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Martin Kružík · Carlos Mora-Corral · Ulisse Stefanelli

Quasistatic elastoplasticity via Peridynamics: existence and localization

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Abstract Peridynamics is a nonlocal continuum mechanical theory based on minimal regularity on the deformations. Its key trait is that of replacing local constitutive relations featuring spacial differential operators with integrals over differences of displacement fields over a suitable positive interaction range. The advantage of such perspective is that of directly including nonregular situations, in which discontinuities in the displacement field may occur. In the linearized elastic setting, the mechanical foundation of the theory and its mathematical amenability have been thoroughly analyzed in the last years. We present here the extension of Peridynamics to linearized elastoplasticity. This calls for considering the time evolution of elastic and plastic variables, as the effect of a combination of elastic energy storage and plastic energy dissipation mechanisms. The quasistatic evolution problem is variationally reformulated and solved by time discretization. In addition, by a rigorous evolutive Γ-convergence argument we prove that the nonlocal peridynamic model converges to classic local elastoplasticity as the interaction range goes to zero.

Keywords Peridynamics · Elastoplasticity · Variational formulation · Existence · Localization

1 Introduction

Peridynamics is a nonlocal mechanical theory based on the formulation of equilibrium systems in integral terms instead of differential relations. Forces acting on a material point are obtained as a combined effect of

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M. Kružík
Institute of Information Theory and Automation, Academy of Sciences, Pod vodárenskou věží 4, 18208 Prague 8, Czech Republic
E-mail: kruzik@utia.cas.cz
http://staff.utia.cas.cz/kruzik

C. Mora-Corral
Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, C/ Tomás y Valiente 7, 28049 Madrid, Spain
E-mail: carlos.mora@uam.es
http://www.uam.es/carlos.mora

U. Stefanelli (✉)
Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
E-mail: ulisse.stefanelli@univie.ac.at
http://www.mat.univie.ac.at/∼stefanelli

U. Stefanelli
Istituto di Matematica Applicata e Tecnologie Informatiche E. Magenes, v. Ferrata 1, 27100 Pavia, Italy

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interactions with other points in a neighborhood. This results in an integral featuring a radial weight which modulates the influence of nearby points in terms of their distance [12].

Introduced by Silling [28], and extended in [29,30], Peridynamics is particularly suited to model situations where displacements tend to develop discontinuities, such as in the case of cracks or dislocations [4,13]. In addition, this nonlocal formulation is capable of integrating discrete and continuous descriptions, possibly serving as a connection between multiple scales [27]. As such, it is particularly appealing in order to model the ever smaller scales of modern technological applications [31].

The starting point of this article is the peridynamic model in linear elasticity analyzed in Mengesha and Du [20] (see also [9,21,29,30] for other related models), which we describe as follows. The elastic equilibrium problem for a linear homogeneous isotropic body subject to the external force of density $\mathbf{b}(\mathbf{x}) \in \mathbb{R}^n$ can be variationally formulated as the minimization of the purely elastic energy

$$E_\rho(\mathbf{u}) = \beta \int_\Omega \mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})^2 \,\mathrm{d}\mathbf{x} + \alpha \int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \left(\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}') - \frac{1}{n}\mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \,\mathrm{d}\mathbf{x}$$

among displacements $\mathbf{u}(\mathbf{x}) \in \mathbb{R}^n$ from a reference configuration $\Omega \subset \mathbb{R}^n$, subject to boundary conditions. Here $\rho : \mathbb{R}^n \to [0,\infty)$ is an integral kernel modeling the strength of interactions with respect to the distance of the points $\mathbf{x}'$ and $\mathbf{x}$, the term $\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}')$ plays the role of a nonlocal elastic strain, projected in the direction $(\mathbf{x}'-\mathbf{x})/|\mathbf{x}'-\mathbf{x}|$, namely

$$\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}') = \frac{\big(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\big) \cdot (\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}, \tag{1.1}$$

and $\mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})$ is a nonlocal analogue of the divergence and is given by

$$\mathfrak{D}_\rho(\mathbf{u})(\mathbf{x}) = \text{p. v.} \int_\Omega \rho(\mathbf{x}'-\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}') \,\mathrm{d}\mathbf{x}', \quad \text{for a.e. } \mathbf{x} \in \Omega, \tag{1.2}$$

where p. v. stands for the principal value. The positive material parameters α and β are related to the shear and bulk moduli of the material, respectively.

The main results of Mengesha and Du [20] are the following. By suitably qualifying assumptions on the kernel ρ, the force $\mathbf{b}$, and by imposing boundary conditions (see below), E_ρ admits a unique minimizer $\mathbf{u}_\rho$. In addition, in [20] it is proved that, in the limit of vanishing interaction range, that is for ρ converging to a Dirac delta function centered at $\mathbf{0}$, the nonlocal solutions $\mathbf{u}_\rho$ converge to the unique solution of the classical local elastic equilibrium system, namely the minimizer of

$$E_0(\mathbf{u}) = \frac{\lambda}{2} \int_\Omega \operatorname{div} \mathbf{u}(\mathbf{x})^2 \,\mathrm{d}\mathbf{x} + \mu \int_\Omega |\nabla^s \mathbf{u}(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \,\mathrm{d}\mathbf{x}.$$

The symbol ∇^s stands for the linearized strain $\nabla^s \mathbf{u} = (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)/2$, and the Lamé coefficients λ and μ are related to α, β, and n via [20, App. A]

$$\lambda = 2\beta - \frac{4\alpha}{n(n+2)}, \quad \mu = \frac{2\alpha}{n+2}. \tag{1.3}$$

Note that $\mu > 0$ and $n\lambda + 2\mu > 0$, making the elastic energy coercive. Indeed, calling $\mathbf{u}_\rho$ and $\mathbf{u}_0$ the minimizers of E_ρ and E_0, respectively, the convergence of $\mathbf{u}_\rho$ to $\mathbf{u}_0$ follows from the Γ-convergence of E_ρ to E_0 [7,8].

The focus of this paper is on extending the elastic theory to encompass plastic effects as well. By moving within the very same frame of classical elastoplasticity [16], one describes the plastic state of the system via an additional internal variable, the plastic strain $\mathbf{P} \in \mathbb{R}^{n\times n}_{\mathrm{s,d}}$ (symmetric and deviatoric tensors), and defines the elastoplastic energy as

$$\begin{aligned} F_\rho(\mathbf{u},\mathbf{P}) = \beta \int_\Omega \mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})^2 \,\mathrm{d}\mathbf{x} + \alpha \int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \left(\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}') - \frac{1}{n}\mathfrak{E}_\rho(\mathbf{u},\mathbf{P})(\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} \\ + \gamma \int_\Omega |\mathbf{P}(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \,\mathrm{d}\mathbf{x}, \end{aligned} \tag{1.4}$$

where the nonlocal elastic strain, projected in direction $(\mathbf{x}'-\mathbf{x})/|\mathbf{x}'-\mathbf{x}|$, features now the additional contribution of the plastic strain as

$$\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}') = \frac{\big(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x})(\mathbf{x}'-\mathbf{x})\big)\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}. \tag{1.5}$$

Correspondingly, we define

$$\mathfrak{E}_\rho(\mathbf{u},\mathbf{P})(\mathbf{x}) = \text{p. v.}\int_\Omega \rho(\mathbf{x}'-\mathbf{x})\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{1.6}$$

which plays the role of a nonlocal divergence of $\mathbf{u}$. Indeed, although it depends on $\mathbf{P}$, one can check that such dependence vanishes when the kernel ρ tends to the Dirac delta function at $\mathbf{0}$ as $\mathbf{P}$ is assumed to be deviatoric, see Lemma 3.5.a.

With respect to the purely elastic case of E_ρ, an additional γ-term is here considered. This models kinematic hardening, and $\gamma > 0$ is the corresponding hardening coefficient. Note that the whole energy F_ρ is quadratic in $(\mathbf{u},\mathbf{P})$. This results in a linearized theory of elastoplasticity, although of a nonlocal nature. By letting the kernel ρ tend to the Dirac delta function at $\mathbf{0}$, the model gets localized and the corresponding localized elastoplastic energy is the classical

$$F_0(\mathbf{u},\mathbf{P}) = \frac{\lambda}{2}\int_\Omega \operatorname{div}\mathbf{u}(\mathbf{x})^2\,\mathrm{d}\mathbf{x} + \mu\int_\Omega |\nabla^s\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{b}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \gamma\int_\Omega |\mathbf{P}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}.$$

In particular, the nonlocal model is consistent with the classical localized theory. In fact, the elastic energy is a function solely of the difference between displacements $\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})$ and the plastic term $\mathbf{P}(\mathbf{x})(\mathbf{x}'-\mathbf{x})$. This is completely analogous to the classical additive decomposition of strains in classical elastoplasticity and in fact converges to it as the kernel ρ tends to the Dirac delta function at $\mathbf{0}$.

Elastoplastic evolution requires the specification of the plastic dissipation mechanism. We follow here the classical von Mises choice: Given some yield stress $\sigma_y > 0$, we specify the energy dissipated in order to pass from the plastic state $\mathbf{P}_0$ to $\mathbf{P}_1$ as

$$H(\mathbf{P}_1-\mathbf{P}_0) = \sigma_y\int_\Omega |\mathbf{P}_1(\mathbf{x})-\mathbf{P}_0(\mathbf{x})|\,\mathrm{d}\mathbf{x}.$$

We let the action of the external force density $\mathbf{b}$ to be depending on time and correspondingly investigate trajectories $t \mapsto (\mathbf{u}_\rho(t),\mathbf{P}_\rho(t))$ solving the quasistatic evolution system

$$\partial_{\mathbf{u}} F_\rho(\mathbf{u}_\rho(t),\mathbf{P}_\rho(t),t) = \mathbf{0}, \tag{1.7}$$

$$\partial_{\dot{\mathbf{P}}} H(\dot{\mathbf{P}}_\rho(t)) + \partial_{\mathbf{P}} F_\rho(\mathbf{u}_\rho(t),\mathbf{P}_\rho(t),t) \ni \mathbf{0}. \tag{1.8}$$

The symbol ∂ above is the subdifferential in the sense of convex analysis, and the dot in (1.8) denotes the time derivative. Relation (1.7) corresponds to the weak formulation of the quasistatic equilibrium system. Relation (1.8) is the plastic flow rule instead. In particular, as H is not smooth in $\mathbf{0}$, relation (1.8) is actually a pointwise inclusion. Quasistatic evolution in the present nonlocal peridynamic elastoplastic context is then driven by the pair of functionals (F_ρ, H), whereas the choice (F_0, H) corresponds to classical localized elastoplasticity.

The two main results of this paper are the following:

- **(Theorem 4.1)** Under suitable assumptions on the data and the kernel ρ, there exists a unique trajectory $t \mapsto (\mathbf{u}_\rho(t),\mathbf{P}_\rho(t))$ solving the nonlocal quasistatic evolution system.
- **(Theorem 4.2)** If ρ converges to the Dirac delta function at $\mathbf{0}$, then the solutions $t \mapsto (\mathbf{u}_\rho(t),\mathbf{P}_\rho(t))$ converge to the unique quasistatic evolution $t \mapsto (\mathbf{u}_0(t),\mathbf{P}_0(t))$ for local classical elastoplasticity.

As mentioned earlier, our elastoplastic model based on the functional F_ρ is a natural extension of the elastic model based on E_ρ by MENGESHA AND DU [20] (see [2,3] for other studies on the existence of minimizers and its relation to local elasticity). In an essential way, we also benefit from the technical tools developed in [20], such as the functional setup (see Sect. 2), the nonlocal Korn inequality (Proposition 2.2) and its δ-uniform version (Proposition 3.8), the L^2-compactness of $\mathcal{S}_\delta$-bounded sequences (Proposition 3.10) and, in general, the overall scheme of the proof of existence, uniqueness, and Γ-convergence (Sect. 3).

To our knowledge, this paper contributes the first variational peridynamic model including internal variables. Note that damage and plastic effects in the frame of Peridynamics have already been considered in

[14,17] and [18], respectively. The analysis of the well-posedness of the quasistatic evolution (Theorem 4.1) and the localization proof (Theorem 4.2) seem unprecedented out of the elastic context.

The well-posedness result is based on time discretization. After explaining the functional setup (Sect. 2), in Sect. 3 we investigate incremental problems of the form

$$\min\big(F_\rho(\mathbf{u},\mathbf{P},t_i)+H(\mathbf{P}-\mathbf{P}_{\rm old})\big),$$

where the previous plastic state $\mathbf{P}_{\rm old}$ and the time t_i are given. These minimization problems are proved to be well-posed (Sect. 3.1) and to converge in the sense of Γ-convergence to the corresponding localized counterparts as the kernel ρ approaches the Dirac delta function at $\mathbf{0}$ (Sect. 3.2). By passing to the limit in the time-discretized problem as the time step goes to zero, one recovers the unique solution to the quasistatic evolution system (Sect. 4). Such limit passage is made possible by the quadratic nature of the energy (Sect. 4.2).

The localization result is derived by applying the general theory of evolutive Γ-convergence for rate-independent evolution from [24]. In particular, such possibility rests upon the Γ-convergence of the energies and the specification of a recovery sequence for a suitable combination of energy and dissipation terms (Sect. 4.3). This again crucially exploits the fact that energies are quadratic. The current peridynamic model is hence consistent with classical localized elastoplasticity under localization.

2 Functional setup

We devote this section to presenting our assumptions and introducing some notations. In the following, we will use lowercase bold letters for vectors in $\mathbb{R}^n$ and capitalized bold letters for tensors in $\mathbb{R}^{n\times n}$. In particular, $\mathbf{a}\cdot\mathbf{b}$ is the standard scalar product. We use the symbol $\mathbf{I}$ for the identity, $\mathbf{A}:\mathbf{B}=\mathrm{tr}(\mathbf{A}^\top\mathbf{B})$ for the standard contraction product, $|\mathbf{A}|^2=\mathbf{A}:\mathbf{A}$ for the norm, and recall that an infinitesimal rigid displacement is a function of the form $\mathbf{x}\mapsto\mathbf{S}\mathbf{x}+\mathbf{v}$ with $\mathbf{S}\in\mathbb{R}^{n\times n}$ skew-symmetric and $\mathbf{v}\in\mathbb{R}^n$.

Let $\Omega\subset\mathbb{R}^n$ (open, bounded and Lipschitz) be the reference configuration of the body. The state of the medium is described by the pair $(\mathbf{u},\mathbf{P})$, where $\mathbf{u}:\Omega\times(0,T)\to\mathbb{R}^n$ is the displacement and $\mathbf{P}:\Omega\times(0,T)\to\mathbb{R}^{n\times n}_{\rm s,d}$ is the plastic strain. Here, $T>0$ is a final reference time and $\mathbb{R}^{n\times n}_{\rm s,d}$ stands for the set of symmetric trace-free (deviatoric) matrices, namely $\mathrm{tr}\,\mathbf{P}(\mathbf{x},t)=0$. We also use the symbol $\mathcal{R}$ for the $L^2(\Omega,\mathbb{R}^n)$ subset of infinitesimal rigid displacements in Ω. We will indicate by $\|\cdot\|_p$ the norm of any L^p space on Ω.

Let an integral kernel $\rho\in L^1(\mathbb{R}^n,[0,\infty))$ with $\|\rho\|_1=n$ be given. We also assume that ρ is radial and $\rho>0$ in some ball centered at the origin. We define for all $(\mathbf{u},\mathbf{P})\in L^2(\Omega,\mathbb{R}^n)\times L^2(\Omega,\mathbb{R}^{n\times n}_{\rm s,d})$ the quantities $\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}')$, $\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}')$, $\mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})$, and $\mathfrak{E}_\rho(\mathbf{u},\mathbf{P})(\mathbf{x})$ from (1.1)–(1.2) and (1.5)–(1.6) for a.e. $\mathbf{x}$ and $\mathbf{x}'$ in Ω. We can hence define the elastoplastic energy F_ρ in (1.4) on the whole of $L^2(\Omega,\mathbb{R}^n)\times L^2(\Omega,\mathbb{R}^{n\times n}_{\rm s,d})$, possibly taking the value ∞.

Note that, by Jensen's (or Hölder's) inequality,

$$\begin{aligned}\mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})^2&\le\int_\Omega\rho(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\mathbf{x}'\ \text{p. v.}\int_\Omega\rho(\mathbf{x}'-\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}')^2\,\mathrm{d}\mathbf{x}'\\&\le n\ \text{p. v.}\int_\Omega\rho(\mathbf{x}'-\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}')^2\,\mathrm{d}\mathbf{x}'\qquad\text{for a.e. }\mathbf{x}\in\Omega.\end{aligned}\tag{2.1}$$

In particular, we have that $\mathfrak{D}_\rho(\mathbf{u})\in L^2(\Omega)$ if

$$\int_\Omega\int_\Omega\rho(\mathbf{x}'-\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}')^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}<\infty.$$

Accordingly, we define

$$|\mathbf{u}|_{\mathcal{S}_\rho}=\left(\int_\Omega\int_\Omega\rho(\mathbf{x}'-\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}')^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\right)^{1/2},\qquad\|\mathbf{u}\|_{\mathcal{S}_\rho}=\left(\|\mathbf{u}\|_2^2+|\mathbf{u}|_{\mathcal{S}_\rho}^2\right)^{1/2},$$

and the space

$$\mathcal{S}_\rho(\Omega)=\left\{\mathbf{u}\in L^2(\Omega,\mathbb{R}^n):|\mathbf{u}|_{\mathcal{S}_\rho}<\infty\right\}.$$

It is immediate to see that $|\cdot|_{\mathcal{S}_\rho}$ is a seminorm and $\|\cdot\|_{\mathcal{S}_\rho}$ is a norm in $\mathcal{S}_\rho(\Omega)$. In fact, $\mathcal{S}_\rho(\Omega)$ is a separable Hilbert space, as shown in [20, Th. 2.1]. It was proved in [10, Lemma 2] that $|\mathbf{u}|_{\mathcal{S}_\rho}=0$ if and only if $\mathbf{u}\in\mathcal{R}$.

In the following, we will impose homogeneous Dirichlet boundary conditions on $\mathbf{u}$ by asking the displacement $\mathbf{u}$ to belong to the closed subspace V of $L^2(\Omega, \mathbb{R}^n)$ given by

$$V = \left\{\mathbf{u} \in L^2(\Omega, \mathbb{R}^n) : \mathbf{u} = \mathbf{0} \text{ a.e. in } \omega\right\},$$

where $\omega \subset \Omega$ is a measurable subset with nonempty interior such that $\Omega \backslash \overline{\omega}$ is Lipschitz. With this choice, it is proved in [10] that $V \cap \mathcal{R} = \{\mathbf{0}\}$ so that (nonnull) infinitesimal rigid-body motions are ruled out; see also [11,15].

Although we stick with this choice of V in the following, let us mention that other boundary conditions can be considered as well. Nonhomogeneous Dirichlet conditions can be easily dealt with, and we refer to [10] for some detail concerning Neumann conditions.

As in (2.1), by Jensen's (or Hölder's) inequality,

$$\mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x})^2 \leq n \text{ p.v.} \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')^2 \, \mathrm{d}\mathbf{x}' \qquad \text{for a.e. } \mathbf{x} \in \Omega. \tag{2.2}$$

In addition, since

$$\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{x}') = \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') + \frac{\mathbf{P}(\mathbf{x})(\mathbf{x}'-\mathbf{x}) \cdot (\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}, \qquad \text{a.e. } \mathbf{x}, \mathbf{x}' \in \Omega, \tag{2.3}$$

we also have the bounds

$$\mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{x}')^2 \leq 2\left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')^2 + |\mathbf{P}(\mathbf{x})|^2\right)$$

and, hence,

$$\int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \mathcal{D}(\mathbf{u})(\mathbf{x}, \mathbf{x}')^2 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq 2 \int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')^2 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} + 2n \int_\Omega |\mathbf{P}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x}. \tag{2.4}$$

In view of (2.1), (2.2), and (2.4), we have that the elastoplastic energy $F_\rho(\mathbf{u}, \mathbf{P})$ is finite in $(\mathbf{u}, \mathbf{P}) \in L^2(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ if and only if

$$\int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')^2 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} < \infty.$$

Accordingly, we define

$$|(\mathbf{u}, \mathbf{P})|_{\mathcal{T}_\rho} = \left(\int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')^2 \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}\right)^{1/2}, \qquad \|(\mathbf{u}, \mathbf{P})\|_{\mathcal{T}_\rho} = \left(\|\mathbf{u}\|_2^2 + \|\mathbf{P}\|_2^2 + |(\mathbf{u}, \mathbf{P})|^2_{\mathcal{T}_\rho}\right)^{1/2}$$

and the space

$$\mathcal{T}_\rho(\Omega) = \left\{(\mathbf{u}, \mathbf{P}) \in L^2(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}}) : |(\mathbf{u}, \mathbf{P})|_{\mathcal{T}_\rho} < \infty\right\}.$$

It is easy to see that $|\cdot|_{\mathcal{T}_\rho}$ is a seminorm and $\|\cdot\|_{\mathcal{T}_\rho}$ is a norm in $\mathcal{T}_\rho(\Omega)$. We have, in fact, the following result.

Lemma 2.1 *We have that $\mathcal{T}_\rho(\Omega) = \mathcal{S}_\rho(\Omega) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$, and the norm $\|\cdot\|_{\mathcal{T}_\rho}$ is equivalent to the product norm in $\mathcal{S}_\rho(\Omega) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. In addition, $\mathcal{T}_\rho(\Omega)$ is a separable Hilbert space.*

Proof By the triangle inequality,

$$|(\mathbf{u}, \mathbf{P})|_{\mathcal{T}_\rho} \leq |(\mathbf{u}, \mathbf{0})|_{\mathcal{T}_\rho} + |(\mathbf{0}, \mathbf{P})|_{\mathcal{T}_\rho} = |\mathbf{u}|_{\mathcal{S}_\rho} + |(\mathbf{0}, \mathbf{P})|_{\mathcal{T}_\rho} \quad \text{and} \quad |\mathbf{u}|_{\mathcal{S}_\rho} = |(\mathbf{u}, \mathbf{0})|_{\mathcal{T}_\rho} \leq |(\mathbf{u}, \mathbf{P})|_{\mathcal{T}_\rho} + |(\mathbf{0}, \mathbf{P})|_{\mathcal{T}_\rho}.$$

Now, $|\mathcal{E}(\mathbf{0}, \mathbf{P})(\mathbf{x}, \mathbf{x}')| \leq |\mathbf{P}(\mathbf{x})|$ for a.e. $\mathbf{x}, \mathbf{x}' \in \Omega$, so $|(\mathbf{0}, \mathbf{P})|^2_{\mathcal{T}_\rho} \leq n\|\mathbf{P}\|_2^2$. This shows the equivalence of norms.

Finally, $\mathcal{T}_\rho(\Omega)$ is a separable Hilbert space because so is $\mathcal{S}_\rho(\Omega)$ (see [20, Th. 2.1]). □

For future reference, recall that the proof of Lemma 2.1 has shown that

$$|\mathbf{u}|_{\mathcal{S}_\rho} \leq |(\mathbf{u}, \mathbf{P})|_{\mathcal{T}_\rho} + \sqrt{n}\|\mathbf{P}\|_2 \quad \text{and} \quad |(\mathbf{u}, \mathbf{P})|_{\mathcal{T}_\rho} \leq |\mathbf{u}|_{\mathcal{S}_\rho} + \sqrt{n}\|\mathbf{P}\|_2. \tag{2.5}$$

A crucial tool in the following is the nonlocal Korn inequality, which we take from [20, Prop. 2.7].

Proposition 2.2 (Nonlocal Korn inequality) *There exists* $C > 0$ *such that* $\|\mathbf{u}\|_2^2 \leq C|\mathbf{u}|_{\mathcal{S}_\rho}^2$ *for all* $\mathbf{u} \in V$.

It is important to remark that the assumption that ρ is strictly positive near the origin is essential in Proposition 2.2. Otherwise, the nonlocal Poincaré inequality can fail (see a counterexample in [1, Remark 6.20]) and consequently, so can the nonlocal Korn inequality.

The following result is proved in [19, Lemma 2.1] (see also [20, Eq. (15)]).

Lemma 2.3 *There exists* $C > 0$ *such that for all* $\mathbf{u} \in H^1(\Omega, \mathbb{R}^n)$,

$$|\mathbf{u}|_{\mathcal{S}_\rho}^2 \leq C\, n \left\| \nabla^s \mathbf{u} \right\|_2^2.$$

We remark that the constant C in Lemma 2.3 does not depend on ρ.

3 Incremental problem

Let us now turn our attention to the incremental elastoplastic problem. Given the plastic strain $\mathbf{P}_{\mathrm{old}} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$, it consists in finding

$$(\mathbf{u}, \mathbf{P}) \in Q = V \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$$

that minimizes the incremental functional

$$F_\rho(\mathbf{u}, \mathbf{P}) + H(\mathbf{P} - \mathbf{P}_{\mathrm{old}}). \tag{3.1}$$

In this section we prove the well-posedness of the incremental problem (Sect. 3.1) as well as the convergence of its solutions the solution of its local counterpart as $\delta \to 0$ (Sect. 3.2).

In order to possibly apply the direct method to the incremental problem (3.1), the coercivity of F_ρ will be instrumental. We check it in the following.

Lemma 3.1 (Coercivity of the energy) *There exists* $c > 0$ *such that for all* $(\mathbf{u}, \mathbf{P}) \in Q$,

$$F_\rho(\mathbf{u}, \mathbf{P}) \geq c\|(\mathbf{u}, \mathbf{P})\|_{\mathcal{T}_\rho}^2 - \frac{1}{c}.$$

Proof Assume with no loss of generality that $(\mathbf{u}, \mathbf{P}) \in \mathcal{T}_\rho$. For any $0 < \eta < 1$ we have

$$\left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n}\mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x})\right)^2 \geq (1-\eta)\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')^2 - (\eta^{-1} - 1)\frac{1}{n^2}\mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x})^2, \tag{3.2}$$

for a.e. $\mathbf{x}, \mathbf{x}' \in \Omega$. On the other hand, thanks to (2.3) we have

$$\mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) = \mathfrak{D}_\rho(\mathbf{u})(\mathbf{x}) - \text{p. v.} \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \frac{\mathbf{P}(\mathbf{x})(\mathbf{x}'-\mathbf{x}) \cdot (\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} \mathrm{d}\mathbf{x}'.$$

In fact,

$$\text{p. v.} \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \frac{\mathbf{P}(\mathbf{x})(\mathbf{x}'-\mathbf{x}) \cdot (\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} \mathrm{d}\mathbf{x}' = \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \frac{\mathbf{P}(\mathbf{x})(\mathbf{x}'-\mathbf{x}) \cdot (\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} \mathrm{d}\mathbf{x}'$$

since

$$\left| \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \frac{\mathbf{P}(\mathbf{x})(\mathbf{x}'-\mathbf{x}) \cdot (\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} \mathrm{d}\mathbf{x}' \right| \leq \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \mathrm{d}\mathbf{x}' |\mathbf{P}(\mathbf{x})| \leq n|\mathbf{P}(\mathbf{x})|.$$

Therefore,

$$\left|\mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x})\right| \leq \left|\mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})\right| + n\,|\mathbf{P}(\mathbf{x})|\,.$$

Consequently,

$$\mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x})^2 \leq 2\mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})^2 + 2n^2\,|\mathbf{P}(\mathbf{x})|^2\,, \qquad \|\mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})\|_2^2 \leq 2\|\mathfrak{D}_\rho(\mathbf{u})\|_2^2 + 2n^2\|\mathbf{P}\|_2^2$$

and

$$\int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x})\mathfrak{E}_\rho(\mathbf{u},\mathbf{P})(\mathbf{x})^2\,\mathrm{d}\mathbf{x} \le n\left\|\mathfrak{E}_\rho(\mathbf{u},\mathbf{P})\right\|_2^2 \le 2n\|\mathfrak{D}_\rho(\mathbf{u})\|_2^2 + 2n^3\|\mathbf{P}\|_2^2. \tag{3.3}$$

Therefore, by (3.2) and (3.3) we have

$$\begin{aligned}\int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) & \left(\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}') - \frac{1}{n}\mathfrak{E}_\rho(\mathbf{u},\mathbf{P})(\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{x} \\ & \ge (1-\eta)|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} - (\eta^{-1}-1)\frac{2}{n}\left(\|\mathfrak{D}_\rho(\mathbf{u})\|_2^2 + n^2\|\mathbf{P}\|_2^2\right). \end{aligned} \tag{3.4}$$

On the other hand, for any $\eta_1 > 0$ we have that

$$\left|\int_\Omega \mathbf{b}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\right| \le \|\mathbf{b}\|_2\|\mathbf{u}\|_2 \le \frac{\|\mathbf{b}\|_2^2}{2\eta_1} + \frac{\eta_1\|\mathbf{u}\|_2^2}{2}. \tag{3.5}$$

Using (3.4) and (3.5), we find that

$$\begin{aligned} F_\rho(\mathbf{u},\mathbf{P}) \ge & \left[\beta - \frac{2}{n}\alpha(\eta^{-1}-1)\right]\|\mathfrak{D}_\rho(\mathbf{u})\|_2^2 + \alpha(1-\eta)|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} \\ & + \left[\gamma - 2n(\eta^{-1}-1)\right]\|\mathbf{P}\|_2^2 - \frac{\eta_1}{2}\|\mathbf{u}\|_2^2 - \frac{1}{2\eta_1}\|\mathbf{b}\|_2^2. \end{aligned}$$

Choosing $0 < \eta < 1$ such that

$$\beta - \frac{2}{n}\alpha(\eta^{-1}-1) \ge 0 \quad \text{and} \quad \gamma - 2n(\eta^{-1}-1) > 0,$$

we have that inequality

$$F_\rho(\mathbf{u},\mathbf{P}) \ge c\left(|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} + \|\mathbf{P}\|_2^2\right) - \frac{\eta_1}{2}\|\mathbf{u}\|_2^2 - \frac{1}{2\eta_1}\|\mathbf{b}\|_2^2 \tag{3.6}$$

is proved for some $c > 0$. By Proposition 2.2 and estimate (2.5), we have

$$\|\mathbf{u}\|_2^2 \le C|\mathbf{u}|^2_{\mathcal{S}_\rho} \le 2C\left(|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} + n\|\mathbf{P}\|_2^2\right) \le 2nC\left(|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} + \|\mathbf{P}\|_2^2\right),$$

so

$$\frac{c}{2}\left(|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} + \|\mathbf{P}\|_2^2\right) + \frac{c}{2}\left(|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} + \|\mathbf{P}\|_2^2\right) \ge \frac{c}{2}\left(|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} + \|\mathbf{P}\|_2^2\right) + \frac{c}{4nC}\|\mathbf{u}\|_2^2. \tag{3.7}$$

Using (3.6) and (3.7) we obtain

$$F_\rho(\mathbf{u},\mathbf{P}) \ge \frac{c}{2}\left(|(\mathbf{u},\mathbf{P})|^2_{\mathcal{T}_\rho} + \|\mathbf{P}\|_2^2\right) + \frac{c}{4nC}\|\mathbf{u}\|_2^2 - \frac{\eta_1}{2}\|\mathbf{u}\|_2^2 - \frac{1}{2\eta_1}\|\mathbf{b}\|_2^2.$$

Choosing $\eta_1 > 0$ so that

$$\frac{c}{4nC} - \frac{\eta_1}{2} > 0$$

we prove the estimate of the statement. □

The semicontinuity of the second term of F_ρ will ensue from the following control on the projected stress.

Lemma 3.2 (Projected stress control) *The transformation T_ρ that assigns each $(\mathbf{u},\mathbf{P})$ to the map*

$$(\mathbf{x},\mathbf{x}') \mapsto \rho(\mathbf{x}'-\mathbf{x})^{\frac{1}{2}}\left[\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}') - \frac{1}{n}\mathfrak{E}_\rho(\mathbf{u},\mathbf{P})(\mathbf{x})\right]$$

is linear and bounded from $\mathcal{T}_\rho(\Omega)$ to $L^2(\Omega\times\Omega)$. Moreover, there exists $C > 0$, not depending on ρ, such that for all $(\mathbf{u},\mathbf{P}) \in \mathcal{T}_\rho(\Omega)$,

$$\left\|T_\rho(\mathbf{u},\mathbf{P})\right\|_{L^2(\Omega\times\Omega)} \le C\,|(\mathbf{u},\mathbf{P})|_{\mathcal{T}_\rho(\Omega)}.$$

Proof The operators $\mathcal{E}$ and $\mathfrak{E}_\rho$ are clearly linear, and, hence, so is T_ρ. The operator

$$(\mathbf{x}, \mathbf{x}') \mapsto \rho(\mathbf{x}'-\mathbf{x})^{\frac{1}{2}} \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')$$

is bounded simply because

$$\int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')^2 \,\mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} = |(\mathbf{u}, \mathbf{P})|^2_{\mathcal{T}_\rho}.$$

Analogously, the operator

$$(\mathbf{x}, \mathbf{x}') \mapsto \rho(\mathbf{x}'-\mathbf{x})^{\frac{1}{2}} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x})$$

is bounded because, thanks to (2.2),

$$\int_\Omega \int_\Omega \rho(\mathbf{x}'-\mathbf{x}) \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x})^2 \,\mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} \le n \int_\Omega \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x})^2 \,\mathrm{d}\mathbf{x} \le n^2 |(\mathbf{u}, \mathbf{P})|^2_{\mathcal{T}_\rho}.$$

This concludes the proof. □

3.1 Well-posedness of the incremental problem

A key feature of the energy functional F_ρ is its strict convexity, which delivers the existence and uniqueness of minimizers.

Proposition 3.3 (Strict convexity of F_ρ) *The functional F_ρ is strictly convex in $(V \cap \mathcal{S}_\rho) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$.*

Proof The operators $\mathfrak{D}_\rho$ and T_ρ (see Lemma 3.2) are linear, which readily implies that F_ρ is convex. Let $(\mathbf{u}_1, \mathbf{P}_1), (\mathbf{u}_2, \mathbf{P}_2) \in (V \cap \mathcal{S}_\rho) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ and $\lambda \in (0, 1)$ satisfy

$$F_\rho(\lambda(\mathbf{u}_1, \mathbf{P}_1) + (1-\lambda)(\mathbf{u}_2, \mathbf{P}_2)) = \lambda F_\rho(\mathbf{u}_1, \mathbf{P}_1) + (1-\lambda) F_\rho(\mathbf{u}_2, \mathbf{P}_2).$$

Since the norms in $L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ and in $L^2(\Omega \times \Omega)$ are strictly convex, we find that $\mathbf{P}_1 = \mathbf{P}_2$ a.e. and $T_\rho(\mathbf{u}_1, \mathbf{P}_1) = T_\rho(\mathbf{u}_2, \mathbf{P}_2)$ a.e. Calling $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$, we infer that $\mathbf{v} \in V$ and $T_\rho(\mathbf{v}, \mathbf{0}) = 0$. Thus, $|\mathbf{v}|_{\mathcal{S}_\rho} = 0$, so, by Proposition 2.2, $\mathbf{v} = \mathbf{0}$ and, hence, $\mathbf{u}_1 = \mathbf{u}_2$ a.e. □

Theorem 3.4 (Well-posedness of the incremental problem) *Let $\mathbf{P}_{\mathrm{old}} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ be given. Then there exists a unique minimizer of $(\mathbf{u}, \mathbf{P}) \mapsto F_\rho(\mathbf{u}, \mathbf{P}) + H(\mathbf{P}-\mathbf{P}_{\mathrm{old}})$ in Q.*

Proof Call $G_\rho : Q \to \mathbb{R} \cup \{\infty\}$ the function $G_\rho(\mathbf{u}, \mathbf{P}) = F_\rho(\mathbf{u}, \mathbf{P}) + H(\mathbf{P}-\mathbf{P}_{\mathrm{old}})$. By Lemma 2.1, it is enough to show existence and uniqueness of minimizers of G_ρ in $(V \cap \mathcal{S}_\rho) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. (Recall that $F_\rho = \infty$ if $\mathbf{u} \notin \mathcal{S}_\rho$.) By Lemma 3.1, G_ρ is bounded from below, so it admits a minimizing sequence $\{(\mathbf{u}_j, \mathbf{P}_j)\}_{j\in\mathbb{N}}$ in $(V \cap \mathcal{S}_\rho) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. By Lemma 3.1 again, $\{(\mathbf{u}_j, \mathbf{P}_j)\}_{j\in\mathbb{N}}$ is bounded in $\mathcal{T}_\rho$. By Lemma 2.1, $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ is bounded in $\mathcal{S}_\rho$ and $\{\mathbf{P}_j\}_{j\in\mathbb{N}}$ is bounded in $L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. As V is a closed subspace of $L^2(\Omega, \mathbb{R}^n)$, it is also a closed subspace of $\mathcal{S}_\rho$. Therefore, there exists $(\mathbf{u}_0, \mathbf{P}_0) \in (V \cap \mathcal{S}_\rho) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ such that, for a subsequence (not relabeled), $\mathbf{u}_j \rightharpoonup \mathbf{u}_0$ in $\mathcal{S}_\rho$ and $\mathbf{P}_j \rightharpoonup \mathbf{P}_0$ in $L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ as $j \to \infty$.

Bound (2.2) tells us that $\mathfrak{E}_\rho$ is a linear bounded operator from $\mathcal{T}_\rho$ to $L^2(\Omega)$. Having in mind that $\mathcal{D}(\mathbf{u}) = \mathcal{E}(\mathbf{u}, \mathbf{0})$ and $\mathfrak{D}_\rho(\mathbf{u}) = \mathfrak{E}_\rho(\mathbf{u}, \mathbf{0})$, we obtain that the operator $\mathfrak{D}_\rho : \mathcal{S}_\rho \to L^2(\Omega)$ is linear and bounded. By Lemma 3.2, the map T_ρ defined therein is linear and bounded. Altogether, G_ρ is the sum of continuous functions with respect to the strong topology of $\mathcal{S}_\rho \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. On the other hand, thanks to Proposition 3.3, G_ρ is strictly convex as a sum of the strictly convex function F_ρ and the convex function $(\mathbf{u}, \mathbf{P}) \mapsto H(\mathbf{P}-\mathbf{P}_{\mathrm{old}})$. Consequently, G_ρ is lower semicontinuous with respect to the weak topology of $\mathcal{S}_\rho \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. Thus,

$$G_\rho(\mathbf{u}_0, \mathbf{P}_0) \le \liminf_{j\to\infty} G_\rho(\mathbf{u}_j, \mathbf{P}_j)$$

and, hence, $(\mathbf{u}_0, \mathbf{P}_0)$ is a minimizer of G_ρ. The uniqueness of minimizers is an immediate consequence of the strict convexity of G_ρ. □

3.2 Localization limit

We shall now check that, as ρ tends to the Dirac delta function at $\mathbf{0}$, the unique solution $(\mathbf{u}_\delta, \mathbf{P}_\delta)$ of the nonlocal incremental problem (3.1) converges to the unique solution of the incremental problem for local classical linearized elastoplasticity. To this aim, let us specify that the local elastoplastic energy $F_0 : Q \to \mathbb{R} \cup \{\infty\}$ is given by

$$\begin{aligned} F_0(\mathbf{u}, \mathbf{P}) =& \beta \int_\Omega \operatorname{div} \mathbf{u}(\mathbf{x})^2 \, \mathrm{d}\mathbf{x} + \alpha\, n \int_\Omega \fint_{\mathbb{S}^{n-1}} \left((\nabla \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\, \mathrm{d}\mathbf{x} \\ & - \int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \gamma \int_\Omega |\mathbf{P}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} \\ =& \frac{\lambda}{2} \int_\Omega \operatorname{div} \mathbf{u}(\mathbf{x})^2 \, \mathrm{d}\mathbf{x} + \mu \int_\Omega |\nabla^s \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \gamma \int_\Omega |\mathbf{P}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} \end{aligned}$$

for $\mathbf{u} \in H^1(\Omega, \mathbb{R}^n)$, and $F_0(\mathbf{u}, \mathbf{P}) = \infty$ otherwise. The numbers λ, μ are given by (1.3). Correspondingly, the local incremental elastoplastic problem reads as follows: Given the previous plastic strain $\mathbf{P}_{\text{old}} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}})$ find $(\mathbf{u}, \mathbf{P}) \in Q$ minimizing

$$F_0(\mathbf{u}, \mathbf{P}) + H(\mathbf{P} - \mathbf{P}_{\text{old}}). \tag{3.8}$$

The proof of existence and uniqueness of the minimizer $(\mathbf{u}, \mathbf{P}) \in (V \cap H^1(\Omega, \mathbb{R}^n)) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}})$ is standard.

We start by computing the Γ-limit of the functional F_δ as ρ tends to the Dirac delta function at $\mathbf{0}$ [7,8]. The precise assumptions of the family of kernels $\{\rho_\delta\}_{\delta>0} \subset L^1(\mathbb{R}^n, [0,\infty))$ with $\|\rho_\delta\|_1 = n$ are as follows: Each ρ_δ is radial, i.e., there exists $\bar\rho_\delta : [0,\infty) \to [0,\infty)$ such that $\rho_\delta(\mathbf{x}) = \bar\rho_\delta(|\mathbf{x}|)$; moreover,

$$\text{the map } (0,\infty) \ni r \mapsto r^{-2} \bar\rho_\delta(r) \text{ is decreasing,} \tag{3.9}$$

$$\text{and } \lim_{\delta\to 0} \int_{\mathbb{R}^n \setminus B(\mathbf{0}, r)} \rho_\delta(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0 \text{ for all } r > 0. \tag{3.10}$$

This set of assumptions (or a slight variant of it) is typical in the analysis of the convergence from a nonlocal functional to a local one; see [5,6,20,25,26]. For ease of notation, in the following the subscript ρ used in the previous sections in F_ρ, $\mathfrak{D}_\rho$, $\mathfrak{E}_\rho$, T_ρ and so on is replaced by the subscript δ, meaning that the kernel involved is ρ_δ.

In this section we prove the Γ-convergence of F_δ to F_0 as $\delta \to 0$ in $L^2(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}})$ endowed with the strong topology in $L^2(\Omega, \mathbb{R}^n)$ and the weak topology in $L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}})$, or, equivalently, in $H^1(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}})$ endowed with the weak topology.

First, we show that $\mathfrak{E}_\delta(\mathbf{u}, \mathbf{P})$ is an approximation of $\operatorname{div} \mathbf{u}$.

Lemma 3.5 (Convergence of the divergence) *Let* $\mathbf{u} \in H^1(\Omega, \mathbb{R}^n)$ *and* $\mathbf{P} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}})$. *The following hold:*

a) $\mathfrak{E}_\delta(\mathbf{u}, \mathbf{P}) \to \operatorname{div} \mathbf{u}$ *as* $\delta \to 0$ *in* $L^2(\Omega)$.

b) For each $\delta > 0$ *let* $\mathbf{u}_\delta \in L^2(\Omega, \mathbb{R}^n)$ *and* $\mathbf{P}_\delta \in L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}})$. *Assume* $\mathbf{u}_\delta \to \mathbf{u}$ *in* $L^2(\Omega, \mathbb{R}^n)$ *and* $\mathbf{P}_\delta \rightharpoonup \mathbf{P}$ *in* $L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}})$ *as* $\delta \to 0$. *Suppose further that* $\sup_{\delta>0} |\mathbf{u}_\delta|_{\mathcal{S}_\delta} < \infty$. *Then* $\mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta) \rightharpoonup \operatorname{div} \mathbf{u}$ *as* $\delta \to 0$ *in* $L^2(\Omega)$.

Proof We start with *a)*. For each $\delta > 0$ we define the operator $\mathfrak{P}_\delta : L^2(\Omega, \mathbb{R}^{n\times n}_{\text{s,d}}) \to L^2(\Omega)$ by

$$\mathfrak{P}_\delta(\mathbf{P})(\mathbf{x}) = \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \frac{\mathbf{P}(\mathbf{x})(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} \, \mathrm{d}\mathbf{x}', \qquad \text{a.e. } \mathbf{x} \in \Omega.$$

Clearly, we have

$$\mathfrak{E}_\delta(\mathbf{u}, \mathbf{P}) = \mathfrak{D}_\delta(\mathbf{u}) - \mathfrak{P}_\delta(\mathbf{P}). \tag{3.11}$$

It was proved in [20, Lemma 3.1] that $\mathfrak{D}_\delta(\mathbf{u}) \to \operatorname{div} \mathbf{u}$ in $L^2(\Omega)$ as $\delta \to 0$. We shall show that $\mathfrak{P}_\delta(\mathbf{P}) \to 0$ in $L^2(\Omega)$. We can express, for a.e. $\mathbf{x} \in \Omega$,

$$\mathfrak{P}_\delta(\mathbf{P})(\mathbf{x}) = \int_{\Omega - \mathbf{x}} \rho_\delta(\tilde{\mathbf{x}}) \frac{\mathbf{P}(\mathbf{x})\tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|^2} \, \mathrm{d}\tilde{\mathbf{x}}, \tag{3.12}$$

so

$$|\mathfrak{P}_\delta(\mathbf{P})(\mathbf{x})| \le n\, |\mathbf{P}(\mathbf{x})| . \tag{3.13}$$

Now let $A \subset\subset \Omega$ and let $0 < r < \operatorname{dist}(A, \partial\Omega)$. Note that $B(\mathbf{0}, r) \subset \Omega - \mathbf{x}$ for any $\mathbf{x} \in A$. By (3.12) and Lemma A.2, we have, for a.e. $\mathbf{x} \in A$,

$$\mathfrak{P}_\delta(\mathbf{P})(\mathbf{x}) = \int_{(\Omega-\mathbf{x})\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}}) \frac{\mathbf{P}(\mathbf{x})\tilde{\mathbf{x}} \cdot \tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|^2} \, \mathrm{d}\tilde{\mathbf{x}},$$

so

$$|\mathfrak{P}_\delta(\mathbf{P})(\mathbf{x})| \le \int_{\mathbb{R}^n\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} \, |\mathbf{P}(\mathbf{x})|$$

and, consequently,

$$\int_A \mathfrak{P}_\delta(\mathbf{P})(\mathbf{x})^2 \, \mathrm{d}\mathbf{x} \le \left(\int_{\mathbb{R}^n\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} \right)^2 \|\mathbf{P}\|_2^2 . \tag{3.14}$$

Thanks to (3.10), we obtain that $\mathfrak{P}_\delta(\mathbf{P}) \to 0$ in $L^2(A)$ as $\delta \to 0$. Now, bound (3.13) implies that the family $\{\mathfrak{P}_\delta(\mathbf{P})^2\}_{\delta>0}$ is equiintegrable, so in fact $\mathfrak{P}_\delta(\mathbf{P}) \to 0$ in $L^2(\Omega)$ as $\delta \to 0$.

Now we show *b)*. In [20, Lemma 3.6] it was proved that $\mathfrak{D}_\delta(\mathbf{u}_\delta) \rightharpoonup \operatorname{div} \mathbf{u}$ in $L^2(\Omega)$ as $\delta \to 0$. Thanks to (3.11), it remains to show that $\mathfrak{P}_\delta(\mathbf{P}_\delta) \rightharpoonup 0$ in $L^2(\Omega)$, and for this we will show that $\{\mathfrak{P}_\delta(\mathbf{P}_\delta)\}_{\delta>0}$ is bounded in $L^2(\Omega)$ and that $\mathfrak{P}_\delta(\mathbf{P}_\delta) \to 0$ in $L^2_{\mathrm{loc}}(\Omega)$.

Let $\delta > 0$. Thanks to (3.13) we have $|\mathfrak{P}_\delta(\mathbf{P}_\delta)| \le n\, |\mathbf{P}_\delta|$, so $\{\mathfrak{P}_\delta(\mathbf{P}_\delta)\}_{\delta>0}$ is bounded in $L^2(\Omega)$. Now let $A \subset\subset \Omega$ and let $0 < r < \operatorname{dist}(A, \partial\Omega)$. By (3.14) we have that

$$\int_A \mathfrak{P}_\delta(\mathbf{P}_\delta)(\mathbf{x})^2 \, \mathrm{d}\mathbf{x} \le \left(\int_{\mathbb{R}^n\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} \right)^2 \|\mathbf{P}_\delta\|_2^2 .$$

Using (3.10) and the fact that $\{\mathbf{P}_\delta\}_{\delta>0}$ is bounded in $L^2(\Omega, \mathbb{R}^{n\times n})$, we conclude that $\mathfrak{P}_\delta(\mathbf{P}_\delta) \to 0$ in $L^2(A)$ as $\delta \to 0$, which finishes the proof. □

As a preparation for the Γ-limit $F_\delta \to F$ as $\delta \to 0$, we start with the pointwise limit.

Proposition 3.6 (Pointwise convergence of F_δ) *Let* $\mathbf{u} \in H^1(\Omega, \mathbb{R}^n)$ *and* $\mathbf{P} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. *Then*

$$\lim_{\delta\to 0} F_\delta(\mathbf{u}, \mathbf{P}) = F_0(\mathbf{u}, \mathbf{P}).$$

Proof Obviously, we only have to show that

$$\lim_{\delta\to 0} \int_\Omega \mathfrak{D}_\delta(\mathbf{u})(\mathbf{x})^2 \, \mathrm{d}\mathbf{x} = \int_\Omega \operatorname{div} \mathbf{u}(\mathbf{x})^2 \, \mathrm{d}\mathbf{x} \tag{3.15}$$

and

$$\begin{aligned} &\lim_{\delta\to 0} \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\ &\quad = n \int_\Omega \fint_{\mathbb{S}^{n-1}} \left((\nabla \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}\mathbf{x}. \end{aligned} \tag{3.16}$$

As mentioned in Lemma 3.5, the limit $\mathfrak{D}_\delta(\mathbf{u}) \to \operatorname{div} \mathbf{u}$ in $L^2(\Omega)$ as $\delta \to 0$ was shown in [20, Lemma 3.1], so we have equality (3.15).

We divide the proof of (3.16) in two steps, according to the regularity of $\mathbf{u}$ and $\mathbf{P}$.

Step 1 We assume additionally that $\mathbf{u} \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and $\mathbf{P} \in C(\bar{\Omega}, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$.

Since $\mathbf{u} \in C^1(\bar{\Omega}, \mathbb{R}^n)$, there exists an increasing bounded function $\sigma : [0, \infty) \to [0, \infty)$ with

$$\lim_{t\to 0} \sigma(t) = 0 \tag{3.17}$$

such that for all $\mathbf{x}, \mathbf{x}' \in \Omega$,

$$\left|\nabla \mathbf{u}(\mathbf{x}') - \nabla \mathbf{u}(\mathbf{x})\right| \leq \sigma(|\mathbf{x}' - \mathbf{x}|).$$

As Ω is a Lipschitz domain, a standard result shows that there exists $c \geq 1$ such that for all $\mathbf{x}, \mathbf{x}' \in \Omega$, we have

$$\left|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\right| \leq c \, \|\nabla \mathbf{u}\|_\infty \left|\mathbf{x}' - \mathbf{x}\right| \tag{3.18}$$

and

$$\left|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) - \nabla \mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x})\right| \leq |\mathbf{x}' - \mathbf{x}| c \, \sigma(|\mathbf{x}' - \mathbf{x}|).$$

For simplicity of notation, we relabel $c\,\sigma$ as σ and, hence, assume that for all $\mathbf{x}, \mathbf{x}' \in \Omega$,

$$\left|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) - \nabla \mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x})\right| \leq |\mathbf{x}' - \mathbf{x}| \sigma(|\mathbf{x}' - \mathbf{x}|). \tag{3.19}$$

Note that (3.18) implies that

$$\left|\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')\right| \leq c \, \|\nabla \mathbf{u}\|_\infty + \|\mathbf{P}\|_\infty . \tag{3.20}$$

Now we show that

$$\lim_{\delta \to 0} \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \left[\left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 - \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 \right] d\mathbf{x}' \, d\mathbf{x} = 0. \tag{3.21}$$

We have

$$\begin{aligned}
&\left| \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \left[\left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 - \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 \right] d\mathbf{x}' \, d\mathbf{x} \right| \\
&= \frac{1}{n^2} \left| \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \left(\operatorname{div} \mathbf{u}(\mathbf{x}) - \mathfrak{E}_\delta(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right) \left(2n^2 \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \mathfrak{E}_\delta(\mathbf{u}, \mathbf{P})(\mathbf{x}) - \operatorname{div} \mathbf{u}(\mathbf{x}) \right) d\mathbf{x}' \, d\mathbf{x} \right| \\
&\leq \frac{1}{n^2} \left(\int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \left(\operatorname{div} \mathbf{u}(\mathbf{x}) - \mathfrak{E}_\delta(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 d\mathbf{x}' \, d\mathbf{x} \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \left(2n^2 \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \mathfrak{E}_\delta(\mathbf{u}, \mathbf{P})(\mathbf{x}) - \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 d\mathbf{x}' \, d\mathbf{x} \right)^{\frac{1}{2}} .
\end{aligned}$$

Thanks to (2.2) and (3.20), the second term of the right-hand side is bounded by a constant times

$$\|\nabla \mathbf{u}\|_\infty + \|\mathbf{P}\|_\infty ,$$

while the first term tends to zero as $\delta \to 0$ thanks to Lemma 3.5. Thus, limit (3.21) is proved.

Now we show

$$\begin{aligned}
&\lim_{\delta \to 0} \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \left[\left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 \right] d\mathbf{x}' \, d\mathbf{x} \\
&= n \int_\Omega \fint_{\mathbb{S}^{n-1}} \left((\nabla \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 d\mathcal{H}^{n-1}(\mathbf{z}) \, d\mathbf{x}.
\end{aligned} \tag{3.22}$$

We express

$$\begin{aligned}
&\int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}')^2 \, d\mathbf{x}' \, d\mathbf{x} \\
&= \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) \left(\frac{(\nabla \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} \right)^2 d\mathbf{x}' \, d\mathbf{x} + \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}' - \mathbf{x}) C(\mathbf{x}, \mathbf{x}') \, d\mathbf{x}' \, d\mathbf{x}
\end{aligned} \tag{3.23}$$

with

$$C(\mathbf{x},\mathbf{x}') = \frac{\big(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})(\mathbf{x}'-\mathbf{x})\big)\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} \times \left(\frac{\big(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) - \nabla\mathbf{u}(\mathbf{x})(\mathbf{x}'-\mathbf{x})\big)\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} + 2\frac{(\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\right).$$

We have, thanks to (3.19),

$$|C(\mathbf{x},\mathbf{x}')| \le \sigma(|\mathbf{x}'-\mathbf{x}|)\,(\|\sigma\|_\infty + 2\|\nabla\mathbf{u}\|_\infty + 2\|\mathbf{P}\|_\infty)\,,$$

so for a.e. $\mathbf{x}\in\Omega$,

$$\left|\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})C(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\right| \le (\|\sigma\|_\infty + 2\|\nabla\mathbf{u}\|_\infty + 2\|\mathbf{P}\|_\infty)\int_{\Omega-\mathbf{x}} \rho_\delta(\tilde{\mathbf{x}})\,\sigma(|\tilde{\mathbf{x}}|)\,\mathrm{d}\tilde{\mathbf{x}} \tag{3.24}$$

and, for any $r>0$,

$$\int_{\Omega-\mathbf{x}} \rho_\delta(\tilde{\mathbf{x}})\,\sigma(|\tilde{\mathbf{x}}|)\,\mathrm{d}\tilde{\mathbf{x}}\,\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^n} \rho_\delta(\tilde{\mathbf{x}})\,\sigma(|\tilde{\mathbf{x}}|)\,\mathrm{d}\tilde{\mathbf{x}} \le n\sigma(r) + \|\sigma\|_\infty \int_{\mathbb{R}^n\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}. \tag{3.25}$$

Bounds (3.24) and (3.25), as well as properties (3.10) and (3.17), imply that

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})C(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = 0. \tag{3.26}$$

Now let $A \subset\subset \Omega$ be measurable and $0<r<\operatorname{dist}(A,\partial\Omega)$. Then, for any $\mathbf{x}\in A$,

$$\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\left(\frac{(\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\right)^2\mathrm{d}\mathbf{x}' = \left[\int_{B(\mathbf{0},r)} + \int_{(\Omega-\mathbf{x})\setminus B(\mathbf{0},r)}\right]\rho_\delta(\tilde{\mathbf{x}})\left(\frac{(\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|^2}\right)^2\mathrm{d}\tilde{\mathbf{x}}, \tag{3.27}$$

with, thanks to Lemma A.2,

$$\int_{B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}})\left(\frac{(\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|^2}\right)^2\mathrm{d}\tilde{\mathbf{x}} = \int_{B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}} ⨍_{\mathbb{S}^{n-1}} ((\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z})^2\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \tag{3.28}$$

and

$$\left|\int_{(\Omega-\mathbf{x})\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}})\left(\frac{(\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|^2}\right)^2\mathrm{d}\tilde{\mathbf{x}}\right| \le 2\left(\|\nabla\mathbf{u}\|_\infty^2 + \|\mathbf{P}\|_\infty^2\right)\int_{\mathbb{R}^n\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}. \tag{3.29}$$

Note that bound (3.20) implies that the family of functions

$$\mathbf{x} \mapsto \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}')^2\,\mathrm{d}\mathbf{x}'$$

is equiintegrable in Ω for $\delta>0$. Hence, property (3.10), together with bound (3.29) and equalities (3.27)–(3.28), shows that

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\left(\frac{(\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\right)^2\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = n\int_\Omega ⨍_{\mathbb{S}^{n-1}} ((\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z})^2\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x},$$

which, together with (3.23) and (3.26), implies

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}')^2\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = n\int_\Omega ⨍_{\mathbb{S}^{n-1}} ((\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z})^2\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x}. \tag{3.30}$$

Now we express

$$\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}')\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}$$
$$= \int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\frac{(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} + \int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})B(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \tag{3.31}$$

with

$$B(\mathbf{x},\mathbf{x}') = \frac{\big(\mathbf{u}(\mathbf{x}')-\mathbf{u}(\mathbf{x})-\nabla\mathbf{u}(\mathbf{x})(\mathbf{x}'-\mathbf{x})\big)\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\,\mathrm{div}\,\mathbf{u}(\mathbf{x}).$$

We have, thanks to (3.19),

$$|B(\mathbf{x},\mathbf{x}')| \le \sigma(|\mathbf{x}'-\mathbf{x}|)\|\,\mathrm{div}\,\mathbf{u}\|_\infty.$$

An analogous reasoning to that of (3.24), (3.25), and (3.26) leads to

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})B(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = 0. \tag{3.32}$$

Now let $A \subset\subset \Omega$ be measurable and $0 < r < \mathrm{dist}(A,\partial\Omega)$. Then, for any $\mathbf{x}\in A$,

$$\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\frac{(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}'$$
$$= \left[\int_{B(\mathbf{0},r)} + \int_{(\Omega-\mathbf{x})\setminus B(\mathbf{0},r)}\right]\rho_\delta(\tilde{\mathbf{x}})\frac{(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))\tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|^2}\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\tilde{\mathbf{x}}, \tag{3.33}$$

with, thanks to Lemma A.2,

$$\int_{B(\mathbf{0},r)}\rho_\delta(\tilde{\mathbf{x}})\frac{(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))\tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|^2}\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\tilde{\mathbf{x}} = \int_{B(\mathbf{0},r)}\rho_\delta(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}\fint_{\mathbb{S}^{n-1}}(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \tag{3.34}$$

and

$$\left|\int_{(\Omega-\mathbf{x})\setminus B(\mathbf{0},r)}\rho_\delta(\tilde{\mathbf{x}})\frac{(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))\tilde{\mathbf{x}}\cdot\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|^2}\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\tilde{\mathbf{x}}\right| \le \|\,\mathrm{div}\,\mathbf{u}\|_\infty\,(\|\nabla\mathbf{u}\|_\infty + \|\mathbf{P}\|_\infty)\int_{\mathbb{R}^n\setminus B(\mathbf{0},r)}\rho_\delta(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}. \tag{3.35}$$

Note that bound (3.20) implies that the family of functions

$$\mathbf{x}\mapsto \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}')\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}'$$

is equiintegrable in Ω for $\delta > 0$. Hence, property (3.10), together with bound (3.35) and equalities (3.33)–(3.34), shows that

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\frac{(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}$$
$$= n\int_\Omega\fint_{\mathbb{S}^{n-1}}(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

which, together with (3.31) and (3.32), implies

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}')\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = n\int_\Omega\fint_{\mathbb{S}^{n-1}}(\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{div}\,\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{3.36}$$

Now let $A \subset\subset \Omega$ be measurable and $0 < r < \mathrm{dist}(A,\partial\Omega)$. Then, for any $\mathbf{x}\in A$,

$$\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\,\mathrm{div}\,\mathbf{u}(\mathbf{x})^2\,\mathrm{d}\mathbf{x}' = \left[\int_{B(\mathbf{0},r)} + \int_{(\Omega-\mathbf{x})\setminus B(\mathbf{0},r)}\right]\rho_\delta(\tilde{\mathbf{x}})\,\mathrm{div}\,\mathbf{u}(\mathbf{x})^2\,\mathrm{d}\tilde{\mathbf{x}}, \tag{3.37}$$

with

$$\left|\int_{(\Omega-\mathbf{x})\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}}) \operatorname{div}\mathbf{u}(\mathbf{x})^2 \,\mathrm{d}\tilde{\mathbf{x}}\right| \le \|\operatorname{div}\mathbf{u}\|_\infty^2 \int_{\mathbb{R}^n\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}. \tag{3.38}$$

Note that the bound

$$\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \operatorname{div}\mathbf{u}(\mathbf{x})^2 \,\mathrm{d}\mathbf{x}' \le n\|\operatorname{div}\mathbf{u}\|_\infty^2$$

implies that the family of functions

$$\mathbf{x} \mapsto \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \operatorname{div}\mathbf{u}(\mathbf{x})^2 \,\mathrm{d}\mathbf{x}'$$

is equiintegrable in Ω for $\delta > 0$. Hence, property (3.10), together with bound (3.38) and equality (3.37) show that

$$\lim_{\delta\to 0}\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \operatorname{div}\mathbf{u}(\mathbf{x})^2 \,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = n\int_\Omega \operatorname{div}\mathbf{u}(\mathbf{x})^2\,\mathrm{d}\mathbf{x}, \tag{3.39}$$

Equalities (3.30), (3.36), and (3.39) show (3.22), while (3.22) and (3.21) yield (3.16) and complete the proof of this step.

Step 2 Now we just assume $\mathbf{u} \in H^1(\Omega, \mathbb{R}^n)$ and $\mathbf{P} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$, as in the statement. Let $\varepsilon > 0$, and let $\bar{\mathbf{u}} \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and $\bar{\mathbf{P}} \in C(\bar{\Omega}, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ be such that

$$\|\bar{\mathbf{u}} - \mathbf{u}\|_{H^1} \le \varepsilon \quad\text{and}\quad \left\|\bar{\mathbf{P}} - \mathbf{P}\right\|_2 \le \varepsilon.$$

This is possible since $\mathbb{R}^{n\times n}_{\mathrm{s,d}}$ is a subspace of $\mathbb{R}^{n\times n}$.

Now, consider Lemma 3.2 and the operator defined therein, which we call T_δ in order to underline the dependence on δ. By Lemmas 2.1, 3.2, and 2.3 there exists $C > 0$ independent of δ such that $\|T_\delta(\mathbf{v}, \mathbf{Q})\|_{L^2(\Omega\times\Omega)} \le C\left(\|\mathbf{v}\|_{H^1} + \|\mathbf{Q}\|_2\right)$ for all $\mathbf{v} \in H^1(\Omega, \mathbb{R}^n)$ and $\mathbf{Q} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. Then,

$$\begin{aligned}
&\left|\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\left[\left(\mathcal{E}(\bar{\mathbf{u}},\bar{\mathbf{P}})(\mathbf{x},\mathbf{x}') - \frac{1}{n}\mathfrak{E}_\delta(\bar{\mathbf{u}},\bar{\mathbf{P}})(\mathbf{x})\right)^2 - \left(\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}') - \frac{1}{n}\mathfrak{E}_\delta(\mathbf{u},\mathbf{P})(\mathbf{x})\right)^2\right]\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\right| \\
&= \left|\left\|T_\delta(\bar{\mathbf{u}},\bar{\mathbf{P}})\right\|^2_{L^2(\Omega\times\Omega)} - \|T_\delta(\mathbf{u},\mathbf{P})\|^2_{L^2(\Omega\times\Omega)}\right| \le \left\|T_\delta(\bar{\mathbf{u}}-\mathbf{u},\bar{\mathbf{P}}-\mathbf{P})\right\|_{L^2(\Omega\times\Omega)}\left\|T_\delta(\bar{\mathbf{u}}+\mathbf{u},\bar{\mathbf{P}}+\mathbf{P})\right\|_{L^2(\Omega\times\Omega)} \\
&\le 4\,C^2\,\varepsilon\left(\varepsilon + \|\mathbf{u}\|_{H^1} + \|\mathbf{P}\|_2\right).
\end{aligned}$$

This concludes the proof. □

Lemma 3.7 (Convergence of F_δ along smooth sequences) *Let $A \subset \Omega$ be a Lipschitz domain. For each $\delta > 0$ let $\mathbf{u}_\delta, \mathbf{u} \in C^1(\bar{A}, \mathbb{R}^n)$, $\mathbf{P}_\delta, \mathbf{P} \in C(\bar{A}, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ and $d_\delta \in C(\bar{A})$ satisfy*

$$\mathbf{u}_\delta \to \mathbf{u} \text{ in } C^1(\bar{A}, \mathbb{R}^n), \quad \mathbf{P}_\delta \to \mathbf{P} \text{ in } C(\bar{A}, \mathbb{R}^{n\times n}_{\mathrm{s,d}}), \quad\text{and}\quad d_\delta \to \operatorname{div}\mathbf{u} \text{ in } C(\bar{A}) \quad \text{as } \delta \to 0.$$

Then

$$\begin{aligned}
&\lim_{\delta\to 0}\int_A\int_A \rho_\delta(\mathbf{x}'-\mathbf{x})\left(\mathcal{E}(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x},\mathbf{x}') - d_\delta(\mathbf{x})\right)^2\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \\
&\quad = n\int_A \fint_{\mathbb{S}^{n-1}} \left((\nabla\mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\,\mathbf{z}\cdot\mathbf{z} - \operatorname{div}\mathbf{u}(\mathbf{x})\right)^2\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Proof We have

$$(\mathcal{E}(\mathbf{u}_\delta,\mathbf{P}_\delta) - d_\delta))^2 - (\mathcal{E}(\mathbf{u},\mathbf{P}) - \operatorname{div}\mathbf{u})^2 = [\mathcal{E}(\mathbf{u}_\delta - \mathbf{u},\mathbf{P}_\delta - \mathbf{P}) + \operatorname{div}\mathbf{u} - d_\delta]\,[\mathcal{E}(\mathbf{u}_\delta + \mathbf{u},\mathbf{P}_\delta + \mathbf{P}) - d_\delta - \operatorname{div}\mathbf{u}]\,.$$

We now use estimates (3.20) to infer that

$$\lim_{\delta\to 0}\left\|(\mathcal{E}(\mathbf{u}_\delta,\mathbf{P}_\delta) - d_\delta)^2 - (\mathcal{E}(\mathbf{u},\mathbf{P}) - \operatorname{div}\mathbf{u})^2\right\|_\infty = 0.$$

Then, by uniform convergence and equality (3.16) we conclude

$$\begin{aligned}&\lim_{\delta\to 0}\int_A\int_A \rho_\delta(\mathbf{x}'-\mathbf{x})\left(\mathcal{E}(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x},\mathbf{x}')-d_\delta(\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\\ &\quad=\lim_{\delta\to 0}\int_A\int_A \rho_\delta(\mathbf{x}'-\mathbf{x})\left(\mathcal{E}(\mathbf{u},\mathbf{P})(\mathbf{x},\mathbf{x}')-\operatorname{div}\mathbf{u}(\mathbf{x})\right)^2 \mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\\ &\quad=n\int_A\fint_{\mathbb{S}^{n-1}}\left((\nabla\mathbf{u}(\mathbf{x})-\mathbf{P}(\mathbf{x}))\,\mathbf{z}\cdot\mathbf{z}-\operatorname{div}\mathbf{u}(\mathbf{x})\right)^2 \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x},\end{aligned}$$

as desired. □

The following nonlocal Korn inequality of [20, Lemma 4.4], with a constant independent of δ, is essential in the proof of the Γ-convergence.

Proposition 3.8 (Uniform nonlocal Korn inequality) *Let $\{\rho_\delta\}_{\delta>0}$ be a family of kernels satisfying (3.9)–(3.10). Then there exist $C>0$ and $\delta_0>0$ such that for all $0<\delta<\delta_0$ and $\mathbf{u}\in V\cap\mathcal{S}_\delta(\Omega)$,*

$$\|\mathbf{u}\|_2^2\le C\,|\mathbf{u}|_{\mathcal{S}_\delta}^2\,.$$

With Proposition 3.8 at hand, we can show the following coercivity bound for F_δ.

Lemma 3.9 (Uniform coercivity of the energy) *Let $\{\rho_\delta\}_{\delta>0}$ be a family of kernels satisfying (3.9)–(3.10). Then there exist $c>0$ and $\delta_0>0$ such that for all $0<\delta<\delta_0$ and $(\mathbf{u},\mathbf{P})\in(V\cap\mathcal{S}_\delta(\Omega))\times L^2(\Omega,\mathbb{R}^{n\times n}_{\mathrm{s,d}})$,*

$$F_\delta(\mathbf{u},\mathbf{P})\ge c\|(\mathbf{u},\mathbf{P})\|_{\mathcal{T}_\delta}^2-\frac{1}{c}.$$

Proof We repeat the proof of Lemma 3.1 until (3.6): We then find that there exists $c_1>0$ such that for all $\delta>0$, all $(\mathbf{u},\mathbf{P})\in\mathcal{T}_\delta$, and all $\eta>0$,

$$F_\delta(\mathbf{u},\mathbf{P})\ge c_1\left(|(\mathbf{u},\mathbf{P})|_{\mathcal{T}_\delta}^2+\|\mathbf{P}\|_2^2\right)+c_1\left(|(\mathbf{u},\mathbf{P})|_{\mathcal{T}_\delta}^2+\|\mathbf{P}\|_2^2\right)-\frac{\eta}{2}\|\mathbf{u}\|_2^2-\frac{1}{2\eta}\|\mathbf{b}\|_2^2.$$

By Proposition 3.8 and estimate (2.5), there exist $C>0$ and $\delta_0>0$ such that for all $0<\delta<\delta_0$,

$$\|\mathbf{u}\|_2^2\le C|\mathbf{u}|_{\mathcal{S}_\delta}^2\le 2nC\left(|(\mathbf{u},\mathbf{P})|_{\mathcal{T}_\delta}^2+\|\mathbf{P}\|_2^2\right).$$

Putting together both inequalities, we find that

$$F_\delta(\mathbf{u},\mathbf{P})\ge c_1\left(|(\mathbf{u},\mathbf{P})|_{\mathcal{T}_\delta}^2+\|\mathbf{P}\|_2^2\right)+\left(\frac{c_1}{2nC}-\frac{\eta}{2}\right)\|\mathbf{u}\|_2^2-\frac{1}{2\eta}\|\mathbf{b}\|_2^2.$$

Choosing $\eta>0$ such that

$$\frac{c_1}{2nC}-\frac{\eta}{2}>0$$

concludes the proof. □

We present the fundamental compactness result of [20, Prop. 4.2].

Proposition 3.10 (Compactness). *Let $\{\rho_\delta\}_{\delta>0}$ be a sequence of kernels satisfying (3.9)–(3.10). Let $\{\mathbf{u}_\delta\}_{\delta>0}$ be a sequence in $L^2(\Omega,\mathbb{R}^n)$ satisfying*

$$\sup_{\delta>0}\|\mathbf{u}_\delta\|_{\mathcal{S}_\delta}<\infty.$$

Then there exists a decreasing sequence $\delta_j\to 0$ and a $\mathbf{u}\in L^2(\Omega,\mathbb{R}^n)$ such that $\mathbf{u}_{\delta_j}\to\mathbf{u}$ in $L^2(\Omega,\mathbb{R}^n)$. Moreover, for any such sequence and any such $\mathbf{u}$ we have that $\mathbf{u}\in H^1(\Omega,\mathbb{R}^n)$.

We now have all ingredients to prove the Γ-limit result. As usual, we divide it into three parts: compactness, lower bound, and upper bound. We label the sequences with δ, the same parameter of F_δ, and, of course, it is implicit that $\delta\to 0$.

Theorem 3.11 (Γ-convergence of the energy) *Let* $\mathcal{V}_\delta = (V \cap \mathcal{S}_\delta) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$.

a) *Let* $(\mathbf{u}_\delta, \mathbf{P}_\delta) \in \mathcal{V}_\delta$ *satisfy* $\sup_\delta F_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta) < \infty$. *Then there exists* $(\mathbf{u}, \mathbf{P}) \in H^1(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ *such that, for a subsequence,* $\mathbf{u}_\delta \to \mathbf{u}$ *in* $L^2(\Omega, \mathbb{R}^n)$ *and* $\mathbf{P}_\delta \rightharpoonup \mathbf{P}$ *in* $L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$.

b) *Let* $(\mathbf{u}_\delta, \mathbf{P}_\delta) \in \mathcal{V}_\delta$ *and* $(\mathbf{u}, \mathbf{P}) \in H^1(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ *satisfy* $\mathbf{u}_\delta \to \mathbf{u}$ *in* $L^2(\Omega, \mathbb{R}^n)$ *and* $\mathbf{P}_\delta \rightharpoonup \mathbf{P}$ *in* $L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. *Then*

$$F_0(\mathbf{u}, \mathbf{P}) \le \liminf_{\delta\to 0} F_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta).$$

c) *Let* $(\mathbf{u}, \mathbf{P}) \in (V \cap H^1(\Omega, \mathbb{R}^n)) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. *Then for each* δ *there exists* $(\mathbf{u}_\delta, \mathbf{P}_\delta) \in \mathcal{V}_\delta$ *such that*

$$F_0(\mathbf{u}, \mathbf{P}) = \lim_{\delta\to 0} F_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta).$$

Proof Part *(a)*. By Lemma 3.9, the set $\{\|(\mathbf{u}, \mathbf{P})\|^2_{\mathcal{I}_\delta}\}_{\delta>0}$ is bounded. We then apply Proposition 3.10 to find the existence of $\mathbf{u}$, and the boundedness of $\{\mathbf{P}_\delta\}_{\delta>0}$ in $L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ for the existence of $\mathbf{P}$.

Part *b)*. Clearly,

$$\|\mathbf{P}\|_2^2 \le \liminf_{\delta\to 0} \|\mathbf{P}_\delta\|_2^2 \quad \text{and} \quad \lim_{\delta\to 0} \int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}_\delta(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Moreover, as mentioned in Lemma 3.5, it was proved in [20, Lemma 3.6] that $\mathfrak{D}_\delta(\mathbf{u}_\delta) \rightharpoonup \operatorname{div} \mathbf{u}$ in $L^2(\Omega)$ as $\delta \to 0$, so $\|\operatorname{div} \mathbf{u}\|_2^2 \le \liminf_\delta \|\mathfrak{D}_\delta(\mathbf{u}_\delta)\|_2^2$. Hence, we are left to the analysis of the remaining term.

Let $\{\varphi_r\}_{r>0}$ be the family of mollifiers defined in Appendix A. Let $A \subset\subset \Omega$ be a Lipschitz domain, and let $0 < r < \operatorname{dist}(A, \partial\Omega)$. By Lemma A.1,

$$\begin{aligned}
&\int_A \int_A \rho_\delta(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\varphi_r \star \mathbf{u}_\delta, \varphi_r \star \mathbf{P}_\delta)(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \varphi_r \star \mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \\
&\quad \le \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.40}$$

Call $\mathbf{u}_r = \varphi_r \star \mathbf{u}$ and $\mathbf{P}_r = \varphi_r \star \mathbf{P}$. Standard properties of mollifiers show that $\varphi_r \star \mathbf{u}_\delta \to \mathbf{u}_r$ in $C^1(\bar{A}, \mathbb{R}^n)$ and $\varphi_r \star \mathbf{P}_{\delta,r} \to \mathbf{P}_r$ in $C(\bar{A}, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ as $\delta \to 0$. Using also Lemma 3.5, we find that $\varphi_r \star \mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta) \to \operatorname{div} \mathbf{u}_r$ in $C(\bar{A})$ as $\delta \to 0$. Thus, letting $\delta \to 0$ in (3.40) and using Lemma 3.7, we obtain

$$\begin{aligned}
&n \int_A \fint_{\mathbb{S}^{n-1}} \left((\nabla \mathbf{u}_r(\mathbf{x}) - \mathbf{P}_r(\mathbf{x}))\, \mathbf{z} \cdot \mathbf{z} - \frac{1}{n} \operatorname{div} \mathbf{u}_r(\mathbf{x}) \right)^2 \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\, \mathrm{d}\mathbf{x} \\
&\quad \le \liminf_{\delta\to 0} \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \left(\mathcal{E}(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.41}$$

Again, standard properties of mollifiers show that $\nabla \mathbf{u}_r \to \nabla \mathbf{u}$ in $L^2(A, \mathbb{R}^{n\times n})$ and a.e., and $\mathbf{P}_r \to \mathbf{P}$ in $L^2(A, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ and a.e., as $r \to 0$. We then let $r \to 0$ and apply dominated convergence in (3.41) to get

$$\begin{aligned}
&n \int_A \fint_{\mathbb{S}^{n-1}} \left((\nabla \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\, \mathrm{d}\mathbf{x} \\
&\quad \le \liminf_{\delta\to 0} \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \left(\mathcal{E}(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.42}$$

Finally, we send $A \nearrow \Omega$ and use monotone convergence in (3.42) to obtain

$$\begin{aligned}
&n \int_\Omega \fint_{\mathbb{S}^{n-1}} \left((\nabla \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right)^2 \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\, \mathrm{d}\mathbf{x} \\
&\quad \le \liminf_{\delta\to 0} \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \left(\mathcal{E}(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}.
\end{aligned}$$

Part *c)*. This follows from Proposition 3.6 by taking $(\mathbf{u}_\delta, \mathbf{P}_\delta) = (\mathbf{u}, \mathbf{P})$. □

We are now ready to present the small-horizon convergence result for the incremental problem.

Corollary 3.12 (Convergence to the local incremental problem) *Let* $\mathbf{P}_{\rm old} \in L^2(\Omega, \mathbb{R}^{n\times n}_{\rm s,d})$ *be given and* $(\mathbf{u}_\delta, \mathbf{P}_\delta)$ *be the solution of the nonlocal incremental problem* (3.1). *Then* $(\mathbf{u}_\delta, \mathbf{P}_\delta) \to (\mathbf{u}_0, \mathbf{P}_0)$ *with the respect to the strong* $\times$ *weak topology in* Q, *where* $(\mathbf{u}_0, \mathbf{P}_0) \in (V \cap H^1(\Omega, \mathbb{R}^n)) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\rm s,d})$ *is the solution of the local incremental problem* (3.8).

Proof For each $\delta > 0$ we have

$$F_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta) + H(\mathbf{P}_\delta - \mathbf{P}_{\rm old}) \le F_\delta(\mathbf{u}_0, \mathbf{P}_0) + H(\mathbf{P}_0 - \mathbf{P}_{\rm old}),$$

so by Proposition 3.6, $\sup_{\delta>0} F_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta) < \infty$. By Theorem 3.11, the sequence $(\mathbf{u}_\delta, \mathbf{P}_\delta)$ is precompact in the strong $\times$ weak topology in Q. Thus, one is left to prove the Γ-convergence of $F_\delta + H(\cdot{-}\mathbf{P}_{\rm old})$ as $\delta \to 0$. The Γ-lim inf follows from the Γ-convergence of F_δ in Theorem 3.11 as H is independent of δ and lower semicontinuous. The existence of a recovery sequence follows by pointwise convergence: See Proposition 3.6. $\square$

4 Quasistatic evolution

Assume now that the body force $\mathbf{b}$ depends on time, namely let $\mathbf{b} \in W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^n))$. Correspondingly, without introducing new notation, we indicate the time-dependent (complementary) energy of the medium via $F_\rho : Q \times [0, T] \to \mathbb{R} \cup \{\infty\}$ given by

$$\begin{aligned} F_\rho(\mathbf{u}, \mathbf{P}, t) = \beta \int_\Omega \mathfrak{D}_\rho(\mathbf{u})(\mathbf{x})^2 \,{\rm d}\mathbf{x} + \alpha \int_\Omega \int_\Omega \rho(\mathbf{x}'{-}\mathbf{x}) \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 {\rm d}\mathbf{x}' \,{\rm d}\mathbf{x} \\ + \gamma \int_\Omega |\mathbf{P}(\mathbf{x})|^2 \,{\rm d}\mathbf{x} - \int_\Omega \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{u}(\mathbf{x}) \,{\rm d}\mathbf{x}. \end{aligned}$$

Note that boundary conditions could be taken to be time dependent as well by letting $\mathbf{u} - \mathbf{u}_{\rm Dir}(t) \in V$, where $\mathbf{u}_{\rm Dir}(t)$ is given. This would originate an additional time-dependent linear term in the energy. We, however, stick to the time-independent condition $\mathbf{u} \in V$, for the sake of simplicity.

The quasistatic elastoplastic evolution of the medium (1.7)–(1.8) can be then specified as

$$\partial_{\mathbf{u}} F_\rho(\mathbf{u}(t), \mathbf{P}(t), t) = \mathbf{0} \quad \text{in } \mathcal{S}^*_\rho, \tag{4.1}$$

$$\partial_{\dot{\mathbf{P}}} H(\dot{\mathbf{P}}(t)) + \partial_{\mathbf{P}} F_\rho(\mathbf{u}(t), \mathbf{P}(t), t) \ni \mathbf{0} \quad \text{in } L^2(\Omega; \mathbb{R}^{n\times n}_{\rm s,d}). \tag{4.2}$$

We have denoted by $\mathcal{S}^*_\rho$ the dual of $\mathcal{S}_\rho$. In particular, relation (4.2) is a pointwise-in-time inclusion in $L^2(\Omega, \mathbb{R}^{n\times n}_{\rm s,d})$.

System (4.1)–(4.2) can be made more explicit by introducing the bilinear form B_ρ associated with the quadratic part of F_ρ, namely

$$\begin{aligned} B_\rho((\mathbf{u}, \mathbf{P}), (\mathbf{v}, \mathbf{Q})) &= \beta \int_\Omega \mathfrak{D}_\delta(\mathbf{u})(\mathbf{x})\, \mathfrak{D}_\delta(\mathbf{v})(\mathbf{x}) \,{\rm d}\mathbf{x} \\ &+ \alpha \int_\Omega \int_\Omega \rho(\mathbf{x}{-}\mathbf{x}') \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right) \left(\mathcal{E}(\mathbf{v}, \mathbf{Q})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{v}, \mathbf{Q})(\mathbf{x}) \right) {\rm d}\mathbf{x}' \,{\rm d}\mathbf{x} \\ &+ \gamma \int_A \mathbf{P}(\mathbf{x}) : \mathbf{Q}(\mathbf{x}) \,{\rm d}\mathbf{x}. \end{aligned}$$

Making use of B_ρ one can equivalently rewrite (4.1)–(4.2) as the nonlocal system

$$2B_\rho((\mathbf{u}(t), \mathbf{P}(t)), (\mathbf{v}, \mathbf{0})) = \int_\Omega \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}) \,{\rm d}\mathbf{x} \quad \forall \mathbf{v} \in \mathcal{S}_\rho,$$

$$2B_\rho((\mathbf{u}(t), \mathbf{P}(t)), (\mathbf{0}, \dot{\mathbf{P}}(t) - \mathbf{w})) \le \int_\Omega \sigma_y |\mathbf{w}(\mathbf{x})| \,{\rm d}\mathbf{x} - \int_\Omega \sigma_y |\dot{\mathbf{P}}(\mathbf{x}, t)| \,{\rm d}\mathbf{x} \quad \forall \mathbf{w} \in L^2(\Omega; \mathbb{R}^{n\times n}_{\rm s,d}).$$

The quasistatic elastoplastic evolution problem consists in finding a strong (in time) solution to system (4.1)–(4.2), starting from the initial state $(\overline{\mathbf{u}}, \overline{\mathbf{P}}) \in (V \cap \mathcal{S}_\rho) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. We equivalently reformulate the problem in energetic terms as that of finding *quasistatic evolution* trajectories $(\mathbf{u}_\rho, \mathbf{P}_\rho) : [0, T] \to Q$ such that, for all $t \in [0, T]$,

$$\mathbf{u}_\rho(t) \in \mathcal{S}_\rho \ \text{ and } \ F_\rho(\mathbf{u}_\rho(t), \mathbf{P}_\rho(t), t) \le F_\rho(\widehat{\mathbf{u}}, \widehat{\mathbf{P}}, t) + H(\widehat{\mathbf{P}} - \mathbf{P}_\rho(t)) \quad \forall (\widehat{\mathbf{u}}, \widehat{\mathbf{P}}) \in Q, \tag{4.3}$$

$$F_\rho(\mathbf{u}_\rho(t), \mathbf{P}_\rho(t), t) + \mathrm{Diss}_{[0,t]}(\mathbf{P}_\rho) = F_\rho(\mathbf{u}_\rho(0), \mathbf{P}_\rho(0), 0) - \int_0^t \int_\Omega \dot{\mathbf{b}}(\mathbf{x}, s) \cdot \mathbf{u}_\rho(\mathbf{x}, s)\, \mathrm{d}\mathbf{x}\, \mathrm{d}s, \tag{4.4}$$

where the dissipation $\mathrm{Diss}_{[0,t]}(\mathbf{P}_\rho)$ is defined as

$$\mathrm{Diss}_{[0,t]}(\mathbf{P}_\rho) = \sup \left\{ \sum_{i=1}^N H(\mathbf{P}_\rho(t_{i-1}) - \mathbf{P}_\rho(t_i)) \right\}$$

and the supremum is taken on all partitions $\{0 = t_0 < t_1 < \cdots < t_N = t\}$ of $[0, t]$. The time-parametrized variational inequality (4.3) is usually called *global stability*. It expresses a minimality of the current state $(\mathbf{u}_\rho(t), \mathbf{P}_\rho(t))$ with respect to possible competitors $(\widehat{\mathbf{u}}, \widehat{\mathbf{P}})$ when the combined effect of energy and dissipation is taken into account. We will call all states $(\mathbf{u}_\rho(t), \mathbf{P}_\rho(t))$ fulfilling (4.3) *stable* and equivalently indicate (4.3) as $(\mathbf{u}_\rho(t), \mathbf{P}_\rho(t)) \in \mathfrak{S}_\rho(t)$, so that $\mathfrak{S}_\rho(t)$ is the set of *stable states* at time t. The scalar relation (4.4) is nothing but the energy balance: The sum of the actual and the dissipated energy (left-hand side of (4.4)) equals the sum of the initial energy and the work done by external actions (right-hand side). Note that systems (4.1)–(4.2) and (4.3)–(4.4) are equivalent as the energy F_ρ is strictly convex (see Proposition 3.3).

This section is devoted to the study of the quasistatic evolution problem (4.3)–(4.4). In particular, we prove that it is well-posed in Sect. 4.2 by passing to the limit into a time discretization discussed in Sect. 4.1. Eventually, we study the localization limit as ρ converges to a Dirac delta function at $\mathbf{0}$ in Sect. 4.3

4.1 Incremental minimization

For the sake of notational simplicity, we drop the subscript ρ from $(\mathbf{u}_\rho, \mathbf{P}_\rho)$ in this subsection. Let a partition $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ of $[0, T]$ be given, and let $(\mathbf{u}_0, \mathbf{P}_0) = (\overline{\mathbf{u}}, \overline{\mathbf{P}})$. The *incremental minimization problem* consists in finding $(\mathbf{u}_i, \mathbf{P}_i) \in Q$ that minimizes

$$F_\rho(\mathbf{u}, \mathbf{P}, t_i) + H(\mathbf{P} - \mathbf{P}_{i-1}) \tag{4.5}$$

for $i = 1, \ldots, N$. Owing to Theorem 3.4, the unique solution $\{(\mathbf{u}_i, \mathbf{P}_i)\}_{i=0}^N$ can be found inductively on i. The minimality in (4.5) and the triangle inequality entail that

$$F_\rho(\mathbf{u}_i, \mathbf{P}_i, t_i) + H(\mathbf{P}_i - \mathbf{P}_{i-1}) \le F_\rho(\widehat{\mathbf{u}}, \widehat{\mathbf{P}}, t_i) + H(\widehat{\mathbf{P}} - \mathbf{P}_i) + H(\mathbf{P}_i - \mathbf{P}_{i-1}) \quad \forall (\widehat{\mathbf{u}}, \widehat{\mathbf{P}}) \in Q. \tag{4.6}$$

This proves in particular that $(\mathbf{u}_i, \mathbf{P}_i)$ is stable for all i. More precisely, $(\mathbf{u}_i, \mathbf{P}_i) \in \mathfrak{S}_\rho(t_i)$ for all $i = 1, \ldots, N$. Again from minimality one has

$$F_\rho(\mathbf{u}_i, \mathbf{P}_i, t_i) + H(\mathbf{P}_i - \mathbf{P}_{i-1}) \le F_\rho(\mathbf{u}_{i-1}, \mathbf{P}_{i-1}, t_i) = F_\rho(\mathbf{u}_{i-1}, \mathbf{P}_{i-1}, t_{i-1}) - \int_\Omega \int_{t_{i-1}}^{t_i} \dot{\mathbf{b}}(\mathbf{x}, s)\, \mathrm{d}s \cdot \mathbf{u}_{i-1}(\mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{4.7}$$

Now, the coercivity of F_ρ from Lemma 3.1 implies the existence of $M > 0$ such that

$$\|(\mathbf{u}, \mathbf{P})\|_{\mathcal{T}_\rho} \le M \left(1 + F_\rho(\mathbf{u}, \mathbf{P})\right), \qquad \forall (\mathbf{u}, \mathbf{P}) \in Q.$$

This and Minkowski's inequality imply

$$\begin{aligned} \int_\Omega \int_{t_{i-1}}^{t_i} \dot{\mathbf{b}}(\mathbf{x}, s)\, \mathrm{d}s \cdot \mathbf{u}_{i-1}(\mathbf{x})\, \mathrm{d}\mathbf{x} &\le \left\| \int_{t_{i-1}}^{t_i} \dot{\mathbf{b}}(\cdot, s)\, \mathrm{d}s \right\|_{L^2(\Omega)} \|\mathbf{u}_{i-1}\|_{L^2(\Omega)} \\ &\le M \int_{t_{i-1}}^{t_i} \left\| \dot{\mathbf{b}}(\cdot, s) \right\|_{L^2(\Omega)} \mathrm{d}s \left(1 + F_\rho(\mathbf{u}_{i-1}, \mathbf{P}_{i-1}, t_{i-1})\right). \end{aligned} \tag{4.8}$$

Fix an integer $m \le N$; by summing (4.7) up for $i = 1, \dots, m$, we get

$$F_\rho(\mathbf{u}_m, \mathbf{P}_m, t_m) + \sum_{i=1}^m H(\mathbf{P}_i - \mathbf{P}_{i-1}) \le F_\rho(\overline{\mathbf{u}}, \overline{\mathbf{P}}, 0) - \sum_{i=1}^m \int_\Omega \int_{t_{i-1}}^{t_i} \dot{\mathbf{b}}(\mathbf{x}, s)\, \mathrm{d}s \cdot \mathbf{u}_{i-1}(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{4.9}$$

while using (4.8) we get

$$F_\rho(\mathbf{u}_m, \mathbf{P}_m, t_m) + \sum_{i=1}^m H(\mathbf{P}_i - \mathbf{P}_{i-1}) \le F_\rho(\overline{\mathbf{u}}, \overline{\mathbf{P}}, 0) + M \, \|\dot{\mathbf{b}}\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}$$
$$+ M \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \left\|\dot{\mathbf{b}}(\cdot, s)\right\|_{L^2(\Omega)} \mathrm{d}s \, F_\rho(\mathbf{u}_{i-1}, \mathbf{P}_{i-1}, t_{i-1}).$$

With the discrete Gronwall's inequality, we deduce that

$$F_\rho(\mathbf{u}_m, \mathbf{P}_m, t_m) + \sum_{i=1}^m H(\mathbf{P}_i - \mathbf{P}_{i-1}) \le C, \tag{4.10}$$

where C depends on $F_\rho(\overline{\mathbf{u}}, \overline{\mathbf{P}}, 0)$ and $\|\dot{\mathbf{b}}\|_{L^1(0,T;L^2(\Omega;\mathbb{R}^n))}$ but not on the time partition. In particular, the incremental minimization problem delivers a stable approximation scheme. This could additionally be combined with a space discretization as well.

4.2 Well-posedness of the quasistatic evolution problem

The aim of this subsection is to check the following well-posedness result.

Theorem 4.1 (Well-posedness of the quasistatic evolution problem) *Let* $\mathbf{b} \in W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^n))$ *and* $(\overline{\mathbf{u}}, \overline{\mathbf{P}}) \in \mathfrak{S}_\rho(0)$. *Then there exists a unique quasistatic evolution* $t \mapsto (\mathbf{u}_\rho(t), \mathbf{P}_\rho(t))$.

Proof This well-posedness argument is quite standard, for the energy F_ρ is quadratic and coercive. Indeed, the statement follows from [23, Thm. 3.5.2] where one finds quasistatic evolutions by passing to the limit in the time-discrete solution of the incremental problem (4.5) as the fineness of the partition goes to 0. Assume for simplicity such partitions to be uniform and given by $t_i^N = iT/N$ (nonuniform partitions can be considered as well) and define $(\mathbf{u}_N, \mathbf{P}_N) : [0, T] \to Q$ to be the backward-in-time piecewise constant interpolant of the solution of the incremental problem (4.5) on the partition.

Bound (4.10) and the coercivity of F_ρ from Lemma 3.1 entail that $\|(\mathbf{u}_N, \mathbf{P}_N)\|_{\mathcal{T}_\rho}$ and $\mathrm{Diss}_{[0,T]}(\mathbf{P}_N)$ are bounded independently of N. This allows for the application of the Helly's selection principle [23, Thm. 2.1.24] which, in combination with Lemma 3.9 and Proposition 3.10, entails that $(\mathbf{u}_N, \mathbf{P}_N)$ converges to $(\mathbf{u}, \mathbf{P})$ with respect to the strong $\times$ weak topology of Q, for all times.

The global stability $(\mathbf{u}(t), \mathbf{P}(t)) \in \mathfrak{S}_\rho(t)$ for all $t \in [0, T]$ follows by passing to the lim sup in (4.6) by means of the so-called *quadratic trick*, see [23, Lem. 3.5.3]: Let $(\widehat{\mathbf{u}}, \widehat{\mathbf{P}}) \in Q$ be given, and define $(\widehat{\mathbf{u}}_N, \widehat{\mathbf{P}}_N) = (\mathbf{u}_N(t_i^N) + \widehat{\mathbf{u}} - \mathbf{u}(t), \mathbf{P}_N(t_i^N) + \widehat{\mathbf{P}} - \mathbf{P}(t))$. By using the short-hand notation $B_\rho(\mathbf{u}, \mathbf{P})$ for $B_\rho((\mathbf{u}, \mathbf{P}), (\mathbf{u}, \mathbf{P}))$, from the fact that $(\mathbf{u}_N(t), \mathbf{P}_N(t)) \in \mathfrak{S}_\rho(t_i^N)$ for $t \in (t_{i-1}^N, t_i^N]$ we deduce that

$$\begin{aligned} 0 &\le F_\rho(\widehat{\mathbf{u}}_N, \widehat{\mathbf{P}}_N, t_i^N) - F_\rho(\mathbf{u}_N(t), \mathbf{P}_N(t), t_i^N) + H(\widehat{\mathbf{P}}_N - \mathbf{P}_N(t)) \\ &= B_\rho(\widehat{\mathbf{u}} - \mathbf{u}(t), \widehat{\mathbf{P}} - \mathbf{P}(t)) + 2B_\rho\Big((\mathbf{u}_N(t), \mathbf{P}_N(t)), (\widehat{\mathbf{u}} - \mathbf{u}(t), \widehat{\mathbf{P}} - \mathbf{P}(t))\Big) \\ &\quad - \int_\Omega \mathbf{b}(\mathbf{x}, t_i^N) \cdot (\widehat{\mathbf{u}}(\mathbf{x}) - \mathbf{u}(\mathbf{x}, t))\, \mathrm{d}\mathbf{x} + H(\widehat{\mathbf{P}} - \mathbf{P}(t)). \end{aligned} \tag{4.11}$$

Take now the limit for $N \to \infty$ in (4.11) and obtain

$$\begin{aligned} 0 &\le B_\rho(\widehat{\mathbf{u}} - \mathbf{u}(t), \widehat{\mathbf{P}} - \mathbf{P}(t)) + 2B_\rho\Big((\mathbf{u}(t), \mathbf{P}(t)), (\widehat{\mathbf{u}} - \mathbf{u}(t), \widehat{\mathbf{P}} - \mathbf{P}(t))\Big) \\ &\quad - \int_\Omega \mathbf{b}(\mathbf{x}, t) \cdot (\widehat{\mathbf{u}}(\mathbf{x}) - \mathbf{u}(\mathbf{x}, t))\, \mathrm{d}\mathbf{x} + H(\widehat{\mathbf{P}} - \mathbf{P}(t)) \\ &= F_\rho(\widehat{\mathbf{u}}, \widehat{\mathbf{P}}, t) - F_\rho(\mathbf{u}(t), \mathbf{P}(t), t) + H(\widehat{\mathbf{P}} - \mathbf{P}(t)). \end{aligned}$$

Since the latter holds for all $(\widehat{\mathbf{u}}, \widehat{\mathbf{P}}) \in Q$, we have proved that $(\mathbf{u}(t), \mathbf{P}(t)) \in \mathfrak{S}_\rho(t)$.

Inequality '$\leq$' in (4.4) follows by passing to the lim inf as $N \to \infty$ in (4.9). The opposite inequality is a consequence of the already checked global stability, see [23, Prop. 2.1.23]. Eventually, uniqueness is a consequence of the strict convexity of F_ρ. $\square$

4.3 Localization limit

The aim of this subsection is to investigate the localization limit for ρ converging to a Dirac delta function at $\mathbf{0}$. Replace ρ by ρ_δ fulfilling assumptions (3.9)–(3.10) of Sect. 3.2, and use δ as subscript instead of ρ wherever relevant. Define

$$\mathcal{S}_0 = \{\mathbf{u} \in H^1(\Omega, \mathbb{R}^n) \ : \ \mathbf{u} = \mathbf{0} \text{ on } \omega\}.$$

We shall check that the quasistatic evolution $(\mathbf{u}_\delta, \mathbf{P}_\delta)$ for the nonlocal model converges to the unique solution $(\mathbf{u}_0, \mathbf{P}_0)$ of the classical local elastoplastic quasistatic problem

$$\partial_{\mathbf{u}} F_0(\mathbf{u}_0(t), \mathbf{P}_0(t), t) = \mathbf{0} \quad \text{in } \mathcal{S}_0^*, \tag{4.12}$$

$$\partial_{\dot{\mathbf{P}}} H(\dot{\mathbf{P}}_0(t)) + \partial_{\mathbf{P}} F_0(\mathbf{u}_0(t), \mathbf{P}_0(t), t) \ni \mathbf{0} \quad \text{in } L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{s,d}}). \tag{4.13}$$

In analogy with (4.1)–(4.2), one can rewrite (4.12)–(4.13) via the bilinear form B_0

$$\begin{aligned} B_0\big((\mathbf{u}, \mathbf{P}), (\mathbf{v}, \mathbf{Q})\big) &= \beta \int_\Omega \operatorname{div} \mathbf{u}(\mathbf{x}) \operatorname{div} \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &+ \alpha n \int_\Omega \fint_{\mathbb{S}^{n-1}} \left((\nabla \mathbf{u}(\mathbf{x}) - \mathbf{P}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} - \frac{1}{n} \operatorname{div} \mathbf{u}(\mathbf{x}) \right) \left((\nabla \mathbf{v}(\mathbf{x}) - \mathbf{Q}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} - \frac{1}{n} \operatorname{div} \mathbf{v}(\mathbf{x}) \right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\, \mathrm{d}\mathbf{x} \\ &+ \gamma \int_\Omega \mathbf{P}(\mathbf{x}) : \mathbf{Q}(\mathbf{x})\, \mathrm{d}\mathbf{x} \end{aligned}$$

as

$$2B_0((\mathbf{u}_0(t), \mathbf{P}_0(t)), (\mathbf{v}, \mathbf{0})) = \int_\Omega \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in \mathcal{S}_0, \tag{4.14}$$

$$2B_0((\mathbf{u}_0(t), \mathbf{P}_0(t)), (\mathbf{0}, \dot{\mathbf{P}}_0(t) - \mathbf{w})) \leq \int_\Omega \sigma_y |\mathbf{w}(\mathbf{x})|\, \mathrm{d}\mathbf{x} - \int_\Omega \sigma_y |\dot{\mathbf{P}}_0(\mathbf{x}, t)|\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{w} \in L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{s,d}}). \tag{4.15}$$

By recalling the expression for the Lamé coefficients (1.3) the latter can be equivalently restated in the classical form

$$\int_\Omega \mathbf{\Sigma}(\mathbf{x}, t) : \nabla^s \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{b}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}) \quad \forall \mathbf{v} \in V, \quad \text{for a.e. } t \in (0, T), \tag{4.16}$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \partial\Omega \setminus \overline{\omega}, \ \text{ for a.e. } t \in (0, T), \tag{4.17}$$

$$\mathbf{\Sigma} = \lambda \operatorname{tr} \left(\nabla^s \mathbf{u} - \mathbf{P}\right) + 2\mu \left(\nabla^s \mathbf{u} - \mathbf{P}\right) \quad \text{a.e. in } \Omega \times (0, T), \tag{4.18}$$

$$\sigma_y \partial |\dot{\mathbf{P}}| + 2\gamma \mathbf{P} \ni \mathbf{\Sigma} \quad \text{a.e. in } \Omega \times (0, T), \tag{4.19}$$

$$\mathbf{P}(0) = \mathbf{P}_0 \quad \text{a.e. in } \Omega. \tag{4.20}$$

Relations (4.12) or (4.14) correspond to the quasistatic equilibrium system (4.16) and the corresponding boundary condition (4.17). Note that, since $\Omega \backslash \overline{\omega}$ is Lipschitz, condition (4.17) can be also read as $\mathbf{u}(t)|_\omega \in H^1_0(\omega, \mathbb{R}^n)$. The isotropic material response is encoded by the constitutive relation (4.18) for the stress $\mathbf{\Sigma}$. (Note, however, that isotropy is here assumed for the sake of definiteness only, for the analysis covers anisotropic cases with no change.) The plastic flow rule (4.13) or (4.15) corresponds to (4.19), to be considered together with initial condition (4.20). Recall that problem (4.16)–(4.20) (equivalently systems (4.12)–(4.13) or (4.14)–(4.15) along with initial conditions) admits a unique strong solution in time [16], which is indeed a quasistatic evolution in the sense of (4.3)–(4.4) [23, Sec. 4.3.1].

Theorem 4.2 (Convergence of quasistatic evolutions) *Let* $\mathbf{b} \in W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^n))$ *and* $(\overline{\mathbf{u}}_\delta, \overline{\mathbf{P}}_\delta) \in \mathfrak{S}_\delta(0)$ *be such that* $(\overline{\mathbf{u}}_\delta, \overline{\mathbf{P}}_\delta) \to (\overline{\mathbf{u}}_0, \overline{\mathbf{P}}_0)$ *with respect to the strong* $\times$ *weak topology of* Q *and* $F_\delta(\overline{\mathbf{u}}_\delta, \overline{\mathbf{P}}_\delta, 0) \to F_0(\overline{\mathbf{u}}_0, \overline{\mathbf{P}}_0, 0)$. *Then, the unique quasistatic evolution of the nonlocal problem* $(\mathbf{u}_\delta, \mathbf{P}_\delta)$ *converges to* $(\mathbf{u}_0, \mathbf{P}_0)$ *with respect to the strong* $\times$ *weak topology of* Q*, for all times, where* $(\mathbf{u}_0, \mathbf{P}_0)$ *is the unique quasistatic evolution of local elastoplasticity.*

Proof This argument follows along the general lines of [24, Thm. 3.8] and hinges on identifying a suitable *mutual recovery sequence* for the functionals F_ρ and H.

The energy balance (4.4) at level ρ, the uniform coercivity of F_ρ from Lemma 3.9, and the fact that $\dot{\mathbf{b}} \in L^1(0, T; L^2(\Omega, \mathbb{R}^n))$ entail that $\sup_{t\in[0,T]} \|(\mathbf{u}_\delta, \mathbf{P}_d)\|_{\mathcal{T}_\delta}$ and $\mathrm{Diss}_{[0,T]}(\mathbf{P}_\delta)$ are bounded independently of δ. By using the generalized Helly's selection principle [24, Thm. A.1], Lemma 3.9, and Proposition 3.10 one extracts a (non-relabeled) subsequence converging to $(\mathbf{u}_0, \mathbf{P}_0)$ strongly $\times$ weakly in Q for all times. By passing to the lim inf as $\delta \to 0$ in the energy balance (4.4), as $F_\delta \to F_0$ in the Γ-convergence sense (Theorem 3.11) one finds that

$$F_0(\mathbf{u}_0(t), \mathbf{P}_0(t), t) + \mathrm{Diss}_{[0,t]}(\mathbf{P}_0) \le F_0(\mathbf{u}_0(0), \mathbf{P}_0(0), 0) - \int_0^t \int_\Omega \dot{\mathbf{b}}(\mathbf{x}, s) \cdot \mathbf{u}_0(\mathbf{x}, s)\, \mathrm{d}\mathbf{x}\, \mathrm{d}s, \tag{4.21}$$

which is the upper energy estimate. Moreover, the initial values of $(\mathbf{u}_0, \mathbf{P}_0)$ can be computed as

$$(\mathbf{u}_0(0), \mathbf{P}_0(0)) = \lim_{\delta\to 0} (\mathbf{u}_\delta(0), \mathbf{P}_\delta(0)) = \lim_{\delta\to 0} (\overline{\mathbf{u}}_\delta, \overline{\mathbf{P}}_\delta) = (\overline{\mathbf{u}}_0, \overline{\mathbf{P}}_0),$$

where the limit is strong $\times$ weak in Q.

We now need to check that $(\mathbf{u}_0, \mathbf{P}_0)$ is globally stable for all times, namely $(\mathbf{u}_0(t), \mathbf{P}_0(t)) \in \mathfrak{S}_0(t)$ for all $t \in [0, T]$, where the latter set of stable states is defined starting from the energy F_0. This is obtained by exploiting once again the quadratic nature of the energy via the *quadratic trick*. As $(\mathbf{u}_\delta(t), \mathbf{P}_\delta(t)) \in \mathfrak{S}_\delta(t)$ for all $t \in [0, T]$, for any $(\widehat{\mathbf{u}}_\delta, \widehat{\mathbf{P}}_\delta) \in Q$ one has that

$$\begin{aligned} 0 &\le F_\delta(\widehat{\mathbf{u}}_\delta, \widehat{\mathbf{P}}_\delta, t) - F_\delta(\mathbf{u}_\delta(t), \mathbf{P}_\delta(t), t) + H(\widehat{\mathbf{P}}_\delta - \mathbf{P}_\delta(t)) \\ &= B_\delta(\widehat{\mathbf{u}}_\delta, \widehat{\mathbf{P}}_\delta) - B_\delta(\mathbf{u}_\delta(t), \mathbf{P}_\delta(t)) - \int_\Omega \mathbf{b}(\mathbf{x}, t) \cdot (\widehat{\mathbf{u}}_\delta - \mathbf{u}_\delta(t))\, \mathrm{d}\mathbf{x} + H(\widehat{\mathbf{P}}_\delta - \mathbf{P}_\delta(t)). \end{aligned} \tag{4.22}$$

Let the competitors $(\widehat{\mathbf{u}}_0, \widehat{\mathbf{P}}_0) \in Q$ be given, and assume for the time being that $(\widehat{\mathbf{u}}_0 - \mathbf{u}_0(t), \widehat{\mathbf{P}}_0 - \mathbf{P}_0(t)) \in C^\infty(\bar{\Omega}; \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. Insert the *mutual recovery sequence*

$$(\widehat{\mathbf{u}}_\delta, \widehat{\mathbf{P}}_\delta) = \big(\mathbf{u}_\delta(t) + \widehat{\mathbf{u}}_0 - \mathbf{u}_0(t), \mathbf{P}_\delta(t) + \widehat{\mathbf{P}}_0 - \mathbf{P}_0(t)\big)$$

into (4.22) getting

$$\begin{aligned} 0 \le\ & B_\delta\big(\widehat{\mathbf{u}}_0 - \mathbf{u}_0(t), \widehat{\mathbf{P}}_0 - \mathbf{P}_0(t)\big) - \int_\Omega \mathbf{b}(\mathbf{x}, t) \cdot (\widehat{\mathbf{u}}_0(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}, t))\, \mathrm{d}\mathbf{x} \\ &+ H(\widehat{\mathbf{P}}_0 - \mathbf{P}_0(t)) + 2B_\delta\big((\mathbf{u}_\delta(t), \mathbf{P}_\delta(t)), (\widehat{\mathbf{u}}_0 - \mathbf{u}_0(t), \widehat{\mathbf{P}}_0 - \mathbf{P}_0(t))\big). \end{aligned} \tag{4.23}$$

We aim now at passing to the limit as $\delta \to 0$ in (4.23). The first two terms in the right-hand side converge by Proposition 3.6, and the dissipation term is independent of δ. One can hence use Lemma B.1 for the last term and conclude that

$$\begin{aligned} 0 \le\ & B_0\big(\widehat{\mathbf{u}}_0 - \mathbf{u}_0(t), \widehat{\mathbf{P}}_0 - \mathbf{P}_0(t)\big) - \int_\Omega \mathbf{b}(\mathbf{x}, t) \cdot (\widehat{\mathbf{u}}_0(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}, t))\, \mathrm{d}\mathbf{x} \\ &+ H(\widehat{\mathbf{P}}_0 - \mathbf{P}_0(t)) + 2B_0\big((\mathbf{u}_0(t), \mathbf{P}_0(t)), (\widehat{\mathbf{u}}_0 - \mathbf{u}_0(t), \widehat{\mathbf{P}}_0 - \mathbf{P}_0(t))\big) \\ =\ & F_0(\widehat{\mathbf{u}}_0, \widehat{\mathbf{P}}_0, t) - F_0(\mathbf{u}_0(t), \mathbf{P}_0(t), t) + H(\widehat{\mathbf{P}}_0 - \mathbf{P}_0(t)). \end{aligned}$$

The stability of $(\mathbf{u}_0(t), \mathbf{P}_0(t))$ is hence checked against all competitors with $(\widehat{\mathbf{u}}_0 - \mathbf{u}_0(t), \widehat{\mathbf{P}}_0 - \mathbf{P}_0(t))$ in $C^\infty(\bar{\Omega}; \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. In order to conclude for the global stability of $(\mathbf{u}_0(t), \mathbf{Q}_0(t))$ at time t one has now to argue by approximation. Let a general competitor $(\widehat{\mathbf{u}}_0, \widehat{\mathbf{P}}_0) \in Q$ with $\widehat{\mathbf{u}}_0 \in H^1(\Omega, \mathbb{R}^n)$ be given, and choose a sequence $(\widehat{\mathbf{u}}_{0j}, \widehat{\mathbf{P}}_{0j}) \in Q$ such that $(\widehat{\mathbf{u}}_{0j}, \widehat{\mathbf{P}}_{0j}) \to (\widehat{\mathbf{u}}_0, \widehat{\mathbf{P}}_0)$ strongly in $H^1(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ and

$(\widehat{\mathbf{u}}_{0j} - \mathbf{u}_0(t), \widehat{\mathbf{P}}_{0j} - \mathbf{P}_0(t)) \in C^\infty(\bar{\Omega}; \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. As F_0 and H are continuous with respect to the strong topology in $H^1(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ and $L^2(\Omega, \mathbb{R}^{n\times n}_{\mathrm{s,d}})^2$, respectively, one gets

$$0 \le \lim_{j\to\infty} \Big(F_0(\widehat{\mathbf{u}}_{0j}, \widehat{\mathbf{P}}_{0j}, t) - F_0(\mathbf{u}_0(t), \mathbf{P}_0(t), t) + H(\widehat{\mathbf{P}}_{0j} - \mathbf{P}_0(t)) \Big)$$
$$= F_0(\widehat{\mathbf{u}}_0, \widehat{\mathbf{P}}_0, t) - F_0(\mathbf{u}_0(t), \mathbf{P}_0(t), t) + H(\widehat{\mathbf{P}}_0 - \mathbf{P}_0(t))$$

which proves $(\mathbf{u}_0(t), \mathbf{Q}_0(t)) \in \mathfrak{S}_0(t)$. Eventually, global stability allows to recover the opposite estimate to (4.21) as in [23, Prop. 2.1.23].

We have hence proved that $(\mathbf{u}_0, \mathbf{P}_0)$ is a quasistatic evolution of the local elastoplastic problem. As F_0 is strictly convex, such solution is unique and convergence holds for the whole sequence. □

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Appendix A: Auxiliary results

We collect here some auxiliary results that have been used in the paper. Let $\varphi \in C^\infty_c(\mathbb{R}^n)$ satisfy $\operatorname{supp}\varphi \subset B(\mathbf{0}, 1)$, $\varphi \ge 0$, and $\int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mathbf{x} = 1$. For each $r > 0$, define the function $\varphi_r \in C^\infty_c(\mathbb{R}^n)$ as $\varphi_r(\mathbf{x}) = r^{-n}\varphi(\mathbf{x}/r)$. Define $\Omega_r = \{\mathbf{x} \in \Omega : \operatorname{dist}(\mathbf{x}, \partial\Omega) > r\}$. As usual, given a function $u : \Omega \to \mathbb{R}$ its mollification $\varphi_r \star u : \Omega_r \to \mathbb{R}$ is defined as

$$(\varphi_r \star u)(\mathbf{x}) = \int_{B(\mathbf{0},r)} \varphi_r(\mathbf{z})\, u(\mathbf{x} - \mathbf{z}) \,\mathrm{d}\mathbf{z}.$$

For vector-valued functions, the mollification is defined componentwise.
The following result was used in Sect. 3.2.

Lemma A.1 (Energy decreases by mollification) *Let* $(\mathbf{u}, \mathbf{P}) \in \mathcal{T}_\rho(\Omega)$. *Let* $A \subset\subset \Omega$ *be measurable, and let* $0 < r < \operatorname{dist}(A, \partial\Omega)$. *Then*

$$\int_A \int_A \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\varphi_r \star \mathbf{u}, \varphi_r \star \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \varphi_r \star \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x}$$
$$\le \int_\Omega \int_\Omega \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x}.$$

Proof For each $\mathbf{x}, \mathbf{x}' \in A$,

$$\mathcal{E}(\varphi_r \star \mathbf{u}, \varphi_r \star \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \varphi_r \star \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) = \int_{B(\mathbf{0},r)} \varphi_r(\mathbf{z}) \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x} - \mathbf{z}, \mathbf{x}' - \mathbf{z}) - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x} - \mathbf{z}) \right) \mathrm{d}\mathbf{z},$$

so, by Jensen's inequality,

$$\left(\mathcal{E}(\varphi_r \star \mathbf{u}, \varphi_r \star \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \varphi_r \star \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2$$
$$\le \int_{B(\mathbf{0},r)} \varphi_r(\mathbf{z}) \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x} - \mathbf{z}, \mathbf{x}' - \mathbf{z}) - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x} - \mathbf{z}) \right)^2 \mathrm{d}\mathbf{z}.$$

Therefore,

$$\int_A \int_A \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\varphi_r \star \mathbf{u}, \varphi_r \star \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \varphi_r \star \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x}$$
$$\le \int_{B(\mathbf{0},r)} \varphi_r(\mathbf{z}) \int_{\Omega_r} \int_{\Omega_r} \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x} - \mathbf{z}, \mathbf{x}' - \mathbf{z}) - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x} - \mathbf{z}) \right)^2 \mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{z}.$$

But, for each $\mathbf{z} \in B(\mathbf{0}, r)$,

$$\begin{aligned}
&\int_A \int_A \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}-\mathbf{z}, \mathbf{x}'-\mathbf{z}) - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}-\mathbf{z}) \right)^2 \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\
&\quad = \int_{A-\mathbf{z}} \int_{A-\mathbf{z}} \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\
&\quad \leq \int_\Omega \int_\Omega \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x},
\end{aligned}$$

so

$$\begin{aligned}
&\int_{B(\mathbf{0},r)} \varphi_r(\mathbf{z}) \int_A \int_A \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}-\mathbf{z}, \mathbf{x}'-\mathbf{z}) - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}-\mathbf{z}) \right)^2 \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{z} \\
&\quad \leq \int_\Omega \int_\Omega \rho(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}, \mathbf{P})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\rho(\mathbf{u}, \mathbf{P})(\mathbf{x}) \right)^2 \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x},
\end{aligned}$$

and the proof is concluded. □

We now show an elementary calculation of some integrals in a ball, where we exploit that the kernel is radial.

Lemma A.2 (Radially symmetric kernels) *Let $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and let $\bar{\rho} : [0, \infty) \to [0, \infty)$ be such that $\rho(\mathbf{x}) = \bar{\rho}(|\mathbf{x}|)$ for a.e. $\mathbf{x} \in \mathbb{R}^n$. Let $r > 0$. The following hold:*

a) *Let $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ be positively homogeneous of degree 0. Then*

$$\int_{B(\mathbf{0},r)} \rho(\mathbf{x}) \, f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{B(\mathbf{0},r)} \rho(\mathbf{x}) \, \mathrm{d}\mathbf{x} \fint_{\mathbb{S}^{n-1}} f(\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).$$

b) *Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then*

$$\int_{B(\mathbf{0},r)} \rho(\mathbf{x}) \frac{\mathbf{A}\mathbf{x} \cdot \mathbf{x}}{|\mathbf{x}|^2} \, \mathrm{d}\mathbf{x} = \frac{1}{n} \int_{B(\mathbf{0},r)} \rho(\mathbf{x}) \, \mathrm{d}\mathbf{x} \, \operatorname{tr} \mathbf{A}.$$

Proof We start with *a)*. We use the coarea formula and the homogeneity of f to find that

$$\int_{B(\mathbf{0},r)} \rho(\mathbf{x}) \, f(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_0^r \bar{\rho}(s) \int_{\partial B(\mathbf{0},s)} f(\mathbf{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) \, \mathrm{d}s = \int_0^r s^{n-1} \, \bar{\rho}(s) \, \mathrm{d}s \int_{\mathbb{S}^{n-1}} f(\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).$$

The above formula applied to the constant function $f = 1$ shows that

$$\int_{B(\mathbf{0},r)} \rho(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \int_0^r s^{n-1} \, \bar{\rho}(s) \, \mathrm{d}s.$$

Putting the two formulas together concludes the proof of *a)*.
For part *b)*, we apply *a)* to the function $f(\mathbf{x}) = \frac{1}{|\mathbf{x}|^2} \mathbf{A}\mathbf{x} \cdot \mathbf{x}$ and obtain that

$$\int_{B(\mathbf{0},r)} \rho(\mathbf{x}) \frac{\mathbf{A}\mathbf{x} \cdot \mathbf{x}}{|\mathbf{x}|^2} \, \mathrm{d}\mathbf{x} = \int_{B(\mathbf{0},r)} \rho(\mathbf{x}) \, \mathrm{d}\mathbf{x} \fint_{\mathbb{S}^{n-1}} \mathbf{A}\mathbf{z} \cdot \mathbf{z} \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).$$

Now let $\mathbf{A}_s = \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)$. Then $\mathbf{A}\mathbf{z} \cdot \mathbf{z} = \mathbf{A}_s \mathbf{z} \cdot \mathbf{z}$ for all $\mathbf{z} \in \mathbb{R}^n$ and $\operatorname{tr} \mathbf{A} = \operatorname{tr} \mathbf{A}_s$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}_s$, let $\mathbf{R} \in O(n)$ and $\mathbf{D} \in \mathbb{R}^{n\times n}$ be such that $\mathbf{A}_s = \mathbf{R}\mathbf{D}\mathbf{R}^\top$ and $\mathbf{D}$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. A change of variables shows that

$$\int_{\mathbb{S}^{n-1}} \mathbf{A}_s \mathbf{z} \cdot \mathbf{z} \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) = \int_{\mathbb{S}^{n-1}} \mathbf{D}\mathbf{z} \cdot \mathbf{z} \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) = \int_{\mathbb{S}^{n-1}} \sum_{i=1}^n \lambda_i z_i^2 \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).$$

Another change of variables shows that for all $i \in \{1, \dots, n\}$,

$$\int_{\mathbb{S}^{n-1}} z_i^2 \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) = \int_{\mathbb{S}^{n-1}} z_1^2 \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}),$$

so

$$\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = \int_{\mathbb{S}^{n-1}} |\mathbf{z}|^2 \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) = n \int_{\mathbb{S}^{n-1}} z_1^2 \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).$$

Thus,

$$\fint_{\mathbb{S}^{n-1}} \sum_{i=1}^{n} \lambda_i z_i^2 \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) = \sum_{i=1}^{n} \lambda_i \fint_{\mathbb{S}^{n-1}} z_1^2 \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i = \frac{1}{n} \operatorname{tr} \mathbf{A},$$

which concludes the proof. □

Appendix B: Convergence lemma

We present here the proof of the key convergence lemma used for passing to the limit in (4.23) in the proof of Theorem 4.2.

Lemma B.1 (Convergence of the bilinear term) *Let* $(\mathbf{u}_\delta, \mathbf{P}_\delta) \to (\mathbf{u}_0, \mathbf{P}_0)$ *strongly* $\times$ *weakly in* Q*,* $(\tilde{\mathbf{u}}, \tilde{\mathbf{P}}) \in C^\infty(\bar{\Omega}; \mathbb{R}^n \times \mathbb{R}^{n\times n}_{\mathrm{s,d}})$*, and* $\|(\mathbf{u}_\delta, \mathbf{p}_\delta)\|_{\mathcal{T}_\delta}$ *be bounded independently of* δ*. Then*

$$B_\delta\big((\mathbf{u}_\delta, \mathbf{P}_\delta), (\tilde{\mathbf{u}}, \tilde{\mathbf{P}})\big) \to B_0\big((\mathbf{u}_0, \mathbf{P}_0), (\tilde{\mathbf{u}}, \tilde{\mathbf{P}})\big).$$

Proof We aim at computing the limit of

$$\begin{aligned} B_\delta\big((\mathbf{u}_\delta, \mathbf{P}_\delta), (\tilde{\mathbf{u}}, \tilde{\mathbf{P}})\big) &= \gamma \int_A \mathbf{P}_\delta(\mathbf{x}) : \tilde{\mathbf{P}}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \beta \int_\Omega \mathfrak{D}_\delta(\mathbf{u}_\delta)(\mathbf{x}) \, \mathfrak{D}_\delta(\tilde{\mathbf{u}})(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ &+ \alpha \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}) \right) \left(\mathcal{E}(\tilde{\mathbf{u}}, \tilde{\mathbf{P}})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{P}})(\mathbf{x}) \right) \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \end{aligned}$$

Passing to the limit in the γ term is straightforward as $\mathbf{P}_\delta \rightharpoonup \mathbf{P}_0$ in $L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{s,d}})$. The β terms goes to the limit as well, for we have that $\mathfrak{D}_\delta(\mathbf{u}_\delta) \rightharpoonup \operatorname{div} \mathbf{u}_0$ in $L^2(\Omega)$ [20, Lemma 3.6] and $\mathfrak{D}_\delta(\tilde{\mathbf{u}}) \to \operatorname{div} \tilde{\mathbf{u}}$ strongly in $L^2(\Omega)$ [20, Lemma 3.1] (see also Lemma 3.5). We will hence focus on the α term, from which, for simplicity of notation, we omit the parameter α:

$$\begin{aligned} &A_\delta\big((\mathbf{u}_\delta, \mathbf{P}_\delta), (\tilde{\mathbf{u}}, \tilde{\mathbf{p}})\big) \\ &= \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}) \right) \left(\mathcal{E}(\tilde{\mathbf{u}}, \tilde{\mathbf{P}})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \mathfrak{E}_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{P}})(\mathbf{x}) \right) \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \end{aligned}$$

The strategy of the proof is that of decomposing A_δ in a sum of integrals and discussing the corresponding limits separately. We proceed in subsequent steps.

Step 1 Let us start by simplifying the problem of computing the limit of A_δ by replacing $\mathfrak{E}_\delta(\mathbf{u}_\delta, \mathbf{P}_\delta)$ and $\mathfrak{E}_\delta(\tilde{\mathbf{u}}, \tilde{\mathbf{P}})$ by $\operatorname{div} \mathbf{u}_0$ and $\operatorname{div} \tilde{\mathbf{u}}$, respectively. In particular, within this step we aim at proving that

$$\lim_{\delta \to 0} \Big[A_\delta\big((\mathbf{u}_\delta, \mathbf{P}_\delta), (\tilde{\mathbf{u}}, \tilde{\mathbf{P}})\big) - \tilde{A}_\delta\big((\mathbf{u}_\delta, \mathbf{P}_\delta), (\tilde{\mathbf{u}}, \tilde{\mathbf{P}}); \mathbf{u}_0\big) \Big] = 0, \tag{B.1}$$

where we have set

$$\begin{aligned} &\tilde{A}_\delta\big((\mathbf{u}_\delta, \mathbf{P}_\delta), (\tilde{\mathbf{u}}, \tilde{\mathbf{P}}); \mathbf{u}_0\big) \\ &= \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}') \left(\mathcal{E}(\mathbf{u}_\delta, \mathbf{P}_\delta)(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \operatorname{div} \mathbf{u}_0(\mathbf{x}) \right) \left(\mathcal{E}(\tilde{\mathbf{u}}, \tilde{\mathbf{P}})(\mathbf{x}, \mathbf{x}') - \frac{1}{n} \operatorname{div} \tilde{\mathbf{u}}(\mathbf{x}) \right) \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \end{aligned}$$

In order to do so, let us write

$$A_\delta\big((\mathbf{u}_\delta,\mathbf{P}_\delta),(\tilde{\mathbf{u}},\tilde{\mathbf{P}})\big)-\tilde{A}_\delta\big((\mathbf{u}_\delta,\mathbf{P}_\delta),(\tilde{\mathbf{u}},\tilde{\mathbf{P}});\mathbf{u}_0\big)=J_\delta^1+J_\delta^2$$

with

$$J_\delta^1=-\frac{1}{n}\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\left(\mathcal{E}(\mathbf{u}_\delta,\mathbf{P}_\delta)\big(\mathbf{x},\mathbf{x}'\big)-\frac{1}{n}\mathfrak{E}_\delta(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x})\right)\left(\mathfrak{E}_\delta(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x})-\operatorname{div}\tilde{\mathbf{u}}(\mathbf{x})\right)\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},$$
$$J_\delta^2=-\frac{1}{n}\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\big(\mathfrak{E}_\delta(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x})-\operatorname{div}\mathbf{u}_0(\mathbf{x})\big)\left(\mathcal{E}(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x},\mathbf{x}')-\frac{1}{n}\operatorname{div}\tilde{\mathbf{u}}(\mathbf{x})\right)\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}$$

and prove that $J_\delta^1\to 0$ and $J_\delta^2\to 0$ as $\delta\to 0$.
As regards J_δ^1, one has the bound

$$\big|J_\delta^1\big|\le\frac{1}{n}\left(\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\left(\mathcal{E}(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x},\mathbf{x}')-\frac{1}{n}\mathfrak{E}_\delta(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x})\right)^2\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\right)^{1/2}$$
$$\times\left(\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\big(\mathfrak{E}_\delta(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x})-\operatorname{div}\tilde{\mathbf{u}}(\mathbf{x})\big)^2\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\right)^{1/2}.$$

The first integral in the right-hand side above is bounded as $\|(\mathbf{u}_\delta,\mathbf{P}_\delta)\|_{\mathcal{T}_\delta}$ is bounded, whereas the second integral tends to 0 because of Lemma 3.5.a.
Next, we rewrite

$$J_\delta^2=-\frac{1}{n}\int_\Omega\big(\mathfrak{E}_\delta(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x})-\operatorname{div}\mathbf{u}_0(\mathbf{x})\big)\left(\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\left(\mathcal{E}(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x},\mathbf{x}')-\frac{1}{n}\operatorname{div}\tilde{\mathbf{u}}(\mathbf{x})\right)\mathrm{d}\mathbf{x}'\right)\mathrm{d}\mathbf{x}.$$

We have that $\mathfrak{E}_\delta(\mathbf{u}_\delta,\mathbf{P}_\delta)\rightharpoonup\operatorname{div}\mathbf{u}_0$ in $L^2(\Omega)$ by Lemma 3.5.b. On the other hand, by arguing as in the proof Proposition 3.6 one gets that the function

$$\mathbf{x}\mapsto\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\left(\mathcal{E}(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x},\mathbf{x}')-\frac{1}{n}\operatorname{div}\tilde{\mathbf{u}}(\mathbf{x})\right)\mathrm{d}\mathbf{x}'$$

is strongly convergent in $L^2(\Omega)$ and $J_\delta^2\to 0$ follows.
Step 2: Decomposition of $\tilde{A}_\delta$ Owing to (B.1) we now argue directly on $\tilde{A}_\delta$ by decomposing it as

$$\tilde{A}_\delta\big((\mathbf{u}_\delta,\mathbf{P}_\delta),(\tilde{\mathbf{u}},\tilde{\mathbf{P}});\mathbf{u}_0\big)=I_\delta^1+I_\delta^2+I_\delta^3+I_\delta^4,\tag{B.2}$$

where

$$I_\delta^1=\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\,\mathcal{E}(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x},\mathbf{x}')\,\mathcal{E}(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},$$
$$I_\delta^2=-\frac{1}{n}\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\,\mathcal{E}(\mathbf{u}_\delta,\mathbf{P}_\delta)(\mathbf{x},\mathbf{x}')\,\operatorname{div}\tilde{\mathbf{u}}(\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},$$
$$I_\delta^3=-\frac{1}{n}\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\,\operatorname{div}\mathbf{u}_0(\mathbf{x})\,\mathcal{E}(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},$$
$$I_\delta^4=\frac{1}{n^2}\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\,\operatorname{div}\mathbf{u}_0(\mathbf{x})\,\operatorname{div}\tilde{\mathbf{u}}(\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}.$$

We discuss each of these integrals in the following steps.
Step 3 Integral I_δ^1 As in (2.3), we decompose the integral as $I_\delta^1=I_\delta^{11}+I_\delta^{12}+I_\delta^{13}$, where

$$I_\delta^{11}=-\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\,\frac{\mathbf{P}_\delta(\mathbf{x})(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\cdot(\mathbf{x}'-\mathbf{x})\,\mathcal{E}(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},$$
$$I_\delta^{12}=\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\,\mathcal{D}(\mathbf{u}_\delta-\mathbf{u}_0)(\mathbf{x},\mathbf{x}')\,\mathcal{E}(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},$$
$$I_\delta^{13}=\int_\Omega\int_\Omega\rho_\delta(\mathbf{x}-\mathbf{x}')\,\mathcal{D}(\mathbf{u}_0)(\mathbf{x},\mathbf{x}')\,\mathcal{E}(\tilde{\mathbf{u}},\tilde{\mathbf{P}})(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},$$

and argue on each term separately.
In order to compute the limit of I_δ^{11}, let us further decompose it as

$$\begin{aligned}
I_\delta^{11} &= I_\delta^{111} + I_\delta^{112} \\
&= -\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}')\,\frac{\mathbf{P}_\delta(\mathbf{x})(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\cdot(\mathbf{x}'-\mathbf{x})\,\frac{(\nabla\tilde{\mathbf{u}}(\mathbf{x})-\tilde{\mathbf{P}}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\cdot(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \\
&\quad -\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}')\,\frac{\mathbf{P}_\delta(\mathbf{x})(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\cdot(\mathbf{x}'-\mathbf{x})\,\frac{(\tilde{\mathbf{u}}(\mathbf{x}')-\tilde{\mathbf{u}}(\mathbf{x})-\nabla\tilde{\mathbf{u}}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))}{|\mathbf{x}'-\mathbf{x}|^2}\cdot(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

The limit of I_δ^{111} can be computed by observing that the integrand is positively homogeneous of degree 0 in $\mathbf{x}'-\mathbf{x}$. In particular, arguing as in Lemma A.2 we can prove that

$$\lim_{\delta\to 0}\left[I_\delta^{111} + n\int_\Omega \fint_{\mathbb{S}^{n-1}} \mathbf{P}_\delta(\mathbf{x})\mathbf{z}\cdot\mathbf{z}\,(\nabla\tilde{\mathbf{u}}(\mathbf{x})-\tilde{\mathbf{P}}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x}\right] = 0$$

and then

$$\begin{aligned}
\lim_{\delta\to 0} -n\int_\Omega \fint_{\mathbb{S}^{n-1}} &\mathbf{P}_\delta(\mathbf{x})\mathbf{z}\cdot\mathbf{z}\,(\nabla\tilde{\mathbf{u}}(\mathbf{x})-\tilde{\mathbf{P}}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x} \\
&= -n\int_\Omega \fint_{\mathbb{S}^{n-1}} \mathbf{P}(\mathbf{x})\mathbf{z}\cdot\mathbf{z}\,(\nabla\tilde{\mathbf{u}}(\mathbf{x})-\tilde{\mathbf{P}}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

In order to handle the integral I_δ^{112} let us firstly observe that, as in (3.19),

$$\left|\frac{(\tilde{\mathbf{u}}(\mathbf{x}')-\tilde{\mathbf{u}}(\mathbf{x})-\nabla\tilde{\mathbf{u}}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))}{|\mathbf{x}'-\mathbf{x}|^2}\cdot(\mathbf{x}'-\mathbf{x})\right| \le \sigma(|\mathbf{x}'-\mathbf{x}|) \tag{B.3}$$

where σ is a modulus of continuity, and that, for all $A \subset\subset \Omega$, $0 < r < \mathrm{dist}(A, \partial\Omega)$, and $\mathbf{x} \in A$ we have, as in (3.25),

$$\int_{\Omega-\mathbf{x}} \rho_\delta(\tilde{\mathbf{x}})\sigma(|\tilde{\mathbf{x}}|)\,\mathrm{d}\tilde{\mathbf{x}} \le n\sigma(r) + \|\sigma\|_\infty \int_{\mathbb{R}^n\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}.$$

Define now the tensor-valued functions

$$\mathbf{x}\mapsto \mathbf{G}_\delta(\mathbf{x}) = \int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}')\frac{(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\,\frac{(\tilde{\mathbf{u}}(\mathbf{x}')-\tilde{\mathbf{u}}(\mathbf{x})-\nabla\tilde{\mathbf{u}}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))}{|\mathbf{x}'-\mathbf{x}|^2}\cdot(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\mathbf{x}'$$

and control them for a.e. $\mathbf{x} \in A$ as follows

$$|\mathbf{G}_\delta(\mathbf{x})| \le \int_{\Omega-\mathbf{x}} \rho_\delta(\tilde{\mathbf{x}})\sigma(|\tilde{\mathbf{x}}|)\,\mathrm{d}\tilde{\mathbf{x}} \le n\sigma(r) + \|\sigma\|_\infty \int_{\mathbb{R}^n\setminus B(\mathbf{0},r)} \rho_\delta(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}.$$

As the right-hand side goes to 0 as $\delta \to 0$, $\sigma(r)$ can be made arbitrarily small by choosing $r \to 0$, and $A \subset\subset \Omega$ is arbitrary we have proved that $\mathbf{G}_\delta(\mathbf{x}) \to \mathbf{0}$ a.e. The above bound proves additionally that $\mathbf{G}_\delta$ are equiintegrable. In particular, $\mathbf{G}_\delta \to \mathbf{0}$ strongly in $L^2(\Omega)$. As $\mathbf{P}_\delta$ is bounded in $L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{s,d}})$ one gets that $I_\delta^{112} \to 0$ as $\delta \to 0$.
The treatment of integral I_δ^{12} requires a nonlocal integration-by-parts formula, see [20, Lemma 2.9]. Indeed, for all $\varphi \in C^\infty(\bar{\Omega}\times\bar{\Omega})$ a direct computation ensures that

$$\int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\mathcal{D}(\mathbf{u})(\mathbf{x},\mathbf{x}')\,\varphi(\mathbf{x},\mathbf{x}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = -\int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathfrak{D}_\delta^*(\varphi)(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{B.4}$$

where the vector-valued operator $\mathfrak{D}_\delta^*$ is given by

$$\mathfrak{D}_\delta^*(\varphi)(\mathbf{x}) = \mathrm{p.v.}\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\frac{\varphi(\mathbf{x},\mathbf{x}')+\varphi(\mathbf{x}'\,\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\mathbf{x}'.$$

Let us apply formula (B.4) to I_δ^{12}, getting

$$I_\delta^{12} = -\int_\Omega (\mathbf{u}_\delta - \mathbf{u}_0)(\mathbf{x}) \cdot \mathfrak{D}_\delta^*(\mathcal{E}(\tilde{\mathbf{u}}, \tilde{\mathbf{P}}))(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Since $\mathbf{u}_\delta \to \mathbf{u}_0$ strongly in $L^2(\Omega;\mathbb{R}^n)$, in order to check that $I_\delta^{12} \to 0$ as $\delta \to 0$ one needs to provide an L^2 bound on $\mathfrak{D}_\delta^*(\mathcal{E}(\tilde{\mathbf{u}}, \tilde{\mathbf{P}}))$. As $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{P}}$ are smooth, this follows along the lines of [22, Formula (2.3)].
Let us now turn to the analysis of integral I_δ^{13}. Once again, some further decomposition is needed. We write $I_\delta^{13} = I_\delta^{131} + I_\delta^{132} + I_\delta^{133}$, where

$$\begin{aligned}
I_\delta^{131} &= \int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\, \frac{\nabla\mathbf{u}_0(\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\, \frac{(\nabla\tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{P}}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},\\
I_\delta^{132} &= \int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\, \frac{\nabla\mathbf{u}_0(\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\, \frac{(\tilde{\mathbf{u}}(\mathbf{x}') - \tilde{\mathbf{u}}(\mathbf{x}) - \nabla\tilde{\mathbf{u}}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},\\
I_\delta^{133} &= \int_\Omega\int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x})\, \frac{(\mathbf{u}_0(\mathbf{x}') - \mathbf{u}_0(\mathbf{x}) - \nabla\mathbf{u}_0(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\\
&\quad\times \frac{(\tilde{\mathbf{u}}(\mathbf{x}') - \tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{P}}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

The integrand of I_δ^{131} is positively homogeneous of degree 0 in $\mathbf{x}'-\mathbf{x}$. By arguing as in Lemma A.2 one can prove that

$$I_\delta^{131} \to n\int_\Omega \fint_{\mathbb{S}^{n-1}} \nabla\mathbf{u}_0(\mathbf{x})\mathbf{z}\cdot\mathbf{z}\,\big(\nabla\tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{P}}(\mathbf{x})\big)\mathbf{z}\cdot\mathbf{z}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x} \quad \text{as } \delta \to 0.$$

Integral I_δ^{132} can be proved to converge to 0 by arguing similarly as in I_δ^{111}, as (compared with (B.3))

$$\left|\frac{\nabla\mathbf{u}_0(\mathbf{x})(\mathbf{x}'-\mathbf{x})\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\, \frac{(\tilde{\mathbf{u}}(\mathbf{x}') - \tilde{\mathbf{u}}(\mathbf{x}) - \nabla\tilde{\mathbf{u}}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\right| \le |\nabla\mathbf{u}_0(\mathbf{x})|\sigma(|\mathbf{x}'-\mathbf{x}|).$$

We aim now at proving that I_δ^{133} goes to 0 as well. As the function

$$(\mathbf{x},\mathbf{x}') \mapsto \frac{(\tilde{\mathbf{u}}(\mathbf{x}') - \tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{P}}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}$$

is bounded, such convergence would follow as soon as we check that the functions

$$(\mathbf{x},\mathbf{x}') \mapsto \rho_\delta(\mathbf{x}'-\mathbf{x})\, \frac{(\mathbf{u}_0(\mathbf{x}') - \mathbf{u}_0(\mathbf{x}) - \nabla\mathbf{u}_0(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}$$

converge to 0 strongly in $L^1(\Omega\times\Omega)$. In case of a smooth function $\mathbf{v}$ this would follow from the bound

$$\left|\rho_\delta(\mathbf{x}'-\mathbf{x})\frac{(\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x}) - \nabla\mathbf{v}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\right| \le C\rho_\delta(\mathbf{x}'-\mathbf{x})\|\mathrm{D}^2\mathbf{v}\|_\infty|\mathbf{x}'-\mathbf{x}|$$

by arguing as for I_δ^{111}. Fix then $\varepsilon > 0$, and choose $\mathbf{v} \in C^\infty(\bar{\Omega};\mathbb{R}^n)$ such that $\mathbf{w} = \mathbf{u}_0 - \mathbf{v}$ fulfills $\|\mathbf{w}\|_{H^1(\Omega;\mathbb{R}^n)} \le \varepsilon$. One has that

$$\begin{aligned}
&\int_\Omega\int_\Omega \left|\rho_\delta(\mathbf{x}'-\mathbf{x})\, \frac{(\mathbf{u}_0(\mathbf{x}') - \mathbf{u}_0(\mathbf{x}) - \nabla\mathbf{u}_0(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\right|\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\\
&\quad\le \int_\Omega\int_\Omega \left|\rho_\delta(\mathbf{x}'-\mathbf{x})\, \frac{(\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x}) - \nabla\mathbf{v}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\right|\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\\
&\qquad+ \int_\Omega\int_\Omega \left|\rho_\delta(\mathbf{x}'-\mathbf{x})\, \frac{(\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x}) - \nabla\mathbf{w}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))\cdot(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2}\right|\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

The first term in the above right-hand side goes to 0 as $\delta \to 0$ because $\mathbf{v}$ is smooth and the second can be treated as follows:

$$\begin{aligned}
&\int_\Omega \int_\Omega \left| \rho_\delta(\mathbf{x}'-\mathbf{x}) \frac{(\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x}) - \nabla \mathbf{w}(\mathbf{x})(\mathbf{x}'-\mathbf{x})) \cdot (\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} \right| \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \\
&\leq \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \, |\nabla \mathbf{w}(\mathbf{x})| \, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} + \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \left| \frac{(\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) \cdot (\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} \right| \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \\
&\leq n \int_\Omega |\nabla \mathbf{w}(\mathbf{x})| \, \mathrm{d}\mathbf{x} + \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}'-\mathbf{x}) \frac{|\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})|}{|\mathbf{x}'-\mathbf{x}|} \, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \\
&\leq c \|\mathbf{w}\|_{H^1(\Omega;\mathbb{R}^n)} \leq c\varepsilon,
\end{aligned}$$

where we have also used [5, Th. 1] (see also [25, Eq. (5)]). As ε is arbitrary, we conclude that I_δ^{131} goes to 0 as $\delta \to 0$.
All in all, we have proved that

$$I_\delta^1 \to n \int_\Omega \fint_{\mathbb{S}^{n-1}} \big(\nabla \mathbf{u}_0(\mathbf{x}) - \mathbf{P}_0(\mathbf{x})\big)\mathbf{z} \cdot \mathbf{z} \, \big(\nabla \tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{P}}(\mathbf{x})\big)\mathbf{z} \cdot \mathbf{z} \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}\mathbf{x} \quad \text{as } \delta \to 0. \tag{B.5}$$

Step 4: Integrals I_δ^2, I_δ^3, and I_δ^4 One can discuss integral I_δ^2 by following the analysis of integral I_δ^1. Indeed, the two integrals correspond to each other upon changing $\mathcal{E}(\tilde{\mathbf{u}}, \tilde{\mathbf{P}})$ there with $\operatorname{div} \tilde{\mathbf{u}}/n$ here. In particular, we have that

$$I_\delta^2 \to -\int_\Omega \fint_{\mathbb{S}^{n-1}} (\nabla \mathbf{u}_0(\mathbf{x}) - \mathbf{P}_0(\mathbf{x})\mathbf{z} \cdot \mathbf{z}) \operatorname{div} \tilde{\mathbf{u}}(\mathbf{x}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}\mathbf{x} \quad \text{as } \delta \to 0. \tag{B.6}$$

As for I_δ^3, we decompose $I_\delta^3 = I_\delta^{31} + I_\delta^{32}$, where

$$\begin{aligned}
I_\delta^{31} &= -\frac{1}{n} \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}') \operatorname{div} \mathbf{u}_0(\mathbf{x}) \frac{(\nabla \tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{P}}(\mathbf{x}))(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^2} \cdot (\mathbf{x}'-\mathbf{x}) \, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}, \\
I_\delta^{32} &= -\frac{1}{n} \int_\Omega \int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}') \operatorname{div} \mathbf{u}_0(\mathbf{x}) \frac{(\tilde{\mathbf{u}}(\mathbf{x}') - \tilde{\mathbf{u}}(\mathbf{x}) - \nabla \tilde{\mathbf{u}}(\mathbf{x})(\mathbf{x}'-\mathbf{x}))}{|\mathbf{x}'-\mathbf{x}|^2} \cdot (\mathbf{x}'-\mathbf{x}) \, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}.
\end{aligned}$$

As the integrand of I_δ^{31} is positively homogeneous of degree 0 in $\mathbf{x}'-\mathbf{x}$, one can use Lemma A.2.a in order to get that

$$I_\delta^{31} \to -\int_\Omega \fint_{\mathbb{S}^{n-1}} \operatorname{div} \mathbf{u}_0(\mathbf{x}) \, (\nabla \tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{P}}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}\mathbf{x} \quad \text{as } \delta \to 0.$$

As regards integral I_δ^{32}, one can simply reproduce the argument of I_δ^{112} in order to check that $I_\delta^{32} \to 0$ as $\delta \to 0$. This allows us to conclude that

$$I_\delta^3 \to -\int_\Omega \fint_{\mathbb{S}^{n-1}} \operatorname{div} \mathbf{u}_0(\mathbf{x}) \, (\nabla \tilde{\mathbf{u}}(\mathbf{x}) - \tilde{\mathbf{P}}(\mathbf{x}))\mathbf{z} \cdot \mathbf{z} \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}\mathbf{x} \quad \text{as } \delta \to 0. \tag{B.7}$$

The treatment of term I_δ^4 is rather straightforward as

$$I_\delta^4 = \frac{1}{n^2} \int_\Omega \operatorname{div} \mathbf{u}_0(\mathbf{x}) \operatorname{div} \tilde{\mathbf{u}}(\mathbf{x}) \left(\int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}') \, \mathrm{d}\mathbf{x}' \right) \mathrm{d}\mathbf{x} \to \frac{1}{n} \int_\Omega \operatorname{div} \mathbf{u}_0(\mathbf{x}) \operatorname{div} \tilde{\mathbf{u}}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \quad \text{as } \delta \to 0, \tag{B.8}$$

where we have used that $\int_\Omega \rho_\delta(\mathbf{x}-\mathbf{x}') \, \mathrm{d}\mathbf{x}' \to n$ as $\delta \to 0$.

Conclusion of the proof By recollecting (B.1), decomposition (B.2), and limits (B.5), (B.6), (B.7), and (B.8) we conclude that

$$\lim_{\delta\to 0} A_\delta\big((\mathbf{u}_\delta,\mathbf{P}_\delta),(\tilde{\mathbf{u}},\tilde{\mathbf{P}})\big) \overset{(B.1)}{=} \lim_{\delta\to 0} \tilde{A}_\delta\big((\mathbf{u}_\delta,\mathbf{P}_\delta),(\tilde{\mathbf{u}},\tilde{\mathbf{P}});\mathbf{u}_0\big) \overset{(B.2)}{=} \lim_{\delta\to 0}\left(I_\delta^1+I_\delta^2+I_\delta^3+I_\delta^4\right)$$
$$\overset{(B.5)}{=} n\int_\Omega \fint_{\mathbb{S}^{n-1}} \big(\nabla\mathbf{u}_0(\mathbf{x})-\mathbf{P}_0(\mathbf{x})\big)\mathbf{z}\cdot\mathbf{z}\,\big(\nabla\tilde{\mathbf{u}}(\mathbf{x})-\tilde{\mathbf{P}}(\mathbf{x})\big)\mathbf{z}\cdot\mathbf{z}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x}$$
$$\overset{(B.6)}{-} \int_\Omega \fint_{\mathbb{S}^{n-1}} (\nabla\mathbf{u}_0(\mathbf{x})-\mathbf{P}_0(\mathbf{x})\mathbf{z}\cdot\mathbf{z})\,\mathrm{div}\,\tilde{\mathbf{u}}(\mathbf{x})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x}$$
$$\overset{(B.7)}{-} \int_\Omega \fint_{\mathbb{S}^{n-1}} \mathrm{div}\,\mathbf{u}_0(\mathbf{x})\,(\nabla\tilde{\mathbf{u}}(\mathbf{x})-\tilde{\mathbf{P}}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x}$$
$$\overset{(B.8)}{+} \frac{1}{n}\int_\Omega \mathrm{div}\,\mathbf{u}_0(\mathbf{x})\;\mathrm{div}\,\tilde{\mathbf{u}}(\mathbf{x})\,\mathrm{d}\mathbf{x}$$
$$= n\int_\Omega \fint_{\mathbb{S}^{n-1}} \left((\nabla\mathbf{u}_0(\mathbf{x})-\mathbf{P}_0(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}-\frac{1}{n}\,\mathrm{div}\,\mathbf{u}_0(\mathbf{x})\right)\left((\nabla\tilde{\mathbf{u}}(\mathbf{x})-\tilde{\mathbf{P}}(\mathbf{x}))\mathbf{z}\cdot\mathbf{z}-\frac{1}{n}\,\mathrm{div}\,\tilde{\mathbf{u}}(\mathbf{x})\right)\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}\mathbf{x},$$

which proves the convergence of the α term of B_δ. This concludes the proof. □

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