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A NOTE ON WEAK SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS

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We revisit the proof of existence of weak solutions of stochastic differential equations with continuous coeficients. In standard proofs, the coefficients are approximated by more regular ones and it is necessary to prove that: i) the laws of solutions of approximating equations form a tight set of measures on the paths space, ii) its cluster points are laws of solutions of the limit equation. We aim at showing that both steps may be done in a particularly simple and elementary manner.

Keywords: stochastic differential equations, continuous coefficients, weak solutions

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Dedicated to the memory of Martin Janžura

1. INTRODUCTION

Let $b\colon \mathbb{R}_{\geq 0}\times\mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $\sigma\colon \mathbb{R}_{\geq 0}\times\mathbb{R}^m \longrightarrow \mathbb{R}^m\otimes\mathbb{R}^n$ be Borel functions. (Notation we use is introduced below.) We shall consider a stochastic differential equation

$$\mathrm{d}X = b(t,X)\,\mathrm{d}t + \sigma(t,X)\,\mathrm{d}W. \tag{1}$$

Recall that a (weak) solution to (1) is a triple $\mathfrak{X} = ((\Omega,\mathscr{F},(\mathscr{F}_t),\boldsymbol{P}),W,X)$, where $(\Omega,\mathscr{F},(\mathscr{F}_t),\boldsymbol{P})$ is a stochastic basis, W an n-dimensional $(\mathscr{F}_t)$-Wiener process and X an $(\mathscr{F}_t)$-progressively measurable process in $\mathbb{R}^m$ such that

$$\int_0^t \left\{\|b(s,X_s)\| + \|\sigma(s,X_s)\|^2\right\}\mathrm{d}s < \infty$$

and

$$X_t = X_0 + \int_0^t b(s,X_s)\,\mathrm{d}s + \int_0^t \sigma(s,X_s)\,\mathrm{d}W_s$$

for any $t\geq 0$ $\boldsymbol{P}$-almost surely. Let ϱ be a Borel probability measure on $\mathbb{R}^m$, we say that the solution $\mathfrak{X}$ satisfies the initial condition

$$X_0 \sim \varrho \tag{2}$$

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if $X(0)_{\#}\boldsymbol{P} = \varrho$, that is, $X(0)$ has the law ϱ. We say that the solution $\mathfrak{X}$ is strong, provided the process X is $(\mathscr{F}_t^{X_0,W})$-adapted, where $(\mathscr{F}_t^{X_0,W})$ is the augmented canonical filtration generated by the Wiener process W and the initial condition X_0.

If the coefficients b and σ are continuous and satisfy a suitable growth hypothesis, then a (weak) solution to the problem (1), (2) exists, as was shown by A. V. Skorokhod already in 1961 ([14]). It is well known that uniqueness may fail even in the deterministic case; furthermore, it can be shown that strong solutions exist only under additional assumptions. Let us recall the basic structure of standard proofs of Skorokhod's result (cf. e. g. [15, Theorem 6.1.7], [11, Theorem IV.2.2], [12, Theorem 5.4.22]). One approximates the coefficients b and σ by functions b_k and σ_k, $k \geq 1$, so that existence of solutions (W_k, X_k) to

$$\mathrm{d}X = b_k(t,X)\,\mathrm{d}t + \sigma_k(t,X)\,\mathrm{d}W, \quad X(0) \sim \varrho,$$

may follow from elementary theory of stochastic differential equations and it may be possible to prove that

i) the set $\mathfrak{L}$ of laws of the processes X_k, $k \geq 1$, on the space of trajectories is relatively weak* compact (equivalently, tight),

ii) cluster points of $\mathfrak{L}$ are laws of a solution to the limit problem (1), (2).

The first step is simpler, usually one finds uniform estimates of moduli of continuity of paths of the processes X_k and invokes the Ascoli–Arzelà theorem. Alternatively, it is possible to use compactness of the Riemann–Liouville operator; such a proof, inspired by infinite-dimensional stochastic analysis, was developed in [8] and [9] for coefficients satisfying the linear growth hypothesis or the hypotheses of Khas'minskiĭ's test for non-explosion, respectively. In the present paper, we propose yet another proof in the latter case, which relies on the embedding theorem for Slobodeckiĭ spaces but otherwise it is quite simple and straightforward (see Theorem 1.2 and Corollary 1.3 below).

The second step is more challenging. We have

$$X_k(t) = X_k(0) + \int_0^t b_k(s, X_k(s))\,\mathrm{d}s + \int_0^t \sigma_k(s, X_k(s))\,\mathrm{d}W_k(s)$$

and the main problem is to show that

$$\int_0^{\cdot} \sigma_k(s, X_k(s))\,\mathrm{d}W_k(s) \underset{k\to\infty}{\longrightarrow} \int_0^{\cdot} \sigma(s, X(s))\,\mathrm{d}W(s) \tag{3}$$

in law for some Wiener process W and a process X which is a limit in law of X_k's. Albeit results providing sufficient conditions for (3) to hold are known (see e. g. [13] for a survey and references), it seems that it is not easy to apply them to a construction of weak solutions. A possible way round is the following: set

$$M_k = X_k - X_k(0) - \int_0^{\cdot} b_k(s, X_k(s))\,\mathrm{d}s,$$

then M_k is a continuous local martingale with a tensor quadratic variation

$$\langle\!\langle M_k \rangle\!\rangle = \int_0^{\cdot} \sigma_k(s, X_k(s))\sigma_k^*(s, X_k(s))\,\mathrm{d}s.$$

Since the formulae for M_k and $\langle\!\langle M_k \rangle\!\rangle$ involve only Lebesgue integrals, recalling that (b_k, σ_k) converge to (b, σ) and the sequence $\{X_k\}$ convergent in law may be represented (on another stochastic basis) by a sequence $\{\tilde{X}_k\}$ convergent almost surely (by the Skorokhod representation theorem, see e. g. [7, Theorem 11.7.2]) it is possible to show that

$$M = \tilde{X} - \tilde{X}(0) - \int_0^{\cdot} b(s, \tilde{X}(s))\,\mathrm{d}s$$

is a continuous local martingale with

$$\langle\!\langle M \rangle\!\rangle = \int_0^{\cdot} \sigma(s, \tilde{X}(s))\sigma^*(s, \tilde{X}(s))\,\mathrm{d}s.$$

The theorem on integral representation of martingales (see e. g. [11, Theorem II.7.1']) yields now a Wiener process $\tilde{W}$ such that

$$M = \int_0^{\cdot} \sigma(s, \tilde{X}(s))\,\mathrm{d}\tilde{W}(s),$$

thus $(\tilde{W}, \tilde{X})$ is the weak solution sought after. (If one works with a solution to the martingale problem instead of a weak solution, then the integral representation theorem is hidden in the proof of equivalence of these two notions; moreover, Skorokhod's representation theorem may be sometimes avoided, e. g. for bounded coefficients.) In the paper [8], wishing to make the proof much more elementary, we eliminated both the integral representation theorem (via a trick coming from the theory of stochastic wave maps) and, more importantly for the present paper, the Skorokhod representation theorem using (virtually) results on preservation of the martingale property under convergence in law (cf. [5] for a survey of such results). Now we aim at making a step further and to prove (3) directly. (We were inspired in part by the paper [1], where a procedure involving (an analogue of) (3) is used to construct a weak solution to a stochastic Navier–Stokes equation, but in conjunction with the Skorokhod representation theorem; see also [6, Lemma 2.1] which is a strengthened version of the basic trick from [1].) The core of our approach is the following observation: set $\gamma_\varepsilon = \varepsilon^{-1}\mathbf{1}_{(0,\varepsilon)}$ and denote by $*$ the convolution. We replace the integral

$$\int_0^{\cdot} \sigma_k(s, X_k(s))\,\mathrm{d}W_k(s)$$

with

$$\int_0^{\cdot} \gamma_\varepsilon * \sigma_k(\cdot, X_k(\cdot))(s)\,\mathrm{d}W_k(s) \tag{4}$$

and note that (4) may be easily expressed as a Lebesgue integral (see Lemma 2.3), hence it is straightforward to pass to the limit $k \to \infty$ and the proof may be completed by taking the limit $\varepsilon \downarrow 0$. That the limit process is the desired one can be checked in a simple way using essentially the Lenglart inequality (see Theorem 1.1 and its proof).

The proof of Theorem 1.1 has an additional merit: it enables us to implement in a very straightforward manner a procedure proposed in [10] to obtain a strong solution provided pathwise uniqueness holds for (1), (2) (see Remark 1.4).

Notation

If X and Y are metric spaces, we denote by $\mathscr{C}(X;Y)$ the space of all continuous mappings from X to Y and by $\mathscr{C}^{0,\beta}(X;Y)$ its subspace of β-Hölder continuous functions. We write $\mathscr{C}(X)$ instead of $\mathscr{C}(X;\mathbb{R})$ and denote by $\mathscr{C}_{\mathrm{b}}(X)$ the subspace of bounded functions from $\mathscr{C}(X)$ equipped with the sup-norm. The space $\mathscr{C}(\mathbb{R}_{\geq 0};\mathbb{R}^n)$ will be endowed with the topology of locally uniform convergence, which turns it into a Polish metric space. Let $\mathscr{P}(X)$ stand for the space of all Borel probability measures on X, this space inherits the weak* topology from $\mathscr{C}_{\mathrm{b}}(X)^*$; a limit with respect to this topology will be denoted by w*-lim. (In probability theory, this topology is usually called weak, but we adhere to the functional analytic terminology here.) If $\mu \in \mathscr{P}(X)$ and $h\colon X \longrightarrow Y$ is a Borel mapping we denote by $h_{\#}\mu$ the image of the measure μ under h, that is, $h_{\#}\mu(B) = \mu(h^{-1}B)$ for all Borel sets $B \subseteq Y$. We shall write $\nu\text{-}\lim_{j\to\infty} f_j = f$ if measurable functions f_j converge to f in probability on a probability space $(N, \mathscr{N}, \nu)$.

The space $\mathbb{R}^d \otimes \mathbb{R}^l$ is identified with the space of all linear mappings from $\mathbb{R}^l$ to $\mathbb{R}^d$ and endowed with the Hilbert-Schmidt norm $\|A\| = \big(\operatorname{Tr}(AA^*)\big)^{1/2}$.

If $h\colon \mathbb{R}_{\geq 0} \longrightarrow Y$ is a function on $\mathbb{R}_{\geq 0}$, we denote by $\pi_t h$ its restriction to the interval $[0,t]$, $t \in \mathbb{R}_{\geq 0}$. In the sequel, any function q defined on $\mathbb{R}_{\geq 0}$ is tacitly extended to $\mathbb{R}$ by setting $q = 0$ on $\mathbb{R}_{<0}$.

Let V be a function having two continuous derivatives on $\mathbb{R}^m$ (in symbol, $V \in \mathscr{C}^2(\mathbb{R}^m)$), by DV and D^2V we denote the first and second Fréchet derivative of V, respectively. Analogously, if V is a function on $\mathbb{R}_{\geq 0} \times \mathbb{R}^m$ and $V(t,\cdot) \in \mathscr{C}^2(\mathbb{R}^m)$, the first and second Fréchet derivatives of $V(t,\cdot)$ are denoted by $D_x V$, $D_x^2 V$, respectively.

Main results

Now we may state our theorems, but let us stress again that these results are well known, the novelty of our approach lies in simplified proofs. The first of them deals with identification of a limit of a sequence of solutions to stochastic differential equations.

Theorem 1.1. Let $F_k\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $G_k\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m \otimes \mathbb{R}^n$, $k \geq 0$, be Borel functions. Assume:

(A1) $F_k(t,\cdot)$ and $G_k(t,\cdot)$ are continuous on $\mathbb{R}^m$ for any $k \geq 0$ and $t \in \mathbb{R}_{\geq 0}$,

(A2) for any $T, R \geq 0$,

$$\int_0^T \sup_{k\geq 0} \sup_{\|x\|\leq R} \big\{\|F_k(t,x)\| + \|G_k(t,x)\|^2\big\}\,\mathrm{d}t < \infty, \tag{5}$$

(A3) for any $t \geq 0$,

$$\lim_{k\to\infty} F_k(t,\cdot) = F_0(t,\cdot), \quad \lim_{k\to\infty} G_k(t,\cdot) = G_0(t,\cdot) \quad \text{locally uniformly on } \mathbb{R}^m,$$

(A4) for any $k \geq 1$ there exists a weak solution $((\Omega_k, \mathscr{F}_k, (\mathscr{F}_t^k), \boldsymbol{Q}_k), W_k, X_k)$ to

$$\mathrm{d}X = F_k(t,X)\,\mathrm{d}t + G_k(t,X)\,\mathrm{d}W, \tag{6}$$

(A5) $\{X_{k\#}\boldsymbol{Q}_k;\ k \geq 1\}$ is a tight set of Borel probability measures on $\mathscr{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$.

Let $\varrho \in \mathscr{P}(\mathbb{R}^m)$ be a cluster point of $\{X_k(0)_{\#}\boldsymbol{Q}_k;\ k \geq 1\}$. Then there exists a weak solution to the problem

$$\mathrm{d}X = F_0(t,X)\,\mathrm{d}t + G_0(t,X)\,\mathrm{d}W, \quad X_0 \sim \varrho. \tag{7}$$

For autonomous equations (whose coefficients do not depend on time) the assumption (A2) follows from (A1) and (A3). Further, the supremum in (5) may be taken over $x \in \mathbb{Q}^m$, $\|x\| \leq R$ by (A1), so the integrand is measurable. The solution $((\Omega,\mathscr{F},(\mathscr{F}_t),\boldsymbol{P}),W, X)$ to (7) constructed in the course of the proof of Theorem 1.1 is such that $X_{\#}\boldsymbol{P}$ is a cluster point of the set $\{X_{k\#}\boldsymbol{Q}_k;\ k \geq 1\}$, as may be checked easily tracing the proof. Therefore, Theorem 1.1 may be restated in the following way (we content ourselves to equations with coefficients continuous in both variables for simplicity): Set

$$\mathscr{Y} = \mathscr{C}(\mathbb{R}_{\geq 0} \times \mathbb{R}^m;\mathbb{R}^m) \times \mathscr{C}(\mathbb{R}_{\geq 0} \times \mathbb{R}^m;\mathbb{R}^m \otimes \mathbb{R}^n) \times \mathscr{P}(\mathscr{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m))$$

and endow the first two factors with the topology of locally uniform convergence and the last one with the weak* topology. Given $(b,\sigma) \in \mathscr{C}(\mathbb{R}_{\geq 0} \times \mathbb{R}^m;\mathbb{R}^m) \times \mathscr{C}(\mathbb{R}_{\geq 0} \times \mathbb{R}^m;\mathbb{R}^m \otimes \mathbb{R}^n)$ we denote by $\mathfrak{L}(b,\sigma)$ the set of all measures ν of the form $\nu = X_{\#}\boldsymbol{Q}$, where $((\Omega,\mathscr{F},(\mathscr{F}_t),\boldsymbol{Q}),W,X)$ is a solution of (1). Then Theorem 1.1 says that the set $\{(b,\sigma,\nu);\ \nu \in \mathfrak{L}(b,\sigma)\} \subseteq \mathscr{Y}$ is closed in $\mathscr{Y}$. (We may replace $\mathfrak{L}(b,\sigma)$ with the set of all measures of the form $(W,X)_{\#}\boldsymbol{Q}$, modifying the space $\mathscr{Y}$ accordingly.)

Our second theorem provides a method for checking tightness of laws of solutions to a family of stochastic differential equations.

Theorem 1.2. Let $F_\alpha\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $G_\alpha\colon \mathbb{R}_{\geq 0} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m \otimes \mathbb{R}^n$, $\alpha \in A$, be Borel functions locally bounded uniformly in $\alpha \in A$, i. e.

$$\forall R,T \in \mathbb{R}_{\geq 0} \qquad \sup_{\alpha\in A}\ \sup_{0\leq t\leq T}\ \sup_{\|x\|\leq R} \big\{\|F_\alpha(t,x)\| + \|G_\alpha(t,x)\|\big\} \equiv \kappa_{R,T} < \infty. \tag{8}$$

Let $\nu \in \mathscr{P}(\mathbb{R}^m)$, suppose that for every $\alpha \in A$ there exists a solution $((\Omega_\alpha,\mathscr{F}_\alpha,(\mathscr{F}^\alpha_t), \boldsymbol{Q}_\alpha),W_\alpha,X_\alpha)$ to the problem

$$\mathrm{d}X = F_\alpha(t,X)\,\mathrm{d}t + G_\alpha(t,X)\,\mathrm{d}W, \quad X(0) \sim \nu.$$

Assume further that $\{X_\alpha;\ \alpha \in A\}$ are bounded in probability on compact intervals uniformly in $\alpha \in A$, i. e.

$$\forall T \in \mathbb{R}_{\geq 0}\ \forall \varepsilon > 0\ \exists R \in \mathbb{R}_{\geq 0} \qquad \sup_{\alpha\in A} \boldsymbol{Q}_\alpha\Big\{\sup_{0\leq t\leq T}\|X_\alpha(t)\| > R\Big\} \leq \varepsilon. \tag{9}$$

Then $\{X_{\alpha\#}\boldsymbol{Q}_\alpha;\ \alpha \in A\}$ is a tight set of Borel probability measures on $\mathscr{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$.

We assume that all equations have the same initial condition only for simplicity, (9) implies that $\{X_\alpha(0)_{\#}\boldsymbol{Q}_\alpha;\alpha \in A\}$ is a tight set and this is sufficient for the proof. Plainly, (9) is a crucial assumption of Theorem 1.2; it need not be clear how to check it for particular families of stochastic differential equations. However, it can be shown easily that (9) holds if there exists a suitable Lyapunov function. Namely:

Corollary 1.3. Theorem 1.2 remains true if instead of (9) it is assumed that there exists a function $V \in \mathscr{C}^2(\mathbb{R}^m)$ satisfying

(i) $V \geq 0$ on $\mathbb{R}^m$,

(ii) $\lim_{\|x\|\to\infty} V(x) = +\infty$, and

(iii) for any $T \geq 0$ there exists a constant $c < \infty$ such that

$$\big\langle DV(x), F_\alpha(t,x)\big\rangle + \frac{1}{2}\operatorname{Tr}\big(D^2V(x)G_\alpha(t,x)G_\alpha^*(t,x)\big) \leq c\big(1+V(x)\big)$$

for all $\alpha \in A$, $t \in [0,T]$, and $x \in \mathbb{R}^m$.

Remark 1.4. If pathwise uniqueness holds for (7) and there exists a weak solutions to this equation, the Yamada-Watanabe theory implies that there exists a strong solution. Combining the idea of the proof of Theorem 1.1 with a simple yet very useful lemma due to I. Gyöngy and N. Krylov (see [10, Lemma 1.1]) we may establish existence of a strong solution in a more elementary way. Namely, varying the proof of Theorem 1.1 we shall prove:

Let the hypotheses (A1), (A2) and (A3) of Theorem 1.1 be satisfied and let $\varrho \in \mathscr{P}(\mathbb{R}^m)$. Assume further:

(A4') there exist a stochastic basis $(G, \mathscr{G}, (\mathscr{G}_t), \boldsymbol{Q})$, a $\mathscr{G}_0$-measurable $\mathbb{R}^m$-valued random variable ψ such that $\psi_\# \boldsymbol{Q} = \varrho$, an n-dimensional $(\mathscr{G}_t)$-Wiener process B and $(\mathscr{G}_t^{\psi,B})$-progressively measurable processes Z_k in $\mathbb{R}^m$, $k \geq 1$, such that $Z_k(0) = \psi$ and $(G, \mathscr{G}, (\mathscr{G}_t^{\psi,B}), \boldsymbol{Q}), B, Z_k)$ solves the problem (6), $k \geq 1$,

(A5') $\{Z_{k\#}\boldsymbol{Q};\ k \geq 1\}$ is a tight subset of $\mathscr{P}(\mathscr{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m))$,

(A6) pathwise uniqueness holds for (7) with the initial condition $X(0) \sim \varrho$.

Then there exists an m-dimensional $(\mathscr{G}_t^{\psi,B})$-progressively measurable processes Z such that $Z(0) = \psi$ $\boldsymbol{Q}$-almost surely and $(G, \mathscr{G}, (\mathscr{G}_t^{\psi,B}), \boldsymbol{Q}), B, Z)$ solves the equation (7).

The Gyöngy–Krylov lemma was applied in [10] to construct a strong solution of a stochastic differential equation with continuous coefficients for which pathwise uniqueness holds and there exists a suitable Lyapunov function. We follow exactly the same pattern of reasoning, but the technicalities of our approach are quite different from those in [10].

Remark 1.5. The proof of Theorem 1.1 yields also another corollary:

Let $((\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P}), W, X)$ be the solution to the problem (7) constructed in the proof of Theorem 1.1, the hypotheses of which are assumed to be satisfied. Let p be a proper regular version of the conditional probability $\boldsymbol{P}(\cdot\,|X_0)$; denote by $(\mathscr{F}_t^y)$ the completion of the filtration $(\mathscr{F}_t)$ with respect to the measure $p(y,\cdot)$, $y \in \mathbb{R}^m$. Then for ϱ-almost all $y \in \mathbb{R}^m$, $((\Omega, \mathscr{F}, (\mathscr{F}_t^y), p(y,\cdot)), W, X)$ is a solution to the equation

$$\mathrm{d}X = F_0(t,X)\,\mathrm{d}t + G_0(t,X)\,\mathrm{d}W, \quad X_0 = y.$$

Let us recall that a proper regular conditional probability p is defined as a system $(p(y,\cdot),\ y\in\mathbb{R}^m)$ of probability measures on $\mathscr{F}$ such that the function $y\longmapsto p(y,A)$ is Borel measurable for any $A\in\mathscr{F}$, $p(y,\{X_0=y\})=1$ for ϱ-almost all $y\in\mathbb{R}^m$, and

$$\int_{\{X_0\in B\}} \boldsymbol{P}(A\,|X_0)\,\mathrm{d}\boldsymbol{P}=\boldsymbol{P}(A\cap\{X_0\in B\})=\int_B p(y,A)\,\mathrm{d}\varrho(y)$$

for every $A\in\mathscr{F}$ and Borel set $B\subseteq\mathbb{R}^m$ (see e. g. [3, Theorem 10.4.8, Corollary 10.4.9]).

Our result is not surprising, of course, one may found much more general results in this direction e. g. in [15, Chapter 6], but our proof requires essentially no extra effort; moreover, whilst in [15] martingale problems are solved, we find weak solutions with the same driving process W for ϱ-almost all $y\in\mathbb{R}^m$.

Example 1.6. We shall indicate how Theorem 1.1 and Corollary 1.3 may be employed to prove the Skorokhod theorem on existence of solutions to stochastic differential equations with continuous coefficients. Let us consider the problem (1), (2) assuming that $b\colon\mathbb{R}_{\geq 0}\times\mathbb{R}^m\longrightarrow\mathbb{R}^m$, $\sigma\colon\mathbb{R}_{\geq 0}\times\mathbb{R}^m\longrightarrow\mathbb{R}^m\otimes\mathbb{R}^n$ are Borel functions, $b(t,\cdot)\in\mathscr{C}(\mathbb{R}^m;\mathbb{R}^m)$, $\sigma(t,\cdot)\in\mathscr{C}(\mathbb{R}^m;\mathbb{R}^m\otimes\mathbb{R}^n)$ for any $t\geq 0$, $\varrho\in\mathscr{P}(\mathbb{R}^m)$ and there exists a function $V\in\mathscr{C}^2(\mathbb{R}^m)$ satisfying the hypotheses (i) and (ii) of Corollary 1.3 and such that for any $T>0$ one has

$$\big\langle DV(x),b(t,x)\big\rangle+\frac{1}{2}\operatorname{Tr}\big(D^2V(x)\sigma(t,x)\sigma^*(t,x)\big)\leq c\big(1+V(x)\big) \tag{10}$$

for some constant $c<\infty$ and all $t\in[0,T]$, $x\in\mathbb{R}^m$. A weak solution of (1), (2) is constructed in two steps.

Step 1. Le us assume that b and σ satisfy, instead of (10), a linear growth hypothesis

$$\sup_{0\leq t\leq T}\ \sup_{x\in\mathbb{R}^m}\frac{\|b(t,x)\|+\|\sigma(t,x)\|}{1+\|x\|}<\infty\quad\text{for any } T>0. \tag{11}$$

Let ϕ be a mollifier, that is, $\phi\in\mathscr{C}^\infty(\mathbb{R}^m)$, $\phi\geq 0$, the support of ϕ is contained in the unit ball of $\mathbb{R}^m$ and $\int_{\mathbb{R}^m}\phi\,\mathrm{d}x=1$. Set $\phi_k(x)=k^m\phi(kx)$, $x\in\mathbb{R}^m$, and

$$b_k(t,\cdot)=b(t,\cdot)*\phi_k,\quad \sigma_k(t,\cdot)=\sigma(t,\cdot)*\phi_k,\quad t\geq 0,\ k\geq 1.$$

Then b_k and σ_k are locally Lipschitz continuous in the space variables and satisfy a linear growth estimate uniformly in $k\geq 1$,

$$\sup_{k\geq 1}\ \sup_{0\leq t\leq T}\ \sup_{x\in\mathbb{R}^m}\frac{\|b_k(t,x)\|+\|\sigma_k(t,x)\|}{1+\|x\|}<\infty\quad\text{for any } T>0. \tag{12}$$

Elementary theory of stochastic differential equations implies that the equations

$$\mathrm{d}X=b_k(t,X)\,\mathrm{d}t+\sigma_k(t,X)\,\mathrm{d}W,\quad X(0)\sim\varrho, \tag{13}$$

$k\geq 1$, have solutions (W_k,X_k) and due to (12) the laws of X_k's are tight by Corollary 1.3 used with the Lyapunov function $V\colon x\longmapsto 1+\|x\|^2$. It follows from the properties of ϕ that the assumptions (A1), (A2) and (A3) of Theorem 1.1 are satisfied as well, hence, under (11), there exists a solution of (1), (2).

Step 2. Suppose now that only (10) holds and set

$$b_k(t,x) = \begin{cases} b(t,x), & t \geq 0,\ \|x\| \leq k, \\ b(t,x)\big(2 - k^{-1}\|x\|\big)^2, & t \geq 0,\ k < \|x\| \leq 2k, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\sigma_k(t,x) = \begin{cases} \sigma(t,x), & t \geq 0,\ \|x\| \leq k, \\ \sigma(t,x)\big(2 - k^{-1}\|x\|\big), & t \geq 0,\ k < \|x\| \leq 2k, \\ 0, & \text{otherwise}. \end{cases}$$

Coefficients defined in this way are bounded, so Step 1 shows that the problem (13) has a solution for any $k \geq 1$ and the laws of these solutions form a tight set by Corollary 1.3, which may be applied with the Lyapunov function V that appears in (10), since it can be established easily that with this choice of V (and b_k, σ_k) the assumption (iii) of Corollary 1.3 is satisfied. To check that the assumptions (A1), (A2) and (A3) of Theorem 1.1 are satisfied is straightforward, thus existence of a solution to (1), (2) follows immediately.

2. PROOFS

The following lemma is essentially a variant of the Lenglart inequality, we provide its simple proof for completeness.

Lemma 2.1. Let M be a continuous local martingale in $\mathbb{R}^d$ defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$, $M_0 = 0$. Then

$$\boldsymbol{P}\Big\{\sup_{t\in[0,T]} \|M_t\| > a \ \&\ \langle M\rangle_T^{1/2} < b\Big\} \leq \frac{b^2}{a^2}$$

for all $T \geq 0$ and any $a, b > 0$.

P r o o f. Choose an arbitrary $\varkappa \in (0, a)$ and set

$$\tau = \inf\{t \geq 0;\ \|M_t\| \geq \varkappa\}, \quad \sigma = \inf\{t \geq 0;\ \langle M\rangle_t^{1/2} \geq b\}.$$

Then

$$\begin{aligned} \boldsymbol{P}\Big\{\sup_{t\in[0,T]} \|M_t\| > a \ \&\ \langle M\rangle_T^{1/2} < b\Big\} &= \boldsymbol{P}\{0 < \tau < T,\, \sigma \geq T\} \\ &\leq \boldsymbol{P}\big\{\|M_{\tau\wedge\sigma\wedge T}\| \geq \varkappa\big\} \leq \frac{1}{\varkappa^2}\boldsymbol{E}\|M_{\tau\wedge\sigma\wedge T}\|^2 \\ &= \frac{1}{\varkappa^2}\boldsymbol{E}\langle M\rangle_{\tau\wedge\sigma\wedge T} \leq \frac{1}{\varkappa^2}\boldsymbol{E}\langle M\rangle_\sigma \leq \frac{b^2}{\varkappa^2}. \end{aligned}$$

(We have used that $(M_{\tau\wedge\cdot})$ is a bounded local martingale, hence an L^2-martingale.) Passing $\varkappa \nearrow a$ we complete the proof. □

The following generalization of the mapping theorem for weak* convergence of probability measures (see e. g. [2, Theorem 2.7]) is surely well known, but since we cannot find a reference, we include a proof. (A more general result is stated in [4, Proposition 3.2], but without proof; moreover, the version given below may be established in a more straightforward way.)

Lemma 2.2. Let X and Y be metric spaces. Assume that $h_n, h\colon X \longrightarrow Y$, $n \in \mathbb{N}$, are continuous mappings satisfying

$$\lim_{n\to\infty} h_n = h \quad \text{uniformly on compact sets in } X$$

and $\mu_n, \mu \in \mathscr{P}(X)$, $n \in \mathbb{N}$, are such that

$$\text{w*-}\lim_{n\to\infty} \mu_n = \mu \tag{14}$$

and the set $\{\mu_n;\ n \in \mathbb{N}\}$ is tight. Then

$$\text{w*-}\lim_{n\to\infty} h_{n\#}\mu_n = h_{\#}\mu. \tag{15}$$

In particular, for every $G \subseteq Y$ open,

$$\liminf_{n\to\infty} \mu_n\{h_n \in G\} \geq \mu\{h \in G\}. \tag{16}$$

Note that if X is Polish, tightness of $\{\mu_n;\ n \in \mathbb{N}\}$ follows from (14) by the Prokhorov theorem.

P r o o f . By the portmanteau theorem, (16) follows immediately from (15). To establish (15), it suffices to show that

$$\lim_{n\to\infty} \int_X u \circ h_n \,\mathrm{d}\mu_n = \lim_{n\to\infty} \int_Y u \,\mathrm{d}h_{n\#}\mu_n = \int_Y u \,\mathrm{d}h_{\#}\mu = \int_X u \circ h \,\mathrm{d}\mu$$

for any bounded uniformly continuous function $u\colon Y \longrightarrow \mathbb{R}$ (see again e. g. [2, Theorem 2.1]). However,

$$\left|\int_X u \circ h_n \,\mathrm{d}\mu_n - \int_X u \circ h \,\mathrm{d}\mu\right| \leq \left|\int_X u \circ h \,\mathrm{d}\mu_n - \int_X u \circ h \,\mathrm{d}\mu\right| + \int_X \bigl|u \circ h - u \circ h_h\bigr| \,\mathrm{d}\mu_n \equiv I_1 + I_2.$$

Choose an arbitrary $\varepsilon > 0$. Since $u \circ h \in \mathscr{C}_{\mathrm{b}}(X)$, $I_1 \leq \varepsilon$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ by (14). By the tightness hypothesis, there exists a compact subset $K \subseteq X$ such that

$$\inf_{n\in\mathbb{N}} \mu_n(K) \geq 1 - \varepsilon.$$

Denote by d_Y the metric of the space Y. Since u is uniformly continuous, we may find $\delta > 0$ such that $|u(x) - u(y)| \leq \varepsilon$ whenever $d_Y(x, y) \leq \delta$. Further, h_n's converge uniformly to h on K, so there exists $n_1 \geq n_0$ such that $\sup_K d_Y(h_n, h) \leq \delta$ for all $n \geq n_1$. Consequently,

$$\begin{aligned} I_2 &\leq \int_K \bigl|u \circ h - u \circ h_n\bigr| \,\mathrm{d}\mu_n + \int_{X \setminus K} \bigl|u \circ h - u \circ h_n\bigr| \,\mathrm{d}\mu_n \\ &\leq \varepsilon + 2 \sup_{n \in \mathbb{N}} \mu_n(X \setminus K) \sup_Y |u|, \end{aligned}$$

hence

$$\left| \int_X u \circ h_n \,\mathrm{d}\mu_n - \int_X u \circ h \,\mathrm{d}\mu \right| \leq 2\varepsilon\bigl(1 + \sup_Y |u|\bigr)$$

for all $n \geq n_1$, which proves our claim. □

Now we prove a lemma containing the main technical trick of our approach. Prior to stating the lemma, we have to introduce a suitable approximation to identity. Let $h \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$, for any $\varepsilon > 0$ define

$$\Pi_\varepsilon h \colon \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R},\ t \longmapsto \frac{1}{\varepsilon} \int_{t-\varepsilon}^t h(r) \,\mathrm{d}r = \bigl(\frac{1}{\varepsilon} \mathbf{1}_{(0,\varepsilon)} * h\bigr)(t).$$

It is well known that $\Pi_\varepsilon h \in L^2(0,T)$ provided $h \in L^2(0,T)$, and

$$\|\Pi_\varepsilon h\|_{L^2(0,T)} \leq \|h\|_{L^2(0,T)}, \quad \lim_{\varepsilon \to 0+} \|\Pi_\varepsilon h - h\|_{L^2(0,T)} = 0$$

(see e. g. [16, Proposition 9.15]).

Lemma 2.3. Let X be a progressively measurable stochastic process defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$ and such that $X \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ $\boldsymbol{P}$-almost surely, W an $(\mathscr{F}_t)$-Wiener process, and $\varepsilon > 0$. Then

$$\int_0^t X(s) \frac{W((s+\varepsilon) \wedge t) - W(s)}{\varepsilon} \,\mathrm{d}s = \int_0^t \Pi_\varepsilon X\,(s) \,\mathrm{d}W(s)$$

for any $t \geq 0$ $\boldsymbol{P}$-almost surely.

P r o o f. Set

$$H(t) = \int_0^t X(s) \,\mathrm{d}s, \quad t \geq 0,$$

then $H(\cdot - \delta)$ is a continuous semimartingale for each $\delta \geq 0$, thus by the product rule for semimartingales we get

$$H(t-\delta)W(t) = \int_0^t H(s-\delta) \,\mathrm{d}W(s) + \int_0^t X(s-\delta)W(s) \,\mathrm{d}s \tag{17}$$

for any $t \geq 0$ $\boldsymbol{P}$-almost surely. We shall use (17) choosing first $\delta = \varepsilon$ and then $\delta = 0$ to obtain

$$\begin{aligned}\int_0^t X(s)W((s+\varepsilon)\wedge t)\,\mathrm{d}s &= W(t)\int_{t-\varepsilon}^t X(s)\,\mathrm{d}s + \int_0^{t-\varepsilon} X(s)W(s+\varepsilon)\,\mathrm{d}s\\ &= W(t)[H(t)-H(t-\varepsilon)] + \int_\varepsilon^t X(s-\varepsilon)W(s)\,\mathrm{d}s\\ &= W(t)[H(t)-H(t-\varepsilon)] + H(t-\varepsilon)W(t) - \int_0^t H(s-\varepsilon)\,\mathrm{d}W(s),\end{aligned}$$

and

$$\int_0^t X(s)W(s)\,\mathrm{d}s = H(t)W(t) - \int_0^t H(s)\,\mathrm{d}W(s).$$

Subtracting the second formula from the first we get

$$\frac{1}{\varepsilon}\int_0^t X(s)\big[W((s+\varepsilon)\wedge t) - W(s)\big]\,\mathrm{d}s = \int_0^t \frac{1}{\varepsilon}\big[H(s)-H(s-\varepsilon)\big]\,\mathrm{d}W(s)$$

and it only remains to note that

$$\frac{1}{\varepsilon}[H(s)-H(s-\varepsilon)] = \frac{1}{\varepsilon}\int_{s-\varepsilon}^s X(r)\,\mathrm{d}r = \Pi_\varepsilon X(s).$$

□

Remark 2.4. (i) Applying Lemma 2.1 to the local martingale $M = \int_0^\cdot (\Pi_\varepsilon X - X)\,\mathrm{d}W$ and invoking Lemma 2.3 we obtain

$$\begin{aligned}\boldsymbol{P}\Big\{\sup_{0\le t\le T}\Big|\int_0^t X(s)\frac{W((s+\varepsilon)\wedge t)-W(s)}{\varepsilon}\,\mathrm{d}s - \int_0^t X(s)\,\mathrm{d}W(s)\Big| > a\\ \&\ \int_0^T \big|\Pi_\varepsilon X(s) - X(s)\big|^2\,\mathrm{d}s < b^2\Big\} \le \frac{b^2}{a^2}\end{aligned} \tag{18}$$

for any $T \in \mathbb{R}_{\geq 0}$, $\varepsilon > 0$ and $a, b > 0$.

(ii) Obviously, Lemma 2.3 remains valid if X is an $\mathbb{R}^m \otimes \mathbb{R}^n$-valued progressively measurable process with $\|X\| \in L^2_{\mathrm{loc}}(\mathbb{R}_{\geq 0})$ $\boldsymbol{P}$-almost surely and W is an n-dimensional Wiener process; the estimate (18) plainly extends to the multidimensional case as well.

Proof of Theorem 1.1. Let us set $\Omega = \mathscr{C}(\mathbb{R}_{\geq 0};\mathbb{R}^m) \times \mathscr{C}(\mathbb{R}_{\geq 0};\mathbb{R}^n)$, endow Ω with the topology of locally uniform convergence (which turns Ω into a Polish metric space) and denote by (X, W) the canonical process on Ω, that is,

$$(X,W)(t)\colon \Omega \longrightarrow \mathbb{R}^m \times \mathbb{R}^n,\ (\zeta,\xi) \longmapsto (\zeta(t),\xi(t)),\quad t \geq 0.$$

Let $\mu_k = (X_k, W_k)_\# \boldsymbol{Q}_k$ be the law of (X_k, W_k) on Ω. Since $\{X_{k\#}\boldsymbol{Q}_k;\ k \geq 1\}$ is tight by (A5), the set $\mathfrak{M} = \{\mu_k;\ k \geq 1\}$ is tight as well. By our hypothesis on ϱ, we may choose a cluster point $\boldsymbol{P}$ of $\mathfrak{M}$ such that $X(0)_\#\boldsymbol{P} = \varrho$. There exists a sequence in $\mathfrak{M}$

converging to $\boldsymbol{P}$; for notational simplicity we shall assume – without loss of generality – that $\boldsymbol{P} = \text{w*-lim}_{k\to\infty} \mu_k$. Let $(\mathscr{F}_t)$ be the $\boldsymbol{P}$-augmentation of the canonical filtration of (X, W).

First, let us check that W is an $(\mathscr{F}_t)$-Wiener process on $(\Omega, \mathscr{F}_\infty, \boldsymbol{P})$. Indeed,

$$W(0)_\# \boldsymbol{P} = \operatorname*{w^*\text{-}lim}_{k\to\infty} W(0)_\# \mu_k = \operatorname*{w^*\text{-}lim}_{k\to\infty} W_k(0)_\# \boldsymbol{Q}_k = \delta_0,$$

hence $W(0) = 0$ $\boldsymbol{P}$-almost surely. Further, W is an $(\mathscr{F}_t)$-martingale on $(\Omega, \mathscr{F}_\infty, \boldsymbol{P})$. Fix $0 \le s \le t$ arbitrarily, let $\varphi\colon \mathscr{C}([0,s];\mathbb{R}^m) \times \mathscr{C}([0,s];\mathbb{R}^n) \longrightarrow \mathbb{R}$ be a bounded continuous function. Then

$$(\zeta, \xi) \longmapsto \varphi(\pi_s\zeta, \pi_s\xi)\big[\xi(t) - \xi(s)\big]$$

is a continuous mapping from Ω to $\mathbb{R}^m$, hence

$$\begin{aligned}
&\int_\Omega \varphi(\pi_s X, \pi_s W)\big[W(t) - W(s)\big]\,\mathrm{d}\boldsymbol{P} \\
&\qquad = \lim_{k\to\infty} \int_\Omega \varphi(\pi_s X, \pi_s W)\big[W(t) - W(s)\big]\,\mathrm{d}\mu_k \\
&\qquad = \lim_{k\to\infty} \int_{\Omega_k} \varphi(\pi_s X_k, \pi_s W_k)\big[W_k(t) - W_k(s)\big]\,\mathrm{d}\boldsymbol{Q}_k \\
&\qquad = 0,
\end{aligned}$$

as $\varphi(\pi_s X_k, \pi_s W_k)$ in $\mathscr{F}^k_s$-measurable and W_k is an $(\mathscr{F}^k_t)$-martingale. Analogously we can check that $W \otimes W - tI$ is an $(\mathscr{F}_t)$-martingale on $(\Omega, \mathscr{F}_\infty, \boldsymbol{P})$, thus our claim follows from the Lévy theorem (see e.g. [12, Theorem 3.3.16]).

Secondly, we aim at proving that the pair (W, X) solves (7). Fix an arbitrary $t \in \mathbb{R}_{\ge 0}$, let $\varepsilon > 0$ be also fixed for the time being. Define

$$\begin{aligned}
H_k\colon \Omega \longrightarrow \mathbb{R}^m,\ (\zeta,\xi) \longmapsto \zeta(t) - \zeta(0) &- \int_0^t F_k(s, \zeta(s))\,\mathrm{d}s \\
&- \int_0^t G_k(s,\zeta(s)) \frac{\xi((s+\varepsilon)\wedge t) - \xi(s)}{\varepsilon}\,\mathrm{d}s, \\
R_k\colon \Omega \longrightarrow \mathbb{R},\ (\zeta,\xi) \longmapsto \Big(\int_0^t &\big\|\Pi_\varepsilon G_k(\cdot, \zeta(\cdot))(s) - G_k(s, \zeta(s))\big\|^2\,\mathrm{d}s\Big)^{1/2}
\end{aligned}$$

for any $k \ge 0$. Using (A1) and (A2) it is easy to check that H_k and R_k are continuous (hence Borel) mappings on Ω. Further,

$$X_k(t) - X_k(0) - \int_0^t F_k(s, X_k(s))\,\mathrm{d}s = \int_0^t G_k(s, X_k(s))\,\mathrm{d}W_k(s)$$

$\boldsymbol{Q}_k$-almost surely, $k \ge 1$, since (W_k, X_k) solves (6); recall that

$$\int_0^t G_k(s, X_k(s)) \frac{W_k((s+\varepsilon)\wedge t) - W_k(s)}{\varepsilon}\,\mathrm{d}s = \int_0^t \Pi_\varepsilon G_k(\cdot, X_k(\cdot))(s)\,\mathrm{d}W_k(s),$$

therefore

$$H_k(X_k,W_k)=\int_0^t\big\{G_k(s,X_k(s))-\varPi_\varepsilon G_k(\cdot,X_k(\cdot))(s)\big\}\,\mathrm{d}W_k(s).$$

From Lemma 2.1 we get (cf. Remark 2.4)

$$\begin{aligned}\mu_k\big\{\|H_k(X,W)\|>a\ \&\ R_k(X)<b\big\}&=\boldsymbol{Q}_k\big\{\|H_k(X_k,W_k)\|>a\ \&\ R_k(X_k)<b\big\}\\&\le\frac{b^2}{a^2}\end{aligned}$$

for all $a,b>0$. Since

$$\begin{aligned}\big|R_k(\zeta)-R_0(\zeta)\big|&\le\Big(\int_0^t\big\|\varPi_\varepsilon G_k(\cdot,\zeta(\cdot))(s)-\varPi_\varepsilon G_0(\cdot,\zeta(\cdot))(s)\big\|\,\mathrm{d}s\Big)^{1/2}\\&\quad+\Big(\int_0^t\big\|G_k(s,\zeta(s))-G_0(s,\zeta(s)\big\|\,\mathrm{d}s\Big)^{1/2}\\&\le 2\Big(\int_0^t\big\|G_k(s,\zeta(s))-G_0(s,\zeta(s)\big\|\,\mathrm{d}s\Big)^{1/2},\end{aligned}$$

the assumptions (A3) and (A2) of Theorem 1.1 and the dominated convergence theorem imply that

$$\lim_{k\to\infty}H_k=H_0,\ \lim_{n\to\infty}R_k=R_0\quad\text{uniformly on compact subsets of }\varOmega.$$

From Lemma 2.2 we obtain

$$\begin{aligned}\boldsymbol{P}\big\{\|H_0(X,W)\|>a\ \&\ R_0(X)<b\big\}&\le\liminf_{n\to\infty}\mu_k\big\{\|H_k(X,W)\|>a\ \&\ R_k(X)<b\big\}\\&\le\frac{b^2}{a^2},\end{aligned}$$

that is

$$\begin{aligned}\boldsymbol{P}\Big\{\Big\|X(t)-X(0)-\int_0^t F_0(s,X(s))\,\mathrm{d}s-\int_0^t G_0(s,X(s))\frac{W((s+\varepsilon)\wedge t)-W(s)}{\varepsilon}\,\mathrm{d}s\Big\|>a\\ \&\ \int_0^t\big\|\varPi_\varepsilon G_0(\cdot,X(\cdot))(s)-G_0(s,X(s))\big\|^2\,\mathrm{d}s<b^2\Big\}\le\frac{b^2}{a^2}\end{aligned}\tag{19}$$

for any $\varepsilon>0$, $a,b>0$. Since

$$\boldsymbol{P}\text{-}\lim_{\varepsilon\to0+}\int_0^t\big\|\varPi_\varepsilon G_0(\cdot,X(\cdot))(s)-G_0(s,X(s))\big\|^2\,\mathrm{d}s=0,$$

we get

$$\begin{aligned}\boldsymbol{P}\text{-}\lim_{\varepsilon\to0+}\int_0^t G_0(s,X(s))\frac{W((s+\varepsilon)\wedge t)-W(s)}{\varepsilon}\,\mathrm{d}s&=\boldsymbol{P}\text{-}\lim_{\varepsilon\to0+}\int_0^t\varPi_\varepsilon G_0(\cdot,X(\cdot))(s)\,\mathrm{d}W(s)\\&=\int_0^t G_0(s,X(s))\,\mathrm{d}W(s).\end{aligned}$$

Convergence in probability implies convergence in law, so using (19) and the portmanteau theorem we arrive at

$$\boldsymbol{P}\Big\{\Big\|X(t)-X(0)-\int_0^t F_0(s,X(s))\,\mathrm{d}s-\int_0^t G_0(s,X(s))\,\mathrm{d}W(s)\Big\|>a\Big\}\le\frac{b^2}{a^2}$$

for all $a,b>0$, thus

$$X(t)-X(0)-\int_0^t F_0(s,X(s))\,\mathrm{d}s-\int_0^t G_0(s,X(s))\,\mathrm{d}W(s)=0$$

$\boldsymbol{P}$-almost surely for any $t\in\mathbb{R}_{\ge 0}$, which means that $((\Omega,\mathscr{F}_\infty,(\mathscr{F}_t),\boldsymbol{P}),W,X)$ is a weak solution of (7). The initial condition $X_0\sim\varrho$ is satisfied by our choice of $\boldsymbol{P}$. □

P r o o f o f T h e o r e m 1.2. It is sufficient to prove that $\{X_{\alpha\#}\boldsymbol{Q}_\alpha;\ \alpha\in A\}$ is a tight set of measures on $\mathscr{C}([0,T];\mathbb{R}^m)$ for any $T\ge 0$. (More precisely, measures $(\pi_T X_\alpha)_\#\boldsymbol{Q}_\alpha$, $\alpha\in A$, are considered, but the simplified notation cannot cause any confusion.) Let us fix a $T>0$ and define

$$\tau_{\alpha,k}=\inf\{t\in[0,T];\ \|X_\alpha(t)\|\ge k\}$$

for $\alpha\in A$ and $k\in\mathbb{R}_{\ge 0}$, where we set $\inf\emptyset=T$. Let us denote by $\boldsymbol{E}_\alpha$ the integral with respect to the measure $\boldsymbol{Q}_\alpha$. Choose an arbitrary $p\in(2,\infty)$, we aim at showing that for any $k\ge 0$ there exists a constant L_k, depending only on T, p and k, such that

$$\sup_{\alpha\in A}\boldsymbol{E}_\alpha\big\|X_\alpha(t\wedge\tau_{\alpha,k})-X_\alpha(s\wedge\tau_{\alpha,k})\big\|^p\le L_k|t-s|^{p/2}\tag{20}$$

whenever $0\le s\le t\le T$. Indeed, using the Hölder and Burkholder-Davis-Gundy inequalities and the local boundedness hypothesis (8) we get

$$\begin{aligned}\boldsymbol{E}_\alpha&\big\|X_\alpha(t\wedge\tau_{\alpha,k})-X_\alpha(s\wedge\tau_{\alpha,k})\big\|^p\\&\le 2^{p-1}\boldsymbol{E}_\alpha\Big\|\int_{s\wedge\tau_{\alpha,k}}^{t\wedge\tau_{\alpha,k}}F_\alpha(u,X_\alpha(u))\,\mathrm{d}u\Big\|^p\\&\qquad+2^{p-1}\boldsymbol{E}_\alpha\Big\|\int_{s\wedge\tau_{\alpha,k}}^{t\wedge\tau_{\alpha,k}}G_\alpha(u,X_\alpha(u))\,\mathrm{d}W_\alpha(u)\Big\|^p\\&\le 2^{p-1}(t-s)^{p-1}\boldsymbol{E}_\alpha\int_{s\wedge\tau_{\alpha,k}}^{t\wedge\tau_{\alpha,k}}\|F_\alpha(u,X_\alpha(u)\|^p\,\mathrm{d}u\\&\qquad+2^{p-1}(t-s)^{\frac{p}{2}-1}C_p\boldsymbol{E}_\alpha\int_{s\wedge\tau_{\alpha,k}}^{t\wedge\tau_{\alpha,k}}\|G_\alpha(u,X_\alpha(u))\|^p\,\mathrm{d}u\\&\le 2^p(T^{p/2}+C_p)\kappa_{k,T}^p(t-s)^{p/2}.\end{aligned}$$

Let an $\varepsilon>0$ be given, we look for a compact set $K\subseteq\mathscr{C}([0,T];\mathbb{R}^m)$ such that

$$\inf_{\alpha\in A}\boldsymbol{Q}_\alpha\{X_\alpha\in K\}\ge 1-2\varepsilon.$$

By (9) we may find $R \in \mathbb{R}_{\geq 0}$ such that

$$\inf_{\alpha \in A} \boldsymbol{Q}_\alpha\{\tau_{\alpha,R} = T\} \geq 1 - \varepsilon.$$

Take $s \in (\frac{1}{p}, \frac{1}{2})$, then

$$\begin{aligned}
&\boldsymbol{Q}_\alpha\Big\{\omega \in \{\tau_{\alpha,R} = T\};\ \int_0^T\int_0^T \frac{\|X_\alpha(t,\omega) - X_\alpha(s,\omega)\|^p}{|t-s|^{1+sp}}\,\mathrm{d}s\,\mathrm{d}t > \varGamma\Big\} \\
&\quad\leq \frac{1}{\varGamma}\boldsymbol{E}_\alpha \mathbf{1}_{\{\tau_{\alpha,R}=T\}} \int_0^T\int_0^T \frac{\|X_\alpha(t) - X_\alpha(s)\|^p}{|t-s|^{1+sp}}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leq \frac{1}{\varGamma}\boldsymbol{E}_\alpha \int_0^T\int_0^T \frac{\|X_\alpha(t\wedge\tau_{\alpha,R}) - X_\alpha(s\wedge\tau_{\alpha,R})\|^p}{|t-s|^{1+sp}}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad\leq \frac{L_R}{\varGamma}\int_0^T\int_0^T \frac{1}{|t-s|^{1+sp-p/2}}\,\mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

by the Chebyshev inequality and (20). Note that the right-hand side of this estimate is finite and independent of $\alpha \in A$, so we may find $\varGamma > 0$ such that for any $\alpha \in A$ there exists $\varOmega'_\alpha \in \mathscr{F}_\alpha$ satisfying $\varOmega'_\alpha \subseteq \{\tau_{\alpha,R} = T\}$, $\boldsymbol{Q}_\alpha(\varOmega'_\alpha) \geq 1 - 2\varepsilon$ and

$$\int_0^T\int_0^T \frac{\|X_\alpha(t) - X_\alpha(s)\|^p}{|t-s|^{1+sp}}\,\mathrm{d}s\,\mathrm{d}t \leq \varGamma \quad \text{on } \varOmega'_\alpha.$$

Plainly,

$$\int_0^T \|X_\alpha(u)\|^p\,\mathrm{d}u \leq R^pT \quad \text{on } \varOmega'_\alpha;$$

we may assume that $\varGamma \geq R^pT$. Denote by K' a closed centered ball with radius $(2\varGamma)^{1/p}$ in the Slobodeckiĭ space $W^{s,p}([0,T];\mathbb{R}^m)$, we showed that $X_\alpha(\omega) \in K'$ for any $\alpha \in A$ and $\omega \in \varOmega'_\alpha$. Let $0 < \beta < s - \frac{1}{p}$, owing to the embedding $W^{s,p}([0,T];\mathbb{R}^m) \hookrightarrow \mathscr{C}^{0,\beta}([0,T];\mathbb{R}^m)$ (see e. g. [17, Theorem 2.8.1]) there exists a closed ball K in $\mathscr{C}^{0,\beta}([0,T];\mathbb{R}^m)$ such that $X_\alpha(\omega) \in K$ for any $\alpha \in A$ and $\omega \in \varOmega'_\alpha$, that is

$$\inf_{\alpha \in A} \boldsymbol{Q}_\alpha\{X_\alpha \in K\} \geq 1 - 2\varepsilon.$$

Since closed balls in $\mathscr{C}^{0,\beta}([0,T];\mathbb{R}^m)$ are compact in $\mathscr{C}([0,T];\mathbb{R}^m)$ by the Arzelà–Ascoli theorem the proof is completed. □

Proof of Corollary 1.3. The proof is fairly standard and we only sketch it. Let $T \geq 0$, we shall show that (9) is satisfied. Set $U(t,x) = e^{-ct}(1 + V(x))$, $t \geq 0$, $x \in \mathbb{R}^m$, then

$$LU(t,x) \equiv \frac{\partial U}{\partial t}(t,x) + \big\langle D_xU(t,x), F_\alpha(t,x)\big\rangle + \frac{1}{2}\operatorname{Tr}\big(D_x^2U(t,x)G_\alpha(t,x)G_\alpha^*(t,x)\big) \leq 0$$

for all $(t,x)\in\mathbb{R}_{\geq 0}\times\mathbb{R}^m$, $\alpha\in A$. Let $\tau_{\alpha,k}$ be the stopping times defined in the proof of Theorem 1.2, then

$$\begin{aligned}U(t\wedge\tau_{\alpha,k},X_\alpha(t\wedge\tau_{\alpha,k}))-U(0,X_\alpha(0))\\&=\int_0^{t\wedge\tau_{\alpha,k}}LU(s,X_\alpha(s))\,\mathrm{d}s+\int_0^{t\wedge\tau_{\alpha,k}}D_xU(s,X_\alpha(s))^*G_\alpha(s,X_\alpha(s))\,\mathrm{d}W_\alpha(s)\\&\leq\int_0^{t\wedge\tau_{\alpha,k}}D_xU(s,X_\alpha(s))^*G_\alpha(s,X_\alpha(s))\,\mathrm{d}W_\alpha(s)\end{aligned}$$

by the Itô formula. Let $\varepsilon>0$ be arbitrary, we can find $K\subseteq\mathbb{R}^m$ compact such that $\nu(K)\geq 1-\varepsilon$. Set $M_\alpha=\{\omega\in\Omega_\alpha;\ X_\alpha(0)\in K\}$, then $M_\alpha\in\mathscr{F}_0^\alpha$ and $\boldsymbol{Q}_\alpha(M_\alpha)\geq 1-\varepsilon$ for all $\alpha\in A$ as $X_\alpha(0)_\#\boldsymbol{Q}_\alpha=\nu$. We have

$$\begin{aligned}\boldsymbol{E}_\alpha\mathbf{1}_{M_\alpha}V(X_\alpha(t\wedge\tau_{\alpha,k}))&\leq e^{cT}\boldsymbol{E}_\alpha\mathbf{1}_{M_\alpha}V(X_\alpha(0))\\&\qquad+e^{cT}\boldsymbol{E}_\alpha\int_0^{t\wedge\tau_{\alpha,k}}\mathbf{1}_{M_\alpha}D_xU(s,X_\alpha(s))^*G_\alpha(s,X_\alpha(s))\,\mathrm{d}W_\alpha(s)\\&=e^{cT}\boldsymbol{E}_\alpha\mathbf{1}_{M_\alpha}V(X_\alpha(0))\leq e^{cT}\sup_K V\equiv\lambda(\varepsilon)<\infty\end{aligned}$$

for any $t\in[0,T]$. Consequently, setting $q_R=\inf_{\|x\|\geq R}V(x)$ we get

$$\begin{aligned}\boldsymbol{Q}_\alpha\{\sup_{0\leq t\leq T}\|X_\alpha(t)\|>R\}&\leq\boldsymbol{Q}_\alpha\{\omega\in M_\alpha;\ \sup_{0\leq t\leq T}\|X_\alpha(t)\|>R\}+\boldsymbol{Q}_\alpha(\Omega_\alpha\setminus M_\alpha)\\&\leq\varepsilon+\boldsymbol{Q}_\alpha\{\omega\in M_\alpha;\ \tau_{\alpha,R}<T\}=\varepsilon+\boldsymbol{E}_\alpha\mathbf{1}_{M_\alpha}\mathbf{1}_{\{\tau_{\alpha,R}<T\}}\\&\leq\varepsilon+\boldsymbol{E}_\alpha\mathbf{1}_{M_\alpha}\mathbf{1}_{\{\tau_{\alpha,R}<T\}}\frac{V(X_\alpha(\tau_{\alpha,R}))}{q_R}\\&\leq\varepsilon+\frac{1}{q_R}\boldsymbol{E}_\alpha\mathbf{1}_{M_\alpha}V(X_\alpha(T\wedge\tau_{\alpha,R}))\leq\varepsilon+\frac{\lambda(\varepsilon)}{q_R}\end{aligned}$$

for any $\alpha\in A$. Noting that $q_R\to\infty$ as $R\to\infty$ we can find $R\geq 0$ such that

$$\sup_{\alpha\in A}\boldsymbol{Q}_\alpha\Big\{\sup_{0\leq t\leq T}\|X_\alpha(t)\|>R\Big\}\leq 2\varepsilon,$$

hence (9) holds. □

Proof of Remark 1.4. First, let us recall the Gyöngy–Krylov lemma. Let $(\Xi,\mathscr{A},q)$ be a probability space, Y a Polish space and $U_k\colon\Xi\longrightarrow Y$ $\mathscr{A}$-measurable mappings, $k\geq 1$. Then there exists an $\mathscr{A}$-measurable $U\colon\Xi\longrightarrow Y$ such that $q\text{-}\lim_{k\to\infty}U_k=U$ if and only if for any two subsequences $\{U_{M_j}\}$, $\{U_{N_j}\}$ of $\{U_k\}$ there exist a subsequence $\{(U_{M_j(l)},U_{N_j(l)})\}_{l=1}^\infty$ of $\{(U_{M_j},U_{N_j})\}$ and a Borel probability measure σ on $Y\times Y$ satisfying $\sigma\{(y,y)\in Y\times Y;\ y\in Y\}=1$ such that

$$\operatorname*{w^*\text{-}lim}_{l\to\infty}(U_{M_j(l)},U_{N_j(l)})_\#q=\sigma.$$

Now we may turn to the proof of Remark 1.4. Let $\{Z_{M_j}\}$, $\{Z_{N_j}\}$ be two arbitrary subsequences of $\{Z_k\}_{k=1}^\infty$, then for any $j\geq 1$ $(B,(Z_{M_j},Z_{N_j}))$ solves the coupled system

$$\mathrm{d}\begin{pmatrix}X_1\\X_2\end{pmatrix}=\begin{pmatrix}F_{M_j}(t,X_1)\\F_{N_j}(t,X_2)\end{pmatrix}\mathrm{d}t+\begin{pmatrix}G_{M_j}(t,X_1)\\G_{N_j}(t,X_2)\end{pmatrix}\mathrm{d}B,\quad\begin{pmatrix}X_1\\X_2\end{pmatrix}(0)=\begin{pmatrix}\psi\\\psi\end{pmatrix}$$

the coefficients of which satisfy the assumptions of Theorem 1.1. By the assumption (A5') the set $\{(Z_{M_j}, Z_{N_j})_\# \boldsymbol{Q};\ j \geq 1\}$ is plainly tight, so the proof of Theorem 1.1 shows that there exist a stochastic basis $((\Theta, \mathscr{R}, (\mathscr{R}_t), \boldsymbol{P})$, an $(\mathscr{R}_t)$-Wiener process W and a subsequence $\{(Z_{M_j(l)}, Z_{N_j(l)}\}_{l=1}^\infty$ converging in law to a pair (U, V) such that $((\Theta, \mathscr{R}, (\mathscr{R}_t), \boldsymbol{P}), W, (U, V))$ is a weak solution to the system

$$\mathrm{d}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F_0(t, X_1) \\ F_0(t, X_2) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} G_0(t, X_1) \\ G_0(t, X_2) \end{pmatrix} \mathrm{d}W \tag{21}$$

and $(U(0), V(0))_\# \boldsymbol{P} = (\psi, \psi)_\# \boldsymbol{Q}$. Consequently, $U(0) \sim \varrho$, $V(0) \sim \varrho$ and the assumption (A6) of pathwise uniqueness yields

$$\boldsymbol{P}\Big\{\sup_{t\geq 0} \|U(t) - V(t)\| = 0\Big\} = 1.$$

(Note that due to the form of (21) both (W, U) and (W, V) are solutions of (7), defined on the same stochastic basis.) Thus, if D denotes the diagonal in $\mathscr{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)^2$, we have

$$(U, V)_\# \boldsymbol{P}(D) = 1.$$

Therefore, setting $\sigma = (U, V)_\# \boldsymbol{P}$ in the Gyöngy–Krylov lemma we see that there exists a random variable $Z\colon G \longrightarrow \mathscr{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ such that

$$\boldsymbol{Q}\text{-}\lim_{k\to\infty} \sup_{0\leq t\leq T} \|Z_k(t) - Z(t)\| = 0 \tag{22}$$

for any $T \in \mathbb{R}_{>0}$. In particular, $Z(0) = \psi$ and Z is a $(\mathscr{G}_t^{\psi,B})$-adapted stochastic process. It is known (see e. g. [7, Theorem 9.2.1] that (22) is equivalent to the property that any subsequence $\{Z_{k(j)}\}$ of $\{Z_k\}$ has a subsequence $\{Z_{k(j(i))}\}$ such that

$$\lim_{i\to\infty} \sup_{0\leq t\leq T} \|Z_{k(j(i))} - Z(t)\| = 0 \quad \boldsymbol{Q}\text{-almost surely.} \tag{23}$$

Using (23) and the assumptions (A2) and (A3) we get

$$\boldsymbol{Q}\text{-}\lim_{k\to\infty} \int_0^T \|F_k(t, Z_k(t)) - F_0(t, Z(t))\| \,\mathrm{d}t = 0,$$
$$\boldsymbol{Q}\text{-}\lim_{k\to\infty} \int_0^T \|G_k(t, Z_k(t)) - G_0(t, Z(t))\|^2 \,\mathrm{d}t = 0$$

for all $T \in \mathbb{R}_{>0}$, so

$$\boldsymbol{Q}\text{-}\lim_{k\to\infty} \int_0^t F_k(s, Z_k(s)) \,\mathrm{d}s = \int_0^t F_0(s, Z(s)) \,\mathrm{d}s,$$
$$\boldsymbol{Q}\text{-}\lim_{k\to\infty} \int_0^t G_k(s, Z_k(s)) \,\mathrm{d}B(s) = \int_0^t G_0(s, Z(s)) \,\mathrm{d}B(s)$$

for any $t \geq 0$, which proves our claim. □

P r o o f o f R e m a r k 1.5. First ve check that there exists a Borel set $\Lambda \subseteq \mathbb{R}^m$ such that $\varrho(\Lambda) = 0$ and W is a Wiener process on $(\Omega, \mathscr{F}, (\mathscr{F}_t), p(y, \cdot))$ for every $y \notin \Lambda$. We shall use the Lévy theorem again. As

$$\infty > \int_\Omega \sup_{0 \le t \le K} |W_t| \, \mathrm{d}\boldsymbol{P} = \int_{\mathbb{R}^m} \int_\Omega \sup_{0 \le t \le K} |W_t| \, \mathrm{d}p(y, \cdot) \, \mathrm{d}\varrho(y)$$

for any $K \in \mathbb{N}$, there exists a ϱ-null set $\Lambda' \subseteq \mathbb{R}^m$ such that $\sup_{[0,T]} |W| \in L^1(p(y, \cdot))$ for all $T \in \mathbb{R}_{\ge 0}$ and $y \notin \Lambda'$. Fix arbitrary $s, t \in \mathbb{Q}_{\ge 0}$, $s < t$, and $A \in \mathscr{F}_s$, then we obtain

$$0 = \int_\Omega \mathbf{1}_{\{X_0 \in B\}} \mathbf{1}_A \big[W_t - W_s \big] \, \mathrm{d}\boldsymbol{P} = \int_B \int_\Omega \mathbf{1}_A \big[W_t - W_s \big] \, \mathrm{d}p(y, \cdot) \, \mathrm{d}\varrho(y)$$

for all Borel sets $B \subseteq \mathbb{R}^m$, since $\{X_0 \in B\} \in \mathscr{F}_0 \subseteq \mathscr{F}_s$ and W is a martingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t), \boldsymbol{P})$. Therefore,

$$\int_A \big[W_t - W_s \big] \, \mathrm{d}p(y, \cdot) = 0 \tag{24}$$

for ϱ-almost all $y \in \mathbb{R}^m$. Taking into account that $\mathscr{F}_s$ is countably generated and employing Dynkin's π/λ-argument we may find $\Lambda'' \supseteq \Lambda'$ such that $\varrho(\Lambda'') = 0$ and (24) holds for all $y \notin \Lambda''$, $s, t \in \mathbb{Q}_{\ge 0}$, $s < t$, and $A \in \mathscr{F}_s$. If $s, t \in \mathbb{R}_{\ge 0}$, $s < t$, we can find $s_n, t_n \in \mathbb{Q}_{\ge 0}$, $s \le s_n < t_n \le t$, $s_n \searrow s$, $t_n \nearrow t$; by (24)

$$\int_A \big[W_{t_n} - W_{s_n} \big] \, \mathrm{d}p(y, \cdot) = 0$$

for all $A \in \mathscr{F}_s \subseteq \mathscr{F}_{s_n}$ and $y \notin \Lambda''$. Since all paths of W are continuous, it follows from the dominated convergence theorem that (24) holds for all $s, t \in \mathbb{R}_{\ge 0}$, $A \in \mathscr{F}_s$ and $y \notin \Lambda''$, so W is a martingale on $(\Omega, \mathscr{F}, (\mathscr{F}_t), p(y, \cdot))$ for any $y \notin \Lambda''$. Analogously, we may study the martingale property of $(W_t \otimes W_t - tI)$.

Further,

$$\lim_{\varepsilon \to 0+} \int_0^t \big\| \Pi_\varepsilon G_0(\cdot, X(\cdot))(s) - G_0(s, X(s)) \big\|^2 \, \mathrm{d}s = 0 \quad \text{for all } t \ge 0 \ \boldsymbol{P}\text{-almost surely,}$$

hence it is easy to check that there exists a ϱ-null set $\Lambda^* \supseteq \Lambda$ such that

$$\lim_{\varepsilon \to 0+} \int_0^t \big\| \Pi_\varepsilon G_0(\cdot, X(\cdot))(s) - G_0(s, X(s)) \big\|^2 \, \mathrm{d}s = 0 \quad \text{for all } t \ge 0 \ p(y, \cdot)\text{-almost surely}$$

for any $y \notin \Lambda^*$, which implies that

$$p(y, \cdot)\text{-}\lim_{\varepsilon \to 0+} \int_0^t G_0(s, X(s)) \frac{W((s+\varepsilon) \wedge t) - W(s)}{\varepsilon} \, \mathrm{d}s = \int_0^t G_0(s, X(s)) \, \mathrm{d}W(s) \tag{25}$$

for all $y \notin \Lambda^*$. (As W is a Wiener process with respect to $p(y, \cdot)$ for $y \notin \Lambda^*$, the stochastic integral makes sense and Lemma 2.3 may be applied.) Fix $t \geq 0$, for $a, b, \varepsilon > 0$ set

$$\begin{aligned} &\Gamma_{a,b,\varepsilon} \\ &= \Big\{ \Big\| X(t) - X(0) - \int_0^t F_0(s, X(s))\,\mathrm{d}s - \int_0^t G_0(s, X(s)) \frac{W((s+\varepsilon)\wedge t) - W(s)}{\varepsilon}\,\mathrm{d}s \Big\| > a \\ &\qquad \& \int_0^t \big\| \Pi_\varepsilon G_0(\cdot, X(\cdot))(s) - G_0(s, X(s)) \big\|^2\,\mathrm{d}s < b^2 \Big\} \end{aligned}$$

and

$$\Gamma_{a,0} = \Big\{ \Big\| X(t) - X(0) - \int_0^t F_0(s, X(s))\,\mathrm{d}s - \int_0^t G_0(s, X(s))\,\mathrm{d}W(s) \Big\| > a \Big\}.$$

By (25) and the portmanteau theorem

$$\liminf_{\varepsilon \to 0+} p(y, \Gamma_{a,b,\varepsilon}) \geq p(y, \Gamma_{a,0}) \quad \text{for any } y \notin \Lambda^*,$$

so by (19) and the Fatou lemma

$$\frac{b^2}{a^2} \geq \liminf_{\varepsilon \to 0+} \boldsymbol{P}(\Gamma_{a,b,\varepsilon}) = \liminf_{\varepsilon \to 0+} \int_{\mathbb{R}^m} p(y, \Gamma_{a,b,\varepsilon})\,\mathrm{d}\varrho(y) \geq \int_{\mathbb{R}^m} p(y, \Gamma_{a,0})\,\mathrm{d}\varrho(y).$$

Since $b > 0$ was arbitrary and the right-hand side does not depend on b we get

$$\int_{\mathbb{R}^m} p(y, \Gamma_{a,0})\,\mathrm{d}\varrho(y) = 0$$

for all $a > 0$ and our claim follows easily. □

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