\sigma;C(\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}))$ [33, Thm. 1.5.25]. Because of this and (22), Λ can be continuously extended to a continuous linear functional on $L^1(\bar\Omega,\sigma;C(\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}))$. However, the dual of this space is isometrically isomorphic to $L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\mathscr{M}(\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}))$. Arguing as in [31, p. 133] we get that there is a family $\lambda := \{\lambda_x\}_{x\in\bar\Omega}$ of probability measures on $\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$ which is σ-weak* measurable, for any $z\in C(\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n})$, the mapping $\bar\Omega\to\mathbb{R} : x\mapsto \int_{\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} z(r,s)\lambda_x(\mathrm{d}r\mathrm{d}s)$ is σ-measurable in the usual sense. Moreover, for σ-almost all $x\in\bar\Omega$ it holds that

$$T_\Lambda(f_0,\psi_0)(x) = \int_{\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} f_0(r)\psi_0(s)\lambda_x(\mathrm{d}r\mathrm{d}s)\ . \tag{23}$$

Altogether, we see that (21) can be rewritten as

$$\lim_{k\to\infty}\int_\Omega g(x)f_0(u_k(x))\psi(w_k(x))\,\mathrm{d}x = \int_{\bar\Omega} g(x)\int_{\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} f_0(r)\psi_0(s)\lambda_x(\mathrm{d}r\mathrm{d}s)\sigma(\mathrm{d}x)\ . \tag{24}$$

Applying the slicing-measure decomposition [12, Thm. 1.5.1] to each λ_x we write $\lambda_x(\mathrm{d}r\mathrm{d}s) = \hat\mu_{s,x}(\mathrm{d}r)\hat\nu_x(\mathrm{d}s)$. As λ_x is a probability measure we get that both $\hat\mu_{s,x}$ as well as $\hat\nu_x$ are probability measures on $\beta_{\mathscr{U}}\mathbb{R}^m$ and $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$, respectively. Plugging this decomposition into (24) and testing it with $f_0 := 1$, we get

$$\lim_{k\to\infty}\int_\Omega g(x)\psi(w_k(x))\,\mathrm{d}x = \int_{\bar\Omega} g(x)\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} \psi_0(s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x)\ . \tag{25}$$

This means that $(\sigma,\hat\nu)$ is the DiPerna-Majda measure *generated by* $\{w_k\}$ [9]. □

In the situation of Theorem 4, passing to a subsequence (not relabeled) if necessary, we may assume in addition that $\{(u_k,w_k)\}$ generates the (classical) Young measure ξ_x. Using the slicing-measure decomposition [12, Thm. 1.5.1] as before, we can always decompose

$\xi_x(\mathrm{d}(r,s)) = \mu_{x,s}(\mathrm{d}r)\nu_x(\mathrm{d}s)$, so that

$$\int_\Omega g(x)f_0(u_k)\psi_0(w_k)\,\mathrm{d}x \to \int_\Omega \int_{\mathbb{R}^m\times\mathbb{R}^{m\times n}} g(x)f_0(r)\psi_0(s)\,\xi_x(\mathrm{d}(r,s))\mathrm{d}x$$
$$= \int_\Omega \int_{\mathbb{R}^{m\times n}} \int_{\mathbb{R}^m} g(x)f_0(r)\psi_0(s)\,\mu_{x,s}(\mathrm{d}r)\nu_x(\mathrm{d}s)\mathrm{d}x,$$

in particular for every $f_0 \in \mathscr{U}$, every $\psi_0 \in \mathscr{R}$ and every $g \in C(\bar\Omega)$. The link between (μ,ν) and $(\hat\mu,\hat\nu)$ is the following:

Corollary 1. *In the situation of Theorem 4, let* $\xi_x(\mathrm{d}(r,s)) = \mu_{x,s}(\mathrm{d}r)\nu_x(\mathrm{d}s)$ *be the Young measure generated by* $\{(u_k,w_k)\}$. *Then* $\mathrm{d}x = \left(\int_{\mathbb{R}^{m\times n}} \frac{1}{1+|t|^p}\hat\nu_x(\mathrm{d}t)\right)\sigma(\mathrm{d}x)$, *and for a.e.* $x \in \Omega$,

$$\nu_x(\mathrm{d}s) = \left(\int_{\mathbb{R}^{m\times n}} \frac{1}{1+|t|^p}\hat\nu_x(\mathrm{d}t)\right)^{-1} \frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p} \tag{26}$$

(this is actually the well known connection between the DiPerna-Majda-measure and the associated Young measure) and

$$\mu_{x,s} = \hat\mu_{x,s} \text{ for } \hat\nu_x\text{-a.e. } s \in \mathbb{R}^{m\times n} \tag{27}$$

Proof. In the following, let $\psi_0 \in C_0(\mathbb{R}^{m\times n})$, i.e., $\psi_0 \in \mathscr{R}$ with the added property that $\psi_0(s) = 0$ for every $s \in \beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}$. Consequently, $\psi(s) := \psi_0(s)(1+|s|^p)$ satisfies $(1+|s|^p)^{-1}\psi(s) \to 0$ as $|s|\to\infty$ ($s \in \mathbb{R}^{m\times n}$) and $\frac{\psi(s)}{1+|s|^p} = 0$ for $s \in \beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}$. In addition, let $g \in C(\bar\Omega)$ and $f_0 \in \mathscr{U}$. From (24), also using the decomposition $\lambda_x(\mathrm{d}r\mathrm{d}s) = \hat\mu_{s,x}(\mathrm{d}r)\hat\nu_x(\mathrm{d}s)$, we get that

$$\lim_{k\to\infty}\int_\Omega g(x)f_0(u_k(x))\psi(w_k(x))\,\mathrm{d}x = \int_{\bar\Omega} g(x)\int_{\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} f_0(r)\hat\mu_{s,x}(\mathrm{d}r)\frac{\psi(s)\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\sigma(\mathrm{d}x)\,. \tag{28}$$

Moreover, since f_0 is bounded, $\{w_k\}$ is bounded in L^p and ψ has less than p-growth, the left hand side can be expressed using the Young measure $\xi_x(\mathrm{d}(r,s)) = \mu_{x,s}(\mathrm{d}r)\nu_x(\mathrm{d}s)$ generated by $\{(u_k,w_k)\}$:

$$\lim_{k\to\infty}\int_\Omega g(x)f_0(u_k(x))\psi(w_k(x))\,\mathrm{d}x = \int_\Omega g(x)\int_{\mathbb{R}^{m\times n}}\int_{\mathbb{R}^m} f_0(r)\mu_{s,x}(\mathrm{d}r)\psi(s)\nu_x(\mathrm{d}s)\mathrm{d}x\,. \tag{29}$$

Since $(\sigma,\hat\nu)$ is a DiPerna-Majda measure (the one generated by $\{w_k\}$), we in particular know that the density of the Lebesgue measure with respect to σ is given by

$$\frac{\mathrm{d}\mathscr{L}^n}{\mathrm{d}\sigma}(x) = \int_{\mathbb{R}^{m\times n}} \frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\,,$$

cf. Proposition 1 (ii). Hence, we can also write the outer integral on right hand side of (29) as an integral with respect to σ, and then compare it to the right hand side of (28). Since g is arbitrary, this implies that for σ-a.e. $x \in \Omega$,

$$\begin{aligned}\left(\int_{\mathbb{R}^{m\times n}}\int_{\mathbb{R}^m} f_0(r)\mu_{s,x}(\mathrm{d}r)\psi(s)\nu_x(\mathrm{d}s)\right)\left(\int_{\mathbb{R}^{m\times n}}\frac{\hat\nu_x(\mathrm{d}t)}{1+|t|^p}\right)\\ = \int_{\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} f_0(r)\hat\mu_{s,x}(\mathrm{d}r)\frac{\psi(s)\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\,.\end{aligned} \tag{30}$$

Here, also notice that it is enough to state (30) for a.e. $x \in \Omega$, because $\mathscr{L}^n$ is absolutely continuous with respect to σ and $\int_{\mathbb{R}^{m\times n}}\frac{\hat\nu_x(\mathrm{d}t)}{1+|t|^p} = 0$ for σ^s-a.e. $x \in \bar\Omega$.

Using the probability measure given by the right hand side of (26), i.e.,

$$\nu_x(\mathrm{d}s) := \Big(\int_{\mathbb{R}^{m\times n}} \frac{\hat{\nu}_x(\mathrm{d}t)}{1+|t|^p}\Big)^{-1} \frac{\hat{\nu}_x(\mathrm{d}s)}{1+|s|^p},$$

we see that (30) is equivalent to

$$\int_{\mathbb{R}^{m\times n}}\int_{\mathbb{R}^m} f_0(r)\mu_{s,x}(\mathrm{d}r)\psi(s)\nu_x(\mathrm{d}s) = \int_{\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} f_0(r)\hat{\mu}_{s,x}(\mathrm{d}r)\psi(s)\tilde{\nu}_x(\mathrm{d}s)\ . \tag{31}$$

Since (31) holds for all $\psi_0 \in C_0(\mathbb{R}^{m\times n})$ (and therefore all ψ with less than p-growth, in particular all bounded ψ) and $\mu_{s,x}$ and $\hat{\mu}_{s,x}$ are probability measures, choosing $f_0 \equiv 1 \in \mathscr{U}$ in (31) yields that $\nu_x = \tilde{\nu}_x$, i.e., (26). Finally, replacing $\tilde{\nu}_x$ by ν_x in (31), and using that the latter holds in particular for all bounded $\psi \in C(\mathbb{R}^{m\times n})$ and all $f_0 \in C_0(\mathbb{R}^m) \subset \mathscr{U}$, we infer (27).

Remark 5. In the situation of Corollary 1, suppose in addition that $u_k \to u$ in L^q for some $q \geq 1$ (for instance by compact embedding, if $\{u_k\}$ is bounded in $W^{1,p}$). We recall that in this case, for the Young measure $\xi_x(\mathrm{d}(r,s)) = \mu_{x,s}(\mathrm{d}r)\nu_x(\mathrm{d}s)$ generated by $\{(u_k,w_k\}$ we have $\mu_{x,s} = \delta_{u(x)}$ for a.e. $x \in \Omega$ (in particular independent of s, cf. [29, Proposition 6.13], e.g.). Consequently, (27) implies that

$$\hat{\mu}_{x,s} = \delta_{u(x)} \ \text{ for a.e. } x \in \Omega \text{ and } \hat{\nu}_x\text{-a.e. } s \in \mathbb{R}^{m\times n} \tag{32}$$

Remark 6. It is left to the interested reader to show that if $u_k \to u$ in $C(\bar{\Omega};\mathbb{R}^m)$ for $k \to \infty$ then $\hat{\mu}_{s,x} = \delta_{u(x)}$ for σ-a.e. $x \in \bar{\Omega}$. Also, $\hat{\mu}_{s,x}$ is then supported only on $\mathbb{R}^m$, so it is independent of the choice of the compactification $\beta_{\mathscr{U}}\mathbb{R}^m$.

The next statement is similar to Theorem 4, but now we consider the limits of the sequence $\int_\Omega f_0(u_k)\psi_0(w_k)(1+|u_k|^q)\,\mathrm{d}x$ where $f_0 \in \mathscr{U}, \psi_0 \in \mathscr{R}$. In particular, the integrand $|u_k|^q$ will thus be admissible. Its proof can easily be deduced by adapting the proof of Theorem 4, essentially interchanging the role of the two sequences.

Theorem 5. *Let $1 \leq q < +\infty$ and $1 \leq p \leq +\infty$. Let $\{u_k\}_{k\in\mathbb{N}}$ be bounded sequence in $L^q(\Omega;\mathbb{R}^m)$ and $\{w_k\}$ a bounded sequence in $L^p(\Omega;\mathbb{R}^{m\times n})$. Then there is a (non-relabeled) subsequence $\{(u_k,w_k)\}$, a positive measure $\sigma^* \in \mathscr{M}(\bar{\Omega})$ and parametrized probability measures $\hat{\nu}^* \in \mathscr{Y}(\bar{\Omega};\beta_{\mathscr{R}}\mathbb{R}^{m\times n})$ (defined σ^*-a.e.) and $\hat{\mu}^* \in \mathscr{Y}(\bar{\Omega}\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n};\beta_{\mathscr{U}}\mathbb{R}^m)$ (defined $\sigma^*\otimes\hat{\nu}^*_x$-a.e.) such that for every $f_0 \in \mathscr{U}$, every $\psi_0 \in \mathscr{R}$ and every $g \in C(\bar{\Omega})$*

$$\begin{aligned}\lim_{k\to\infty}&\int_\Omega g(x)f(u_k(x))\psi_0(w_k(x))\,\mathrm{d}x\\ &=\int_{\bar{\Omega}}\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} g(x)f_0(r)\psi_0(s)\hat{\mu}^*_{s,x}(\mathrm{d}r)\hat{\nu}^*_x(\mathrm{d}s)\sigma^*(\mathrm{d}x)\ ,\end{aligned} \tag{33}$$

where $f(r) := f_0(r)(1+|r|^q)$. Moreover, $(\sigma^*,\overline{\hat{\mu}^*}_x) \in \mathscr{DM}^q_{\mathscr{U}}(\bar{\Omega};\mathbb{R}^m)$ is the the DiPerna-Majda measure generated by $\{u_k\}$, where $\overline{\hat{\mu}^*}_x$ is given as follows:

$$\int_{\beta_{\mathscr{U}}\mathbb{R}^m} f_0(r)\overline{\hat{\mu}^*}_x(\mathrm{d}r) = \int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} f_0(r)\hat{\mu}^*_{s,x}(\mathrm{d}r)\hat{\nu}^*_x(\mathrm{d}s) \tag{34}$$

for all $f_0 \in \mathscr{U}$ and σ^*-a.e. $x \in \bar{\Omega}$.

Analogously to Corollary 1, we have

Corollary 2. *In the situation of Theorem 5, let $\xi_x(\mathrm{d}(r,s)) = \mu_{x,s}(\mathrm{d}r)\nu_x(\mathrm{d}s)$ be the Young measure generated by $\{(u_k,w_k)\}$. Then $\mathrm{d}x = \Big(\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\int_{\mathbb{R}^m}\frac{1}{1+|z|^q}\hat{\mu}^*_{x,t}(\mathrm{d}z)\hat{\nu}^*_x(\mathrm{d}t)\Big)\,\sigma^*(\mathrm{d}x)$, and for a.e. $x \in \Omega$,*

$$\nu_x(\mathrm{d}s) = \left(\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}} \int_{\mathbb{R}^m} \frac{1}{1+|z|^q}\hat{\mu}^*_{x,t}(\mathrm{d}z)\hat{\nu}^*_x(\mathrm{d}t)\right)^{-1}\left(\int_{\mathbb{R}^m}\frac{1}{1+|z|^q}\hat{\mu}^*_{x,s}(\mathrm{d}z)\right)\hat{\nu}^*_x(\mathrm{d}s), \quad (35)$$

$$\mu_{x,s}(\mathrm{d}r) = \left(\int_{\mathbb{R}^m}\frac{1}{1+|z|^q}\hat{\mu}^*_{x,s}(\mathrm{d}z)\right)^{-1}\frac{\hat{\mu}_{x,s}(\mathrm{d}r)}{1+|r|^q} \quad \textit{for } \hat{\nu}_x\textit{-a.e. } s\in\beta_{\mathscr{R}}\mathbb{R}^{m\times n}. \quad (36)$$

Analogous to the case of Young measures or DiPerna-Majda-measures, we say that $(\sigma,\hat{\nu},\hat{\mu})$ [or $(\sigma^*,\hat{\nu}^*,\hat{\mu}^*)$, respectively] *is generated by* $\{(u_k,w_k)\}$ whenever (20) [((33)] holds for all $(g,f_0,\psi_0)\in C(\bar{\Omega})\times\mathscr{U}\times\mathscr{R}$.

Theorem 4 and Theorem 5 can be combined, leading to the following statement. It provides a representation for limits of rather general nonlinear functionals along a given sequence. The suitable class of integrands is

$$\mathbb{H}^{q,p}(\Omega,\mathscr{U},\mathscr{R}) = \left\{ h \;\middle|\; \begin{array}{l} h(x,r,s) = h_0^{(1)}(x,r,s)(1+|s|^p) + h_0^{(2)}(x,r,s)(1+|r|^q) \\ h_0^{(1)},h_0^{(2)}\in C(\bar{\Omega}\times\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n}) \end{array}\right\}. \quad (37)$$

Theorem 6 (representation theorem). *Let* $1\le q\le+\infty$ *and* $1\le p<+\infty$. *Let* $\{u_k\}_{k\in\mathbb{N}}$ *be bounded sequence in* $L^q(\Omega;\mathbb{R}^m)$ *and* $\{w_k\}$ *a bounded sequence in* $L^p(\Omega;\mathbb{R}^{m\times n})$. *Then there is a (non-relabeled) subsequence* $\{(u_k,w_k)\}$ *generating the measures* $(\sigma,\hat{\nu},\hat{\mu})$ *and* $(\sigma^*,\hat{\nu}^*,\hat{\mu}^*)$ *(in the sense of* (20) *and* (33)*, respectively), and in addition, for every* $h_0^{(1)},h_0^{(2)}\in C(\bar{\Omega}\times\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n})$,

$$\begin{aligned}&\lim_{k\to\infty}\int_\Omega \left(h_0^{(1)}(x,u_k,w_k)(1+|w_k|^p)+h_0^{(2)}(x,u_k,w_k)(1+|u_k|^q))\right)\mathrm{d}x\\ &\quad=\int_{\bar{\Omega}}\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} h_0^{(1)}(x,r,s)\hat{\mu}_{s,x}(\mathrm{d}r)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x)\\ &\qquad+\int_{\bar{\Omega}}\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} h_0^{(2)}(x,r,s)\hat{\mu}^*_{s,x}(\mathrm{d}r)\hat{\nu}^*_x(\mathrm{d}s)\sigma^*(\mathrm{d}x)\,.\end{aligned} \quad (38)$$

Remark 7. As a special case, we recover a representation of the limit for functionals with integrands in $Y^{q,p}(\Omega;\mathscr{U};\mathscr{R})$ as in Theorem 3, since

$$\tilde{h}_0(x,r,s)(1+|r|^q+|s|^p) = h_0(x,r,s)(1+|r|^q)+h_0(x,r,s)(1+|s|^p),$$

where

$$h_0(x,r,s) := \frac{1+|r|^q+|s|^p}{2+|r|^q+|s|^p}\tilde{h}_0(x,r,s)$$

The quotient which appears here does not matter, because $(r,s)\mapsto\frac{1+|r|^q+|s|^p}{2+|r|^q+|s|^p}$ converges to the constant 1 as $|(r,s)|\to\infty$, and therefore it is an element of $\overline{\mathscr{U}\otimes\mathscr{R}} = C(\beta_{\mathscr{U}}\mathbb{R}^m\times\beta_{\mathscr{R}}\mathbb{R}^{m\times n})$.

Remark 8. Notice that $(\sigma,\hat{\nu},\hat{\mu})$ and $(\sigma^*,\hat{\nu}^*,\hat{\mu}^*)$ are not independent, because they share the same underlying Young measure $\xi_x(\mathrm{d}(r,s)) = \mu_{x,s}(\mathrm{d}r)\nu_x(\mathrm{d}s)$, see Corollary 1 and Corollary 2. Using that, we get yet another representation: For $h\in\mathbb{H}^{q,p}(\Omega,\mathscr{U},\mathscr{R})$ (cf. (37)),

$$\begin{aligned}&\lim_{k\to\infty}\int_\Omega h(x,u_k,w_k)\,\mathrm{d}x\\ &=\int_{\bar{\Omega}}\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m} h_0^{(1)}(x,r,s)\hat{\mu}_{s,x}(\mathrm{d}r)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x)\\ &\quad+\int_{\bar{\Omega}}\int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}}\int_{\beta_{\mathscr{U}}\mathbb{R}^m\setminus\mathbb{R}^m} h_0^{(2)}(x,r,s)\hat{\mu}^*_{s,x}(\mathrm{d}r)\hat{\nu}^*_x(\mathrm{d}s)\sigma^*(\mathrm{d}x)\\ &\quad+\int_\Omega\int_{\mathbb{R}^{m\times n}}\int_{\mathbb{R}^m} h(x,r,s)\mu_{s,x}(\mathrm{d}r)\nu_x(\mathrm{d}s)\mathrm{d}x.\end{aligned} \quad (39)$$

Remark 9. If either $\{|u_k|^q\}$ or $\{|w_k|^q\}$ is equi-integrable, then (39) can be further simplified. For instance, if $\{u_k\}$ is bounded in $L^{\tilde{q}}$ for some $\tilde{q} > q$), then $\{|u_k|^q\}$ is equi-integrable, and it that case, it is known (e.g., see [31, Lemma 3.2.14]) that for the associated DiPerna-Majda measure $(\sigma^*, \overline{\hat{\mu}^*}_x)$, we have that σ^* is absolutely continuous with respect to $\mathscr{L}^n$ and $\overline{\hat{\mu}^*}_x(\beta_{\mathscr{U}}\mathbb{R}^m \setminus \mathbb{R}^m) = 0$ for a.e. $x \in \Omega$. Due to (34), the latter implies that $\hat{\mu}^*_{x,s}(\beta_{\mathscr{U}}\mathbb{R}^m \setminus \mathbb{R}^m) = 0$ for a.e. $x \in \Omega$ and $\hat{\nu}^*_x$-a.e. $s \in \beta_{\mathscr{R}}\mathbb{R}^{m\times n}$. Accordingly, for $h \in \mathbb{H}^{q,p}(\Omega, \mathscr{U}, \mathscr{R})$ (cf. (37)),

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega h(x,u_k,w_k)\,\mathrm{d}x = & \int_{\bar{\Omega}} \int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \int_{\beta_{\mathscr{U}}\mathbb{R}^m} h_0^{(1)}(x,r,s)\hat{\mu}_{s,x}(\mathrm{d}r)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) \\ & + \int_\Omega \int_{\mathbb{R}^{m\times n}} \int_{\mathbb{R}^m} h(x,r,s)\mu_{s,x}(\mathrm{d}r)\nu_x(\mathrm{d}s)\mathrm{d}x.\end{aligned} \tag{40}$$

2.2 Analysis for couples $\{(u_k, \nabla u_k)\}$

For the rest of the article, we are mainly interested in sequences of the form $(u_k, w_k) = (u_k, \nabla u_k)$, with a bounded sequence $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, $1 \le p < \infty$, and integrands $h \in \mathbb{H}^{q,p}(\Omega, \mathscr{U}, \mathscr{R})$ (cf. (37)) for some $q < p^*$. Here, p^* is the exponent of the Sobolev embedding, i.e.,

$$p^* := \begin{cases} pn/(n-p) & \text{if } 1 \le p < n, \\ +\infty & \text{otherwise.}\end{cases}$$

In particular, such integrands satisfy

$$|h(x,r,s)| \le C(1+|r|^q+|s|^p) \quad \text{for all } x \in \bar{\Omega},\ r \in \mathbb{R}^m,\ s \in \mathbb{R}^{m\times n}, \tag{41}$$

with a constant $C \ge 0$.

Since we assume that $q < p^*$, we can represent limits using (40), with the added observation that the Young measure generated by $\{u_k\}$ is given by $\delta_{u(x)}$ (whence $\mu_{x,s} = \delta_{u(x)}$ for all s), where u denotes the weak limit of $\{u_k\}$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. This gives the following result.

Theorem 7. *Let $(u_k, w_k) := (u_k, \nabla u_k)$, with a bounded sequence $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, $1 \le p < \infty$, such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$, $\{(\nabla u_k)\}$ generates the (classical) Young measure ν_x in the sense of* (9) *and $\{(u_k, \nabla u_k)\}$ generates the measure $(\sigma, \hat{\nu}, \hat{\mu})$ in the sense of* (20). *Then for every $h \in \mathbb{H}^{q,p}(\Omega, \mathscr{U}, \mathscr{R})$ (cf.* (37)*),*

$$\begin{aligned}&\lim_{k\to\infty} \int_\Omega h(x,u_k(x),\nabla u_k(x))\,\mathrm{d}x \\ &= \int_\Omega \int_{\mathbb{R}^{m\times n}} h(x,u(x),s)\nu_x(\mathrm{d}s)\,\mathrm{d}x \\ &\quad + \int_{\bar{\Omega}} \int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \int_{\beta_{\mathscr{U}}\mathbb{R}^m} h_0^{(1)}(x,r,s)\hat{\mu}_{s,x}(\mathrm{d}r)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x)\,.\end{aligned} \tag{42}$$

Remark 10. If $h(x,u(x),\cdot)$ is quasiconvex, we can further calculate in (42) as follows:

$$\int_\Omega \int_{\mathbb{R}^{m\times n}} h(x,u(x),s)\nu_x(\mathrm{d}s)\,\mathrm{d}x \ge \int_\Omega h(x,u(x),\nabla u(x))\,\mathrm{d}x. \tag{43}$$

Remark 11. If $p > n$, $W^{1,p}(\Omega;\mathbb{R}^m)$ is compactly embedded in $C(\bar{\Omega};\mathbb{R}^m)$, and therefore $u_k \to u$ uniformly on $\bar{\Omega}$. In view of Remark 6, we then have that $\hat{\mu}_{s,x} = \delta_{u(x)}$ for σ-a.e. $x \in \bar{\Omega}$, for $\hat{\nu}_x$-a.e. $s \in \beta_{\mathscr{R}}\mathbb{R}^{m\times n}$. Hence,

$$\int_{\beta_{\mathscr{U}}\mathbb{R}^m} h_0^{(1)}(x,r,s)\hat{\mu}_{s,x}(\mathrm{d}r) = h_0^{(1)}(x,u(x),s)$$

in the right hand side of (42).

2.3 Examples

Below, we give a couple of examples of sequences and measures from Theorem 4 generated by them.

Example 3. Let $u_k \in W^{1,1}(0,2)$ be such that

$$u_k(x) := \begin{cases} 0 & \text{if } 0 \le x \le 1-1/k, \\ kx-k+1 & \text{if } 1-1/k \le x \le 1, \\ -2kx+2k+1 & \text{if } 1 \le x \le 1+1/k, \\ -1 & \text{if } 1+1/k \le x \le 2. \end{cases}$$

Let $w_k := u_k'$, i.e.,

$$w_k(x) := \begin{cases} 0 & \text{if } 0 \le x \le 1-1/k, \\ k & \text{if } 1-1/k \le x \le 1, \\ -2k & \text{if } 1 \le x \le 1+1/k, \\ 0 & \text{if } 1+1/k \le x \le 2. \end{cases}$$

Let $f_0 \in C(\mathbb{R})$ be bounded with its primitive denoted by F, i.e., $F' = f_0$, $g \in C(\bar{\Omega})$, and let $\psi = \psi_0(1+|\cdot|)$ where $\psi_0 \in \mathscr{R}$ corresponding to the two-point (or sphere) compactification $\beta_{\mathscr{R}}\mathbb{R} = \mathbb{R} \cup \{\pm\infty\}$, i.e., $\psi_0 \in C(\mathbb{R})$ is such that $\lim_{s\to\pm\infty} \psi_0(s) =: \psi_0(\pm\infty) \in \mathbb{R}$. Then

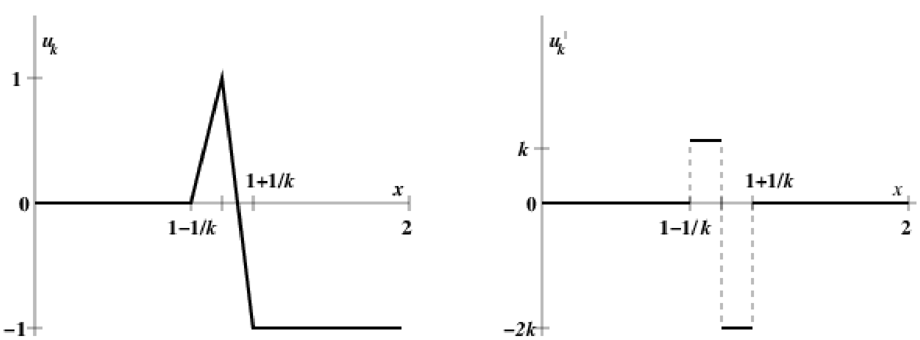

Fig. 1 Sequence $\{u_k, u_k'\}_{k\in\mathbb{N}}$ from Example 3.

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^2 f_0(u_k(x))\psi(w_k(x))g(x)\,\mathrm{d}x\\
&=\lim_{k\to\infty}\Big(\int_0^{1-1/k} f_0(0)\psi_0(0)g(x)\,\mathrm{d}x+\int_{1+1/k}^2 f_0(-1)\psi_0(0)g(x)\,\mathrm{d}x\Big)\\
&\quad+\lim_{k\to\infty}\Big(\int_{1-1/k}^1 f_0(kx-k+1)\psi_0(k)(1+k)g(x)\,\mathrm{d}x\\
&\qquad\qquad+\int_1^{1+1/k} f_0(-2kx+2k+1)\psi_0(-2k)(1+2k)g(x)\,\mathrm{d}x\Big)\\
&=\psi_0(0)(f_0(0)\int_0^1 g(x)\,\mathrm{d}x+f_0(-1)\int_1^2 g(x)\,\mathrm{d}x)\\
&\quad+\lim_{k\to\infty}\Big(\int_{1-1/k}^1 [F(kx-k+1)]'\psi_0(k)\frac{(1+k)}{k}g(x)\,\mathrm{d}x\\
&\qquad\qquad+\int_1^{1+1/k}[F(-2kx+2k+1)]'\psi_0(-2k)\frac{(1+2k)}{-2k}g(x)\,\mathrm{d}x\Big)\\
&=f_0(0)\psi_0(0)\int_0^1 g(x)\,\mathrm{d}x+f_0(-1)\psi_0(0)\int_1^2 g(x)\,\mathrm{d}x\\
&\quad+g(1)(F(1)-F(0))\psi_0(+\infty)+g(1)(F(1)-F(-1))\psi_0(-\infty)\\
&=\int_0^2\int_{\beta_{\mathscr{U}}\mathbb{R}}\int_{\beta_{\mathscr{R}}\mathbb{R}} g(x)f_0(r)\psi_0(s)\hat{\mu}_{s,x}(\mathrm{d}r)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x)\ ,
\end{aligned}$$

where $\sigma=\mathscr{L}^1+3\delta_1$,

$$\hat{\nu}_x=\begin{cases}\delta_0 & \text{if } x\in[0,1)\cup(1;2],\\ \frac{1}{3}\delta_\infty+\frac{2}{3}\delta_{-\infty} & \text{if } x=1,\end{cases}$$

and

$$\hat{\mu}_{s,x}=\begin{cases}\delta_0 & \text{if } 0\le x<1,\\ \delta_{-1} & \text{if } 1<x\le 2,\\ \mathscr{L}^1\llcorner_{(0,1)} & \text{if } s=+\infty \text{ and } x=1,\\ \frac{1}{2}\mathscr{L}^1\llcorner_{(-1,1)} & \text{if } s=-\infty \text{ and } x=1.\end{cases}$$

Changing the previous sequence slightly we get the same measure $(\sigma,\hat{\nu})$, the same limit of $\{u_k\}$ but a different measure $\hat{\mu}$.

Example 4. Let $u_k\in W^{1,1}(0,2)$ be such that

$$u_k(x):=\begin{cases}0 & \text{if } 0\le x\le 1-2/k,\\ -kx+k-2 & \text{if } 1-2/k\le x\le 1-1/k,\\ kx-k & \text{if } 1-1/k\le x\le 1,\\ -kx+k & \text{if } 1\le x\le 1+1/k,\\ -1 & \text{if } 1+1/k\le x\le 2.\end{cases}$$

Let $w_k:=u_k'$, i.e.,

$$w_k(x):=\begin{cases}0 & \text{if } 0\le x\le 1-2/k,\\ -k & \text{if } 1-2/k\le x\le 1-1/k,\\ k & \text{if } 1-1/k\le x\le 1,\\ -k & \text{if } 1\le x\le 1+1/k,\\ 0 & \text{if } 1+1/k\le x\le 2.\end{cases}$$

Then a computation analogous to the one above shows that

$\sigma = \mathscr{L}^1 + 3\delta_1$,

$$\hat{\nu}_x = \begin{cases} \delta_0 & \text{if } x \in [0,1) \cup (1;2], \\ \frac{1}{3}\delta_\infty + \frac{2}{3}\delta_{-\infty} & \text{if } x = 1, \end{cases}$$

and

$$\hat{\mu}_{s,x} = \begin{cases} \delta_0 & \text{if } 0 \le x < 1, \\ \delta_{-1} & \text{if } 1 < x \le 2, \\ \mathscr{L}^1 \llcorner_{(-1,0)} & \text{if } s = -\infty \text{ and } x = 1, \\ \mathscr{L}^1 \llcorner_{(-1,0)} & \text{if } s = +\infty \text{ and } x = 1, \end{cases}$$

These two examples show that $\hat{\mu}$ captures behavior of $\{u_k\}$ and cannot be read off either from $(\sigma, \hat{\nu})$ and/or from u.

Example 5. In the next example, we just set $u_k := u$, where $u(x) := 0$ if $x \in [0,1]$ and $u(x) = -1$ if $x \in (1;2]$, and $\{w_k\}_{k\in\mathbb{N}}$ for all $k \in \mathbb{N}$ as before. This gives us

$$\begin{aligned}
&\lim_{k\to\infty} \int_0^2 f_0(u(x))\psi(w_k(x))g(x)\,\mathrm{d}x \\
&= \lim_{k\to\infty} \Big(\int_0^{1-1/k} f_0(0)\psi_0(0)g(x)\,\mathrm{d}x + \int_{1+1/k}^2 f_0(-1)\psi_0(0)g(x)\,\mathrm{d}x \Big) \\
&\quad + \lim_{k\to\infty} \Big(\int_{1-1/k}^1 f_0(0)\psi_0(k)(1+k)g(x)\,\mathrm{d}x + \int_1^{1+1/k} f_0(-1)\psi_0(-2k)(1+2k)g(x)\,\mathrm{d}x \Big) \\
&= f_0(0)\psi_0(0)\int_0^1 g(x)\,\mathrm{d}x + f_0(-1)\psi_0(0)\int_1^2 g(x)\,\mathrm{d}x \\
&\quad + \lim_{k\to\infty} \Big(k\int_{1-1/k}^1 f_0(0)\psi_0(k)\frac{1+k}{k}g(x)\,\mathrm{d}x + k\int_1^{1+1/k} f_0(-1)\psi_0(-2k)\frac{1+2k}{k}g(x)\,\mathrm{d}x \Big) \\
&= f_0(0)\psi_0(0)\int_0^1 g(x)\,\mathrm{d}x + f_0(-1)\psi_0(0)\int_1^2 g(x)\,\mathrm{d}x \\
&\quad + g(1)f_0(0)\psi_0(+\infty) + 2g(1)f_0(-1)\psi_0(-\infty)) \\
&= \int_0^2 \int_{\beta_{\mathscr{R}}\mathbb{R}} \int_{\beta_{\mathscr{U}}} g(x)f_0(r)\psi_0(s)\nu_{s,x}(\mathrm{d}r)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x)\ ,
\end{aligned}$$

where $\sigma = \mathscr{L}^1 + 3\delta_1$,

$$\hat{\nu}_x = \begin{cases} \delta_0 & \text{if } x \in [0,1) \cup (1;2], \\ \frac{1}{3}\delta_\infty + \frac{2}{3}\delta_{-\infty} & \text{if } x = 1, \end{cases}$$

and

$$\hat{\mu}_{s,x} = \begin{cases} \delta_0 & \text{if } 0 \le x < 1, \\ \delta_0 & \text{if } x = 1 \text{ and } s = +\infty, \\ \delta_{-1} & \text{if } x = 1 \text{ and } s = -\infty, \\ \delta_{-1} & \text{if } 1 < x \le 2. \end{cases}$$

In the example below, we calculate the measure $\hat{\mu}$ of the strongly converging sequence.

Example 6. Let $p = 1$, consider the one-point compactification $\beta_{\mathscr{R}}\mathbb{R} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$, and let $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,1}(0,2)$, $u_k \rightharpoonup u$, be a sequence of nondecreasing functions such that $u_k(0) = 0$ and $u_k(2) = 1$ for all $k \in \mathbb{N}$. In addition, suppose that $\{u_k'\}_{k\in\mathbb{N}} \subset L^1(0,2)$ converges to zero in measure and it concentrates at $x = 1$, i.e., $\{u_k'\}$ generates $(\sigma, \hat{\nu}) \in \mathscr{DM}^p_{\mathscr{R}}(\Omega;\mathbb{R}^{m\times n})$ given by

$$\sigma = \mathscr{L}^1 + \delta_1, \quad \hat{\nu}_x = \begin{cases} \delta_0 & \text{if } x \in [0,1) \cup (1,2], \\ \delta_\infty & \text{if } x = 1. \end{cases}$$

Moreover, let $\alpha \geq 0$, let $f_0(r) \in C_0(\mathbb{R})$ be such that

$$f_0(r) = \begin{cases} r^\alpha & \text{if } 0 \leq r \leq 1 \\ 1 & \text{for } r \geq 1 \end{cases}$$

and let $\psi(s) := |s|$. As u_k is nondecreasing it must always satisfy $u_k \in [0,1]$, so that $f_0(u_k) = u_k^\alpha$, and $u_k' \geq 0$. Consequently, in view of Theorem 4

$$\begin{aligned} \lim_{k\to\infty} \int_0^2 f_0(u_k(x))\psi(u_k'(x))\,\mathrm{d}x &= \int_0^2 \int_{\beta_{\mathscr{R}}\mathbb{R}} \int_{\beta_{\mathscr{U}}\mathbb{R}} r^\alpha \hat{\mu}_{s,x}(\mathrm{d}r) \frac{s}{1+|s|} \hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) \\ &= \int_{\beta_{\mathscr{U}}\mathbb{R}} r^\alpha \hat{\mu}_{\infty,1}(\mathrm{d}r)\ . \end{aligned}$$

On the other hand,

$$\begin{aligned} \lim_{k\to\infty} \frac{1}{\alpha+1}(u_k^{\alpha+1}(2) - u_k^{\alpha+1}(0)) &= \lim_{k\to\infty} \int_0^2 \frac{1}{\alpha+1}(u_k^{\alpha+1}(x))'\,\mathrm{d}x = \lim_{k\to\infty} \int_0^2 u_k^\alpha(x) u_k'(x)\,\mathrm{d}x \\ &= \int_{\beta_{\mathscr{U}}\mathbb{R}} r^\alpha \hat{\mu}_{\infty,1}(\mathrm{d}r) = \lim_{k\to\infty} \int_{u_k(0)}^{u_k(2)} r^\alpha\,\mathrm{d}r = \int_0^1 r^\alpha\,\mathrm{d}r\ . \end{aligned}$$

Since $\alpha \geq 0$ is arbitrary and the polynomials are dense in the continuous functions on all compact subsets of $\mathbb{R}$, we infer that

$$\hat{\mu}_{s,x} = \begin{cases} \delta_{u(x)} & \text{if } x \in [0,1) \cup (1,2], \\ \mathscr{L}^1 \llcorner_{(0,1)} & \text{if } x = 1 \text{ and } s = \infty. \end{cases}$$

This means that only values of limits of u_k at $x = 0$ and $x = 2$ influence $\hat{\mu}_{\infty,1}$, i.e., the measure at the point where σ concentrates.

3 Applications to weak lower semicontinuity in Sobolev spaces

We here focus on weak lower semikcontinuity of "signed" integral functionals in $W^{1,p}$, i.e., functional whose integrand may have a negative part which has p-growth in the gradient variable. The case of non-negative integrands (or weaker growth in the negative direction) is well-known, see e.g. [1].

Throughout this section, let $\mathscr{U}$ and $\mathscr{R}$ denote rings of bounded continuous functions corresponding to suitable metrizable compactifications $\beta_{\mathscr{U}}\mathbb{R}^m$ and $\beta_{\mathscr{R}}\mathbb{R}^{m\times n}$ of $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$, respectively, as before. The choice of these rings can be adapted to the particular integrand h at hand in the results presented below. Compactifications by the sphere are sufficiently rich for most practical purposes.

If $p > n$, we can exploit the embedding of $W^{1,p}(\Omega;\mathbb{R}^m)$ into continuous functions on $\bar{\Omega}$. Still, even for quasiconvex integrands concentration effects near the boundary of the domain can prevent lower semicontinuity. However, as it turns out this is the only remaining obstacle. Unlike in the related result of Ball and Zhang [5] where small measurable (but otherwise pretty unknown) sets are removed from the domain, for us it is enough to "peel" away a layer near $\partial\Omega$:

Lemma 1. *(Peeling lemma for $p > n$) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a boundary of class C^1, let $\infty > p > n$ and let $h \in \mathbb{H}^{q,p}(\Omega, \mathscr{U}, \mathscr{R})$ (cf. (37)). Moreover, assume that $h(x,r,\cdot)$ is quasiconvex for a.e. $x \in \Omega$ (and therefore all $x \in \bar{\Omega}$, by continuity) and every $r \in \rho^m$, and let $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be a bounded sequence with $u_k \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then there exists an increasing sequence of open set Ω_j (possibly depending on the subsequence of*

$\{u_k\}$) with boundary of class C^∞, $\bar{\Omega}_j \subset \Omega$ and $\bigcup_j \Omega_j = \Omega$ such that

$$\liminf_{k\to\infty} \int_{\Omega_j} h(x, u_k(x), \nabla u_k(x))\, \mathrm{d}x \geq \int_{\Omega_j} h(x, u(x), \nabla u(x))\, \mathrm{d}x.$$

Proof. We select a subsequence of $\{u_k\}$ so that "$\liminf = \lim$" and such that $\{(u_k)\}$ generates a Young measure ν, and $\{(u_k, \nabla u_k)\}$ generates a measure $(\sigma, \hat{\nu}, \hat{\mu})$ in the sense of (20). Now let $\Omega_0 := \emptyset$. For each j, we choose an open set Ω_j with smooth boundary such that

$$K_j := \bar{\Omega}_{j-1} \cup \{x \in \Omega : \mathrm{dist}(x; \partial\Omega) \geq \tfrac{1}{j}\} \subset \Omega_j \subset \bar{\Omega}_j \subset \Omega$$

and

$$\sigma(\partial\Omega_j) = 0 \tag{44}$$

Here, notice that since the distance of the compact set K_j to $\partial\Omega$ is positive, we can find uncountably many pairwise disjoint candidates for Ω_j. Since σ is a finite measure, all but countably many of them must satisfy (44). Clearly, the measure generated by $\{(u_k, \nabla u_k)\}$ on Ω_j coincides with $(\sigma, \hat{\nu}, \hat{\mu})$ on the open set Ω_j, and due to (44) even on $\bar{\Omega}_j$. Hence, by Theorem 7, Remark 11,

$$\begin{aligned}
\lim_{k\to\infty} \int_{\Omega_j} h(x, u_k(x), \nabla u_k(x))\, \mathrm{d}x &= \int_{\Omega_j} \int_{\mathbb{R}^{m\times n}} h(x, u(x), s)\, \nu_x(\mathrm{d}s)\, \mathrm{d}x \\
&\quad + \int_{\bar{\Omega}_j} \int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n} \setminus \mathbb{R}^{m\times n}} h_0^{(1)}(x, u(x), s)\, \hat{\nu}_x(\mathrm{d}s)\, \sigma(\mathrm{d}x) \\
&\geq \int_{\Omega_j} h(x, u(x), \nabla u(x))\, \mathrm{d}x.
\end{aligned}$$

Here, the inequality above is due to Remark 10 and (17) with $\psi_0(s) := h_0^{(1)}(x, u(x), s)$ (separately applied for each x); for $\psi(s) := (1 + |s|^p)\psi_0(s)$ and its quasiconvex hull $Q\psi$ we have $Q\psi > -\infty$ because $h(x, u(x), \cdot)$ is quasiconvex and $\psi(s) - h(x, u(x), s) = h_0^{(2)}(x, u(x), s)(1 + |u(x)|^q)$ is bounded.

To get lower semicontinuity for all sequences and on the whole domain, we need an extra condition on the integrand on the boundary, namely, p-quasisubcritical growth from below, as in the case of integrands without explicit dependence on u (cf. Theorem 2).

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a boundary of class C^1, let $\infty > p > n$ and let $h \in \mathbb{H}^{q,p}(\Omega, \mathscr{U}, \mathscr{R})$ (cf. (37)). Then, if $h(x, r, \cdot)$ is quasiconvex for a.e. $x \in \Omega$ (and therefore all $x \in \bar{\Omega}$, by continuity) and all $r \in \mathbb{R}^m$ and $\tilde{h}(x, s) := h(x, u(x), s)$ has p-quasisubcritical growth from below (see (19)) for all $x \in \partial\Omega$ and all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, $w \mapsto \int_\Omega h(x, w(x), \nabla w(x))\, \mathrm{d}x$ is weakly lower semicontinuous in $W^{1,p}(\Omega; \mathbb{R}^m)$.*

Proof. Let $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$. In view of Remark 11, the measures generated by (subsequences of) $\{(u, \nabla u_k)\}$ and $\{(u_k, \nabla u_k)\}$ in the sense of (20) always coincide. As a consequence of (40) and (42), it therefore suffices to show that for each $u \in W^{1,p}(\Omega; \mathbb{R}^m) \subset C(\bar{\Omega}; \mathbb{R}^m)$, $w \mapsto \int_\Omega h(x, u(x), \nabla w(x))\, \mathrm{d}x$ is weakly lower semicontinuous. The latter follows from Theorem 7.

Remark 12. In Theorem 8, quasiconvexity of $h(x, u(x), \cdot)$ in Ω and p-qscb of $h(x, u(x), \cdot)$ at every $x \in \partial\Omega$ are also necessary for weak lower semicontinuity. We omit the details.

As already briefly pointed out in the introduction, the situation becomes significantly more complicated if $p \leq n$. Using our measures to express the limit as in Theorem 7, we can at least reduce the problem to a property of an integrand without explicit dependence on u, for each given sequence:

Proposition 2. *Let $p \le n$, suppose that $h(x,r,\cdot)$ is quasiconvex, $h \in \mathbb{H}^{q,p}(\Omega,\mathscr{U},\mathscr{R})$, and let $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ be a bounded sequence such that $u_k \rightharpoonup u$ and $\{(u_k,\nabla u_k)\}$ generates a measure $(\sigma,\hat{\nu},\hat{\mu})$ in the sense of* (20)*. Then*

$$\liminf_{k\to\infty} \int_\Omega h(x,u_k,\nabla u_k)\,\mathrm{d}x \ge \int_\Omega h(x,u,\nabla u)\,\mathrm{d}x,$$

provided that for σ-a.e. $x \in \bar{\Omega}$,

$$\int_{\bar{\Omega}} \int_{\beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \tilde{h}(x,s)\,\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) \ge 0, \tag{45}$$

where $\tilde{h}(x,s) := (1+|s|^p)\int_{\beta_{\mathscr{U}}} h_0^{(1)}(x,r,s)\,\hat{\mu}_{x,s}(\mathrm{d}r)$. Here, recall that $h(x,r,s) = h_0^{(1)}(x,r,s)(1+|s|^p) + h_0^{(2)}(x,r,s)(1+|r|^q)$, cf. (37).

Proof. This is a straightforward consequence of Theorem 7 and Remark 10.

Remark 13. Given $h \in \mathbb{H}^{q,p}(\Omega,\mathscr{U},\mathscr{R})$, $h_0^{(1)}(x,r,s)$ is uniquely determined for $s \in \beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}$, but not for $s \in \mathbb{R}^{m\times n}$. Of course, (45) actually is only a condition on the restriction of $h_0^{(1)}$ to $\bar{\Omega}\times\beta_{\mathscr{U}}\mathbb{R}^m\times(\beta_{\mathscr{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n})$.

4 Concluding remarks

We have seen that generalized DiPerna-Majda measures introduced here can be helpful in proofs of weak lower semicontinuity. Other applications are, for example, in impulsive control problems where the concentration of controls typically results in discontinuity of the state variable [14]. An open challenging problem is to find some explicit characterization of generalized Diperna-Majda measures generated by pairs of functions and their gradients, namely $\{(u_k,\nabla u_k)\} \subset W^{1,p}(\Omega;\mathbb{R}^m)\times L^p(\Omega;\mathbb{R}^{m\times n})$. This could then help us to find necessary and sufficient conditions for weak lower semicontinuity of $u \mapsto \int_\Omega h(x,u(x),\nabla u(x))\,\mathrm{d}x$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ for $1 < p < +\infty$ and for $h \in \mathbb{H}^p$.

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