SIAM J. MATH. ANAL.
Vol. 50, No. 4, pp. 4426–4456

ON THE PASSAGE FROM NONLINEAR TO LINEARIZED VISCOELASTICITY*

MANUEL FRIEDRICH† AND MARTIN KRUŽÍK‡

Abstract. We formulate a quasistatic nonlinear model for nonsimple viscoelastic materials at a finite-strain setting in the Kelvin–Voigt rheology where the viscosity stress tensor complies with the principle of time-continuous frame indifference. We identify weak solutions in the nonlinear framework as limits of time-incremental problems for vanishing time increment. Moreover, we show that linearization around the identity leads to the standard system for linearized viscoelasticity and that solutions of the nonlinear system converge in a suitable sense to solutions of the linear one. The same property holds for time-discrete approximations, and we provide a corresponding commutativity result. Our main tools are the theory of gradient flows in metric spaces and Γ-convergence.

Key words. viscoelasticity, metric gradient flows, Γ-convergence, dissipative distance, curves of maximal slope, minimizing movements

AMS subject classifications. 74D05, 74D10, 35A15, 35Q74, 49J45

DOI. 10.1137/17M1131428

1. Introduction.

Neglecting inertia, a nonlinear viscoelastic material in Kelvin–Voigt rheology obeys the following system of equations:

$$-\mathrm{div}\Big(\partial_F W(\nabla y) + \partial_{\dot F} R(\nabla y, \partial_t \nabla y)\Big) = f \text{ in } [0,T]\times\Omega. \tag{1}$$

Here, $[0,T]$ is a process time interval with $T>0$, $\Omega\subset\mathbb{R}^d$ ($d=2$ or $d=3$) is a smooth bounded domain representing the reference configuration, and $y:[0,T]\times\Omega\to\mathbb{R}^d$ is a deformation mapping with corresponding deformation gradient ∇y. Further, $W:\mathbb{R}^{d\times d}\to[0,\infty]$ is a stored energy density, which represents a potential of the first Piola–Kirchhoff stress tensor T^E, i.e., $T^E := \partial_F W := \partial W/\partial F$, and $F\in\mathbb{R}^{d\times d}$ is the placeholder of ∇y. Finally, $R:\mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}\to[0,\infty)$ denotes a (pseudo)potential of dissipative forces, where $\dot F\in\mathbb{R}^{d\times d}$ is the placeholder of $\partial_t\nabla y$, and $f:[0,T]\times\Omega\to\mathbb{R}^d$ is a volume density of external forces acting on Ω. In the present contribution, we consider a version of (1) for nonsimple materials where the elastic stored energy density depends also on the second gradient of y. In this case, we get

$$-\mathrm{div}\Big(\partial_F W(\nabla y) + \varepsilon\mathcal{L}_P(\nabla^2 y) + \partial_{\dot F} R(\nabla y, \partial_t \nabla y)\Big) = f \text{ in } [0,T]\times\Omega, \tag{2}$$

where $\varepsilon>0$ is small and $\mathcal{L}_P$ is a first order differential operator which is associated to an additional term $\int_\Omega P(\nabla^2 y)$ in the stored elastic energy; e.g., for $P(G) := \frac{1}{2}|G|^2$ with

*Received by the editors May 22, 2017; accepted for publication (in revised form) June 12, 2018; published electronically August 14, 2018.
http://www.siam.org/journals/sima/50-4/M113142.html
Funding: The work of the authors was funded by the Vienna Science and Technology Fund (WWTF) through project MA14-009. The first author acknowledges support by the Alexander von Humboldt Stiftung and thanks for the warm hospitality at ÚTIA AVČR, where this project was initiated. The second author acknowledges support by the GAČR-FWF project 16-34894L. Both authors were also supported by the MŠMT ČR mobility project 7AMB16AT015.
†Institute for Computational and Applied Mathematics, University of Münster, D-48149 Münster, Germany (manuel.friedrich@uni-muenster.de).
‡Institute of Information Theory and Automation, Czech Academy of Sciences, CZ-182 08 Praha 8, Czechia, and Faculty of Civil Engineering, Czech Technical University, CZ-166 29 Praha 6, Czechia (kruzik@utia.cas.cz).

$G \in \mathbb{R}^{d\times d\times d}$, we get $-\mathrm{div}\mathcal{L}_P(\nabla^2 y) = \Delta^2 y$. We refer to (12) for more details. Thus, we resort to the so-called nonsimple materials, the stored energy density (and the first Piola–Kirchhoff stress tensor, too) of which depends also on the second gradient of the deformation. This idea was first introduced by Toupin [30, 31] and proved to be useful in mathematical elasticity (see, e.g., [6, 8, 12, 23, 24, 26]) because it brings additional compactness to the problem. The first Piola–Kirchhoff stress tensor, T^E, then reads for all $i, j \in \{1, \dots, d\}$

$$T^E{}_{ij}(F,G) := \partial_{F_{ij}} W(F) + \varepsilon\big(\mathcal{L}_P(G)\big)_{ij} = \partial_{F_{ij}} W(F) - \varepsilon \sum_{k=1}^{d} \partial_k\big(\partial_{G_{ijk}} P(G)\big),$$

where $G \in \mathbb{R}^{d\times d\times d}$ is the placeholder for the second gradient of y. The term $\varepsilon\partial_G P(G)$ is usually called hyperstress.

We standardly assume that W as well as P are frame-indifferent functions, i.e., that $W(F) = W(QF)$ and $P(G) = P(QG)$ for every proper rotation $Q \in \mathrm{SO}(d)$, every $F \in \mathbb{R}^{d\times d}$, and every $G \in \mathbb{R}^{d\times d\times d}$. This implies that W depends on the right Cauchy–Green strain tensor $C := F^\top F$; see, e.g., [11]. We wish to emphasize that, in the case of nonsimple materials, no convexity properties of W are needed; in particular, we do not have to assume that W is polyconvex; see [5, 11]. Moreover, it is shown in [21] that if W satisfies suitable and physically relevant growth conditions (as $W(F) \to \infty$ if $\det F \to 0$), then every minimizer of the elastic energy is a weak solution to the corresponding Euler–Lagrange equations.

The second term on the left-hand side of (1) is the viscous stress tensor $S(F, \dot F) := \partial_{\dot F} R(F, \dot F)$ which has its origin in viscous dissipative mechanisms of the material. Notice that its potential R plays an analogous role as W in the case of purely elastic, i.e., nondissipative processes. Naturally, we require that $R(F, \dot F) \geq R(F, 0) = 0$. The viscous stress tensor must comply with the time-continuous frame-indifference principle, meaning that for all F,

$$S(F, \dot F) = F\tilde S(C, \dot C),$$

where $\tilde S$ is a symmetric matrix-valued function. This condition constrains R so that (see [3, 4, 22], also [17])

$$R(F, \dot F) = \tilde R(C, \dot C) \tag{3}$$

for some nonnegative function $\tilde R$. In other words, R must depend on the right Cauchy–Green strain tensor C and its time derivative $\dot C$.

In this work, we are interested in the case of small strains, i.e., when $\nabla u := \nabla y - \mathbf{Id}$ is of order δ for some small $\delta > 0$. Here, $u := y - \mathbf{id}$ is the displacement corresponding to y with $\mathbf{id}$ and $\mathbf{Id}$ standing for the identity map and identity matrix, respectively. Such a property is certainly meaningful if one considers initial values y_0 with $\|\nabla y_0 - \mathbf{Id}\|_{L^2(\Omega)} \leq \delta$. Therefore, it is convenient to define the rescaled displacement $u = \delta^{-1}(y - \mathbf{id})$. Introducing a proper scaling in the above equation we get

$$-\mathrm{div}\Big(\delta^{-1}\partial_F W(\mathbf{Id} + \delta\nabla u) + \tilde\varepsilon\mathcal{L}_P(\delta\nabla^2 u) + \delta^{-1}\partial_{\dot F} R(\mathbf{Id} + \delta\nabla u, \delta\partial_t\nabla u)\Big) = f \tag{4}$$

for $\tilde\varepsilon = \tilde\varepsilon(\delta)$ appropriate. Note that to obtain (4) from (1) we write the latter equation for $f := \delta f$ and then divide the whole equation by δ. Formally, we can pass to the limit and obtain the equation (for $\tilde\varepsilon \to 0$ as $\delta \to 0$)

$$-\mathrm{div}\Big(\mathbb{C}_W e(u) + \mathbb{C}_D e(\partial_t u)\Big) = f, \tag{5}$$

where $\mathbb{C}_W := \partial^2_{F^2} W(\mathbf{Id})$ is the tensor of elastic constants, $\mathbb{C}_D := \partial^2_{\dot{F}^2} R(\mathbf{Id}, 0)$ is the tensor of viscosity coefficients, and $e(u) := (\nabla u + (\nabla u)^\top)/2$ denotes the linear strain tensor.

The goal of this contribution is twofold: we first show existence of solutions to the nonlinear system of equations (4). Afterwards, we make the limit passage rigorous; i.e., we show that solutions to the nonlinear equations converge to the unique solution of the linear systems as $\delta \to 0$. Interestingly, although the nonlinear viscoelastisity system is written for a nonsimple material, in the limit we obtain the standard linear equations without spatial gradients of $e(u)$.

Our general strategy is to treat the system of quasistatic viscoelasticity in the abstract setting of metric gradient flows (see [2]) which was, to our best knowledge, formulated for the first time in [22] for simple materials (i.e., only the first gradient of y is considered). However, in their setting, a passage from time-discrete problems to a continuous one is only possible in a specific one-dimensional case. See also [7] for a related approach in materials undergoing phase transition. This, in our opinion, also supports models of nonsimple materials as their linearization leads to the usual small-strain viscoelasticity model which seems unreachable (or at least rather difficult) in the case of simple materials.

An abstract framework for the study of metric gradient flows along a sequence of energies and metric spaces has been developed in [27, 28]. In practice, for each specific problem the challenge lies in proving that the additional conditions needed to ensure convergence of gradient flows are satisfied (we refer to [28] for some examples in that direction). Our aim is to show that the passage of nonlinear to linearized viscoelasticity can be formulated in this setting. Let us also mention that a rigorous analysis of the static, purely elastic case without viscosity goes back to [15].

Heuristically, the idea of gradient flows in metric spaces stems from the observation that, having a Hilbert space (equipped with the dot product $\langle\cdot,\cdot\rangle$), the inequality

$$|u'|^2 + 2\langle u', \nabla\phi(u)\rangle + |\nabla\phi(u)|^2 \geq 0$$

becomes equality if and only if

$$u' = -\nabla\phi(u),$$

i.e., if u solves the gradient flow equation. This approach can be extended to metric spaces provided we are able to find analogies to $|u'|$ and $|\nabla\phi|$ in metric spaces. These are called the metric derivative and the upper gradient (or slope), respectively. Precise definitions can be found in section 3.1 below.

The plan of the paper is as follows. In section 2, we introduce the nonlinear and linear systems of viscoelasticity in more detail and state our main results. In particular, Theorems 2.1 and 2.2 show the existence of solutions to the nonlinear and linear problems, respectively. These solutions can be identified with so-called *curves of maximal slope* introduced in [16]. Proofs of existence rely on semidiscretization in time and on the theory of *generalized minimizing movements* and gradient flows in metric spaces (see [2]), where the underlying metric is given by a *dissipation distance* suitably related to the potential R (see (10)). Finally, Theorem 2.3 shows the relationship between the two systems. Besides convergence of solutions of (2) to solutions of (5), we also get analogous convergences for semidiscretized problems. Moreover, convergences for vanishing time step and $\delta \to 0$ commute; see Figure 1. (For a related commutativity result in an abstract setting we refer to [10].)

Section 3 is devoted to definitions of generalized minimizing movements and curves of maximal slope. Here we also collect the necessary existence results proved in [2]. Moreover, we present a statement similar to [25, 28] about sequences of curves of maximal slope and their limits as well as a corresponding result for minimizing movements.

Further, section 4 shows interesting properties of dissipation distances related to our viscous dissipation. It turns out that by frame indifference (3) the dissipation distances are genuinely nonconvex. However, due to the presence of the higher order gradient we are able to obtain sufficiently good convexity properties in order to apply the abstract theory (see [2, 28]). Finally, proofs of our results can be found in section 5. In particular, we relate curves of maximal slope for the nonlinear system with limiting curves of maximal slope as $\delta \to 0$ and identify these configurations as weak solutions of (2) and (5).

In what follows, we use standard notation for Lebesgue spaces, $L^p(\Omega)$, which are measurable maps on $\Omega \subset \mathbb{R}^d$ integrable with the pth power (if $1 \le p < +\infty$) or essentially bounded (if $p = +\infty$). Sobolev spaces, i.e., $W^{k,p}(\Omega)$, denote the linear spaces of maps which, together with their derivatives up to the order $k \in \mathbb{N}$, belong to $L^p(\Omega)$. Further, $W^{k,p}_0(\Omega)$ contains maps from $W^{k,p}(\Omega)$ having zero boundary conditions (in the sense of traces). In order to emphasize its Hilbert structure, we write $H^1(\Omega) := W^{1,2}(\Omega)$. We also work with the dual space to $H^1_0(\Omega)$ denoted by $H^{-1}(\Omega)$. We refer to [1] for more details on Sobolev spaces and their duals.

If $A \in \mathbb{R}^{d\times d\times d\times d}$ and $e \in \mathbb{R}^{d\times d}$, then $Ae \in \mathbb{R}^{d\times d}$ such that for $i, j \in \{1, \dots, d\}$ we define $(Ae)_{ij} := A_{ijkl}e_{kl}$, where we use Einstein's summation convention. An analogous convention is used in similar occasions in what follows. Finally, at many spots, we follow closely notation introduced in [2] to ease readability of our work, because the theory developed there is one of the main tools of our analysis.

2. The model and main results.

2.1. The nonlinear setting. We adopt the usual setting of nonlinear elasticity: consider $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Fix $\delta > 0$ (small), $p > d$, and $0 < \alpha < 1$. The parameter $\tilde{\varepsilon}(\delta)$ introduced in (4) is defined as $\tilde{\varepsilon}(\delta) := \delta^{1-p\alpha}$.

Stored elastic energy and body forces. We introduce the nonlinear elastic energy $\phi_\delta : W^{2,p}(\Omega;\mathbb{R}^d) \to [0,\infty]$ by

$$\phi_\delta(y) = \frac{1}{\delta^2}\int_\Omega W(\nabla y(x))\,dx + \frac{1}{\delta^{p\alpha}}\int_\Omega P(\nabla^2 y(x))\,dx - \frac{1}{\delta}\int_\Omega f(x)\cdot y(x)\,dx \tag{6}$$

for a *deformation* $y : W^{2,p}(\Omega;\mathbb{R}^d) \to \mathbb{R}^d$. Here, $W : \mathbb{R}^{d\times d} \to [0,\infty]$ is a single-well, frame-indifferent stored energy functional with the usual assumptions in nonlinear elasticity. Altogether, we suppose that there exists $c > 0$ such that the following holds:

(7)
(i) W is continuous and C^3 in a neighborhood of $SO(d)$.
(ii) Frame indifference: $W(QF) = W(F)$ for all $F \in \mathbb{R}^{d\times d}, Q \in SO(d)$.
(iii) $W(F) \ge c\,\mathrm{dist}^2(F, SO(d)),\ W(F) = 0$ iff $F \in SO(d)$,

where $SO(d) = \{Q \in \mathbb{R}^{d\times d} : Q^\top Q = \mathbf{Id}, \det Q = 1\}$. Moreover, $P : \mathbb{R}^{d\times d\times d} \to [0,\infty]$ denotes a higher order perturbation satisfying the following:

$$
(8) \qquad \begin{aligned}
&\text{(i)} \quad \text{Frame indifference: } P(QG) = P(G) \text{ for all } G \in \mathbb{R}^{d\times d\times d}, Q \in SO(d).\\
&\text{(ii)} \quad P \text{ is convex and } C^1.\\
&\text{(iii)} \quad \text{Growth condition: for all } G \in \mathbb{R}^{d\times d\times d} \text{ we have}\\
&\qquad\qquad c_1|G|^p \le P(G) \le c_2|G|^p, \qquad |\partial_G P(G)| \le c_2|G|^{p-1}
\end{aligned}
$$

for $0 < c_1 < c_2$. Finally, $f \in L^\infty(\Omega;\mathbb{R}^d)$ denotes a volume force. From now on we always drop the target space $\mathbb{R}^d$ for notational convenience when no confusion arises. We remark that by minor adaptions of our arguments we can also treat potentials with additional dependence on the material point $x \in \Omega$. We scale the energy appropriately with a (small) positive parameter δ as we will eventually be interested in the behavior in the small strain limit $\delta \to 0$.

Dissipation potential and viscous stress. Consider a time-dependent deformation $y : [0,T]\times\Omega \to \mathbb{R}^d$. Viscosity is related not only to the strain $\nabla y(t,x)$ but also to the strain rate $\partial_t\nabla y(t,x)$ and can be expressed in terms of a dissipation potential $R(\nabla y, \partial_t\nabla y)$, where $R : \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d} \to [0,\infty)$. An admissible potential has to satisfy frame indifference in the sense (see [3, 22])

$$
(9) \qquad R(F,\dot F) = R(QF, Q(\dot F + AF)) \quad \forall Q \in SO(d), A \in \mathrm{Skew}(d)
$$

for all $F \in GL_+(d)$ and $\dot F \in \mathbb{R}^{d\times d}$, where $GL_+(d) = \{F \in \mathbb{R}^{d\times d} : \det F > 0\}$ and $\mathrm{Skew}(d) = \{A \in \mathbb{R}^{d\times d} : A = -A^\top\}$.

Following the discussion in [22, section 2.2], from the point of modeling it is much more convenient to postulate the existence of a (smooth) global distance $D : GL_+(d) \times GL_+(d) \to [0,\infty)$ satisfying $D(F,F) = 0$ for all $F \in GL_+(d)$, from which an associated dissipation potential R can be calculated by

$$
(10) \qquad R(F,\dot F) := \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon^2}D^2(F+\varepsilon\dot F, F) = \frac{1}{4}\partial^2_{F_1^2}D^2(F,F)[\dot F,\dot F]
$$

for $F \in GL_+(d)$, $\dot F \in \mathbb{R}^{d\times d}$, where $\partial^2_{F_1^2}D^2(F_1,F_2)$ denotes the Hessian of D^2 in direction of F_1 at (F_1,F_2), being a fourth order tensor. We have the following assumptions on D for some $c > 0$.

(11)

$$
\begin{aligned}
&\text{(i)} \quad D(F_1,F_2) > 0 \text{ if } F_1^\top F_1 \neq F_2^\top F_2,\\
&\text{(ii)} \quad D(F_1,F_2) = D(F_2,F_1),\\
&\text{(iii)} \quad D(F_1,F_3) \le D(F_1,F_2) + D(F_2,F_3),\\
&\text{(iv)} \quad D(\cdot,\cdot) \text{ is } C^3 \text{ in a neigborhood of } SO(d)\times SO(d),\\
&\text{(v)} \quad \text{separate frame indifference: } D(Q_1F_1,Q_2F_2) = D(F_1,F_2)\\
&\qquad \text{for all } Q_1,Q_2 \in SO(d), \text{ for all } F_1,F_2 \in GL_+(d),\\
&\text{(vi)} \quad D(F,\mathbf{Id}) \ge c\,\mathrm{dist}(F,SO(d)) \ \text{ for all } F \in \mathbb{R}^{d\times d} \text{ in a neighborhood of } SO(d).
\end{aligned}
$$

Note that conditions (i) and (iii) state that D is a true distance when restricted to symmetric matrices. We cannot expect more due to the separate frame indifference (v). We also note that (v) implies (9) as shown in [22, Lemma 2.1]. Note that in our model we do not require any conditions of polyconvexity for either W or D [5]. For examples of admissible dissipation distances we refer the reader to [22, section 2.3].

Equations of nonlinear viscoelasticity. We will impose the boundary conditions $y(t,x) = x$ for $(t,x) \in [0,T] \times \partial\Omega$, and for convenience we define the set $W^{2,p}_{\mathbf{id}}(\Omega) = \{y = \mathbf{id} + u \in W^{2,p}(\Omega) : u \in W^{2,p}_0(\Omega)\}$, where **id** denotes the identity function on Ω. We remark that our results can be extended to more general Dirichlet boundary conditions, too, which we do not include here for the sake of maximizing simplicity rather than generality. We now introduce a differential operator associated to the perturbation P (cf. (8)). To this end, we use the notation $(\nabla y)_{ik} = \partial_k y_i$ and $(\nabla^2 y)_{ijk} = \partial^2_{jk} y_i$ for $i,j,k \in \{1,\dots,d\}$ and define

$$\big(\mathcal{L}_P(\nabla^2 y)\big)_{ij} = -\sum\nolimits_{k=1}^{d} \partial_k (\partial_G P(\nabla^2 y))_{ijk}, \qquad i,j \in \{1,\dots,d\}, \tag{12}$$

for $y \in W^{2,p}_{\mathbf{id}}(\Omega)$, where the derivatives have to be understood in the sense of distributions. The equations of nonlinear viscoelasticity then read as (respecting the different scalings of the terms in (6))

$$\begin{cases} -\mathrm{div}\Big(\partial_F W(\nabla y) + \delta^{2-p\alpha}\mathcal{L}_P(\nabla^2 y) + \partial_{\dot F} R(\nabla y, \partial_t \nabla y)\Big) = \delta f & \text{in } [0,\infty) \times \Omega, \\ y(0,\cdot) = y_0 & \text{in } \Omega, \\ y(t,\cdot) \in W^{2,p}_{\mathbf{id}}(\Omega) & \text{for } t \in [0,\infty) \end{cases} \tag{13}$$

for some $y_0 \in W^{2,p}_{\mathbf{id}}(\Omega)$, where $\partial_F W(\nabla y(t,x))$ denotes the first *Piola–Kirchhoff stress tensor* and $\partial_{\dot F} R(\nabla y(t,x), \partial_t \nabla y(t,x))$ the *viscous stress* with R as introduced in (10). The first goal of the present contribution is to prove the existence of weak solutions to (13). More precisely, we say that $y \in L^\infty([0,\infty); W^{2,p}_{\mathbf{id}}(\Omega)) \cap W^{1,2}([0,\infty); H^1(\Omega))$ is a *weak solution* of (13) if $y(0,\cdot) = y_0$ and for a.e. $t \geq 0$

$$\begin{aligned} &\int_\Omega \Big(\partial_F W(\nabla y(t,x)) + \partial_{\dot F} R(\nabla y(t,x), \partial_t \nabla y(t,x))\Big) : \nabla\varphi(x)\,dx \\ &\qquad + \int_\Omega \delta^{2-p\alpha} \partial_G P(\nabla^2 y(t,x)) : \nabla^2 \varphi(x)\,dx = \delta \int_\Omega f(x)\cdot\varphi(x)\,dx \end{aligned} \tag{14}$$

for all $\varphi \in W^{2,p}_0(\Omega)$. In particular, we note that the first term in the second line is well defined for a weak solution by (8)(iii) and Hölder's inequality.

2.2. The linear problem. After rescaling with δ^{-1} and introducing the rescaled displacement field $u(t,x) = \delta^{-1}(y(t,x) - x)$, the partial differential equation (13) can be written as

$$-\mathrm{div}\Big(\delta^{-1}\partial_F W(\mathbf{id} + \delta\nabla u) + \delta^{1-p\alpha}\mathcal{L}_P(\delta\nabla^2 u) + \delta^{-1}\partial_{\dot F} R(\mathbf{id} + \delta\nabla u, \delta\partial_t \nabla u)\Big) = f$$

with an initial datum $u_0 = \delta^{-1}(y_0 - \mathbf{id})$. For α small, letting $\delta \to 0$ we obtain, at least formally, the equation

$$\begin{cases} -\mathrm{div}\Big(\mathbb{C}_W e(u) + \mathbb{C}_D e(\partial_t u)\Big) = f & \text{in } [0,\infty) \times \Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \\ u(t,\cdot) \in H^1_0(\Omega) & \text{for } t \in [0,\infty), \end{cases} \tag{15}$$

where $\mathbb{C}_W := \partial^2_{F^2} W(\mathbf{Id})$ and $\mathbb{C}_D := \frac{1}{2}\partial^2_{F_1^2} D^2(\mathbf{Id}, \mathbf{Id})$ (cf. (10)). Note that the frame indifference of the energy and the dissipation (see (7)(ii) and (11)(v), respectively) imply that the contributions only depend on the symmetric part of the strain $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ and the strain rate $e(\partial_t u) = \frac{1}{2}(\partial_t \nabla u + \partial_t (\nabla u)^\top)$. Let us also mention that the stress tensor is related to the linearized elastic energy $\bar{\phi}_0 : H^1_0(\Omega) \to [0,\infty)$ given by

$$\bar{\phi}_0(u) = \int_\Omega \frac{1}{2}\mathbb{C}_W[e(u)(x), e(u)(x)]\,dx - \int_\Omega f(x)\cdot u(x)\,dx \tag{16}$$

for $u \in H^1_0(\Omega)$. The goal of this article is to show that the above reasoning can be made rigorous: we will prove that (15) admits a unique weak solution and that solutions of (13) converge to the solution of (15) in a suitable sense. Here, similarly as before, we say $u \in W^{1,2}([0,\infty); H^1_0(\Omega))$ is a *weak solution* of (15) if $u(0,\cdot) = u_0$ and for a.e. $t \geq 0$ and all $\varphi \in H^1_0(\Omega)$ we have

$$\int_\Omega (\mathbb{C}_W e(u) + \mathbb{C}_D e(\partial_t u)) : \nabla\varphi = \int_\Omega f\cdot\varphi$$

2.3. Main results. Let us introduce the *global dissipation distance* between two deformations for the nonlinear and linear setting by

$$\begin{aligned} \mathcal{D}_\delta(y_0,y_1) &= \delta^{-1}\left(\int_\Omega D^2(\nabla y_0,\nabla y_1)\right)^{1/2}, \\ \bar{\mathcal{D}}_0(u_0,u_1) &= \left(\int_\Omega \mathbb{C}_D[\nabla u_0 - \nabla u_1, \nabla u_0 - \nabla u_1]\right)^{1/2} \end{aligned} \tag{17}$$

for $y_0, y_1 \in W^{2,p}_{\mathbf{id}}(\Omega)$ and $u_0, u_1 \in H^1_0(\Omega)$, respectively. (In many notations we include an overline to indicate that the notion is related to the linear setting.) We also define the sublevel sets $\mathscr{S}^M_\delta := \{y \in W^{2,p}_{\mathbf{id}}(\Omega) : \phi_\delta(y) \leq M\}$. (For convenience we do not include Ω in the notation.) Our general strategy will be to show that the spaces $(\mathscr{S}^M_\delta, \mathcal{D}_\delta)$ and $(H^1_0(\Omega), \bar{\mathcal{D}}_0)$ are complete metric spaces and to follow the approach in [2] (see Theorems 4.5 and 4.6 below).

In particular, to show existence of solutions to (13) and (15), we will apply an approximation scheme solving suitable time-incremental minimization problems and show that time-continuous limits are curves of maximal slope for the elastic energies $\phi_\delta, \bar{\phi}_0$, respectively. Finally, using the property that in Hilbert spaces curves of maximal slope can be related to gradient flows, we find solutions to (13) and (15).

Moreover, to study the relation between the nonlinear and linear problems we will apply some results about the limit of sequences of curves of maximal slope proved in section 3.3.

For the main definitions and notation for discrete solutions, (generalized) minimizing movements (abbreviated by MM and GMM; see Definition 3.2), and curves of maximal slope we refer to section 3.1. In particular, we define Φ_δ and $\bar{\Phi}_0$, respectively, as in (20), replacing $\phi, \mathcal{D}$ by $\phi_\delta, \mathcal{D}_\delta$ and $\bar{\phi}_0, \bar{\mathcal{D}}_0$, respectively. Moreover, we write $|\partial\phi_\delta|_{\mathcal{D}_\delta}$, $|\partial\bar{\phi}_0|_{\bar{\mathcal{D}}_0}$ for the (local) slopes and $|y'|_{\mathcal{D}_\delta}$, $|u'|_{\bar{\mathcal{D}}_0}$ for the metric derivatives, respectively (see Definition 3.1). Finally, discrete solutions for time step $\tau > 0$ will be denoted by $\tilde{Y}^\delta_\tau$ and $\tilde{U}^0_\tau$, respectively.

Our first main result addresses the existence of solutions to the nonlinear problem.

THEOREM 2.1 (solutions to the nonlinear problem). *Let $M > 0$ and $\mathscr{S}^M_\delta = \{y \in W^{2,p}_{\mathbf{id}}(\Omega) : \phi_\delta(y) \leq M\}$. Then for $\delta > 0$ sufficiently small only depending on M the following holds:*

(i) *Existence of GMM. $GMM(\Phi_\delta; y_0) \neq \emptyset$ for all $y_0 \in \mathscr{S}^M_\delta$.*

(ii) *Curves of maximal slope. For all $y_0 \in \mathscr{S}^M_\delta$ each $y \in GMM(\Phi_\delta; y_0)$ is a curve of maximal slope for ϕ_δ with respect to the strong upper gradient $|\partial\phi_\delta|_{\mathcal{D}_\delta}$; in particular for all $T > 0$ we have the energy identity*

$$\frac{1}{2}\int_0^T |y'|^2_{\mathcal{D}_\delta}(t)\,dt + \frac{1}{2}\int_0^T |\partial\phi_\delta|^2_{\mathcal{D}_\delta}(y(t))\,dt + \phi_\delta(y(T)) = \phi_\delta(y_0). \tag{18}$$

(iii) *Relation to PDE. For all* $y_0 \in \mathscr{S}_\delta^M$ *each* $y \in GMM(\Phi_\delta; y_0)$ *is a weak solution of the partial differential equations of nonlinear viscoelasticity* (13) *in the sense of* (14).

For the linearized model we obtain the following results.

THEOREM 2.2 (solutions to the linear problem). *The limiting linear problem has the following properties.*

(i) *Existence/uniqueness of MM. For all* $u_0 \in H^1_0(\Omega)$ *there exists a unique* $u \in MM(\bar{\Phi}_0; u_0)$.

(ii) *Curves of maximal slope. For all* $u_0 \in H^1_0(\Omega)$ *the minimizing movement* $u \in MM(\bar{\Phi}_0; u_0)$ *is the unique curve of maximal slope for* $\bar{\phi}_0$ *with respect to the strong upper gradient* $|\partial\bar{\phi}_0|_{\bar{\mathcal{D}}_0}$.

(iii) *Relation to PDE. For all* $u_0 \in H^1_0(\Omega)$ *the unique* $u \in MM(\Phi_\delta; u_0)$ *is a weak solution of the partial differential equations of linear viscoelasticity* (15).

In contrast to Theorem 2.1, we get that the weak solution to (15) for given initial value $u_0 \in H^1_0(\Omega)$ is uniquely determined and a minimizing movement (and not simply a generalized one). Finally, we study the relation of the solutions to (13) and (15).

THEOREM 2.3 (relation between nonlinear and linear problems). *Fix a null sequence* $(\delta_k)_k$ *and a sequence of initial data* $(y_0^k)_{k\in\mathbb{N}} \subset W^{2,p}_{\mathbf{id}}(\Omega)$ *such that*

$$\sup_{k\in\mathbb{N}} \phi_{\delta_k}(y_0^k) < \infty, \qquad \delta_k^{-p\alpha}\int_\Omega P(\nabla^2 y_0^k) \to 0, \qquad \delta_k^{-1}(y_0^k - \mathbf{id}) \to u_0 \in H^1_0(\Omega).$$

Let u *be the unique element of* $MM(\bar{\Phi}_0; u_0)$. *Then the following holds:*

(i) *Convergence of discrete solutions. For all* $\tau > 0$ *and all discrete solutions* $\tilde{Y}_\tau^{\delta_k}$ *as in* (21) *below there is a discrete solution* $\tilde{U}_\tau^0$ *for the linearized system such that* $\delta_k^{-1}(\tilde{Y}_\tau^{\delta_k}(t) - \mathbf{id}) \to \tilde{U}_\tau^0(t)$ *strongly in* $H^1(\Omega)$ *for all* $t \in [0,\infty)$.

(ii) *Convergence of continuous solutions. Each sequence* $y_k \in GMM(\Phi_{\delta_k}; y_0^k)$, $k \in \mathbb{N}$, *satisfies* $\delta_k^{-1}(y_k(t) - \mathbf{id}) \to u(t)$ *strongly in* $H^1(\Omega)$ *for all* $t \in [0,\infty)$.

(iii) *Convergence at specific scales. For each null sequence* $(\tau_k)_k$ *and each sequence of discrete solutions* $\tilde{Y}_{\tau_k}^{\delta_k}$ *as in* (21) *we have* $\delta_k^{-1}(\tilde{Y}_{\tau_k}^{\delta_k}(t) - \mathbf{id}) \to u(t)$ *strongly in* $H^1(\Omega)$ *for all* $t \in [0,\infty)$.

We remark that, in the formulation of [9, 10], property (iii) states that the configuration u is a minimizing movement along ϕ_{δ_k} at scale τ_k. Let us emphasize that the convergence in Theorem 2.3 is with respect to the strong $H^1(\Omega)$-topology. In particular, Theorems 2.1–2.3 imply a commutativity result which we illustrate in Figure 1. From now on we set $f \equiv 0$ for convenience. The general case indeed follows with minor modifications, which are standard.

3. Preliminaries: Generalized minimizing movements and curves of maximal slope. In this section we first recall the relevant definitions and also give a convergence result for discrete solutions to curves of maximal slope proved in [2]. In section 3.3 we then present a result about the limit of sequences of curves of maximal slope being a variant of results presented in [13, 28].

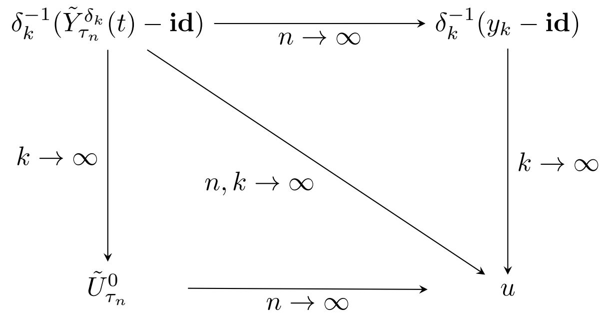

Fig. 1. *Illustration of the commutativity result given in Theorems* 2.1–2.3. *The horizontal arrows are addressed in Theorems* 2.1 *and* 2.2*, respectively. For the vertical and diagonal arrows we refer to Theorem* 2.3.

3.1. Definitions. We consider a complete metric space $(\mathscr{S}, \mathcal{D})$. We say a curve $u : (a, b) \to \mathscr{S}$ is *absolutely continuous* with respect to $\mathcal{D}$ if there exists $m \in L^1(a, b)$ such that

$$\mathcal{D}(u(s), u(t)) \leq \int_s^t m(r)\, dr \quad \forall\ a \leq s \leq t \leq b. \tag{19}$$

The smallest function m with this property, denoted by $|u'|_{\mathcal{D}}$, is called the *metric derivative* of u and satisfies for a.e. $t \in (a, b)$ (see [2, Theorem 1.1.2] for the existence proof)

$$|u'|_{\mathcal{D}}(t) := \lim_{s \to t} \frac{\mathcal{D}(u(s), u(t))}{|s - t|}.$$

We now define the notion of a *curve of maximal slope*. We only give the basic definition here and refer to [2, sections 1.2 and 1.3] for motivations and more details. By $h^+ := \max(h, 0)$ we denote the positive part of a function h.

Definition 3.1 (upper gradients, slopes, curves of maximal slope). *We consider a complete metric space $(\mathscr{S}, \mathcal{D})$ with a functional $\phi : \mathscr{S} \to (-\infty, +\infty]$.*

(i) *A function $g : \mathscr{S} \to [0, \infty]$ is called a strong upper gradient for ϕ if for every absolutely continuous curve $v : (a, b) \to \mathscr{S}$ the function $g \circ v$ is Borel and*

$$|\phi(v(t)) - \phi(v(s))| \leq \int_s^t g(v(r))|v'|_{\mathcal{D}}(r)\, dr \quad \forall\ a < s \leq t < b.$$

(ii) *For each $u \in \mathscr{S}$ the local slope of ϕ at u is defined by*

$$|\partial\phi|_{\mathcal{D}}(u) := \limsup_{w \to u} \frac{(\phi(u) - \phi(w))^+}{\mathcal{D}(u, w)}.$$

(iii) *An absolutely continuous curve $u : (a, b) \to \mathscr{S}$ is called a curve of maximal slope for ϕ with respect to the strong upper gradient g if for a.e. $t \in (a, b)$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) \leq -\frac{1}{2}|u'|^2_{\mathcal{D}}(t) - \frac{1}{2}g^2(u(t)).$$

We now introduce minimizing movements. In the following we will use an approximation scheme solving suitable time-incremental minimization problems: consider a fixed time step $\tau > 0$, and suppose that an initial datum U^0_τ is given. Whenever $U^0_\tau, \ldots, U^{n-1}_\tau$ are known, U^n_τ is defined as (if existent)

$$U^n_\tau = \mathrm{argmin}_{v\in\mathcal{S}}\Phi(\tau, U^{n-1}_\tau; v), \quad \Phi(\tau, u; v) := \frac{1}{2\tau}\mathcal{D}(v, u)^2 + \phi(v). \tag{20}$$

Supposing that for a choice of τ a sequence $(U^n_\tau)_{n\in\mathbb{N}}$ solving (20) exists, we define the piecewise constant interpolation by

$$\tilde{U}_\tau(0) = U^0_\tau, \quad \tilde{U}_\tau(t) = U^n_\tau \text{ for } t \in ((n-1)\tau, n\tau], \; n \geq 1. \tag{21}$$

In the following, $\tilde{U}_\tau$ will be called a *discrete solution.* Note that the existence of discrete solutions is usually guaranteed by the direct method of the calculus of variations under suitable compactness, coercivity, and lower semicontinuity assumptions. Finally, we introduce the *modulus of the derivative*

$$|\tilde{U}'_\tau|_{\mathcal{D}}(t) = \frac{\mathcal{D}(U^n_\tau, U^{n-1}_\tau)}{\tau} \quad \text{for } t \in ((n-1)\tau, n\tau], \; n \geq 1.$$

DEFINITION 3.2 (minimizing movements). (i) *We say a curve $u : [0,\infty) \to \mathscr{S}$ is a minimizing movement for Φ as defined in* (20)*, starting from the initial datum $u_0 \in \mathscr{S}$, if for every sequence of time steps $(\tau_k)_k$ with $\tau_k \to 0$ there exist discrete solutions defined in* (21) *such that*

$$\begin{aligned} &\lim_{k\to\infty} \phi(U^0_{\tau_k}) = \phi(u_0), \qquad \limsup_{k\to\infty} \mathcal{D}(U^0_{\tau_k}, u_0) < \infty, \\ &\lim_{k\to\infty} \mathcal{D}(\tilde{U}_{\tau_k}(t), u(t)) = 0 \quad \forall t \in [0,\infty). \end{aligned} \tag{22}$$

By $MM(\Phi; u_0)$ we denote the collection of all minimizing movements for Φ starting from u_0.

(ii) *Likewise, we say a curve $u : [0,\infty) \to \mathscr{S}$ is a generalized minimizing movement for Φ starting from $u_0 \in \mathscr{S}$ if there exists a sequence of time steps $(\tau_k)_k$ with $\tau_k \to 0$ and corresponding discrete solutions such that* (22) *holds. The collection of all such curves is denoted by $GMM(\Phi; u_0)$.*

3.2. Compactness of discrete solutions and convergence to curves of maximal slope. Suppose again that $(\mathscr{S}, \mathcal{D})$ is a complete metric space. As discussed in [2, Remark 2.0.5], it is convenient to introduce a weaker topology on $\mathscr{S}$ to have more flexibility in the derivation of compactness properties. Assume that there is a Hausdorff topology σ on $\mathscr{S}$, which is compatible with $\mathcal{D}$ in the sense that σ is weaker than the topology induced by $\mathcal{D}$ and satisfies

$$u_n \overset{\sigma}{\to} u, \;\; v_n \overset{\sigma}{\to} v \quad \Rightarrow \quad \liminf_{n\to\infty} \mathcal{D}(u_n, v_n) \geq \mathcal{D}(u, v). \tag{23}$$

Consider a functional $\phi : \mathscr{S} \to [0, +\infty)$ with the following properties:

$$\begin{aligned} &\text{(i)} \;\; u_n \overset{\sigma}{\to} u, \quad \sup\nolimits_{n,m} \mathcal{D}(u_n, u_m) < \infty \quad \Rightarrow \quad \liminf_{n\to\infty} \phi(u_n) \geq \phi(u); \\ &\text{(ii)} \;\; \text{for all } N \in \mathbb{N} \text{ there is a } \sigma\text{-sequentially compact set } K_N \text{ such that} \\ &\qquad \{u \in \mathscr{S} : \phi(u) + \mathcal{D}(u, u_*) \leq N\} \subset K_N \text{ for some point } u_* \in \mathscr{S}. \end{aligned} \tag{24}$$

Note that nonnegativity of ϕ can be generalized to a suitable *coerciveness* condition, (see [2, Equation (2.1.2b)]), which we do not include here for the sake of simplicity. From [2, Proposition 2.2.3, Theorem 2.3.3, Remark 2.3.4(i)] we obtain the following compactness and convergence result.

THEOREM 3.3. *Suppose that ϕ satisfies* (24) *and $v \in \mathscr{S} \mapsto |\partial\phi|_{\mathcal{D}}(v)$ is a strong upper gradient for ϕ and σ-lower semicontinuous. Then the following holds:*

(i) *Suppose that there is a sequence of initial data $(U^0_{\tau_k})_{k\in\mathbb{N}}$ and $u_0 \in \mathscr{S}$ with $\sup_k \mathcal{D}(U^0_{\tau_k}, u_0) < +\infty$, $U^0_{\tau_k} \overset{\sigma}{\to} u_0$, and $\phi(U^0_{\tau_k}) \to \phi(u_0)$. Then there is an absolutely continuous curve $u : [0,\infty) \to \mathscr{S}$ and a subsequence, not relabeled, of $(\tau_k)_{k\in\mathbb{N}}$ such that a sequence of discrete solutions $(\tilde{U}_{\tau_k})_{k\in\mathbb{N}}$ defined in* (21) *satisfies $\tilde{U}_{\tau_k}(t) \overset{\sigma}{\to} u(t)$ for all $t \in [0,\infty)$.*

(ii) *Every $u \in GMM(\Phi; u_0)$ for each $u_0 \in \mathscr{S}$ is a curve of maximal slope for ϕ with respect to $|\partial\phi|_{\mathcal{D}}$, and in particular u satisfies the energy identity*

$$\frac{1}{2}\int_0^T |u'|^2_{\mathcal{D}}(t)\,dt + \frac{1}{2}\int_0^T |\partial\phi|^2_{\mathcal{D}}(u(t))\,dt + \phi(u(T)) = \phi(u_0) \quad \forall T > 0. \tag{25}$$

Moreover, for a sequence of discrete solutions $(\tilde{U}_{\tau_k})_{k\in\mathbb{N}}$ as in (i) *we have*

$$\lim_{k\to\infty} \phi(\tilde{U}_{\tau_k}(t)) = \phi(u(t)) \quad \forall t \in [0,\infty),$$
$$\lim_{k\to\infty} |\partial\phi|_{\mathcal{D}}(\tilde{U}_{\tau_k}) = |\partial\phi|_{\mathcal{D}}(u) \quad \textit{in} \quad L^2_{\mathrm{loc}}([0,\infty)),$$
$$\lim_{k\to\infty} |\tilde{U}'_{\tau_k}|_{\mathcal{D}} = |u'|_{\mathcal{D}} \quad \textit{in} \quad L^2_{\mathrm{loc}}([0,\infty)).$$

In particular, Theorem 3.3(i) states that the limit u is a generalized minimizing movement, provided that σ coincides with the topology induced by $\mathcal{D}$. We remark that $GMM(\Phi; u_0)$ could also be defined with respect to the weaker topology σ; see [2, Definition 2.0.6]. For our purposes, however, a definition in terms of $\mathcal{D}$ is more convenient.

The result can be considerably improved if Φ satisfies suitable convexity properties (see [2, Theorems 4.0.4 and 4.0.7]).

THEOREM 3.4. *Suppose that ϕ is $\mathcal{D}$-lower semicontinuous and $\phi \geq 0$. Moreover, assume that for all $\tau > 0$ and for all $w, v_0, v_1 \in \mathscr{S}$ there exists a curve $(\gamma_t)_{t\in[0,1]} \subset \mathscr{S}$ with $\gamma_0 = v_0$ and $\gamma_1 = v_1$ such that*

$$\Phi(\tau, w; \gamma_t) \leq (1-t)\Phi(\tau, w; v_0) + t\Phi(\tau, w; v_1) - \frac{t(1-t)}{2\tau}\mathcal{D}(v_0, v_1)^2 \quad \forall t \in [0,1].$$

Then for each $u_0 \in \mathscr{S}$ there exists a unique $u \in MM(\Phi; u_0)$. Moreover, the assertion of Theorem 3.3 *(with σ being the topology induced by $\mathcal{D}$) holds, and for a discrete solution $\tilde{U}_\tau$ with $U^0_\tau = u_0$ we have $\mathcal{D}(\tilde{U}_\tau(t), u(t))^2 \leq \frac{1}{2}\tau^2|\partial\phi|^2_{\mathcal{D}}(u_0)$ for all $t > 0$.*

Note that in contrast to Theorem 3.3, Theorem 3.4 yields also a uniqueness result for minimizing movements. Observe that (24)(ii) is not necessary for Theorem 3.4 since the solvability of the problem $\mathrm{argmin}_{v\in\mathscr{S}}\Phi(\tau, u; v)$ for $\tau > 0$ and $u \in \mathscr{S}$ (cf. (20)) follows from a convexity argument. In this setting, much more refined results can be established, and we refer to [2, section 4] for more details.

3.3. Limits of curves of maximal slopes. We now consider a set $\mathscr{S}$ and a sequence of metrics $(\mathcal{D}_n)_n$ on $\mathscr{S}$ as well as a limiting metric $\mathcal{D}$. We again assume that all metric spaces are complete. Moreover, let $(\phi_n)_n$ be a sequence of functionals with $\phi_n : \mathscr{S} \to [0,\infty]$. Suppose that there is a Hausdorff topology σ on $\mathscr{S}$ which is weaker than the topology induced by each $\mathcal{D}_n, \mathcal{D}$ and satisfies similarly to (23)

$$u_n \overset{\sigma}{\to} u, \;\; v_n \overset{\sigma}{\to} v \quad \Rightarrow \quad \liminf_{n\to\infty} \mathcal{D}_n(u_n, v_n) \geq \mathcal{D}(u,v). \tag{26}$$

Moreover, assume that $(\phi_n)_n$ satisfy (24)(ii), i.e., for all $N \in \mathbb{N}$ there is a σ-sequentially compact set K_N and $u_* \in \mathscr{S}$ such that for all $n \in \mathbb{N}$

$$\{u \in \mathscr{S} : \phi_n(u) + \mathcal{D}_n(u, u_*) \le N\} \subset K_N. \tag{27}$$

To ensure the existence of limiting curves of maximal slope, we will apply the following refined version of the Arzelà–Ascoli theorem.

THEOREM 3.5. *Let $T > 0$, and let metrics $\mathcal{D}_n$, $\mathcal{D}$ and functionals $(\phi_n)_n$ be given such that* (26) *holds with respect to the topology σ. Let $K \subset \mathscr{S}$ be a σ-sequentially compact set. Let $u_n : [0,T] \to \mathscr{S}$ be curves such that*

$$u_n(t) \in K \ \forall n \in \mathbb{N}, t \in [0,T], \qquad \limsup_{n\to\infty} \mathcal{D}_n(u_n(s), u_n(t)) \le \omega(s,t) \quad \forall s,t \in [0,T]$$

for a symmetric function $\omega : [0,T]^2 \to [0,\infty)$ with

$$\lim_{(s,t)\to(r,r)} \omega(s,t) = 0 \qquad \forall r \in [0,T] \setminus \mathscr{C},$$

where $\mathscr{C}$ is an at most countable subset of $[0,T]$. Then there exists a (not relabeled) subsequence and a limiting curve $u : [0,T] \to \mathscr{S}$ such that

$$u_n(t) \overset{\sigma}{\to} u(t) \quad \forall t \in [0,T], \quad u \text{ is } \mathcal{D}\text{-continuous in } [0,T] \setminus \mathscr{C}.$$

Proof. We follow the proof of [2, Proposition 3.3.1] with the only difference being that the lower semicontinuity condition for the metric is replaced by our condition (26) along the sequence of metrics. ∎

Now consider also a limiting functional $\phi : \mathscr{S} \to [0,\infty]$. We suppose lower semicontinuity of the functionals and the slopes in the following sense: for all $u \in \mathscr{S}$ and $(u_k)_k \subset \mathscr{S}$ we have

$$u_k \overset{\sigma}{\to} u \quad \Rightarrow \quad \liminf_{k\to\infty} |\partial\phi_k|_{\mathcal{D}_k}(u_k) \ge |\partial\phi|_{\mathcal{D}}(u), \quad \liminf_{k\to\infty} \phi_k(u_k) \ge \phi(u). \tag{28}$$

We now obtain the following result about limits of curves of maximal slope.

THEOREM 3.6. *Consider a set $\mathscr{S}$, metrics $(\mathcal{D}_n)_{n\in\mathbb{N}}$ and functionals $\phi_n : \mathscr{S} \to [0,\infty]$, $n \in \mathbb{N}$, as well as $\mathcal{D}$ and $\phi : \mathscr{S} \to [0,\infty]$. Suppose that there is a weaker topology σ on $\mathscr{S}$ such that* (26), (27), *and the implication* (28) *hold. Moreover, assume that $|\partial\phi_n|_{\mathcal{D}_n}$, $|\partial\phi|_{\mathcal{D}}$ are strong upper gradients for ϕ_n, ϕ with respect to $\mathcal{D}_n$, $\mathcal{D}$, respectively.*

Let $T > 0$ and $\bar{u} \in \mathscr{S}$. For all $n \in \mathbb{N}$ let u_n be a curve of maximal slope for ϕ_n with respect to $|\partial\phi_n|_{\mathcal{D}_n}$ such that

$$\begin{aligned} &\text{(i)} \quad \sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \big(\phi_n(u_n(t)) + \mathcal{D}_n(u_n(t), \bar{u})\big) < \infty, \\ &\text{(ii)} \quad u_n(0) \overset{\sigma}{\to} \bar{u}, \quad \phi_n(u_n(0)) \to \phi(\bar{u}). \end{aligned} \tag{29}$$

Then there exists a limiting function $u : [0,T] \to \mathscr{S}$ such that up to a subsequence, not relabeled,

$$u_n(t) \overset{\sigma}{\to} u(t), \qquad \phi_n(u_n(t)) \to \phi(u(t)) \quad \forall t \in [0,T]$$

as $n \to \infty$ and u is a curve of maximal slope for ϕ with respect to $|\partial\phi|_{\mathcal{D}}$.

The result is an adaption of a statement in [28] where condition (26) is replaced by a lower bound condition on the metric derivatives along the sequence. We also refer to [13], where a similar result is proved without the assumption that the slopes are *strong* upper gradients (cf. [2, Definitions 1.2.1 and 1.2.2] for the definition of strong and weak upper gradients), which comes at the expense that a suitable continuity condition along $(\phi_k)_k$ for sequences $(u_k)_k$ converging with respect to the metric has to be imposed.

Proof. From the properties of a curve of maximal slope we have (cf. (25))

$$\frac{1}{2}\int_0^t |u_n'|^2_{\mathcal{D}_n}(s)\,ds + \frac{1}{2}\int_0^t |\partial\phi_n|^2_{\mathcal{D}_n}(u_n(s))\,ds + \phi_n(u_n(t)) = \phi_n(u_n(0)) \tag{30}$$

for all $t \in [0,T]$. (Here, we have used that $|\partial\phi_n|_{\mathcal{D}_n}$ are strong upper gradients for ϕ_n with respect to $\mathcal{D}_n$.) From (30) and the equiboundedness of $\phi_n(u_n(t))$ (see (29)(i)) we get

$$\sup_{n\in\mathbb{N}}\int_0^T |u_n'|^2_{\mathcal{D}_n}(t)\,dt + \sup_{n\in\mathbb{N}}\int_0^T |\partial\phi_n|^2_{\mathcal{D}_n}(u_n(t))\,dt < \infty.$$

Consequently, there is a function $A \in L^2((0,T))$ such that $|u_n'|_{\mathcal{D}_n} \rightharpoonup A$ weakly in $L^2((0,T))$ up to a subsequence, not relabeled. In particular, this yields

$$\limsup_{n\to\infty}\mathcal{D}_n(u_n(s),u_n(t)) \le \limsup_{n\to\infty}\int_s^t |u_n'|_{\mathcal{D}_n} \le \omega(s,t) := \int_s^t A(r)\,dr \tag{31}$$

for all $0 \le s \le t \le T$ by (19). Using (27), (29)(i), and (31), we can apply Theorem 3.5 and obtain an absolutely continuous curve $u : [0,T] \to \mathscr{S}$ as well as a further subsequence (not relabeled) such that $u_n(t) \stackrel{\sigma}{\to} u(t)$ for all $t \in [0,T]$. Moreover, recalling (26) we get $\mathcal{D}(u(s),u(t)) \le \int_s^t A(r)\,dr$, which gives $|u'| \le A$. By (28) we get

$$|\partial\phi|_{\mathcal{D}}(u(t)) \le \liminf_{n\to\infty}|\partial\phi_n|_{\mathcal{D}_n}(u_n(t)), \quad \phi(u(t)) \le \liminf_{n\to\infty}\phi_n(u_n(t))$$

for $t \in [0,T]$. This together with the fact that $|u_n'|_{\mathcal{D}_n} \rightharpoonup A$ weakly in $L^2((0,T))$ and $|u'| \le A$ gives

$$\begin{aligned}
&\frac{1}{2}\int_0^t |u'|^2_{\mathcal{D}}(s)\,ds + \frac{1}{2}\int_0^t |\partial\phi|^2_{\mathcal{D}}(u(s))\,ds + \phi(u(t))\\
&\le \frac{1}{2}\int_0^t A^2(s)\,ds + \frac{1}{2}\int_0^t \liminf_{n\to\infty}|\partial\phi_n|^2_{\mathcal{D}_n}(u_n(s))\,ds + \liminf_{n\to\infty}\phi_n(u_n(t))\\
&\le \liminf_{n\to\infty}\left(\frac{1}{2}\int_0^t |u_n'|^2_{\mathcal{D}_n}(s)\,ds + \frac{1}{2}\int_0^t |\partial\phi_n|^2_{\mathcal{D}_n}(u_n(s))\,ds + \phi_n(u_n(t))\right)
\end{aligned}$$

for all $t \in [0,T]$, where in the second step we used Fatou's lemma. Using (29)(ii), (30), and $\bar{u} = u(0)$ we get

$$\frac{1}{2}\int_0^t |u'|^2_{\mathcal{D}}(s)\,ds + \frac{1}{2}\int_0^t |\partial\phi|^2_{\mathcal{D}}(u(s))\,ds + \phi(u(t)) \le \liminf_{n\to\infty}\phi_n(u_n(0)) = \phi(u(0)).$$

On the other hand, as $|\partial\phi|_{\mathcal{D}}$ is a strong upper gradient for ϕ with respect to $\mathcal{D}$, we obtain (recall Definition 3.1)

$$\phi(u(0)) \le \phi(u(t)) + \int_0^t |\partial\phi|_{\mathcal{D}}(u(s))|u'|_{\mathcal{D}}(s)\,ds.$$

Therefore, combining the previous estimates and using Young's inequality we derive

$$|u'|_{\mathcal{D}}(t) = |\partial\phi|_{\mathcal{D}}(u(t)), \quad \phi(u(0)) - \phi(u(t)) = \int_0^t |\partial\phi|_{\mathcal{D}}(u(s))|u'|_{\mathcal{D}}(s)\,ds$$

for a.e. $t \in [0,T]$ and $\lim_{n\to\infty} \phi_n(u_n(t)) = \phi(u(t))$ for all $t \in [0,T]$. It follows that u is absolutely continuous, and for a.e. $t \in [0,T]$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) = -|\partial\phi|_{\mathcal{D}}(u(t))|u'|_{\mathcal{D}}(t).$$

This concludes the proof. $\square$

We now study discrete solutions along the sequence of functionals $(\phi_n)_n$.

THEOREM 3.7. *Consider a set $\mathscr{S}$, metrics $(\mathcal{D}_n)_{n\in\mathbb{N}}$, and functionals $\phi_n : \mathscr{S} \to [0,\infty)$, $n \in \mathbb{N}$, as well as $\mathcal{D}$ and $\phi : \mathscr{S} \to [0,\infty)$. Suppose that there is a weaker topology σ on $\mathscr{S}$ such that (26), (27), and the implication (28) hold. Moreover, assume that $|\partial\phi|_{\mathcal{D}}$ is a strong upper gradient for ϕ with respect to $\mathcal{D}$.*

Let $T > 0$. Consider a null sequence $(\tau_k)_k$ and initial data $(U^0_{\tau_k})_k$, $\bar{u}$ with

$$\sup_k \mathcal{D}_k(U^0_{\tau_k}, \bar{u}) < +\infty, \qquad U^0_{\tau_k} \overset{\sigma}{\to} \bar{u}, \qquad \phi_k(U^0_{\tau_k}) \to \phi(\bar{u}).$$

Then for each sequence of discrete solutions $(\tilde{U}_{\tau_k})_k$ starting from $(U^0_{\tau_k})_k$ there is a curve u of maximal slope for ϕ with respect to $|\partial\phi|_{\mathcal{D}}$ such that up to a subsequence, not relabeled, $\tilde{U}_{\tau_k}(t) \overset{\sigma}{\to} u(t)$ and $\phi_k(\tilde{U}_{\tau_k}(t)) \to \phi(u(t))$ for $t \in [0,T]$.

For the proof we refer to [25, section 2]. Let us also mention the recently obtained variant [10] where, similarly to [13], the lower semicontinuity along the sequence $(\phi_n)_n$ (see (28)) is replaced by a continuity condition. Note that in their setting it is not necessary to require that $|\partial\phi|_{\mathcal{D}}$ is a strong upper gradient.

4. Properties of energies and dissipation distances. In this section we prove several properties about the energies and dissipation distances. Let $\delta > 0$ and $0 < \alpha < 1$ and recall the definition of the nonlinear energy in (6)–(8) as well as (11). We recall that $\mathscr{S}^M_\delta = \{y \in W^{2,p}_{\mathbf{id}}(\Omega) : \phi_\delta(y) \le M\}$. In the whole section, $C \ge 1$ and $0 < c \le 1$ indicate generic constants, which may vary from line to line and depend on M, Ω, the exponent $p > d$ (see (8)), and on the constants in (7), (8), and (11), but are always independent of the small parameter δ.

4.1. Basic properties. We start with some properties about the Hessian of W and D. By $\partial^2 D^2$ we denote the Hessian and by $\partial^2_{F_1^2} D^2, \partial^2_{F_2^2} D^2$ the Hessian in the direction of the first or second entry of D^2, respectively. Moreover, we define $\mathrm{sym}(F) = \frac{F+F^\top}{2}$ for $F \in \mathbb{R}^{d\times d}$ and recall the definition of $\mathbb{C}_W, \mathbb{C}_D$ in (15). By $\mathbf{Id} \in \mathbb{R}^{d\times d}$ we again denote the identity matrix.

LEMMA 4.1 (properties of the Hessian). *Let $F_1, F_2 \in \mathbb{R}^{d\times d}$ and $Y \in \mathbb{R}^{d\times d}$ in a neighborhood of $\mathbf{Id}$ such that $\partial^2 D^2(Y,Y)$ exists.*

(i) *We have $\partial^2 D^2(Y,Y)[(F_1,F_2),(F_1,F_2)] = \partial^2_{F_1^2} D^2(Y,Y)[F_1 - F_2, F_1 - F_2] = \partial^2_{F_2^2} D^2(Y,Y)[F_1 - F_2, F_1 - F_2]$.*

(ii) *We have $\partial^2 D^2(\mathbf{Id},\mathbf{Id})[(F_1,F_2),(F_1,F_2)] = \mathbb{C}_D[\mathrm{sym}(F_1 - F_2), \mathrm{sym}(F_1 - F_2)]$.*

(iii) *There is a constant $c > 0$ independent of F such that $\mathbb{C}_W[F,F] \ge c|\mathrm{sym}(F)|^2$, $\mathbb{C}_D[F,F] \ge c|\mathrm{sym}(F)|^2$.*

Proof. (i) Set $H = \partial^2 D^2(Y,Y)$ for brevity. By symmetry (11)(ii) we find two fourth order tensors $H_1, H_2 : \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d} \to \mathbb{R}$ such that $H[(F_1,F_2),(F_1,F_2)] = H_1[F_1,F_1] + 2H_2[F_1,F_2] + H_1[F_2,F_2]$ and $H_2[F_1,F_2] = H_2[F_2,F_1]$. Note that $H_1 = \partial^2_{F_1^2} D^2(Y,Y) = \partial^2_{F_2^2} D^2(Y,Y)$. As $D(F,F) = 0$ for all $F \in GL_+(d)$, we get that $H[(F,F),(F,F)] = 0$ for all $F \in \mathbb{R}^{d\times d}$. Thus, we obtain $H_1[F,F] = -H_2[F,F]$ for all $F \in \mathbb{R}^{d\times d}$, and we compute

$$\begin{aligned} H_1[F_1 - F_2, &F_1 - F_2] \\ &= -H_2[F_1 - F_2, F_1 - F_2] = -H_2[F_1,F_1] + 2H_2[F_1,F_2] - H_2[F_2,F_2] \\ &= H_1[F_1,F_1] + 2H_2[F_1,F_2] + H_1[F_2,F_2] = H[(F_1,F_2),(F_1,F_2)]. \end{aligned}$$

Property (ii) follows from frame indifference (11)(v) by an elementary computation. Finally, the growth condition for $\mathbb{C}_W$ and $\mathbb{C}_D$ stated in (iii) follow from (7)(iii) and (11)(vi), respectively. ∎

In the following, by **id** we again denote the identity function.

LEMMA 4.2 (rigidity). *There is constant $C > 1$ independent of δ such that for δ sufficiently small for all $y \in \mathscr{S}_\delta^M$ we have*

(i) $\|y - \mathbf{id}\|_{H^1(\Omega)} \le C\|\operatorname{dist}(\nabla y, SO(d))\|_{L^2(\Omega)}$,

(ii) $\|\nabla y - \mathbf{Id}\|_{L^\infty(\Omega)} \le C\delta^\alpha, \quad \|y - \mathbf{id}\|_{L^\infty(\Omega)} \le C\delta^\alpha$.

Proof. (i) is a typical geometric rigidity argument, see, e.g., [15, 19]. By [19, Theorem 3.1] and Poincaré's inequality we find a rotation $Q \in SO(d)$ and $b \in \mathbb{R}^d$ such that

$$\|y - (Q\cdot + b)\|_{H^1(\Omega)} \le C\|\operatorname{dist}(\nabla y, SO(d))\|_{L^2(\Omega)}. \tag{32}$$

Passing to a trace estimate and using $y = \mathbf{id}$ on $\partial\Omega$, we get $\|\mathbf{id} - (Q\cdot + b)\|_{L^2(\partial\Omega)} \le C\|\operatorname{dist}(\nabla y, SO(d))\|_{L^2(\Omega)}$. Using [15, Lemma 3.3] we then find $|b| + |Q - \mathbf{Id}| \le C\|\mathbf{id} - (Q\cdot + b)\|_{L^2(\partial\Omega)}$ for a constant only depending on Ω. This together with (32) implies (i).

We now prove (ii). By the definition of ϕ_δ and (8)(iii) we get $\|\nabla^2 y\|^p_{L^p(\Omega)} \le CM\delta^{p\alpha}$ for all $y \in \mathscr{S}_\delta^M$. As $p > d$, Poincaré's inequality yields some $F \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$ such that

$$\|y - (F\cdot + b)\|_{W^{1,\infty}(\Omega)} \le C\delta^\alpha \tag{33}$$

for a constant additionally depending on Ω, M, and p. Using $\phi_\delta(y) \le M$, (7)(iii), and (i) we compute

$$\|(F\cdot + b) - \mathbf{id}\|^2_{H^1(\Omega)} \le C\|\operatorname{dist}(\nabla y, SO(d))\|^2_{L^2(\Omega)} + C|\Omega|\delta^{2\alpha} \le C\delta^2 M + C|\Omega|\delta^{2\alpha}.$$

Since $\alpha \le 1$, this gives $|b| + |F - \mathbf{Id}| \le C\delta^\alpha$, which together with (33) yields (ii). ∎

In the following we set for shorthand $H_Y := \frac{1}{2}\partial^2_{F_1^2} D^2(Y,Y) = \frac{1}{2}\partial^2_{F_2^2} D^2(Y,Y)$ for $Y \in GL_+(d)$, and given a deformation $y \in W^{2,p}_{\mathbf{id}}(\Omega)$ we also introduce the mapping $H_{\nabla y} : \Omega \to \mathbb{R}^{d\times d\times d\times d}$ by $H_{\nabla y}(x) = H_{\nabla y(x)}$ for $x \in \Omega$. Recall the definition of $\mathcal{D}_\delta, \bar{\mathcal{D}}_0$ in (17) and $\mathbb{C}_W$ below (15).

LEMMA 4.3 (dissipation and energy). *There are constants $0 < c < 1$, $C > 1$ independent of δ such that for all $y, y_0, y_1 \in \mathscr{S}_\delta^M$ for δ sufficiently small we have*

(i) $\left|\delta^2\mathcal{D}_\delta(y_0,y_1)^2 - \int_\Omega H_{\nabla y_0}[\nabla(y_1 - y_0), \nabla(y_1 - y_0)]\right| \le C\|\nabla(y_1 - y_0)\|^3_{L^3(\Omega)}$,

(ii) $c\|y_1 - y_0\|_{H^1(\Omega)} \le \delta \mathcal{D}_\delta(y_0, y_1) \le C\|y_1 - y_0\|_{H^1(\Omega)}$,
(iii) $\left|\mathcal{D}_\delta(y_0, y_1)^2 - \bar{\mathcal{D}}_0(u_0, u_1)^2\right| \le C\delta^\alpha$,
(iv) $\left|\delta^{-2}\int_\Omega W(\nabla y) - \int_\Omega \frac{1}{2}\mathbb{C}_W[e(u), e(u)]\right| \le C\delta^\alpha$,
where $u = \delta^{-1}(y - \mathbf{id})$ *and* $u_i = \delta^{-1}(y_i - \mathbf{id})$, $i = 0, 1$. *In particular,* (ii) *shows that the topologies induced by* $\mathcal{D}_\delta$ *and* $\|\cdot\|_{H^1(\Omega)}$ *coincide.*

Proof. Recall that D^2 is C^3 in a neighborhood of $(\mathbf{Id}, \mathbf{Id})$. In view of the uniform bound on $\nabla y_0, \nabla y_1$ (see Lemma 4.2(ii)) and a Taylor expansion of D^2 at $(\nabla y_0, \nabla y_0)$, we derive by Lemma 4.1

$$\int_\Omega D^2(\nabla y_0, \nabla y_1) = \int_\Omega H_{\nabla y_0}[\nabla(y_1 - y_0), \nabla(y_1 - y_0)] + O(\|\nabla(y_1 - y_0)\|^3_{L^3(\Omega)}).$$

This gives (i). We obtain $\|H_{\nabla y_0} - \mathbb{C}_D\|_{L^\infty(\Omega)} \le C\delta^\alpha$ by regularity of D and Lemma 4.2(ii). This together with (i), Lemma 4.2(ii), and Lemma 4.1 yields

$$\int_\Omega D^2(\nabla y_0, \nabla y_1) = \int_\Omega \mathbb{C}_D[e(y_1) - e(y_0), e(y_1) - e(y_0)] + O(\delta^\alpha \|\nabla y_1 - \nabla y_0\|^2_{L^2(\Omega)}). \tag{34}$$

Now by (34), Lemma 4.1(iii), and Korn's inequality we derive for δ small enough

$$\begin{aligned}\int_\Omega D^2(\nabla y_0, \nabla y_1) &\ge c\|e(y_1) - e(y_0)\|^2_{L^2(\Omega)} + O(\delta^\alpha \|\nabla y_1 - \nabla y_0\|^2_{L^2(\Omega)}) \\ &\ge c\|\nabla y_1 - \nabla y_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Here we used that $y_1 - y_0 = 0$ on $\partial\Omega$. The first inequality in (ii) follows from Poincaré's inequality. The other inequality can be seen along similar lines. By Lemma 4.2(i), (7)(iii), and the fact that $y_0, y_1 \in \mathscr{S}^M_\delta$ we get

$$\|\nabla y_i - \mathbf{Id}\|^2_{L^2(\Omega)} \le C\|\operatorname{dist}(\nabla y_i, SO(d))\|^2_{L^2(\Omega)} \le C\phi_\delta(y_i) \le CM\delta^2 \tag{35}$$

for $i = 0, 1$. Recalling the definition of $\mathcal{D}_\delta, \bar{\mathcal{D}}_0$, we now obtain (iii) by (34).

Finally, to see (iv), an argument very similar to (i), essentially relying on a Taylor expansion and Lemma 4.3(ii), yields

$$\left|\delta^{-2}\int_\Omega W(\nabla y) - \int_\Omega \frac{1}{2}\mathbb{C}_W[e(u), e(u)]\right| \le C\delta^{\alpha-2}\|\nabla y - \mathbf{Id}\|^2_{L^2(\Omega)},$$

which together with (35) implies the claim. ∎

We close this section with proving differentiablity of $\int_\Omega W(\nabla y)$.

LEMMA 4.4 (differentiablity of $\int_\Omega W(\nabla y)$). *For* $(y_n)_n \subset \mathscr{S}^M_\delta$ *and* $y \in \mathscr{S}^M_\delta$ *with* $\mathcal{D}_\delta(y_n, y) \to 0$*, we have*

$$\lim_{n\to\infty} \frac{\int_\Omega W(\nabla y_n) - \int_\Omega W(\nabla y) - \int_\Omega \partial_F W(\nabla y) : (\nabla y_n - \nabla y)}{\mathcal{D}_\delta(y_n, y)} = 0.$$

Proof. By a Taylor expansion we find a universal constant $C' > 0$ such that $|W(F_2) - W(F_1) - \partial_F W(F_1) : (F_2 - F_1)| \le C'|F_1 - F_2|^2$ for all F_1, F_2 with $|F_1 - \mathbf{Id}|, |F_2 - \mathbf{Id}| \le C\delta^\alpha$, where C is the constant in Lemma 4.2(ii). This together with Lemmas 4.2(ii) and 4.3(ii) gives the result. ∎

4.2. Metric spaces and convexity. In this section we show that $(\mathscr{S}_\delta^M, \mathcal{D}_\delta)$, $(H_0^1(\Omega), \bar{\mathcal{D}}_0)$ are complete metric spaces and derive convexity properties for the energies and dissipation distances.

THEOREM 4.5 (properties of $(\mathscr{S}_\delta^M, \mathcal{D}_\delta)$ and ϕ_δ). *For $\delta > 0$ small enough we have the following:*

(i) *$(\mathscr{S}_\delta^M, \mathcal{D}_\delta)$ is a complete metric space.*

(ii) *Compactness: If $(y_n)_n \subset \mathscr{S}_\delta^M$, then $(y_n)_n$ admits a subsequence converging weakly in $W^{2,p}(\Omega)$, strongly in $W^{1,\infty}(\Omega)$, and with respect to $\mathcal{D}_\delta$.*

(iii) *Lower semicontinuity: $\mathcal{D}_\delta(y_n, y) \to 0 \quad \Rightarrow \quad \liminf_{n\to\infty} \phi_\delta(y_n) \geq \phi_\delta(y)$.*

Proof. First, recalling (6) and (8)(iii), we have $\|\nabla^2 y\|^p_{L^p(\Omega)} \leq CM\delta^{p\alpha}$ for all $y \in \mathscr{S}_\delta^M$, which together with Lemma 4.2(ii) shows $\sup_{y\in\mathscr{S}_\delta^M} \|y\|_{W^{2,p}(\Omega)} < \infty$. This implies (ii) recalling $p > d$ and also using Lemma 4.3(ii). In particular, for a sequence $(y_n)_n$ converging to y with respect to $\mathcal{D}_\delta$ we have $y_n \rightharpoonup y$ weakly in $W^{2,p}(\Omega)$ and $y_n \to y$ strongly in $W^{1,\infty}(\Omega)$. Then (iii) follows from Fatou's lemma and the fact that $\liminf_{n\to\infty} \int_\Omega P(\nabla^2 y_n) \geq \int_\Omega P(\nabla^2 y)$ by (8)(ii).

We now finally show (i). Apart from the positivity, all properties of a metric follow directly from (11) and (17). To show that if $\mathcal{D}_\delta(y_0, y_1) = 0$ for $y_0, y_1 \in \mathscr{S}_\delta^M$, then $y_0 = y_1$, we apply Lemma 4.3(ii). Finally, it remains to show that $(\mathscr{S}_\delta^M, \mathcal{D}_\delta)$ is complete. Let $(y_k)_k$ be a Cauchy sequence with respect to $\mathcal{D}_\delta$. By (ii) we find $y \in W^{2,p}(\Omega)$ and a subsequence (not relabeled) such that $y_k \to y$ in $W^{1,\infty}(\Omega)$. Then also $\lim_{k\to\infty} \mathcal{D}_\delta(y_k, y) = 0$ by Lemma 4.3(ii). By (iii) we get $y \in \mathscr{S}_\delta^M$. The fact that $(y_k)_k$ is a Cauchy sequence now implies that the whole sequence y_k converges to y with respect to $\mathcal{D}_\delta$. This concludes the proof. ∎

Similar properties can be derived in the linear setting. Recall the definition of $\bar{\mathcal{D}}_0$ in (17).

THEOREM 4.6 (properties of $(H_0^1(\Omega), \bar{\mathcal{D}}_0)$ and $\bar{\phi}_0$). *We have the following:*

(i) *$(H_0^1(\Omega), \bar{\mathcal{D}}_0)$ is a complete metric space.*

(ii) *Continuity: $\bar{\mathcal{D}}_0(u_n, u) \to 0 \quad \Rightarrow \quad \lim_{n\to\infty} \bar{\phi}_0(u_n) = \bar{\phi}_0(u)$.*

Proof. By Lemma 4.1(iii) we find a constant $c > 0$ such that

$$\bar{\mathcal{D}}_0(u_0, u_1)^2 \geq c\|e(u_0) - e(u_1)\|^2_{L^2(\Omega)} \geq \|u_0 - u_1\|^2_{H^1(\Omega)},$$

where the last step follows from Korn's and Poincaré's inequality. This show that $(H_0^1(\Omega), \bar{\mathcal{D}}_0)$ is a complete metric space, where $\bar{\mathcal{D}}_0$ is equivalent to the metric induced by $\|\cdot\|_{H^1(\Omega)}$. Recalling (16) we find that $\bar{\phi}_0$ is continuous with respect to $\bar{\mathcal{D}}_0$. ∎

The following properties are crucial to use the theory in [2].

THEOREM 4.7 (convexity and generalized geodesics in the nonlinear setting). *There is a constant $C \geq 1$ independent of δ such that, for δ small and for all $y_0, y_1 \in \mathscr{S}_\delta^M$,*

$$\text{(i)} \quad \mathcal{D}_\delta(y_s, y_0)^2 \leq s^2 \mathcal{D}_\delta(y_1, y_0)^2 (1 + C\|\nabla y_1 - \nabla y_0\|_{L^\infty(\Omega)}),$$
$$\text{(ii)} \quad \phi_\delta(y_s) \leq (1-s)\phi_\delta(y_0) + s\phi_\delta(y_1),$$

where $y_s := (1-s)y_0 + sy_1$, $s \in [0,1]$.

Note that y_s is not a geodesic in the sense of [2, Definition 2.4.2], but y_s can be understood as a generalized geodesic. We also refer to [22, sections 3.2 and 3.4] for a discussion about generalized geodesics in a related setting.

Proof. Let $y_s = (1-s)y_0 + sy_1$. By Lemma 4.3(i) we obtain

$$\delta^2 \mathcal{D}_\delta(y_1, y_0)^2 \geq \int_\Omega H_{\nabla y_0}[\nabla(y_1 - y_0), \nabla(y_1 - y_0)] - C\int_\Omega |\nabla y_1 - \nabla y_0|^3.$$

Likewise, we get

$$\delta^2 \mathcal{D}_\delta(y_s, y_0)^2 \leq s^2 \int_\Omega H_{\nabla y_0}[\nabla(y_1 - y_0), \nabla(y_1 - y_0)] + Cs^3 \int_\Omega |\nabla y_1 - \nabla y_0|^3.$$

Combining the two estimates, we therefore obtain

$$\mathcal{D}_\delta(y_s, y_0)^2 \leq s^2\big(\mathcal{D}_\delta(y_1, y_0)^2 + C\delta^{-2}\|\nabla y_1 - \nabla y_0\|^3_{L^3(\Omega)}\big),$$

which together with Lemma 4.3(ii) shows (i). To see (ii), it suffices to show that $\int_\Omega W(\nabla y_s) \leq (1-s)\int_\Omega W(\nabla y_0) + s\int_\Omega W(\nabla y_1)$ since P is convex (see (8)(ii)). A Taylor expansion gives $\int_\Omega W(\nabla y) = \frac{1}{2}\int_\Omega \mathbb{C}_W[\nabla y, \nabla y] + \omega(\nabla y)$ for a (regular) function $\omega : \mathbb{R}^{d\times d} \to \mathbb{R}$ with $\partial_F \omega(0) = 0$ and $\partial^2_{F^2}\omega(0) = 0$. We get

$$\begin{aligned}\int_\Omega \mathbb{C}_W[\nabla y_s, \nabla y_s] = (1-s)\int_\Omega \mathbb{C}_W[\nabla y_0, \nabla y_0] + s\int_\Omega \mathbb{C}_W[\nabla y_1, \nabla y_1] \\ - s(1-s)\int_\Omega \mathbb{C}_W[\nabla(y_1 - y_0), \nabla(y_1 - y_0)].\end{aligned} \tag{36}$$

Denote by $B_{2C\delta^\alpha}(\mathbf{Id}) \subset \mathbb{R}^{d\times d}$ the ball with center $\mathbf{Id}$ and radius $2C\delta^\alpha$ with the constant C from Lemma 4.2(ii). Since $F \mapsto \omega(F) + \frac{1}{2}\|\partial^2_{F^2}\omega\|_{L^\infty(B_{2C\delta^\alpha}(\mathbf{Id}))}|F|^2$ is convex on $B_{2C\delta^\alpha}(\mathbf{Id})$, we get by Lemma 4.2(ii)

$$\begin{aligned}\int_\Omega \omega(\nabla y_s) \leq s\int_\Omega \omega(\nabla y_0) + (1-s)\int_\Omega \omega(\nabla y_1) \\ + \frac{1}{2}s(1-s)\|\partial^2_{F^2}\omega\|_{L^\infty(B_{2C\delta^\alpha}(\mathbf{Id}))}\int_\Omega |\nabla y_1 - \nabla y_0|^2.\end{aligned}$$

By the fact that $\partial^2_{F^2} w(0) = 0$ and the regularity of ω we find $\|\partial^2_{F^2}\omega\|_{L^\infty(B_{2C\delta^\alpha}(\mathbf{Id}))} \leq C\delta^\alpha$. Combining the previous three estimates and recalling that

$$\int_\Omega W(\nabla y) = \frac{1}{2}\int_\Omega \mathbb{C}_W[\nabla y, \nabla y] + \omega(\nabla y),$$

we conclude

$$\begin{aligned}&\int_\Omega W(\nabla y_s) - (1-s)\int_\Omega W(\nabla y_0) - s\int_\Omega W(\nabla y_1) \\ &\leq -s(1-s)\int_\Omega \mathbb{C}_W[\nabla(y_1 - y_0), \nabla(y_1 - y_0)] + \frac{1}{2}s(1-s)C\delta^\alpha\int_\Omega |\nabla(y_1 - y_0)|^2 \leq 0\end{aligned}$$

for δ small enough, where the last step follows from Lemma 4.1(iii) and Korn's inequality. ∎

We note without proof that by a similar reasoning as in (ii) one can show that for given $w \in \mathscr{S}^M_\delta$

$$\mathcal{D}_\delta(y_s, w)^2 \leq (1-s)\mathcal{D}_\delta(y_0, w)^2 + s\mathcal{D}_\delta(y_1, w)^2 - s(1-s)(1 - C\delta^\alpha)\mathcal{D}_\delta(y_1, y_0)^2.$$

This implies that $\mathcal{D}_\delta$ is $2(1 - C\delta^\alpha)$-convex in the sense of [2, Assumption 4.0.1]. Note that this property is not strong enough to apply directly the results in [2, sections 2.4 and 4]. Nevertheless, we will be able to derive representations and lower semicontinuity properties for the slopes by direct computations (see Lemmas 4.9 and 5.3 below.) However, in the linear setting we obtain 2-convexity, as the following result shows.

Lemma 4.8 (convexity in the linear setting). *For all $u_0, u_1 \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$ with $u_s := (1-s)u_0 + su_1$ we have*

$$\bar{\mathcal{D}}_0(u_s, v)^2 \le (1-s)\bar{\mathcal{D}}_0(u_0, v)^2 + s\bar{\mathcal{D}}_0(u_1, v)^2 - s(1-s)\bar{\mathcal{D}}_0(u_1, u_0)^2.$$

Proof. The property follows from an elementary computation as in (36) taking into account that $\bar{\mathcal{D}}_0^2$ is quadratic. ∎

4.3. Properties of local slopes. We now derive representations and properties of the slopes corresponding to ϕ_δ and $\bar{\phi}_0$. Recall Definition 3.1.

Lemma 4.9 (slopes). (i) *For $\delta > 0$ small enough the local slopes in the nonlinear setting admit the representation*

$$|\partial \phi_\delta|_{\mathcal{D}_\delta}(y) = \sup_{w \neq y} \frac{(\phi_\delta(y) - \phi_\delta(w))^+}{\mathcal{D}_\delta(y, w)(1 + C\|\nabla y - \nabla w\|_{L^\infty(\Omega)})^{1/2}} \qquad \forall y \in \mathscr{S}_\delta^M,$$

where C is the constant from Theorem 4.7. The slopes are lower semicontinuous with respect to both $H^1(\Omega)$ and $\mathcal{D}_\delta$ and are strong upper gradients for ϕ_δ.

(ii) *The local slope for the linear energy $\bar{\phi}_0$ admits the representation*

$$|\partial \bar{\phi}_0|_{\bar{\mathcal{D}}_0}(u) = \sup_{v \neq u} \frac{(\bar{\phi}_0(u) - \bar{\phi}_0(v))^+}{\bar{\mathcal{D}}_0(u, v)}$$

and is a strong upper gradient for $\bar{\phi}_0$.

Proof. Before we start with the actual proof, let us recall from [2, Lemma 1.2.5] that in a complete metric space $(\mathscr{S}, \mathcal{D})$ with energy ϕ one has that $|\partial\phi|_{\mathcal{D}}$ is a weak upper gradient for ϕ in the sense of [2, Definition 1.2.2]. We do not repeat the definition of weak upper gradients but only mention that weak upper gradients are also strong upper gradients if for each absolutely continuous curve $z : (a, b) \to \mathscr{S}$ with $|\partial\phi|_{\mathcal{D}}(z)|z'|_{\mathcal{D}} \in L^1(a, b)$, the function $\phi \circ z$ is absolutely continuous.

Moreover, [2, Lemma 1.2.5] also states that if ϕ is $\mathcal{D}$-lower semicontinuous, then the global slope

$$\mathcal{S}_\phi(v) := \sup_{w \neq v} \frac{(\phi(v) - \phi(w))^+}{\mathcal{D}(v, w)} \tag{37}$$

is a strong (and thus also weak) upper gradient for ϕ.

We now give the proof of (i). We partially follow the proofs of Theorem 2.4.9 and Corollary 2.4.10 in [2]. To confirm the representation of $|\partial\phi_\delta|_{\mathcal{D}_\delta}$, we use the definition of the local slope in Definition 3.1 and obtain, with C being the constant from Theorem 4.7(i),

$$\begin{aligned} |\partial \phi_\delta|_{\mathcal{D}_\delta}(y) &= \limsup_{w \to y} \frac{(\phi_\delta(y) - \phi_\delta(w))^+}{\mathcal{D}_\delta(y, w)} = \limsup_{w \to y} \frac{(\phi_\delta(y) - \phi_\delta(w))^+}{\mathcal{D}_\delta(y, w)(1 + C\|\nabla y - \nabla w\|_\infty)^{1/2}} \\ &\le \sup_{w \neq y} \frac{(\phi_\delta(y) - \phi_\delta(w))^+}{\mathcal{D}_\delta(y, w)(1 + C\|\nabla y - \nabla w\|_\infty)^{1/2}}, \end{aligned}$$

where in the second equality we used that $w \to y$ (with respect to $\mathcal{D}_\delta$) implies $\|\nabla w - \nabla y\|_{L^\infty(\Omega)} \to 0$ by Theorem 4.5(ii). To see the other inequality, it is not restrictive to suppose that $y \neq w$ and

$$\phi_\delta(y) - \phi_\delta(w) > 0. \tag{38}$$

By Theorem 4.7(ii) with $y_0 = y$ and $y_1 = w$ we get

$$\frac{\phi_\delta(y) - \phi_\delta(y_s)}{\mathcal{D}_\delta(y, y_s)} \geq \frac{\phi_\delta(y) - \phi_\delta(w)}{\mathcal{D}_\delta(y, w)} \frac{s\mathcal{D}_\delta(y, w)}{\mathcal{D}_\delta(y, y_s)}$$

for all $s \in [0, 1]$, where $y_s = (1 - s)y + sw$. Then we derive by (38) and Theorem 4.7(i)

$$|\partial\phi_\delta|_{\mathcal{D}_\delta}(y) \geq \frac{\phi_\delta(y) - \phi_\delta(w)}{\mathcal{D}_\delta(y, w)(1 + C\|\nabla y - \nabla w\|_\infty)^{1/2}}.$$

The claim now follows by taking the supremum with respect to w. To confirm the lower semicontinuity, we consider $y_h \to y$ in $\mathcal{D}_\delta$ or equivalently in $H^1(\Omega)$ (see Lemma 4.3(ii)). If $w \neq y$, then $w \neq y_h$ for h large enough, and thus

$$\begin{aligned}\liminf_{h\to\infty} |\partial\phi_\delta|_{\mathcal{D}_\delta}(y_h) &\geq \liminf_{h\to\infty} \frac{(\phi_\delta(y_h) - \phi_\delta(w))^+}{\mathcal{D}_\delta(y_h, w)(1 + C\|\nabla y_h - \nabla w\|_\infty)^{1/2}} \\ &\geq \frac{(\phi_\delta(y) - \phi_\delta(w))^+}{\mathcal{D}_\delta(y, w)(1 + C\|\nabla y - \nabla w\|_\infty)^{1/2}},\end{aligned}$$

where we used Theorem 4.5(ii),(iii). By taking the supremum with respect to w the lower semicontinuity follows.

It remains to show that $|\partial\phi_\delta|_{\mathcal{D}_\delta}$ is a strong upper gradient. With Lemma 4.2(ii), for δ small enough we find $\mathcal{S}_{\phi_\delta}(y) \leq 2|\partial\phi_\delta|_{\mathcal{D}_\delta}(y)$ with $\mathcal{S}_{\phi_\delta}$ as introduced in (37). Recalling the remarks at the beginning of the proof, to show that $|\partial\phi_\delta|_{\mathcal{D}_\delta}$ is a strong upper gradient we have to check that for all absolutely continuous $z : (a, b) \to \mathscr{S}_\delta^M$ with $|\partial\phi_\delta|_{\mathcal{D}_\delta}(z)|z'|_{\mathcal{D}_\delta} \in L^1(a, b)$, the function $\phi_\delta \circ z$ is absolutely continuous. First, it follows $\mathcal{S}_{\phi_\delta}(z)|z'|_{\mathcal{D}_\delta} \in L^1(a, b)$ as $\mathcal{S}_{\phi_\delta} \leq 2|\partial\phi_\delta|_{\mathcal{D}_\delta}$. Since ϕ_δ is $\mathcal{D}_\delta$-lower semicontinous, $\mathcal{S}_{\phi_\delta}$ is a strong upper gradient. Thus, we indeed get that $\phi_\delta \circ z$ is absolutely continuous; see Definition 3.1.

We now concern ourselves with (ii). The representation of the local slope follows from the convexity property in Lemma 4.8 as was shown in [2, Theorem 2.4.9]. Therefore, $\mathcal{S}_{\bar{\phi}_0} = |\partial\bar{\phi}_0|_{\bar{\mathcal{D}}_0}$, which is $\bar{\mathcal{D}}_0$ lower semicontinous by Lemma 4.6(ii), and thus $|\partial\bar{\phi}_0|_{\bar{\mathcal{D}}_0}$ is a strong upper gradient. ∎

5. Proof of the main results. In this section we give the proof of Theorems 2.1–2.3.

5.1. Existence of curves of maximal slope. In this section we prove the first two parts of Theorems 2.1 and 2.2, which essentially follow from the properties of the metric spaces established in sections 4.2 and 4.3 by applying the general results recalled in section 3.2.

Proof of Theorem 2.1(i),(ii). First, we note that the assumptions of Theorem 3.3 are satisfied by Lemmas 4.9(i) and 4.5(ii),(iii), where we let $\mathscr{S} = \mathscr{S}_\delta^M$ and let σ be the topology induced by $\mathcal{D}_\delta$.

(i) Fix $y_0 \in \mathscr{S}_\delta^M$. Define the initial data $U_\tau^0 = y_0$ for all $\tau > 0$. Applying Theorem 3.3(i) we find a curve y which is the limit of a sequence of discrete solutions with $y(0) = y_0$. Thus, in view of Definition 3.2, $y \in GMM(\Phi_\delta; y_0)$, which is therefore nonempty.

(ii) To see that generalized minimizing movements are curves of maximal slope, it suffices to apply Theorem 3.3(ii). ∎

Proof of Theorem 2.2(i),(ii). In the linear setting the convexity property given in Lemma 4.8 holds, and $\bar\phi_0$ is convex by (16) and Lemma 4.1(iii). Thus, Theorem 3.4 is applicable. Apart from uniqueness, the result then follows from Theorem 3.4. It remains to show that the unique minimizing movement is also the unique curve of maximal slope for $\bar\phi_0$ with respect to the strong upper gradient $|\partial\bar\phi_0|_{\bar{\mathcal{D}}_0}$. To this end, we follow an idea used, e.g., in [20].

We first observe that the metric derivative $|u'|^2_{\bar{\mathcal{D}}_0}$ is convex. Indeed, let $u^1, u^2 : [0,\infty) \to H^1_0(\Omega)$ be two curves. We get for $u^3 = \frac{1}{2}(u^1 + u^2)$ by Young's inequality (define $v^i = u^i(s) - u^i(t)$, $i = 1, 2$, for brevity)

$$\begin{aligned}
\bar{\mathcal{D}}_0(u^3(s), u^3(t))^2 &= \int_\Omega \mathbb{C}_D[e((v^1+v^2)/2), e((v^1+v^2)/2)] \\
&= \sum\nolimits_{i=1,2} \frac{1}{4}\int_\Omega \mathbb{C}_D[e(v^i), e(v^i)] + \frac{1}{2}\int_\Omega \mathbb{C}_D[e(v^1), e(v^2)] \\
&\le \sum\nolimits_{i=1,2} \frac{1}{2}\int_\Omega \mathbb{C}_D[e(v^i), e(v^i)] = \frac{1}{2}\bar{\mathcal{D}}_0(u^1(s), u^1(t))^2 + \frac{1}{2}\bar{\mathcal{D}}_0(u^2(s), u^2(t))^2.
\end{aligned}$$

Dividing by $|s-t|^2$ and letting s go to t we obtain the claim. We also anticipate from Lemma 5.4 below that $u \mapsto |\partial\bar\phi_0|^2_{\bar{\mathcal{D}}_0}(u)$ is convex.

Assume there were two different curves of maximal slope u^1, u^2 starting from u_0; i.e., we find some T such that $e(u^1(T)) \neq e(u^2(T))$ since otherwise the curves would coincide by Korn's inequality. Set $u^3 = \frac{1}{2}(u^1 + u^2)$, and compute by the strict convexity of $\mathbb{C}_W$ on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ (see Lemma 4.1(iii)), the convexity properties of the slope and metric derivative, and (25)

$$\begin{aligned}
\bar\phi_0(u_0) &= \frac{1}{2}\sum_{i=1,2}\left(\frac{1}{2}\int_0^T |(u^i)'|^2_{\bar{\mathcal{D}}_0}(t)\,dt + \frac{1}{2}\int_0^T |\partial\bar\phi_0|^2_{\bar{\mathcal{D}}_0}(u^i(t))\,dt + \bar\phi_0(u^i(T))\right) \\
&> \frac{1}{2}\int_0^T |(u^3)'|^2_{\bar{\mathcal{D}}_0}(t)\,dt + \frac{1}{2}\int_0^T |\partial\bar\phi_0|^2_{\bar{\mathcal{D}}_0}(u^3(t))\,dt + \bar\phi_0(u^3(T)),
\end{aligned}$$

which contradicts the fact that $|\partial\bar\phi_0|_{\bar{\mathcal{D}}_0}$ is an upper gradient (see Definition 3.1(i), and use Young's inequality). This contradiction establishes uniqueness and concludes the proof. ∎

5.2. Γ-convergence and lower semicontinuity. As a preparation for the passage to the linear problem, we recall and prove Γ-convergence results for the energies and lower semicontinuity for the slopes. In the following it is convenient to express all quantities in terms of the linear setting. To this end, recalling (6) and (17), for $u, v \in W^{2,p}_0(\Omega)$ and $\tau, \delta > 0$ we define

$$\begin{aligned}
&\bar\phi_\delta(u) = \phi_\delta(\mathbf{id} + \delta u), \quad \bar\phi_{\delta,P}(u) = \delta^{-p\alpha}\int_\Omega P(\delta\nabla^2 u), \quad \bar\phi_{\delta,W}(u) = \bar\phi_\delta(u) - \bar\phi_{\delta,P}(u), \\
&\bar{\mathcal{D}}_\delta(u,v) = \mathcal{D}_\delta(\mathbf{id} + \delta u, \mathbf{id} + \delta v), \quad \bar\Phi_\delta(\tau, v; u) = \bar\phi_\delta(u) + \frac{1}{2\tau}\bar{\mathcal{D}}_\delta(u,v)^2, \\
&|\partial\bar\phi_\delta|_{\bar{\mathcal{D}}_\delta}(u) = |\partial\phi_\delta|_{\mathcal{D}_\delta}(\mathbf{id} + \delta u).
\end{aligned}$$

We extend $\bar\phi_\delta$ to a functional defined on $H^1_0(\Omega)$ by setting $\bar\phi_\delta(u) = +\infty$ for $u \in H^1_0(\Omega)\setminus W^{2,p}_0(\Omega)$. Likewise, we extend $\bar\Phi_\delta$. Moreover, we say $u \in \bar{\mathscr{S}}^M_\delta$ if $\mathbf{id}+\delta u \in \mathscr{S}^M_\delta$. We obtain the following Γ-convergence results. (For an exhaustive treatment of Γ-convergence we refer the reader to [14].)

THEOREM 5.1 (Γ-convergence). *Let $(\delta_n)_n$ be a null sequence.*

(i) *The functionals $\bar{\phi}_{\delta_n} : H^1_0(\Omega) \to [0,\infty]$ Γ-converge to $\bar{\phi}_0$ in the weak $H^1(\Omega)$-topology.*

(ii) *For each $\tau > 0$, $M > 0$, and each sequence $(\bar{v}_n)_n$ with $\bar{v}_n \in \bar{\mathscr{S}}^M_{\delta_n}$ and $\bar{v}_n \to \bar{v}$ strongly in $H^1(\Omega)$, the functionals $\bar{\Phi}_{\delta_n}(\tau, \bar{v}_n; \cdot) : H^1_0(\Omega) \to [0,\infty]$ Γ-converge to $\bar{\Phi}_0(\tau, \bar{v}; \cdot)$ in the weak $H^1(\Omega)$-topology.*

Proof. (i) The result is essentially proved in [15], and we only give a short sketch highlighting the relevant adaptions. Since $\bar{\phi}_{\delta_n,P} \geq 0$, for the lower bound it suffices to prove $\liminf_{n\to\infty} \bar{\phi}_{\delta_n,W}(u_n) \geq \bar{\phi}_0(u)$ whenever $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$. This was proved under more general assumptions in [15, Proposition 4.4]. In our setting it follows readily by using Lemma 4.3(iv) and the lower semicontinuity of $\bar{\phi}_0$ (see Lemma 4.1(iii)).

By a general approximation argument in the theory of Γ-convergence it suffices to establish the upper bound for smooth functions u; cf. [15, Proposition 4.1]. For such a function, setting $u_n = u$, we find $\lim_n \bar{\phi}_{\delta_n,W}(u_n) = \bar{\phi}_0(u)$ (see Lemma 4.3(iv) or [15, Proposition 4.1]), and moreover it is not hard to see that $\bar{\phi}_{\delta_n,P}(u_n) \to 0$ by the growth of P and the fact that $\alpha < 1$. This concludes the proof of (i).

(ii) We first suppose that the sequence $(\bar{v}_n)_n$ is constantly $\bar{v}$. Then $\bar{\Phi}_{\delta_n}(\tau, \bar{v}; \cdot)$ Γ-converges to $\bar{\Phi}_0(\tau, \bar{v}; \cdot)$ repeating exactly the proof of (i), where, in addition to Lemma 4.3(iv), we also use Lemma 4.3(iii). To obtain the general case, it now suffices to prove that for every sequence $(u_n)_n$ uniformly bounded in $H^1_0(\Omega)$ and $u_n \in \bar{\mathscr{S}}^M_{\delta_n}$ for some M large enough we obtain

$$\lim_{n\to\infty} |\bar{\mathcal{D}}_{\delta_n}(u_n, \bar{v}_n)^2 - \bar{\mathcal{D}}_{\delta_n}(u_n, \bar{v})^2| = 0.$$

In view of Lemma 4.3(iii), it suffices to show $\lim_{n\to\infty} |\bar{\mathcal{D}}_0(u_n, \bar{v}_n)^2 - \bar{\mathcal{D}}_0(u_n, \bar{v})^2| = 0$. To this end, we note that (recall (17))

$$\begin{aligned}\bar{\mathcal{D}}_0(u_n, \bar{v}_n)^2 - \bar{\mathcal{D}}_0(u_n, \bar{v})^2 &= \int_\Omega \mathbb{C}_D[\nabla \bar{v}_n, \nabla \bar{v}_n] - \int_\Omega \mathbb{C}_D[\nabla \bar{v}, \nabla \bar{v}] \\ &\quad - 2\int_\Omega \mathbb{C}_D[\nabla u_n, \nabla \bar{v}_n - \nabla \bar{v}],\end{aligned}$$

which by the assumption on $(\bar{v}_n)_n$ and $(u_n)_n$ converges to zero. ◻

We remark that by a general result in the theory of Γ-convergence we get that (almost) minimizers associated to the sequence of functionals converge to minimizers of the limiting functional. We obtain the following strong convergence result for recovery sequences which in various settings has been derived in, e.g., [15, 18, 29].

LEMMA 5.2 (strong convergence of recovery sequences). *Suppose that the assumptions of Theorem 5.1 hold. Let $M > 0$, and let $(u_n)_n$ be a sequence with $u_n \in \bar{\mathscr{S}}^M_{\delta_n}$. Let $u \in H^1_0(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ and*

$$\text{(i)}\quad \bar{\phi}_{\delta_n}(u_n) \to \bar{\phi}_0(u) \quad \textit{or} \quad \text{(ii)}\quad \bar{\Phi}_{\delta_n}(\tau, \bar{v}_n; u_n) \to \bar{\Phi}_0(\tau, \bar{v}; u).$$

Then $u_n \to u$ strongly in $H^1(\Omega)$.

Proof. If $\bar{\phi}_{\delta_n}(u_n) \to \bar{\phi}_0(u)$, we find $\bar{\phi}_0(u_n) \to \bar{\phi}_0(u)$ by Lemma 4.3(iv), and thus by Lemma 4.1(iii)

$$\begin{aligned}&\|e(u_n - u)\|^2_{L^2(\Omega)} \leq C \int_\Omega \mathbb{C}_W[e(u_n - u), e(u_n - u)] \\ &\quad = C\left(\int_\Omega \mathbb{C}_W[e(u_n), e(u_n)] + \int_\Omega \mathbb{C}_W[e(u), e(u)] - 2\int_\Omega \mathbb{C}_W[e(u_n), e(u)]\right) \to 0\end{aligned}$$

as $n \to \infty$. The assertion of (i) follows from Korn's inequality. The proof of (ii) is similar, where one additionally takes Lemma 4.3(iii) into account. ∎

We close this section with a lower semicontinuity result for the slopes.

LEMMA 5.3 (lower semicontinuity of slopes). *For each sequence* $(u_n)_n \subset \bar{\mathscr{S}}^M_{\delta_n}$ *with* $u_n \rightharpoonup u$ *weakly in* $H^1(\Omega)$ *we have* $\liminf_{n\to\infty} |\partial\bar\phi_{\delta_n}|_{\bar{\mathcal{D}}_{\delta_n}}(u_n) \geq |\partial\bar\phi_0|_{\bar{\mathcal{D}}_0}(u)$.

Proof. For $\varepsilon > 0$ fix $u' \in C^\infty_c(\Omega;\mathbb{R}^d)$ with $\|u'-u\|_{H^1(\Omega)} \leq \varepsilon$. Fix $v \in C^\infty_c(\Omega;\mathbb{R}^d)$, $v \neq u', u$. We first note that with $w_n := u_n - u' + v$ we have by Lemma 4.9(i)

$$
\begin{aligned}
|\partial\bar\phi_{\delta_n}|_{\bar{\mathcal{D}}_{\delta_n}}(u_n) &= \sup_{w\neq u_n} \frac{(\bar\phi_{\delta_n}(u_n) - \bar\phi_{\delta_n}(w))^+}{\bar{\mathcal{D}}_{\delta_n}(u_n,w)(1 + C\|\mathbf{Id} + \delta_n\nabla u_n - (\mathbf{Id} + \delta_n \nabla w)\|_{L^\infty(\Omega)})^{1/2}} \\
&\geq \frac{(\bar\phi_{\delta_n}(u_n) - \bar\phi_{\delta_n}(w_n))^+}{\bar{\mathcal{D}}_{\delta_n}(u_n,w_n)(1 + C_v\delta_n)^{1/2}},
\end{aligned}
$$

where C_v is a constant depending also on v and u'. Note that, since u', v are smooth, we indeed get $w_n = u_n - u' + v \in \bar{\mathscr{S}}^M_{\delta_n}$ for n large enough for some possibly larger $M > 0$. Consequently, by Lemma 4.3(iii),(iv) we get

$$
\liminf_{n\to\infty} |\partial\bar\phi_{\delta_n}|_{\bar{\mathcal{D}}_{\delta_n}}(u_n) \geq \liminf_{n\to\infty} \frac{(\bar\phi_0(u_n) - \bar\phi_0(w_n) + \bar\phi_{\delta_n,P}(u_n) - \bar\phi_{\delta_n,P}(w_n))^+}{\bar{\mathcal{D}}_0(u_n,w_n)}. \tag{39}
$$

Recalling (16) (for $f \equiv 0$) we obtain by a direct computation

(40)
$$
\begin{aligned}
\lim_{n\to\infty} \big(\bar\phi_0(u_n) - \bar\phi_0(u_n - u' + v)\big) &= \lim_{n\to\infty}\left(-\bar\phi_0(v-u') - 2\int_\Omega \mathbb{C}_W[e(u_n), e(v-u')]\right) \\
&= -\bar\phi_0(v-u') - 2\int_\Omega \mathbb{C}_W[e(u), e(v-u')] \\
&= \bar\phi_0(u) - \bar\phi_0(v) - \bar\phi_0(u'-u) + 2\int_\Omega \mathbb{C}_W[e(u'-u), e(v)].
\end{aligned}
$$

Moreover, by convexity of P and the definition $w_n := u_n - u' + v$ we find

$$
\bar\phi_{\delta_n,P}(u_n) - \bar\phi_{\delta_n,P}(u_n - u' + v) \geq \delta_n^{-p\alpha}\int_\Omega \partial_G P(\delta_n\nabla^2 w_n) : \delta_n(\nabla^2 u' - \nabla^2 v), \tag{41}
$$

which vanishes as $n \to \infty$ by (8)(iii), Hölder's inequality, $1 + \alpha(p-1) - \alpha p > 0$, and the fact that $\|\delta_n\nabla^2 w_n\|^p_{L^p(\Omega)} \leq CM\delta_n^{p\alpha}$. (The latter follows from $w_n \in \bar{\mathscr{S}}^M_{\delta_n}$.) Combining (39)–(41), using $\bar{\mathcal{D}}_0(u_n,w_n) = \bar{\mathcal{D}}_0(v,u')$, and recalling $u_n \rightharpoonup u$, we get after some calculations

$$
\begin{aligned}
\liminf_{n\to\infty} |\partial\bar\phi_{\delta_n}|_{\bar{\mathcal{D}}_{\delta_n}}(u_n) &\geq \frac{(\bar\phi_0(u) - \bar\phi_0(v) - \bar\phi_0(u'-u) + 2\int_\Omega \mathbb{C}_W[e(u'-u), e(v)])^+}{\bar{\mathcal{D}}_0(v,u')} \\
&\geq \frac{(\bar\phi_0(u) - \bar\phi_0(v))^+}{\bar{\mathcal{D}}_0(v,u)} - C\varepsilon
\end{aligned}
$$

for some $C > 0$ depending only on u, u', and v. Letting first $\varepsilon \to 0$ and taking then the supremum with respect to v we get

$$
\liminf_{n\to\infty} |\partial\bar\phi_{\delta_n}|_{\bar{\mathcal{D}}_{\delta_n}}(u_n) \geq \sup_{v\in C^\infty_c(\Omega), v\neq u} \frac{(\bar\phi_0(u) - \bar\phi_0(v))^+}{\bar{\mathcal{D}}_0(v,u)}.
$$

In view of Lemma 4.9(ii), the claim now follows by approximating each $v \in H^1_0(\Omega)$ by a sequence of smooth functions noting that the right-hand side is continuous with respect to $H^1(\Omega)$-convergence. ∎

5.3. Passage from nonlinear to linear viscoelasticity. In this section we now give the proof of Theorem 2.3. For the whole section we fix a null sequence $(\delta_k)_k$ and sequence of initial data $(y_0^k)_{k\in\mathbb{N}} \subset W^{2,p}_{\mathbf{id}}(\Omega)$ such that $\delta_k^{-1}(y_0^k - \mathbf{id}) \to u_0 \in H^1_0(\Omega)$. Moreover, we fix $M > 0$ so large that $y_0^k \in \mathscr{S}^M_{\delta_k}$ for $k \in \mathbb{N}$.

Proof of Theorem 2.3(i). Let $\tau > 0$, and let $\tilde{Y}^{\delta_k}_\tau$ as in (21) be a discrete solution. For each $k \in \mathbb{N}$ we then have the sequence $(U_k^n)_{n\in\mathbb{N}}$ with $U_k^n = \delta_k^{-1}(\tilde{Y}^{\delta_k}_\tau(n\tau) - \mathbf{id}) \in \bar{\mathscr{S}}^M_{\delta_k}$ for $n \in \mathbb{N}$. We need to show that there exists a sequence $(U_0^n)_{n\in\mathbb{N}}$ with $U_0^0 = u_0$ such that

$$\text{(i)}\ \ U_0^n = \mathrm{argmin}_{v\in H^1_0(\Omega)} \bar{\Phi}_0(\tau, U_0^{n-1}; v), \qquad \text{(ii)}\ \ U_k^n \to U_0^n \text{ strongly in } H^1(\Omega)$$

for all $n \in \mathbb{N}$. We show this property by induction.

Suppose $(U_0^i)_{i=0}^n$ have been found such that the above properties hold. In particular, we note that (ii) holds for $n = 0$ by assumption. We now pass from step n to $n+1$.

As $U_k^n \to U_0^n$ strongly in $H^1(\Omega)$ and thus by Theorem 5.1(ii) $\bar{\Phi}_{\delta_k}(\tau, U_k^n; \cdot)$ Γ-converges to $\bar{\Phi}_0(\tau, U_0^n; \cdot)$, we derive by properties of Γ-convergence that the (unique) minimizer of $\bar{\Phi}_0(\tau, U_0^n; \cdot)$, denoted by U_0^{n+1}, is the limit of minimizers of $\bar{\Phi}_{\delta_k}(\tau, U_k^n; \cdot)$. Consequently, we obtain $U_k^{n+1} \rightharpoonup U_0^{n+1}$ weakly in $H^1(\Omega)$ and $\bar{\Phi}_{\delta_k}(\tau, U_k^n; U_k^{n+1}) \to \bar{\Phi}_0(\tau, U_0^n; U_0^{n+1})$. Thus, Lemma 5.2 implies that the sequence even converges strongly in $H^1(\Omega)$. This concludes the induction step. ∎

In the following let u be the unique element of $MM(\bar{\Phi}_0; u_0)$.

Proof of Theorem 2.3(ii). We let σ be the weak $H^1(\Omega)$-topology. We consider the sequence of metrics $\mathcal{D}_k = \bar{\mathcal{D}}_{\delta_k}$ on $H^1_0(\Omega)$ and the functionals $\phi_k = \bar{\phi}_{\delta_k}$ as well as the limiting objects $\bar{\mathcal{D}}_0$ and $\bar{\phi}_0$. We note that (26) is satisfied due to Lemma 4.3(iii) and the fact that $\bar{\mathcal{D}}_0$ is quadratic and convex (see Lemma 4.1(iii)). Moreover, (28) is also satisfied by the Γ-liminf inequality in Lemmas 5.1(i) and 5.3.

Finally, (27) holds also. In fact, by the rigidity estimate in Lemma 4.2(i) and (6), (7)(iii) we find for all $k \in \mathbb{N}$ and $u \in \bar{\mathscr{S}}^M_{\delta_k}$ letting $y = \mathbf{id} + \delta_k u$

$$\begin{aligned} \|u\|^2_{H^1(\Omega)} &= \delta_k^{-2}\|y - \mathbf{id}\|^2_{H^1(\Omega)} \le C\delta_k^{-2}\|\operatorname{dist}(\nabla y, SO(d))\|^2_{L^2(\Omega)} \\ &\le C\delta_k^{-2}\phi_{\delta_k}(y) \le CM. \end{aligned} \tag{42}$$

Now consider a sequence $(y_k)_k$ of generalized minimizing movements starting from y_0^k with $\delta_k^{-1}(y_0^k - \mathbf{id}) \to u_0$ in $H^1(\Omega)$. For convenience we also introduce the curves $u_k = \delta_k^{-1}(y_k - \mathbf{id})$. Fix $M > 0$ so large that $y_0^k \in \mathscr{S}^M_{\delta_k}$ for $k \in \mathbb{N}$. As $\bar{\phi}_{\delta_k}(u_k(t)) \le \phi_{\delta_k}(y_k^0)$ for all $t \ge 0$, we get $\sup_k \sup_t(\phi_{\delta_k}(u_k(t)) + \mathcal{D}_k(u_k(t), u_0)) < \infty$ by (42) and Lemma 4.3(iii).

Consequently, (29)(i) also holds, and (29)(ii) is satisfied by the assumption on the initial data and Lemma 4.3(iv). Since the slopes are strong upper gradients by Lemma 4.9, we can apply Theorem 3.6, and the existence of a limiting curve of maximal slope follows. As this curve is uniquely given by u (see Theorem 2.2(ii)), we indeed obtain $u_k(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega)$ for all $t \in [0, \infty)$ up to a subsequence. Since the limit is unique, we see that the whole sequence converges to u by Urysohn's subsequence principle.

It remains to observe that the convergence is actually strong. This follows from the fact that $\lim_{k\to\infty} \bar{\phi}_{\delta_k}(u_k(t)) = \bar{\phi}_0(u(t))$ for all $t \in [0, \infty)$ (see Theorem 3.6) and Lemma 5.2. ∎

Proof of Theorem 2.3(iii). Proceeding as in the previous proof, we see that all assumptions of Theorem 3.7 are satisfied. Therefore, we get that for any sequence of discrete solutions there is a subsequence converging pointwise weakly in $H^1(\Omega)$ to a curve of maximal slope for $\bar{\phi}_0$ which can again be identified as u. The strong convergence as well as the convergence of the whole sequence follow exactly as in the previous proof. ◻

5.4. Fine representation of the slopes and solutions to the equations. In this section we derive fine representations for the slopes which will allow us to relate curves of maximal slope with solutions to (13) and (15).

Recall that $\mathbb{C}_D$ as defined in (15) is a fourth order symmetric tensor inducing a quadratic form $(F_1, F_2) \mapsto \mathbb{C}_D[F_1, F_2]$ which is positive definite on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ (cf. Lemma 4.1). Moreover, it maps $\mathbb{R}^{d\times d}$ to $\mathbb{R}^{d\times d}_{\mathrm{sym}}$, denoted by $F \mapsto \mathbb{C}_D F$ in the following. More precisely, the mapping $F \mapsto \mathbb{C}_D F$ from $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ to $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ is bijective. By $\sqrt{\mathbb{C}_D}$ we denote its (unique) root and by $\sqrt{\mathbb{C}_D}^{-1}$ the inverse of $\sqrt{\mathbb{C}_D}$, both mappings defined on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. We start with a fine representation of the slope in the linear setting.

LEMMA 5.4 (slope in the linear setting). *There exists a linear differential operator $\mathcal{L}_0 : H^1_0(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ satisfying $\mathrm{div}\mathcal{L}_0(u) = 0$ in $H^{-1}(\Omega;\mathbb{R}^d)$ such that for all $u \in H^1_0(\Omega)$ we have*

$$|\partial\bar{\phi}_0|_{\bar{\mathcal{D}}_0}(u) = \|\sqrt{\mathbb{C}_D}^{-1}\big(\mathbb{C}_W e(u) + \mathcal{L}_0(u)\big)\|_{L^2(\Omega)}.$$

Particularly, we note that $|\partial\bar{\phi}_0|^2_{\bar{\mathcal{D}}_0}$ is convex on $H^1_0(\Omega)$.

Proof. Recalling (16) (for $f \equiv 0$), (17), Definition 3.1(ii), and Lemma 4.1, we have

$$\begin{aligned}(43)\qquad |\partial\bar{\phi}_0|_{\bar{\mathcal{D}}_0}(u) &= \limsup_{v\to u} \frac{(\bar{\phi}_0(u) - \bar{\phi}_0(v))^+}{\bar{\mathcal{D}}_0(u,v)}\\ &= \limsup_{v\to u} \frac{\big(\int_\Omega \mathbb{C}_W[e(u), e(u-v)] - \frac{1}{2}\mathbb{C}_W[e(v-u), e(v-u)]\big)^+}{(\int_\Omega \mathbb{C}_D[e(u-v), e(u-v)])^{1/2}}\\ &= \limsup_{v\to u} \frac{\int_\Omega \mathbb{C}_W[e(u), e(u-v)]}{\|\sqrt{\mathbb{C}_D}e(u-v)\|_{L^2(\Omega)}} = \sup_{w\neq 0} \frac{\int_\Omega \mathbb{C}_W[e(u), e(w)]}{\|\sqrt{\mathbb{C}_D}e(w)\|_{L^2(\Omega)}},\end{aligned}$$

where in the second step we used $\int_\Omega \mathbb{C}_W[e(v-u), e(v-u)]/\|\sqrt{\mathbb{C}_D}e(u-v)\|_{L^2(\Omega)} \to 0$ as $v \to u$. Let $\bar{w}$ be the unique solution to the minimization problem

$$\min_{v\in H^1_0(\Omega)} \int_\Omega \left(\frac{1}{2}|\sqrt{\mathbb{C}_D}e(v)|^2 - \int_\Omega \mathbb{C}_W[e(u), e(v)]\right).$$

Clearly, $\bar{w}$ necessarily satisfies

$$\int_\Omega \Big(\sqrt{\mathbb{C}_D}e(\bar{w}) : \sqrt{\mathbb{C}_D}e(\varphi) - \mathbb{C}_W[e(u), e(\varphi)]\Big) = 0$$

for all $\varphi \in H^1_0(\Omega)$. This condition can also be formulated as

$$(44)\qquad \mathcal{L}_0(u) : e(\varphi) = 0 \quad \forall\varphi \in H^1_0(\Omega), \quad \text{where} \quad \mathcal{L}_0(u) := \mathbb{C}_D e(\bar{w}) - \mathbb{C}_W e(u).$$

As the solution $\bar{w}$ depends linearly on u, we also get that $\mathcal{L}_0$ is a linear operator. By (43) and the property of $\mathcal{L}_0$ we now find

$$
\begin{aligned}
|\partial\bar\phi_0|_{\bar{\mathcal{D}}_0}(u) &= \sup_{w\neq 0}\frac{\int_\Omega(\mathbb{C}_W e(u)+\mathcal{L}_0(u)):e(w)}{\|\sqrt{\mathbb{C}_D}e(w)\|_{L^2(\Omega)}}\\
&= \sup_{w\neq 0}\frac{\int_\Omega\left(\sqrt{\mathbb{C}_D}^{-1}(\mathbb{C}_W e(u)+\mathcal{L}_0(u))\right):\sqrt{\mathbb{C}_D}e(w)}{\|\sqrt{\mathbb{C}_D}e(w)\|_{L^2(\Omega)}}\\
&\le \|\sqrt{\mathbb{C}_D}^{-1}(\mathbb{C}_W e(u)+\mathcal{L}_0(u))\|_{L^2(\Omega)},
\end{aligned}
$$

where in the last step we used the Cauchy–Schwarz inequality. On the other hand, by definition of $\mathcal{L}_0$ in (44), we get

$$
\begin{aligned}
|\partial\bar\phi_0|_{\bar{\mathcal{D}}_0}(u) &\ge \frac{\int_\Omega\left(\sqrt{\mathbb{C}_D}^{-1}(\mathbb{C}_W e(u)+\mathcal{L}_0(u))\right):\sqrt{\mathbb{C}_D}e(\bar w)}{\|\sqrt{\mathbb{C}_D}e(\bar w)\|_{L^2(\Omega)}}\\
&= \|\sqrt{\mathbb{C}_D}e(\bar w)\|_{L^2(\Omega)} = \|\sqrt{\mathbb{C}_D}^{-1}(\mathbb{C}_W e(u)+\mathcal{L}_0(u))\|_{L^2(\Omega)}.
\end{aligned}
$$

This concludes the proof. ∎

Recall the definition of the symmetric fourth order tensor $H_Y = \frac{1}{2}\partial^2_{F_1^2}D^2(Y,Y)$ for $Y\in GL_+(d)$ (see before Lemma 4.3). Let $Y\in\mathbb{R}^{d\times d}$ be in a small neighborhood of $\mathbf{Id}$ such that Y^{-1} exists. Similarly to the discussion before Lemma 5.4, we get that H_Y induces a bijective mapping from $Y^{-\top}\mathbb{R}^{d\times d}_{\rm sym}$ to $Y\mathbb{R}^{d\times d}_{\rm sym}$ by using frame indifference (11)(v) and the growth assumption (11)(vi). We then introduce $\sqrt{H_Y}$ as a bijective mapping from $Y^{-\top}\mathbb{R}^{d\times d}_{\rm sym}$ to $Y\mathbb{R}^{d\times d}_{\rm sym}$. In a similar fashion, we introduce the inverse $\sqrt{H_Y}^{-1}$.

For a given deformation $y:\Omega\to\mathbb{R}^d$ we introduce a mapping $H_{\nabla y}:\Omega\to\mathbb{R}^{d\times d\times d\times d}$ by $H_{\nabla y}(x) = H_{\nabla y(x)}$ for $x\in\Omega$. We note by Lemma 4.2(ii), the fact that $D\in C^3$, and a continuity argument that

$$
\left\|\sqrt{H_{\mathbf{Id}}}-\sqrt{H_{\nabla y}}\right\|_{L^\infty(\Omega)}\le C\delta^\alpha \tag{45}
$$

for all $y\in\mathscr{S}^M_\delta$ for a sufficiently large constant $C>0$. Moreover, recall the definition of the operator $\mathcal{L}_P:\{\nabla^2 u: u\in W^{2,p}_{\mathbf{id}}(\Omega)\}\to W^{-1,\frac{p}{p-1}}(\Omega;\mathbb{R}^{d\times d})$ in (12). We write $\beta=\delta^{2-\alpha p}$ in the following for convenience. Note that $\int_\Omega \partial_G P(\nabla^2 y):\nabla^2\varphi = \mathcal{L}_P(y):\nabla\varphi$ for all $y\in W^{2,p}_{\mathbf{id}}(\Omega)$ and $\varphi\in W^{2,p}_0(\Omega)$, where the boundary term vanishes due to $\nabla\varphi=0$ on $\partial\Omega$. We now obtain the following result.

Lemma 5.5 (slope in the nonlinear setting). *There exists a differential operator $\mathcal{L}^*_P:\{y\in W^{2,p}_{\mathbf{id}}(\Omega):{\rm div}\mathcal{L}_P(\nabla^2 y)\in H^{-1}(\Omega;\mathbb{R}^d)\}\to L^2(\Omega;\mathbb{R}^{d\times d})$ satisfying ${\rm div}\mathcal{L}^*_P(y)={\rm div}\mathcal{L}_P(\nabla^2 y)$ in $H^{-1}(\Omega;\mathbb{R}^d)$ such that for $\delta>0$ small enough and for all $y\in\mathscr{S}^M_\delta$ we have*

$$
|\partial\phi_\delta|_{\mathcal{D}_\delta}(y)=\begin{cases}\frac{1}{\delta}\|\sqrt{H_{\nabla y}}^{-1}\big(\partial_F W(\nabla y)+\beta\mathcal{L}^*_P(y)\big)\|_{L^2(\Omega)} & \text{if } {\rm div}\mathcal{L}_P(\nabla^2 y)\in H^{-1}(\Omega),\\ +\infty & \text{else.}\end{cases}
$$

Remark 5.6. We remark that the expression is well defined in the following sense: if $\nabla y(x)=Y(x)$ in the above notation, then we indeed have $\partial_F W(\nabla y(x))+\beta\mathcal{L}^*_P(y(x)) \in Y(x)\mathbb{R}^{d\times d}_{\rm sym}$ for a.e. $x\in\Omega$.

Proof. We prove the lower bond (i) first in the case ${\rm div}\mathcal{L}_P(\nabla^2 y)\in H^{-1}(\Omega)$ and (ii) afterwards if ${\rm div}\mathcal{L}_P(\nabla^2 y)\notin H^{-1}(\Omega)$. Finally, (iii) we establish the upper bound.

(i) Suppose that $\mathrm{div}\mathcal{L}_P(\nabla^2 y) \in H^{-1}(\Omega)$. Consider the minimization problem

$$\min_{w\in H^1_0(\Omega)} \int_\Omega \left(\frac{1}{2}|\sqrt{H_{\nabla y}}\nabla w|^2 - (\partial_F W(\nabla y) + \beta\mathcal{L}_P(\nabla^2 y)\right) : \nabla w.$$

By (45), the fact that $\sqrt{H_{\mathbf{Id}}} = \sqrt{\mathbb{C}_D}$, Lemma 4.1(iii), and Korn's inequality we have

$$\begin{aligned}\|\sqrt{H_{\nabla y}}\nabla w\|^2_{L^2(\Omega)} &\geq \|\sqrt{H_{\mathbf{Id}}}\nabla w\|^2_{L^2(\Omega)} - C\delta^{2\alpha}\|\nabla w\|^2_{L^2(\Omega)}\\ &\geq C\|e(w)\|^2_{L^2(\Omega)} - C\delta^{2\alpha}\|\nabla w\|^2_{L^2(\Omega)} \geq C\|\nabla w\|^2_{L^2(\Omega)}\end{aligned}$$

for δ sufficiently small for all $w \in H^1_0(\Omega)$. Moreover, we have $|\int_\Omega \mathcal{L}_P(\nabla^2 y) : \nabla w| \leq \|\mathrm{div}\mathcal{L}_P(\nabla^2 y)\|_{H^{-1}(\Omega)}\|w\|_{H^1(\Omega)}$ for all $w \in H^1_0(\Omega)$. Thus, the solution $\bar{w}$ of the problem exists, is unique, and satisfies

$$(\partial_F W(\nabla y) + \beta\mathcal{L}_P(\nabla^2 y)) : \nabla\varphi = \sqrt{H_{\nabla y}}\nabla\bar{w} : \sqrt{H_{\nabla y}}\nabla\varphi = H_{\nabla y}\nabla\bar{w} : \nabla\varphi$$

for all $\varphi \in H^1_0(\Omega)$. Define $\mathcal{L}^*_P(y) := \beta^{-1}(H_{\nabla y}\nabla\bar{w} - \partial_F W(\nabla y))$ and note that

$$\mathcal{L}^*_P(y) : \nabla\varphi = \mathcal{L}_P(\nabla^2 y) : \nabla\varphi \quad \forall\, \varphi \in H^1_0(\Omega) \tag{46}$$

as well as $\mathcal{L}^*_P(y) \in L^2(\Omega)$. Moreover, since $\beta\mathcal{L}^*_P(y) + \partial_F W(\nabla y) = H_{\nabla y}\nabla\bar{w}$, recalling the properties of $H_{\nabla y}$ we see that Remark 5.6 applies. Fix $\varepsilon > 0$, and choose $w_\varepsilon \in C^\infty_c(\Omega;\mathbb{R}^d)$ with $\|\bar{w} - w_\varepsilon\|_{H^1(\Omega)} \leq \varepsilon$. Letting $w_n = y - \frac{1}{n}w_\varepsilon$ we get by a Taylor expansion

$$\begin{aligned}n\delta^2(\phi_\delta(w_n) - \phi_\delta(y)) &= n\int_\Omega \partial_F W(\nabla y) : (\nabla w_n - \nabla y) + nO\left(\|\nabla w_n - \nabla y\|^2_{L^2(\Omega)}\right)\\ &\quad + n\beta\int_\Omega \partial_G P(\nabla^2 y) : (\nabla^2 w_n - \nabla^2 y) + n\beta O\left(\|\nabla^2 w_n - \nabla^2 y\|^2_{L^2(\Omega)}\right)\\ &= -\int_\Omega \partial_F W(\nabla y) : \nabla w_\varepsilon - \beta\partial_G P(\nabla^2 y) : \nabla^2 w_\varepsilon + O(1/n),\end{aligned}$$

where $O(1/n)$ depends on the choice of w_ε. Similarly, we get by Lemma 4.3(i)

$$\begin{aligned}n^2\delta^2\mathcal{D}_\delta(y,w_n)^2 &= n^2\int_\Omega H_{\nabla y}[\nabla(y - w_n), \nabla(y - w_n)] + n^2 O\left(\|\nabla w_n - \nabla y\|^3_{L^3(\Omega)}\right)\\ &= \|\sqrt{H_{\nabla y}}\nabla w_\varepsilon\|^2_{L^2(\Omega)} + O(1/n).\end{aligned}$$

For brevity we introduce

$$\Phi(w) = \left(\int_\Omega (\partial_F W(\nabla y) + \beta\mathcal{L}_P(\nabla^2 y)) : \nabla w\right)\|\sqrt{H_{\nabla y}}\nabla w\|^{-1}_{L^2(\Omega)}.$$

Since $\mathcal{D}_\delta(y,w_n) \to 0$, we now obtain

$$\begin{aligned}\delta|\partial\phi_\delta|_{\mathcal{D}_\delta}(y) &\geq \limsup_{n\to\infty}\frac{\delta(\phi_\delta(y) - \phi_\delta(w_n))^+}{\mathcal{D}_\delta(y,w_n)}\\ &\geq \frac{\int_\Omega \partial_F W(\nabla y) : \nabla w_\varepsilon + \int_\Omega \beta\partial_G P(\nabla^2 y) : \nabla^2 w_\varepsilon}{\|\sqrt{H}_{\nabla y}\nabla w_\varepsilon\|_{L^2(\Omega)}} = \Phi(w_\varepsilon),\end{aligned}$$

where in the last step we used the definition of $\mathcal{L}_P$ in (12). Recalling the definition of $\mathcal{L}^*_P$ and (46) we now derive

$$\Phi(\bar{w}) - \Phi(w_\varepsilon) + \delta|\partial\phi_\delta|_{\mathcal{D}_\delta}(y) \geq \Phi(\bar{w}) = \frac{\int_\Omega H_{\nabla y}\nabla\bar{w} : \nabla\bar{w}}{\|\sqrt{H_{\nabla y}}\nabla\bar{w}\|_{L^2(\Omega)}}$$
$$= \frac{\int_\Omega \sqrt{H_{\nabla y}}\nabla\bar{w} : \sqrt{H_{\nabla y}}\nabla\bar{w}}{\|\sqrt{H_{\nabla y}}\nabla\bar{w}\|_{L^2(\Omega)}} = \|\sqrt{H_{\nabla y}}\nabla\bar{w}\|_{L^2(\Omega)}$$
$$= \|\sqrt{H_{\nabla y}}^{-1}\big(\partial_F W(\nabla y) + \beta\mathcal{L}_P^*(y)\big)\|_{L^2(\Omega)}.$$

By definition of w_ε we get $|\Phi(\bar{w}) - \Phi(w_\varepsilon)| \to 0$ as $\varepsilon \to 0$ and the lower bound in the case $\mathrm{div}\mathcal{L}_P(\nabla^2 y) \in H^{-1}(\Omega)$ follows.

(ii) Now suppose that $\mathrm{div}\mathcal{L}_P(\nabla^2 y) \notin H^{-1}(\Omega)$. Let $(y_n)_n$ be a sequence of smooth functions converging to y in $W^{2,p}(\Omega)$. Then $\mathcal{L}_P^*(y_n)$ is not bounded in $L^2(\Omega)$. Indeed, otherwise we would get by the definition of $\mathcal{L}_P$, (8)(iii), and (46) that

$$\Big|\int_\Omega \mathcal{L}_P(\nabla^2 y) : \nabla\varphi\Big| = \Big|\int_\Omega \partial_G P(\nabla^2 y) : \nabla^2\varphi\Big| = \lim_{n\to\infty}\Big|\int_\Omega \partial_G P(\nabla^2 y_n) : \nabla^2\varphi\Big|$$
$$= \lim_{n\to\infty}\Big|\int_\Omega \mathcal{L}_P^*(y_n) : \nabla\varphi\Big| \leq C\|\nabla\varphi\|_{L^2(\Omega)}$$

for all $\varphi \in W_0^{2,p}(\Omega)$. This, however, contradicts the assumption $\mathrm{div}\mathcal{L}_P(\nabla^2 y) \notin H^{-1}(\Omega)$. As energy and dissipation are $W^{2,p}(\Omega)$-continuous (see (7), (8), and Lemma 4.3(ii)), we find for some fixed $\varepsilon > 0$ and n large enough by Lemma 4.9(i)

$$\varepsilon + |\partial\phi_\delta|_{\mathcal{D}_\delta}(y) \geq \sup_{w\neq y_n} \frac{(\phi_\delta(y_n) - \phi_\delta(w))^+}{\mathcal{D}_\delta(y_n, w)(1 + C\|\nabla y_n - \nabla w\|_{L^\infty(\Omega)})^{1/2}} = |\partial\phi_\delta|_{\mathcal{D}_\delta}(y_n).$$

By the representation of the slope at y_n and the fact that $\mathcal{L}_P^*(y_n)$ is not bounded in $L^2(\Omega)$, the right-hand side tends to infinity for $n \to \infty$, as desired.

(iii) For the upper bound, we first use Lemmas 4.3(i),(ii) and 4.5(ii) to get

$$1 = \lim_{w\to v} \frac{\mathcal{D}_\delta(v,w)^2}{\mathcal{D}_\delta(v,w)^2} \geq \limsup_{w\to v} \frac{\|\sqrt{H_{\nabla v}}\nabla(w-v)\|^2_{L^2(\Omega)} - C\|\nabla v - \nabla w\|^3_{L^3(\Omega)}}{\delta^2\mathcal{D}_\delta(v,w)^2}$$
$$\geq \limsup_{w\to v} \frac{\|\sqrt{H_{\nabla v}}\nabla(w-v)\|^2_{L^2(\Omega)}}{\delta^2\mathcal{D}_\delta(v,w)^2} - C\limsup_{w\to v}\|\nabla v - \nabla w\|_{L^\infty(\Omega)}$$
$$= \limsup_{w\to v} \frac{\|\sqrt{H_{\nabla v}}\nabla(w-v)\|^2_{L^2(\Omega)}}{\delta^2\mathcal{D}_\delta(v,w)^2}.$$

This together with Lemma 4.4 and the convexity of P gives

$$\delta|\partial\phi_\delta|_{\mathcal{D}_\delta}(y) = \limsup_{w\to y} \frac{\delta^2(\phi_\delta(y) - \phi_\delta(w))^+}{\delta\mathcal{D}_\delta(y,w)}$$
$$\leq \limsup_{w\to y} \frac{\int_\Omega \partial_F W(\nabla y) : \nabla(y-w) + \int_\Omega \beta\partial_G P(\nabla^2 y) : \nabla^2(y-w)}{\|\sqrt{H_{\nabla y}}(\nabla w - \nabla y)\|_{L^2(\Omega)}}.$$

Recalling the definition of $\mathcal{L}_P$ and using (46) as in the lower bound, we get

$$\delta|\partial\phi_\delta|_{\mathcal{D}_\delta}(y) \leq \limsup_{w\to y} \frac{\int_\Omega(\partial_F W(\nabla y) + \beta\mathcal{L}_P^*(y)) : \nabla(y-w)}{\|\sqrt{H_{\nabla y}}(\nabla w - \nabla y)\|_{L^2(\Omega)}}.$$

Finally, the Cauchy–Schwarz inequality gives

$$\delta|\partial\phi_\delta|_{\mathcal{D}_\delta}(y) \leq \limsup_{w\to y} \frac{\int_\Omega \sqrt{H_{\nabla y}}^{-1}(\partial_F W(\nabla y) + \beta\mathcal{L}_P^*(y)) : \sqrt{H_{\nabla y}}\nabla(y-w)}{\|\sqrt{H_{\nabla y}}(\nabla w - \nabla y)\|_{L^2(\Omega)}}$$
$$\leq \|\sqrt{H_{\nabla y}}^{-1}(\partial_F W(\nabla y) + \beta\mathcal{L}_P^*(y))\|_{L^2(\Omega)}. \qquad \square$$

Finally, following [2, section 1.4] we relate curves of maximal slope with solutions to (13) and (15). Similar to [2, Corollary 1.4.5], this relies on the fact that the stored energy can be written as a sum of a convex functional and a C^1 functional on $H^1(\Omega)$.

Proof of Theorem 2.1(iii) *and Theorem* 2.2(iii). We only give the proof for the nonlinear equation. The proof for the linear equation is easier and can be seen along similar lines.

First, the fact that $\phi_\delta(y(t))$ is decreasing in time, together with (6)–(8), gives $y \in L^\infty([0,\infty); W^{2,p}_{\mathbf{id}}(\Omega))$. Moreover, since $|y'|_{\mathcal{D}_\delta} \in L^2([0,\infty))$ by (18) and $\mathcal{D}_\delta$ is equivalent to the $H^1(\Omega)$-norm (see Lemma 4.3(ii)), we observe that y is an absolutely continuous curve in the Hilbert space $H^1(\Omega)$. By [2, Remark 1.1.3] this implies that y is differentiable for a.e. t with $\partial_t \nabla y(t) \in L^2(\Omega)$ for a.e. t, that

$$\nabla y(t) - \nabla y(s) = \int_s^t \partial_t \nabla y(r)\, dr \quad \text{a.e. in } \Omega \quad \forall\, 0 \le s < t, \tag{47}$$

and that $y \in W^{1,2}([0,\infty); H^1(\Omega))$. More precisely, by Fatou's lemma and Lemma 4.3(i) we get for a.e. t

(48)

$$\begin{aligned}
|y'|_{\mathcal{D}_\delta}(t) &= \lim_{s\to t} \frac{\mathcal{D}_\delta(y(s),y(t))}{|s-t|} = \lim_{s\to t} \delta^{-1} \left(\frac{\delta^2 \mathcal{D}_\delta(y(s),y(t))^2}{|s-t|^2} \right)^{1/2} \\
&\ge \delta^{-1} \left(\int_\Omega \liminf_{s\to t} \left(H_{\nabla y(t)} \left[\frac{\nabla y(s) - \nabla y(t)}{|s-t|}, \frac{\nabla y(s) - \nabla y(t)}{|s-t|} \right] \right.\right. \\
&\qquad\qquad \left.\left. + |s-t|^{-2} O(|\nabla y(t) - \nabla y(s)|^3) \right) \right)^{1/2} \\
&= \delta^{-1} \left(\int_\Omega H_{\nabla y(t)}[\partial_t \nabla y(t), \partial_t \nabla y(t)] \right)^{1/2} = \delta^{-1} \|\sqrt{H_{\nabla y(t)}} \partial_t \nabla y(t)\|_{L^2(\Omega)}.
\end{aligned}$$

We now determine the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(y(t))$ of the absolutely continuous curve $\phi_\delta \circ y$. Fix t such that $\lim_{s\to t} \frac{\mathcal{D}_\delta(y(s),y(t))}{|s-t|}$ exists, which holds for a.e. t. Then by Lemma 4.4 we find

$$\lim_{s\to\infty} \frac{\int_\Omega W(\nabla y(s)) - \int_\Omega W(\nabla y(t)) - \int_\Omega \partial_F W(\nabla y(t)) : (\nabla y(s) - \nabla y(t))}{s-t} = 0.$$

The previous estimate together with the convexity of P yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(y(t)) &= \lim_{s\to t} \frac{\phi_\delta(y(s)) - \phi_\delta(y(t))}{s-t} \\
&\ge \liminf_{s\to t} \frac{1}{\delta^2(s-t)} \int_\Omega \Big(\partial_F W(\nabla y(t)) : (\nabla y(s) - \nabla y(t)) \\
&\qquad\qquad + \beta \partial_G P(\nabla^2 y(t)) : (\nabla^2 y(s) - \nabla^2 y(t)) \Big) \\
&= \liminf_{s\to t} \frac{1}{\delta^2(s-t)} \int_\Omega \big(\partial_F W(\nabla y(t)) + \beta \mathcal{L}^*_P(y(t)) \big) : (\nabla y(s) - \nabla y(t)),
\end{aligned}$$

where as before $\beta = \delta^{2-\alpha p}$. In the last step we integrated by parts and used $\mathrm{div}(\mathcal{L}^*_P(y(t))) = \mathrm{div}(\mathcal{L}_P(\nabla^2 y(t)))$ by Lemma 5.5. Note that the last term is well defined as $\mathcal{L}^*_P(y(t)) \in L^2(\Omega)$ for a.e. t by Lemma 5.5 and (18). Now (47) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(y(t)) \geq \delta^{-2} \int_\Omega \sqrt{H_{\nabla y(t)}}^{-1} \big(\partial_F W(\nabla y(t)) + \beta \mathcal{L}_P^*(y(t))\big) : \sqrt{H_{\nabla y(t)}}\partial_t \nabla y(t).$$

We find by Lemma 5.5, (48), and Young's inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(y(t)) \geq -\frac{1}{2}\big(|\partial\phi_\delta|^2_{\mathcal{D}_\delta}(y(t)) + |y'|^2_{\mathcal{D}_\delta}(t)\big) \geq \frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(y(t)),$$

where the last step is a consequence of the fact that y is a curve of maximal slope with respect to ϕ_δ. Consequently, all inequalities employed in the proof are in fact equalities, and we get

$$\sqrt{H_{\nabla y(t)}}^{-1} \big(\partial_F W(\nabla y(t)) + \beta \mathcal{L}_P^*(y(t))\big) = -\sqrt{H_{\nabla y(t)}}\partial_t \nabla y(t)$$

pointwise a.e. in Ω. Equivalently, recalling $\partial_{\dot F} R(F, \dot F) = \frac{1}{2}\partial^2_{F_1^2} D^2(F,F)\dot F = H_F \dot F$ from (10), we obtain

$$(\partial_F W(\nabla y(t)) + \beta \mathcal{L}_P^*(y(t))) + \partial_{\dot F} R(\nabla y(t), \partial_t \nabla y(t)) = 0$$

pointwise a.e. in Ω. Multiplying the equation with $\nabla\varphi$ for $\varphi \in W_0^{2,p}(\Omega)$, using again $\int_\Omega \mathcal{L}_P^*(y(t)) : \nabla\varphi = \int_\Omega \mathcal{L}_P(\nabla^2 y(t)) : \nabla\varphi$ by Lemma 5.5 and the definition of $\mathcal{L}_P(\nabla^2 y(t))$, we conclude that y is a weak solution (see (14)). ∎

Acknowledgment. We wish to thank Ulisse Stefanelli for turning our attention to this problem.

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Elsevier, Amsterdam, 2003.

[2] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures Math ETH Zürich, Birkhäuser, Basel, 2005.

[3] S. S. Antman, *Physically unacceptable viscous stresses*, Z. Angew. Math. Phys., 49 (1998), pp. 980–988.

[4] S. S. Antman, *Nonlinear Problems of Elasticity*, Springer, New York, 2004.

[5] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal., 63 (1977), pp. 337–403.

[6] J. M. Ball, J. C. Currie, and P. L. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal., 41 (1981), pp. 135–174.

[7] J. M. Ball and Y. Şengül, *Quasistatic nonlinear viscoelasticity and gradient flows*, J. Dynam. Differential Equations, 27 (2015), pp. 405–442.

[8] R. C. Batra, *Thermodynamics of non-simple elastic materials*, J. Elasticity, 6 (1976), pp. 451–456.

[9] A. Braides, *Local Minimization, Variational Evolution and Γ-convergence*, Springer-Verlag, Berlin, 2014.

[10] A. Braides, M. Colombo, M. Gobbino, and M. Solci, *Minimizing movements along a sequence of functionals and curves of maximal slope*, C. R. Math. Acad. Sci. Paris, 354 (2016), pp. 685–689.

[11] P. G. Ciarlet, *Mathematical Elasticity, Vol.* I: *Three-Dimensional Elasticity*, North-Holland, Amsterdam, 1988.

[12] P. J. Chen, M. E. Gurtin, and W. O. Williams, *On the thermodynamics of non-simple elastic materials with two temperatures*, Z. Angew. Math. Phys., 20 (1969), pp. 107–112.

[13] M. Colombo and M. Gobbino, *Passing to the limit in maximal slope curves: From a regularized Perona-Malik equation to the total variation flow*, Math. Models Methods Appl. Sci., 22 (2012), 1250017.

[14] G. Dal Maso, *An Introduction to Γ-Convergence*, Birkhäuser, Basel, 1993.

[15] G. Dal Maso, M. Negri, and D. Percivale, *Linearized elasticity as Γ-limit of finite elasticity*, Set-Valued Var. Anal., 10 (2002), pp. 165–183.

[16] E. De Giorgi, A. Marino, and M. Tosques, *Problems of evolution in metric spaces and maximal decreasing curve*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 68 (1980), pp. 180–187.

[17] S. Demoulini, *Weak solutions for a class of nonlinear systems of viscoelasticity*, Arch. Ration. Mech. Anal., 155 (2000), pp. 299–334.

[18] M. Friedrich and B. Schmidt, *An atomistic-to-continuum analysis of crystal cleavage in a two-dimensional model problem*, J. Nonlinear Sci., 24 (2014), pp. 145–183.

[19] G. Friesecke, R. D. James, and S. Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity*, Comm. Pure Appl. Math., 55 (2002), pp. 1461–1506.

[20] N. Gigli, *On the heat flow on metric measure spaces: Existence, uniqueness and stability*, Calc. Var. Partial Differential Equations, 39 (2010), pp. 101–120.

[21] T. J. Healey and S. Krömer, *Injective weak solutions in second-gradient nonlinear elasticity*, ESAIM Control Optim. Calc. Var., 15 (2009), pp. 863–871.

[22] A. Mielke, C. Ortner, and Y. Şengül, *An approach to nonlinear viscoelasticity via metric gradient flows*, SIAM J. Math. Anal., 46 (2014), pp. 1317–1347.

[23] A. Mielke and T. Roubíček, *Rate-independent elastoplasticity at finite strains and its numerical approximation*, Math. Models Methods Appl. Sci., 26 (2016), pp. 2203–2236.

[24] A. Mielke and T. Roubíček, *Thermoviscoelasticity in Kelvin-Voigt rheology at large strains*, manuscript in preparation.

[25] C. Ortner, *Two Variational Techniques for the Approximation of Curves of Maximal Slope*, Technical report NA05/10, Oxford University Computing Laboratory, Oxford, UK, 2005.

[26] P. Podio-Guidugli, *Contact interactions, stress, and material symmetry for nonsimple elastic materials*, Theor. Appl. Mech., 28–29, (2002), pp. 261–276.

[27] E. Sandier and S. Serfaty, *Gamma-convergence of gradient flows with applications to Ginzburg-Landau*, Comm. Pure Appl. Math., 57 (2004), pp. 1627–1672.

[28] S. Serfaty, *Gamma-convergence of gradient flows on Hilbert and metric spaces and applications*, Discrete Contin. Dyn. Syst., 31 (2011), pp. 1427–1451.

[29] B. Schmidt, *Linear Γ-limits of multiwell energies in nonlinear elasticity theory*, Contin. Mech. Thermodyn., 20 (2008), pp. 375–396.

[30] R. A. Toupin, *Elastic materials with couple stresses*, Arch. Ration. Mech. Anal., 11 (1962), pp. 385–414.

[31] R. A. Toupin, *Theory of elasticity with couple stress*, Arch. Ration. Mech. Anal., 17 (1964), pp. 85–112.