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A FAST NUMERICAL TEST OF MULTIVARIATE POLYNOMIAL POSITIVENESS WITH APPLICATIONS

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The paper presents a simple method to check a positiveness of symmetric multivariate polynomials on the unit multi-circle. The method is based on the sampling polynomials using the fast Fourier transform. The algorithm is described and its possible applications are proposed. One of the aims of the paper is to show that presented algorithm is significantly faster than commonly used method based on the semi-definite programming expression.

Keywords: multidimensional systems, positive polynomials, fast Fourier transforms, stability, numerical algorithm

Classification: 12D10, 47N70, 65T50, 65Y20

1. INTRODUCTION

Multidimensional (n-D) systems are a very active topic in the last decades with applications in many areas and, as it was stated by [28], the n-D systems approach does bring advantages in solving control problems. The n-D systems can be used to formulate a large number of problems in, e. g., digital image processing, biomedicine, and control of systems described by partial differential equations. The interested reader is referred to [8, 9] for fundamentals and basic informations. The essential difference of n-D systems from the classical (1-D) ones is that an information propagates in more than one independent directions. The class of n-D systems includes, e. g., repetitive processes [25, 27], spatially invariant systems [10, 11], positive n-D systems [20], or n-D systems of the fractional order [21], and iterative learning control [10, 11, 12]. The n-D systems approach is applied in control of ladder circuits [32, 31], and in modelling of complex systems [5, 6]. Methods for identification of n-D systems are also developed in, for instance, [1, 26]. Linear n-D systems can be described by rational functions or matrices of several independent variables. The variables represent different space coordinates or mixed time and space coordinates.

In the stability and stabilisation of n-D systems, positive and sum-of-square polynomials [16, 18, 19] play a role. Roughly speaking, a system is stable if (and only if) a certain polynomial is positive. Numerical algorithms for stability analysis based on the checking polynomial positiveness are discussed in, e. g., [14, 15, 17]. The paper [14] reviews stability tests for n-D discrete-time systems. In [15], a stability test

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for n-D discrete-time systems is based on Gram matrix associated with a polynomial whose positiveness on the unit n-circle is checked using the sum-of-squares decomposition. In [17], a stability test is proposed based on positivstellensatz for trigonometric polynomials.

In this paper we show an another approach to test the positiveness of polynomials. A very simple method is based on sampling a polynomial at points on the unit circle. Our numerical simulations and experiments show that if a great many points are used, it can be stated that a polynomial is positive on the unit circle if and only if all samples are positive. It is shown that the implementation using the fast Fourier transform (FFT) gives an efficient tool to check the positiveness. Computation times are compared with one of the methods given in [15]. One of the aims of this paper is to show that the proposed test is dramatically faster than the method of [15].

The algorithm was originally published in [2]. This paper brings an extension to the multivariate case and proposes its applications in n-D systems theory, in particular, in stability analysis of repetitive processes and spatially invariant systems.

The paper is organised as follows. In Sec. 2, the problem of positiveness of polynomials on the unit n-circle is outlined and the algorithm of [15] is briefly described. The numerical algorithm based on fast sampling using FFT is given in Sec. 3. Applications of the proposed algorithm are discussed in Sec. 4. Numerical experiments and simulations are performed in Sec. 5. Some remarks conclude the paper in Sec. 6.

2. POSITIVENESS OF POLYNOMIALS

We shall concentrate on stability of linear discrete-time repetitive processes and time-invariant spatially distributed systems. We make clear later that this stability relates to a multivariate polynomial, say, $f(z_1, z_2, \ldots, z_n)$ in n indeterminates $z_1 \in \mathbb{C}, z_2 \in \mathbb{C}, \ldots, z_n \in \mathbb{C}$. Let f be of the form

$$f = \sum_{k_1=0}^{d_1} \sum_{k_2=0}^{d_2} \cdots \sum_{k_n=0}^{d_n} f_{k_1,k_2,\ldots,k_n} \, z_1^{k_1} \, z_2^{k_2} \, \cdots \, z_n^{k_n}, \qquad f_{k_1,k_2,\ldots,k_n} \in \mathbb{R}. \tag{1}$$

Let us study when f has no zero inside the unit n-circle, that is, when

$$f \neq 0 \quad \text{for all} \quad |z_1| \leq 1, |z_2| \leq 1, \ldots, |z_n| \leq 1. \tag{2}$$

It was proved in [29, Theorem 3] that (2) is equivalent to the set of conditions

1. for some $\beta_1, \ldots, \beta_n$ such that $|\beta_r| = 1, r = 1, \ldots, n$ and for all $i, i = 1, \ldots, n$,
$$f \neq 0 \qquad \text{when } z_r = \beta_r, r \neq i \quad \text{and} \quad |z_i| \leq 1, \tag{3}$$
2. $$f \neq 0 \qquad \text{when } |z_1| = |z_2| = \cdots = |z_n| = 1. \tag{4}$$

The univariate conditions (3) can be verified by well-known methods. The condition (4) is multivariate one. It can be formulated using the positiveness of a polynomial as follows, see [15, 16]. Let

$$g(z_1, z_2, \ldots, z_n) = f(z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}) \, f(z_1, z_2, \ldots, z_n), \tag{5}$$

so, it can be expressed as

$$g = \sum_{k_1=-d_1}^{d_1} \sum_{k_2=-d_2}^{d_2} \cdots \sum_{k_n=-d_n}^{d_n} g_{k_1,k_2,\ldots,k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}, \qquad g_{k_1,k_2,\ldots,k_n} \in \mathbb{R} \tag{6}$$

with coefficients symmetric about the origin, i. e., $g_{k_1,k_2,\ldots,k_n} = g_{-k_1,-k_2,\ldots,-k_n}$ and similarly. The following lemma follows from, e. g., [15].

Lemma 2.1. The condition (2) is satisfied if and only if (3) holds and (5) is strictly positive for all $|z_1| = |z_2| = \cdots = |z_n| = 1$.

In [15], a numerical test of positiveness of (5) is implemented by taking an appropriately small $\varepsilon > 0$ and testing if

$$g(z_1, z_2, \ldots, z_n) - \varepsilon \text{ is sum-of-squares.} \tag{7}$$

Using the semi-definite programming (SDP), (7) can be formulated as a feasibility problem

$$\begin{aligned} &\text{find } Q \\ &\text{s. t. } \; g_{k_1,\ldots,k_n} = \operatorname{trace}\left[T_{k_1,\ldots,k_n} \cdot Q\right] \\ &Q \geq \varepsilon I, \end{aligned} \tag{8}$$

where $T_{k_1,\ldots,k_n} = T_{k_n} \otimes \cdots \otimes T_{k_1}$ and $T_{k_i} \in \mathbb{R}^{(d_i+1)\times(d_i+1)}$ are elementary Toeplitz matrices with ones only on the k_ith diagonal, the symbol $\otimes$ denotes Kronecker product, and trace X is the trace of the matrix X. In what follows the above approach is called *SDP-based method.*

3. FAST NUMERICAL ALGORITHM

The positiveness of (5) for all $|z_1| = |z_2| = \cdots = |z_n| = 1$ can be checked directly by substituting points from the unit n-circle for $z_1, z_2, \ldots, z_n$. Let $N_1, N_2, \ldots, N_n$ denote numbers of points on the unit n-circle. These points are

$$e^{-2\pi j \frac{i_1}{N_1}}, \quad e^{-2\pi j \frac{i_2}{N_2}}, \ldots, \quad e^{-2\pi j \frac{i_n}{N_n}}, \qquad \begin{aligned} i_1 &= 0, \ldots, N_1 - 1, \\ i_2 &= 0, \ldots, N_2 - 1, \\ &\vdots \\ i_n &= 0, \ldots, N_n - 1. \end{aligned}$$

Substituting these points into (5) results in $N_1 \cdot N_2 \cdot \cdots \cdot N_n$ numerical values. This operation is exactly the same what the *discrete Fourier transform* (DFT) algorithm performs. Let us recall the basic definitions. For an n-D array of complex numbers g, its direct DFT is an n-D array G, where

$$G_{k_1,k_2,\ldots,k_n} = \sum_{i_1=0}^{N_1-1} \left(e^{-2\pi j \frac{i_1 k_1}{N_1}} \sum_{i_2=0}^{N_2-1} \left(e^{-2\pi j \frac{i_2 k_2}{N_2}} \cdots \sum_{i_n=0}^{N_n-1} e^{-2\pi j \frac{i_n k_n}{N_n}} g_{i_1,i_2,\ldots,i_n} \right)\right).$$

If an n-D array of complex numbers G is given, its inverse DFT is an n-D array of complex numbers g

$$g_{i_1,i_2,\ldots,i_n} = \frac{1}{\prod_{i=1}^{n} N_i} \sum_{k_1=0}^{N_1-1} \left(e^{2\pi j \frac{i_1 k_1}{N_1}} \sum_{k_2=0}^{N_2-1} \left(e^{2\pi j \frac{i_2 k_2}{N_2}} \cdots \sum_{k_n=0}^{N_n-1} e^{2\pi j \frac{i_n k_n}{N_n}} G_{k_1,k_2,\ldots,k_n} \right)\right).$$

The both above definitions are well known. The DFT is in common use in many engineering areas, most frequently in the signal processing. To accelerate the numerical computation, the efficient fast Fourier transform (FFT) algorithms are available. The most common one is the Cooley-Tukey algorithm derived and described by [13]. Since FFT algorithms play an important role, they are available as built-in functions in many computing packages.

The stability test can be performed as follows. Convert coefficients $g_{k_1,k_2,\ldots,k_n}$ of the polynomial (5) to the n-D array. In the case $n = 1$, we have coefficients g_{k_1} and the array is of the form

$$\begin{pmatrix} g_0 & g_1 & g_2 & \cdots & g_{d_1} & 0 & \cdots & 0 & g_{d_1} & \cdots & g_2 & g_1 \end{pmatrix}$$

and of the size $1 \times N_1$. For example, let $N_1 = 8$ and $G(z_1) = 10+3\,(z_1+z_1^{-1})+4\,(z_1^2+z_2^{-2})$. The above array reads

$$\begin{pmatrix} 10 & 3 & 4 & 0 & 0 & 0 & 4 & 3 \end{pmatrix}.$$

Applying DFT algorithm gives 8 samples $24, 14.2426, 2, 5.7574, 12, 5.7574, 2, 14.2426$. One can insure that substituting points

$$e^{-2\pi j \frac{i_1}{8}}, \quad i_1 = 0, \ldots, 7,$$

that is, points

$$1, \sqrt{2}\left(\frac{1}{2} - \frac{j}{2}\right), -j, \sqrt{2}\left(-\frac{1}{2} - \frac{j}{2}\right), -1, \sqrt{2}\left(-\frac{1}{2} + \frac{j}{2}\right), j, \sqrt{2}\left(\frac{1}{2} + \frac{j}{2}\right)$$

into $G(z_1)$ for z_1 results in the same values as the DFT before.

In case $n = 2$, the array is of the form

$$\begin{pmatrix}
g_{0,0} & g_{0,1} & \cdots & g_{0,d_2} & 0 & \cdots & 0 & g_{0,d_2} & \cdots & g_{0,1} \\
g_{1,0} & g_{1,1} & \cdots & g_{1,d_2} & 0 & \cdots & 0 & g_{1,d_2} & \cdots & g_{1,1} \\
\vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 & \vdots & \vdots & \vdots \\
g_{d_1,0} & g_{d_1,1} & \cdots & g_{d_1,d_2} & 0 & \cdots & 0 & g_{d_1,d_2} & \cdots & g_{d_1,1} \\
0 & 0 & \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & 0 \\
g_{d_1,0} & g_{d_1,1} & \cdots & g_{d_1,d_2} & 0 & \cdots & 0 & g_{d_1,d_2} & \cdots & g_{d_1,1} \\
\vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 & \vdots & \vdots & \vdots \\
g_{1,0} & g_{1,1} & \cdots & g_{1,d_2} & 0 & \cdots & 0 & g_{1,d_2} & \cdots & g_{1,1}
\end{pmatrix}$$

and of the size $N_1 \times N_2$. Similarly for $n > 2$.

Applying the Fourier transform to the n-D array, we got $N_1 \cdot N_2 \cdot \dots \cdot N_n$ samples on the unit n-circle. The stability condition can be based on the following fact: If (5) is positive for all $|z_1| = |z_2| = \dots = |z_n| = 1$ then all $N_1 \cdot N_2 \cdot \dots \cdot N_n$ samples on the unit n-circle obtained above by FFT are positive. The below lemma follows immediately.

Lemma 3.1. If the condition (2) is satisfied then all $N_1 \cdot N_2 \cdot \dots \cdot N_n$ samples of (5) on the unit n-circle obtained by FFT are positive.

The above condition is necessary, not sufficient, since we consider only the finite number of samples. However, our experiments confirm that if the number of samples is sufficiently high, say 10–50 times higher than d_i, this method analyses the stability correctly. In what follows, the approach described in this section is called *FFT-based method.*

4. APPLICATIONS

The above proposed method can be used in stability analysis of n-D systems. Two particular examples are shown in this section. The first one deals with stability of repetitive processes, the second one with stability of spatially invariant systems.

4.1. Stability of linear repetitive processes

Linear repetitive processes are a special class of n-D systems. Their comprehensive description is provided by [27]. A discrete linear repetitive process can be described by the state-space model of the form

$$\begin{aligned} x_{k+1}(p+1) &= A\, x_{k+1}(p) + B_0\, y_k(p) + B\, u_{k+1}(p) \\ y_{k+1}(p) &= C\, x_{k+1}(p) + D_0\, y_k(p) + D\, u_{k+1}(p) \end{aligned} \tag{9}$$

over $0 \le p \le \alpha$, α constant, $k \ge 0$, with the boundary conditions given by

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \ge 0, \\ y_0(p) &= f(p), \quad p = 0, 1, \dots, \alpha - 1, \end{aligned} \tag{10}$$

where d_{k+1} is a vector with constant entries and $f(p) \in \mathbb{R}^m$ is a vector whose entries are known function of p. Let shift operators z_1, z_2 in the along the pass (p) and pass-to-pass (k) direction, respectively, be defined as

$$\begin{aligned} x_k(p) &= z_1\, x_k(p+1) \\ y_k(p) &= z_2\, y_{k+1}(p). \end{aligned} \tag{11}$$

The following lemma holds, see, e. g. [24].

Lemma 4.1. A discrete linear repetitive process described by (9) is stable along the pass if, and only if,

$$c(z_1, z_2) = \det \left(\begin{bmatrix} I - z_1\, A & -z_1\, B_0 \\ -z_2\, C & I - z_2\, D_0 \end{bmatrix} \right) \neq 0 \tag{12}$$

for all $|z_1| \le 1$, $|z_2| \le 1$.

Using the same principle as in Sec. 2, we obtain (12) in the equivalent form of the set of conditions

1. for some β_1, β_2 such that $|\beta_r| = 1, r = 1, \dots, 2$ and for all $i, i = 1, 2,$

$$c \neq 0 \qquad \text{when } z_r = \beta_r, r \neq i \quad \text{and} \quad |z_i| \leq 1, \tag{13}$$

2.

$$c \neq 0 \qquad \text{when } |z_1| = |z_2| = 1. \tag{14}$$

The univariate conditions (13) can be verified by well-known methods. The condition (14) is multivariate one. Let

$$G(z_1, z_2) = c(z_1^{-1}, z_2^{-1})\, c(z_1, z_2). \tag{15}$$

Applying Lemma 2.1 we can state the following.

Lemma 4.2. A discrete linear repetitive process described by (11) is stable if (13) holds and (15) is strictly positive for all $|z_1| = |z_2| = 1$.

Proof. It follows from Lemmas 2.1 and 4.1. □

Now, we can formulate the following lemma.

Lemma 4.3. If a repetitive process described by (11) is stable then all $N_1 \cdot N_2$ samples of (15) on the unit bi-circle obtained by FFT are positive.

Once again we note that the above condition is necessary, not sufficient. However, if the number of samples is sufficiently high, say 10–50 times higher than N_i, this method analyses the stability correctly.

4.2. Stability of spatially invariant systems

In the second example, we shall concentrate on linear time-invariant spatially invariant systems with one temporal and two spatial variables. For instance, papers [7, 10, 11] deals with control and stabilisation of the spatial-temporal dynamics. Mathematically, such systems are described by partial differential equations. Stability of spatially invariant systems is discussed in, e. g., [4, 7, 9]. Here, we briefly recall the basic knowledge. The reader is referred to [4] and references therein for more details.

A $(2+1)$-D linear spatially invariant system discrete in both time and space can be described by the transfer function of the form

$$P(z, w) = \frac{b(z, z_1, z_2)}{a(z, z_1, z_2)}, \tag{16}$$

where a and b are bivariate polynomials, variable z corresponds to time delay and variables z_1, z_2 correspond to shift along the spatial coordinate axis. Since (16) describes a system causal in time and non-causal in space, a and b are *one-sided* in z and *two-sided* in z_1, z_2. Furthermore, for physical systems, it is reasonable to assume spatial symmetry, hence, a can be assumed in the form

$$a(z, z_1, z_2) = \sum_{k=0}^{m} \sum_{l_1=0}^{s_1} \sum_{l_2=0}^{s_2} a_{k,l_1,l_2}\, z^k \left(z_1^{l_1} + z_1^{-l_1}\right) \left(z_2^{l_2} + z_2^{-l_2}\right), \tag{17}$$

$a_{k,l_1,l_2} \in \mathbb{R}$, and (17) can be written in the form

$$a[z_1, z_2](z) = \sum_{k=0}^{m} a_k(z_1, z_2)\, z^k, \tag{18}$$

where $a_k(z_1, z_2), k = 0, \ldots, m$ are two-sided bivariate polynomials. Similarly for b.

A system described by (16) with a and b free of a common factor is structurally stable if and only if

$$a(z, z_1, z_2) \neq 0 \qquad \text{for all} \quad \{|z| \leq 1\} \cap \{|z_1| = 1\} \cap \{|z_2| = 1\}\,. \tag{19}$$

The criterion (19) is equivalent to the positiveness on the unit circle of the Schur–Cohn matrix corresponding to $a(z, z_1, z_2)$. The Schur–Cohn matrix [33] reads

$$H(z_1, z_2) = S_1^{\mathrm{T}} S_1 - S_2^{\mathrm{T}} S_2, \tag{20}$$

where

$$S_1 = \begin{pmatrix} a_0(z_1, z_2) & a_1(z_1, z_2) & \cdots & a_{m-1}(z_1, z_2) \\ 0 & a_0(z_1, z_2) & \cdots & a_{m-2}(z_1, z_2) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_0(z_1, z_2) \end{pmatrix},$$

$$S_2 = \begin{pmatrix} a_m(z_1, z_2) & a_{m-1}(z_1, z_2) & \cdots & a_1(z_1, z_2) \\ 0 & a_m(z_1, z_2) & \cdots & a_2(z_1, z_2) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_m(z_1, z_2) \end{pmatrix},$$

where $a_k(z_1, z_2), k = 0, \ldots, m$ are given in (18). The Schur–Cohn matrix (20) is the symmetric polynomial matrix of size m and can be written in the form

$$H(z_1, z_2) = \sum_{k_1=-2\,s_1}^{2\,s_1} \sum_{k_2=-2\,s_2}^{2\,s_2} H_{k_1,k_2}\, z_1^{k_1}\, z_2^{k_2}, \tag{21}$$

that is, in the form of (6). Hence, we can formulate the following lemma.

Lemma 4.4. The criterion (19) is satisfied if and only if the Schur–Cohn matrix (20) corresponding to (18) is positive definite on the unit circle, that is, $H(z_1, z_2) \succ 0$ for all $|z_1| = 1$, $|z_2| = 1$.

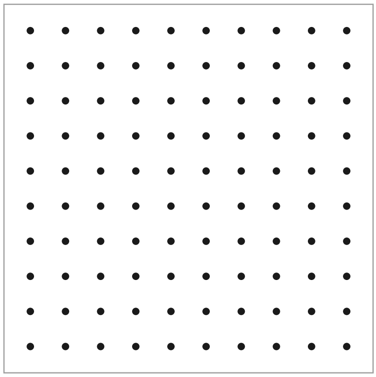


Fig. 1. A plate with nodes placed in the square grid.

For example, consider a system of a controlled heat conduction in a plate equipped with an array of temperature sensors and heaters as it is depicted in Figure 1. Suppose the input heat to be the input and the temperature to be the output. This system was described by the transfer function of the form (16) by [3]. Suppose a plate of the dimension 1 m×1 m equipped with 9 nodes×9 nodes and the sampling time period equal to 1 s. The transfer function corresponding to these values is

$$P = \frac{b(z, z_1, z_2)}{a(z, z_1, z_2)} = \frac{z}{1 - 0.4\,z - 0.15\,z\,(z_1 + z_1^{-1} + z_2 + z_2^{-1})}. \tag{22}$$

The Schur–Cohn matrix corresponding to $a(z, z_1, z_2)$ of (22) is scalar and reads

$$\begin{aligned} H_a(z_1, z_2) &= 0.75 + 0.12\,(z_1 + z_1^{-1} + z_2 + z_2^{-1}) \\ &\quad + 0.045\,(z_1\,z_2 + z_1\,z_2^{-1} + z_1^{-1}\,z_2 + z_1^{-1}\,z_2^{-1}) + 0.0225\,(z_1^2 + z_1^{-2} + z_2^2 + z_2^{-2}). \end{aligned} \tag{23}$$

Let $N_1 = N_2 = 8$ and form the $N_1 \times N_2$ array

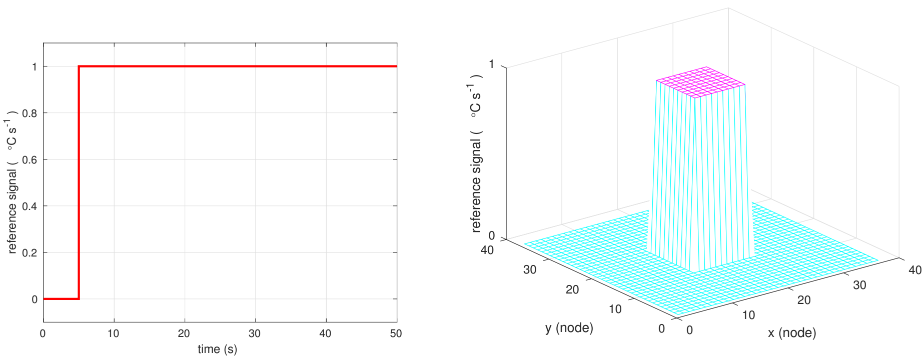
$$\begin{pmatrix} 0.75 & -0.12 & -0.0225 & 0 & 0 & 0 & -0.0225 & -0.12 \\ -0.12 & -0.045 & 0 & 0 & 0 & 0 & 0 & -0.045 \\ -0.0225 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0225 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.12 & -0.045 & 0 & 0 & 0 & 0 & 0 & -0.045 \end{pmatrix}.$$

Applying the FFT on the above array gives the $N_1 \times N_2$ array of samples

$$\begin{pmatrix} 0 & 0.168 & 0.51 & 0.762 & 0.84 & 0.762 & 0.51 & 0.168 \\ 0.168 & 0.3206 & 0.6253 & 0.84 & 0.9026 & 0.84 & 0.6253 & 0.3206 \\ 0.51 & 0.6253 & 0.84 & 0.9647 & 0.99 & 0.9647 & 0.84 & 0.6253 \\ 0.762 & 0.84 & 0.9647 & 0.9994 & 0.9874 & 0.9994 & 0.9647 & 0.84 \\ 0.84 & 0.9026 & 0.99 & 0.9874 & 0.96 & 0.9874 & 0.99 & 0.9026 \\ 0.762 & 0.84 & 0.9647 & 0.9994 & 0.9874 & 0.9994 & 0.9647 & 0.84 \\ 0.51 & 0.6253 & 0.84 & 0.9647 & 0.99 & 0.9647 & 0.84 & 0.6253 \\ 0.168 & 0.3206 & 0.6253 & 0.8400 & 0.9026 & 0.84 & 0.6253 & 0.3206 \end{pmatrix}.$$

One can see that there is a sample that is not positive. Hence, the system is not stable. Applying the criterion (19) directly leads to the same result. One can check that $a(z, z_1, z_2)$ of (22) has a root on the stability margin ($z_1 = 1, z_2 = 1$ and $z = 1$).

Numerical simulations confirm the above result. Consider the input signal given in Figure 2. The response of the system is given in Figure 3. Clearly, the system is not stable since the temperature goes to infinity.



(a) (b)

Fig. 2. Input signal of the system of temperature conduction in a plate, (a) at the middle of the plate, (b) in a plate from 5 s.

Now, consider the control scheme of Figure 4 and the controller

$$R = \frac{z}{1 + 0.5\,z + 0.15\,z\,(z_1 + z_1^{-1} + z_2 + z_2^{-1})}.$$

The closed-loop characteristic polynomial is

$$c(z, z_1, z_2) = -0.0225\,z^2\,(z_1 + z_1^{-1} + z_2 + z_2^{-1})^2 - 0.135\,z^2\,(z_1 + z_1^{-1} + z_2 + z_2^{-1}) + 0.8\,z^2 + 0.1\,z + 1$$

and the corresponding Schur–Cohn matrix is given by (20) with

$$S_1 = \begin{pmatrix} 1 & 0.1 \\ 0 & 1 \end{pmatrix}, \qquad S_2 \begin{pmatrix} * & 0.1 \\ 0 & * \end{pmatrix},$$

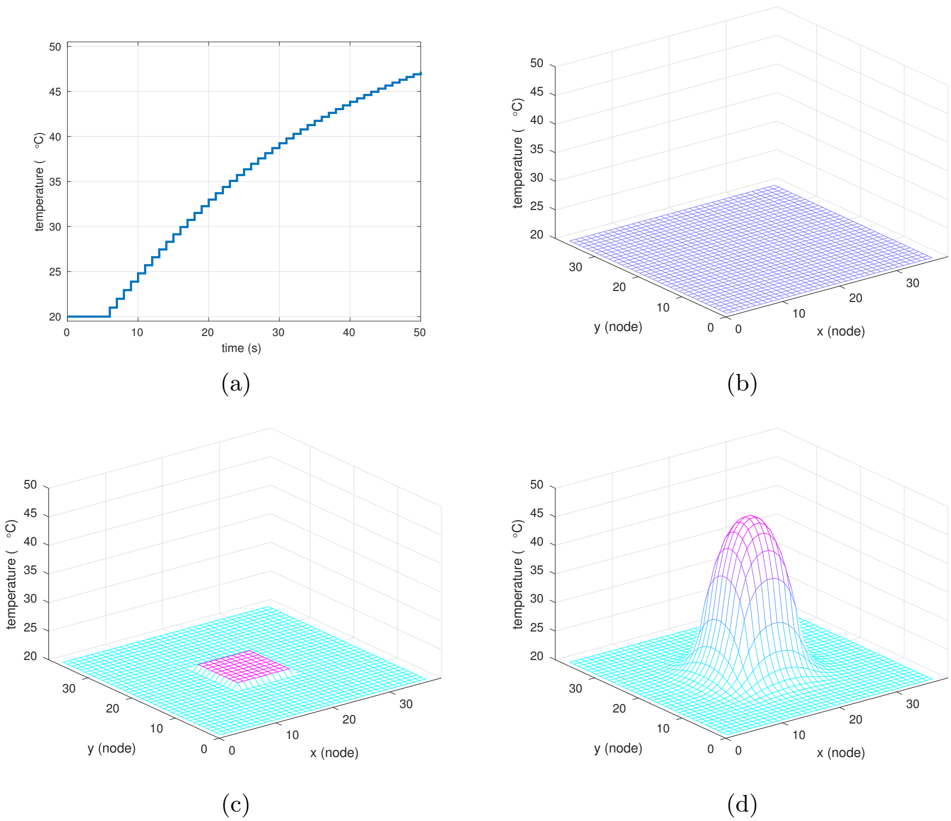


Fig. 3. Response of the system to the input signal of Figure 2, temperature (a) at the middle of the plate, (b) at the time 0 s, (c) at the time 6 s, (d) at the time 50 s.

where $* = -0.0225\left(z_1 + z_1^{-1} + z_2 + z_2^{-1}\right)^2 - 0.135\left(z_1 + z_1^{-1} + z_2 + z_2^{-1}\right) + 0.8$, so it 2×2 matrix. The stability test can be performed as follows. Three polynomials arising in the (symmetric) Schur–Cohn matrix can be sampled as independent scalars using the 2-D

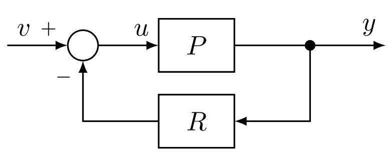
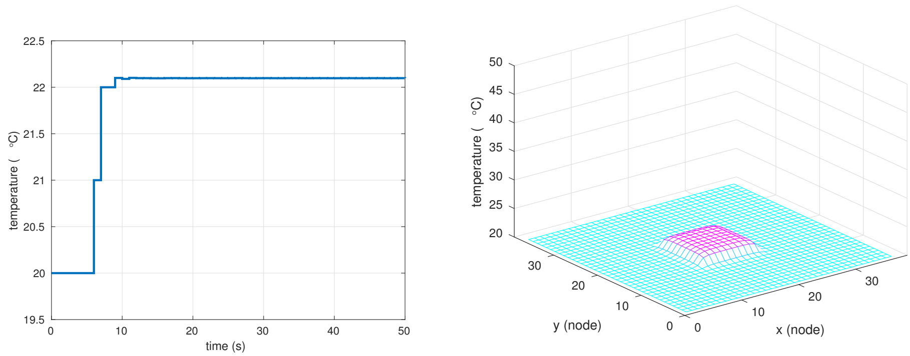


Fig. 4. Standard feedback configuration.

FFT algorithm. All samples can be given back into matrices. To obtain the resulting feedback system stable, all $N_1 \cdot N_2$ matrices have to be definitely positive.

One can insure that control P by R leads to stable closed-loop system. Our numerical simulations confirms this fact, see Figure 5.



(a) (b)

Fig. 5. Response of the controlled system to the input signal of Figure 2, (a) temperature at the middle of the plate, (b) temperature at the time 50 s.

5. NUMERICAL EXPERIMENTS

In this section we perform experiments and tests of the above described approaches. The FFT-based method proposed in the previous section is compared with the SDP-based method by [15] in the sense of the computational time. We also show how the correctness of results returned by FFT-based method depends on numbers of samples $N_1, N_2, \ldots, N_n$.

5.1. Computational time

SDP-based method and the FFT-based method are compared in the sense of the computational time. Other attributes are not taken into account in this subsection. Stable polynomials $A(z_1, z_2)$ in two variables z_1 and z_2 are generated. The corresponding polynomials (5) is computed and its positiveness on the unit bi-circle is checked. This is repeated many times and the mean time needed for the calculation is taken. The FFT-based method is applied in the first case with 2^5-point FFT, in the second one with 2^7-point FFT.

The results of the experiments are shown in Table 1. The degrees in z_1 and z_2 of the polynomial are given in the first and the second columns of the table, respectively. Computing times in seconds taken by the SDP-based method, 2^5-point FFT and 2^7-point FFT are shown in the next three columns. The star (*) means that computation takes over 60 seconds.

$\deg_z$ (−)	$\deg_v$ (−)	SDP (s)	FFT (s) $N = 2^5$	FFT (s) $N = 2^7$
1	1	0.28	0.03	0.16
2	2	0.28	0.03	0.17
5	5	1.3	0.04	0.17
10	10	*	0.05	0.18
10	1	0.33	0.03	0.17
10	2	1.1	0.04	0.17
10	5	24	0.05	0.18

Tab. 1. Computational times the methods.

All experiments were made on a supercomputer within the Matlab Software by [23]. The implementation of the SDP-based algorithm uses SeDuMi [30] and Yalmip [22]. Standard Matlab implementation of FFT was used for implementation of the method based on FFT. The times were taken using `tic` and `toc` Matlab commands.

5.2. False results for small number of samples

This subsection shows that FFT-based method can return false result if the polynomial is sampled at too small number of points. Consider the polynomial

$$G(z_1) = 4 + (z_1 + z_1^{-1}) + (z_1^2 + z_1^{-2}) - (z_1^3 + z_1^{-3}).$$

The graph of $G(z_1)|_{z_1=e^{j\omega}}$ for $0 \le \omega \le 2\pi$ is in Figure 6 and Figure 7, where it is depicted that $G(z_1)$ is not positive for all values z_1 on the unit circle. Choose the number of interpolation points $N_1 = 8$. It can be seen from Figure 6, that all 8 samples are positive. For this value of N_1, the above polynomial is classified as stable. This is not the correct result.

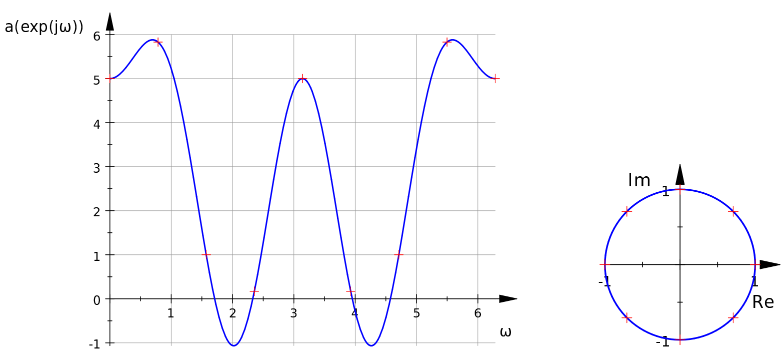


Fig. 6. Graph of $G(z_1)$ for all $|z_1| = 1$, $N = 8$ interpolation points are marked by the red crosses.

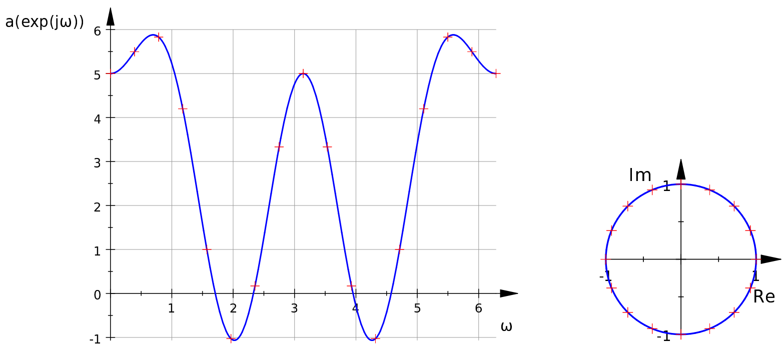


Fig. 7. Graph of $G(z_1)$ for all $|z_1| = 1$, $N = 16$ interpolation points are marked by the red crosses.

Now, choose $N_1 = 16$. One can see from Figure 7, that there are negative samples. In this case the polynomial is classified correctly as unstable.

6. CONCLUSIONS

A numerical test of positiveness of symmetric polynomial was presented in this paper. The test is simply based on sampling the matrix polynomial using the fast Fourier transform. It was shown that this method is dramatically faster than one proposed in the literature based on the semi-definite programming expression. Our method returns results within a few milliseconds. The adopted method required a longer time for computing in all considered cases.

In spite of the fact that the proposed method provides necessary but not sufficient condition it can find applications in stability analysis of multidimensional systems, repetitive processes, or spatially distributed systems. It can serve as a very fast numerical stability test. The number of variables is not limited since for test of positiveness on the unit n-circle the algorithm for n-dimensional FFT is available.

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