Brazilian Journal of Probability and Statistics
2015, Vol. 29, No. 4, 878–896
DOI: 10.1214/14-BJPS251

Generalizations of some probability inequalities and L^p convergence of random variables for any monotone measure

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Abstract. This paper has three specific aims. First, some probability inequalities, including Hölder's inequality, Lyapunov's inequality, Minkowski's inequality, concentration inequalities and Fatou's lemma for Choquet-like expectation based on a monotone measure are shown, extending previous work of many researchers. Second, we generalize some theorems about the convergence of sequences of random variables on monotone measure spaces for Choquet-like expectation. Third, we extend the concept of uniform integrability for Choquet-like expectation. These results are useful for the solution of various problems in machine learning and made it possible to derive new efficient algorithms in any monotone system. Corresponding results are valid for capacities, the usefulness of which has been demonstrated by the rapidly expanding literature on generalized probability theory.

1 Choquet-like expectation

Choquet (1954) extended the idea of probability measure to the concept of "capacity" so defining the "Choquet expectation", which has been shown to have many applications in statistics, economics, and finance, among other fields (Huber and Strassen, 1973, 1974; Maccheroni and Marinacci, 2005; Wasserman and Kadane, 1990). For example, a new concept of nonadditive limit laws based on Choquet expectation has been recently discussed in Maccheroni and Marinacci (2005). In Bayesian statistics, as imprecise probabilities, Bayes' theorem for Choquet capacities was proposed by Wasserman and Kadane (1990). Furthermore, Huber and Strassen (1973, 1974) have established the minimax tests and the Neyman–Pearson lemma for capacities.

As a rather new theory, pseudo-analysis has proved itself to be a vast source of powerful tools that are being successfully applied in many mathematical theories as well as in various practical problems (see Maslov and Samborskij, 1992; Pap, 2002, 2005; Pap and Vivona, 2000). Mesiar (1995) introduced two classes of Choquet-like integrals based on pseudo-analysis. The first class is called "Choquet-like expectation" which further generalizes the Choquet expectation

Key words and phrases. Capacities, probability inequalities, Choquet-like expectation.
Received July 2012; accepted May 2014.

concept to any monotone measure based on pseudo-analysis. In this class, pseudo-operations are defined by a continuous strictly increasing function g. Notice that the main motivation for Choquet-like expectation lies in the possibility of expanding the applicability of Choquet expectation by combining the properties of pseudo-analysis. Another one concerns the Choquet-like integrals based on the special operator "sup" and a pseudo-multiplication $\otimes$. Note that the second class is not an extension of the Lebesgue integral.

Probability inequalities are at the heart of the mathematical analysis of various problems (Boucheron et al., 2003; Finner, 1992; Matkowski, 1996). For instance, concentration inequalities are important in the mathematical analysis of various problems in machine learning for finding new efficient algorithms (Boucheron et al., 2003). Probability inequalities play an important role in various proofs of limit theorems. However, the additivity assumption in the proofs of these problems seems to be illogical in many uncertain phenomena (Maccheroni and Marinacci, 2005). The aim of this paper is to generalize some probability inequalities of random variables for any monotone measures using the Choquet-like expectation.

The rest of the paper is organized as follows. In the next section, we give some basic concepts that will be used in this paper. In Section 2, we present generalizations of some probability inequalities, including Hölder's inequality, Lyapunov's inequality, Minkowski's inequality, concentration inequalities and Fatou's lemma based on a monotone measure. Generalizations of convergence of sequences of random variables are given in Section 3.

For a fixed measurable space $(\Omega, \mathcal{F})$, that is, a nonempty set Ω equipped with a σ-algebra $\mathcal{F}$, recall that a random variable $X : \Omega \to \mathbb{R}$ is called $\mathcal{F}$-measurable if, for each $B \in \mathcal{B}(\mathbb{R})$, the σ-algebra of Borel subsets of $\mathbb{R}$, the preimage $X^{-1}(B)$ is an element of $\mathcal{F}$.

1.1 Definitions and fundamental properties

Definition 1.1. A monotone measure μ on a measurable space $(\Omega, \mathcal{F})$ is a set function $\mu : \mathcal{F} \to [0, \infty]$ satisfying:

(i) $\mu(\varnothing) = 0$,
(ii) $\mu(\Omega) > 0$,
(iii) $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. The triple $(\Omega, \mathcal{F}, \mu)$ is also called a *monotone measure space*.

A monotone measure μ is called *real* if $\|\mu\| = \mu(\Omega) < \infty$ and μ is said to be an additive measure if $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \varnothing$. We call a monotone measure $\mu : \mathcal{F} \to [0, 1]$ a *capacity*, if $\mu(\Omega) = 1$ (Choquet, 1954). Note that the monotone measure μ satisfying:

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$$

is also *submodular (2-alternating)* (Choquet, 1954). Such submodular measures have interesting applications to probability and statistics. For example, in capacities, Huber and Strassen (1973) obtained that the submodular property is necessary and sufficient for generalizing the Neyman–Pearson lemma to sets of probabilities.

A real monotone measure μ on $\mathcal{F}$ is *continuous from below* if $\mu(B_n) \nearrow \mu(B)$ for all sequences $B_n \in \mathcal{F}$, $B_n \nearrow B$. μ is *continuous from above* if $\mu(B_n) \searrow \mu(B)$ for all sequences $B_n \in \mathcal{F}$, $B_n \searrow B$. A monotone measure being continuous both from below and from above is called *continuous*. In particular, a real monotone measure μ is *order-continuous* if $\lim_{n\to\infty} \mu(A_n) = 0$ whenever $A_n \searrow \varnothing$. Throughout this paper, $\mathbb{I}_A$ denotes the indicator function of the set A.

If $\upsilon : \mathcal{F} \to [0, \infty]$ is a submodular continuous measure, then the triple $(\Omega, \mathcal{F}, \upsilon)$ is also called a *submodular continuous (SC-) measure space*.

Given the monotone space $(\Omega, \mathcal{F}, \mu)$, we shall denote by ω any element of Ω and we put $\{X \geq t\} = \{\omega : X(\omega) \geq t\}$ for any $t > 0$. The Choquet expectation of X over $A \in \mathcal{F}$ w.r.t. the real monotone measure μ is defined as

$$\mathbb{E}_C^{\mu}[X\mathbb{I}_A] = \int_0^{\infty} \mu\big(A \cap \{X \geq t\}\big)\, dt - \int_{-\infty}^{0} \big[\|\mu\| - \mu\big(A \cap \{X \geq t\}\big)\big]\, dt. \tag{1.1}$$

In particular, if $A = \Omega$, then

$$\mathbb{E}_C^{\mu}[X] = \int_0^{\infty} \mu\big(\{X \geq t\}\big)\, dt - \int_{-\infty}^{0} \big[\|\mu\| - \mu\big(\{X \geq t\}\big)\big]\, dt.$$

Note that for nonnegative X, the Choquet expectation of X over $A \in \mathcal{F}$ is defined as

$$\mathbb{E}_C^{\mu}[X\mathbb{I}_A] = \int_0^{\infty} \mu\big(A \cap \{X \geq t\}\big)\, dt,$$

which works for any monotone measure μ.

A Choquet-like expectation (Mesiar, 1995) may be based on pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$ defined as follows.

Definition 1.2 (Mesiar, 1995). An operation $\oplus : [0, \infty]^2 \to [0, \infty]$ is called a pseudo-addition if the following properties are satisfied:

(P1) $a \oplus 0 = 0 \oplus a = a$ (neutral element);
(P2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);
(P3) $a \leq c$ and $b \leq d$ imply that $a \oplus b \leq c \oplus d$ (monotonicity);
(P4) $a_n \to a$ and $b_n \to b$ imply that $a_n \oplus b_n \to a \oplus b$ (continuity).

Definition 1.3 (Mesiar, 1995). Let $\oplus$ be a given pseudo-addition on $[0, \infty]$. Another binary operation $\otimes$ on $[0, \infty]$ is said to be a pseudo-multiplication corresponding to $\oplus$ if the following properties are satisfied:

(M1) $a \otimes (x \oplus y) = (a \otimes x) \oplus (a \otimes y)$;
(M2) $a \leq b$ implies $(a \otimes x) \leq (b \otimes x)$ and $(x \otimes a) \leq (x \otimes b)$;

(M3) $a \otimes x = 0 \Leftrightarrow a = 0$ or $x = 0$;
(M4) $\exists e \in (0, \infty]$ such that $e \otimes x = x \otimes e = x$ for any $x \in [0, \infty]$ (i.e., there exists the neutral element e);
(M5) $a_n \to a \in (0, \infty)$ and $x_n \to x$ imply $(a_n \otimes x_n) \to (a \otimes x)$ and $\infty \otimes x = \lim_{a \to \infty}(a \otimes x)$;
(M6) $a \otimes x = x \otimes a$;
(M7) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$.

As it is shown by Mesiar (1995), if $\otimes$ is a pseudo-multiplication corresponding to a given pseudo-addition $\oplus$ fulfilling axioms (M1)–(M7) and if its identity element e is not an idempotent of $\oplus$, then there is a unique continuous strictly increasing function $g : [0, \infty] \to [0, \infty]$ with $g(0) = 0$ and $g(\infty) = \infty$, such that $g(e) = 1$ and

$$a \oplus b = g^{-1}\big(g(a) + g(b)\big) \qquad \oplus \text{ is called a } g\text{-addition,}$$

$$a \otimes b = g^{-1}\big(g(a) \cdot g(b)\big) \qquad \odot \text{ is called a } g\text{-multiplication.}$$

On the other hand, if the identity element e of the pseudo-multiplication is also an idempotent of $\oplus$ (i.e., $e \oplus e = e$), then $\oplus = \vee$ (= sup, i.e., the logical addition).

For $x \in [0, \infty]$ and $p \in (0, \infty)$, we will introduce the pseudo-power $x_{\otimes}^{(p)}$ as follows: If $p = n$ is a natural number, then $x_{\otimes}^{(n)} = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n\text{-times}}$. If p is not a natural number, then the corresponding power is defined by $x_{\otimes}^{(p)} = \sup\{y_{\otimes}^{(m)} | y_{\otimes}^{(n)} \le x,$ where m, n are natural numbers such that $\frac{m}{n} \le p\}$. Evidently, if $x \otimes y = g^{-1}(g(x) \cdot g(y))$, then $x_{\otimes}^{(p)} = g^{-1}(g^p(x))$. If $\oplus$ is a pseudo-addition with $\oplus \ge +$ (the usual addition) and $\otimes$ is a corresponding pseudo-multiplication with $\otimes \ge \cdot$ (the usual multiplication), we call such a pair of $(\oplus, \otimes)$ a magnifying pair of pseudo-arithmetic operations. Clearly, $\oplus \ge +$ means that $g(c) + g(d) \ge g(c + d)$ for any c, d, that is, that g is subadditive. Similarly, $\otimes \ge \cdot$ means that $g(c)g(d) \ge g(cd)$, that is, that g is submultiplicative. Accordingly, the function $h : [0, \infty] \to [0, \infty]$ given by $h(x) = \log(g(e^x))$ satisfies

$$h(x) + h(y) = \log\big(g\big(e^x\big) \cdot g\big(e^y\big)\big) \ge \log\big(g\big(e^x e^y\big)\big) = h(x + y),$$

that is, h is subadditive. A typical example of such a generator is $g(x) = x^p$ with $p \in (0, 1]$, that is, $a \oplus b = (a^p + b^p)^{1/p}$ and $\otimes$ is the usual multiplication.

There are two classes of Choquet-like integrals. The first we call the Choquet-like expectation, based on a g-addition and g-multiplicatio. The second we call the $\mathbb{S}_{\mu}^{\otimes}$ integral, based on sup and a corresponding pseudo multiplication.

Definition 1.4. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mu : \mathcal{F} \to [0, \infty]$ be a monotone measure. Let $\oplus$ and $\otimes$ be generated by a generator g. The Choquet-like expectation of a nonnegative measurable function X over $A \in \mathcal{F}$ w.r.t. the real monotone measure μ can be represented as

$$\mathbb{E}^{\mu}_{Cl,g}[X\mathbb{I}_A] = g^{-1}(\mathbb{E}^{g(\mu)}_{C}[g(X)\mathbb{I}_A]) = g^{-1}\left(\int_0^\infty g\mu(A \cap \{g(X) \geq t\})\,dt\right).$$

In particular, if $A = \Omega$, then

$$\mathbb{E}^{\mu}_{Cl,g}[X] = g^{-1}(\mathbb{E}^{g(\mu)}_{C}[g(X)]). \tag{1.2}$$

We define $\zeta^p_{Cl,g} = \{X \,|\, (\mathbb{E}^{\mu}_{Cl,g}[|X|^{(p)}_{\otimes}])^{(1/p)}_{\otimes} < \infty\}$ and $\zeta^p_C = \{X \,|\, (\mathbb{E}^{\mu}_C[|X|^p])^{1/p} < \infty\}$ for all $p \geq 1$.

Definition 1.5 (Mesiar, 1995). Let $\otimes$ be a pseudo-multiplication corresponding to sup and fulfilling (M1)–(M7). Then the $\mathbb{S}^{\otimes}_{\mu}$ integral of a nonnegative measurable function X w.r.t. the real monotone measure μ can be represented as

$$\mathbb{S}^{\otimes}_{\mu}[X] = \sup_{a\in[0,\infty]} (a \otimes \mu(\{X \geq a\}).$$

Remark 1.6. Some nonadditive integrals such as the Sugeno integral (Sugeno, 1974), the Shilkret integral (Shilkret, 1971) and the seminormed integral (Suárez García and Gil Álvarez, 1986) are special cases of $\mathbb{S}^{\otimes}_{\mu}$ integral. The $\mathbb{S}^{\otimes}_{\mu}$ integral is the Sugeno integral whenever $\otimes = \min$. If $\otimes$ is the standard product, then the Shilkret integral can be recognized. Restricting now to the unit interval [0, 1] we shall consider the measurable function $X : \Omega \to [0, 1]$ with $\|\mu\| = 1$. In this case, we have the restriction of the pseudo-multiplication $\otimes$ to $[0, 1]^2$ (called a semicopula $\circledast$ (Bassan and Spizzichino, 2005; Durante and Sempi, 2005)). The $\mathbb{S}^{\otimes}_{\mu}$ integral on the [0, 1] scale related to the semicopula $\circledast$ was called the seminormed integral. Recently, Agahi et al. and others proved some inequalities for the seminormed integral which are in the second class of Choquet-like integrals, see Agahi et al. (2012) and Ouyang and Mesiar (2009). Notice that in the present paper, we focus on the first class of Choquet-like integrals, that is, Choquet-like expectation.

Definition 1.7 (Maccheroni and Marinacci, 2005). Random variables ξ and ζ are called comonotonic if

$$[\xi(\omega) - \xi(\omega')][\zeta(\omega) - \zeta(\omega')] \geq 0 \qquad \forall \omega, \omega' \in \Omega.$$

The following results summarize the basic properties of the Choquet-like expectation (Mesiar, 1995).

Proposition 1.8. *The Choquet-like expectation* (1.2) *has the following properties*:

(P1) $X \leq Y$ *implies that* $\mathbb{E}^{\mu}_{Cl,g}[X] \leq \mathbb{E}^{\mu}_{Cl,g}[Y]$ (*monotonicity*);

(P2) $\mathbb{E}^{\mu}_{Cl,g}[X \oplus Y] = \mathbb{E}^{\mu}_{Cl,g}[X] \oplus \mathbb{E}^{\mu}_{Cl,g}[Y]$ *whenever* X *and* Y *are comonotonic* (*comonotonic* $\oplus$-*additivity*);

(P3) $\mathbb{E}^{\mu}_{Cl,g}[c \otimes X] = c \otimes \mathbb{E}^{\mu}_{Cl,g}[Y]$ *for all* $c > 0$ ($\otimes$-*homogeneity*);
(P4) $\mathbb{E}^{\mu}_{Cl,g}[X]$ *is* $\oplus$-*additivity if and only if* μ *is* $\oplus$-*additive* (*coincidentity*).

2 Some probability inequalities

The present section aims to provide some advanced type inequalities involving Choquet-like expectation. In this section, we consider $\oplus$ and $\otimes$ to be generated by g.

2.1 Key inequalities

2.1.1 *Hölder's inequality.*

Theorem 2.1. *Let* X, Y *be two nonnegative random variables. For the Choquet-like expectation* (1.2), *the Hölder inequality*

$$\mathbb{E}^{\mu}_{Cl,g}[X \otimes Y] \le \big(\mathbb{E}^{\mu}_{Cl,g}[X^{(p)}_{\otimes}]\big)^{(1/p)}_{\otimes} \otimes \big(\mathbb{E}^{\mu}_{Cl,g}[Y^{(q)}_{\otimes}]\big)^{(1/q)}_{\otimes} \tag{2.1}$$

holds if $g(\mu)$ *is submodular and* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p > 1$.

Proof. Observe that

$$\begin{aligned}\mathbb{E}^{\mu}_{Cl,g}[X \otimes Y] &= g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}[g(g^{-1}((g\circ X)(g\circ Y)))]\big)\\ &= g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}[(g\circ X)(g\circ Y)]\big).\end{aligned} \tag{2.2}$$

From (2.2) and using the Hölder inequality for Choquet expectation (Mesiar et al., 2010), we have

$$\begin{aligned}
&g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}[(g\circ X)(g\circ Y)]\big)\\
&\quad\le \big[g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[(g\circ X)^p])^{1/p} \times (\mathbb{E}^{g(\mu)}_{C}[(g\circ Y)^q])^{1/q}\big)\big]\\
&\quad= \big[g^{-1}\big(g\big(g^{-1}((\mathbb{E}^{g(\mu)}_{C}[(g\circ X)^p])^{1/p})\big) \times g\big(g^{-1}((\mathbb{E}^{g(\mu)}_{C}[(g\circ Y)^q])^{1/q})\big)\big)\big]\\
&\quad= g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}(g\circ X)^p)^{1/p}\big) \otimes g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}(g\circ Y)^q)^{1/q}\big)\\
&\quad= \big[g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g(g^{-1}((g\circ X)^p))])^{1/p}\big)\\
&\qquad\quad \otimes g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g(g^{-1}((g\circ Y)^q))])^{1/q}\big)\big]\\
&\quad= g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g(X^{(p)}_{\otimes})])^{1/p}\big) \otimes g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g(Y^{(q)}_{\otimes})])^{1/q}\big)\\
&\quad= \big[g^{-1}\big((g(g^{-1}(\mathbb{E}^{g(\mu)}_{C}[g(X^{(p)}_{\otimes})])))^{1/p}\big)\\
&\qquad\quad \otimes g^{-1}\big((g(g^{-1}(\mathbb{E}^{g(\mu)}_{C}[g(Y^{(q)}_{\otimes})])))^{1/q}\big)\big]
\end{aligned}$$

$$= g^{-1}((g(\mathbb{E}^{\mu}_{Cl,g}[X^{(p)}_{\otimes}]))^{1/p}) \otimes g^{-1}((g(\mathbb{E}^{\mu}_{Cl,g}[Y^{(q)}_{\otimes}]))^{1/q})$$
$$= (\mathbb{E}^{\mu}_{Cl,g}[X^{(p)}_{\otimes}])^{(1/p)}_{\otimes} \otimes (\mathbb{E}^{\mu}_{Cl,g}[Y^{(q)}_{\otimes}])^{(1/q)}_{\otimes}.$$

Hence, (2.1) is valid. This completes the proof. □

Corollary 2.2. *Let $X_1, X_2, \dots, X_n$ be random variables. For the Choquet-like expectation (1.2), the Hölder inequality*

$$\mathbb{E}^{\mu}_{Cl,g}\left[\bigotimes_{i=1}^{n} |X_i|\right] \le \bigotimes_{i=1}^{n} (\mathbb{E}^{\mu}_{Cl,g}[|X_i|^{(p_i)}_{\otimes}])^{(1/p_i)}_{\otimes}$$

holds if $g(\mu)$ is submodular and $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $p_i > 1$, $n \ge 2$.

Theorem 2.3. *Let X, Y be two comonotonic nonnegative random variables defined on a real monotone measure space $(\Omega, \mathcal{F}, \mu)$. For the Choquet-like expectation (1.2), the Hölder inequality*

$$\mathbb{E}^{\mu}_{Cl,g}[X \otimes Y] \le (\mathbb{E}^{\mu}_{Cl,g}[X^{(p)}_{\otimes}])^{(1/p)}_{\otimes} \otimes (\mathbb{E}^{\mu}_{Cl,g}[Y^{(q)}_{\otimes}])^{(1/q)}_{\otimes}$$

holds if $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Proof. Recall the Hölder inequality for Choquet expectation (Zhu and Ouyang, 2011) which asserts that if X, Y are comonotonic and nonnegative, then

$$\mathbb{E}^{g(\mu)}_{C}[XY] \le (\mathbb{E}^{g(\mu)}_{C}[X^p])^{1/p} (\mathbb{E}^{g(\mu)}_{C}[Y^q])^{1/q}, \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. From (2.2) and using (2.3), we complete the proof via a similar argument as in the proof of Theorem 2.1. □

Corollary 2.4 (Lyapunov's inequality). *Let X be a nonnegative random variable defined on a real monotone measure space $(\Omega, \mathcal{F}, \mu)$. For $s \ge r \ge 1$ and the Choquet-like expectation (1.2), the following inequality*

$$(\mathbb{E}^{\mu}_{Cl,g}[X^{(r)}_{\otimes}])^{(1/r)}_{\otimes} \le (\|\mu\|)^{((s-r)/rs)}_{\otimes} \otimes (\mathbb{E}^{\mu}_{Cl,g}[X^{(s)}_{\otimes}])^{(1/s)}_{\otimes} \tag{2.4}$$

holds.

2.1.2 *Minkowski's inequality.*

Theorem 2.5. *Let X, Y be two nonnegative random variables. For $s \ge 1$ and the Choquet-like expectation (1.2), the Minkowski inequality*

$$(\mathbb{E}^{\mu}_{Cl,g}[(X \oplus Y)^{(s)}_{\otimes}])^{(1/s)}_{\otimes} \le (\mathbb{E}^{\mu}_{Cl,g}[X^{(s)}_{\otimes}])^{(1/s)}_{\otimes} \oplus (\mathbb{E}^{\mu}_{Cl,g}[Y^{(s)}_{\otimes}])^{(1/s)}_{\otimes} \tag{2.5}$$

holds if $g(\mu)$ is submodular.

Proof. Observe that

$$\begin{aligned}(\mathbb{E}^{\mu}_{Cl,g}[(X \oplus Y)^{(s)}])^{(1/s)}_{\otimes} &= g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g((X \oplus Y)^{(s)}_{\otimes})])^{1/s}\big)\\ &= g^{-1}\big((\mathbb{E}^{g(\mu)}_{C} g(g^{-1}((g(X \oplus Y))^{s})))^{1/s}\big)\\ &= g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[((g \circ X) + (g \circ Y))^{s}])^{1/s}\big).\end{aligned}$$

By using the Minkowski inequality for Choquet expectation (Mesiar et al., 2010), we have

$$\begin{aligned}&g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[((g \circ X) + (g \circ Y))^{s}])^{1/s}\big)\\ &\quad\le g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[(g \circ X)^{s}])^{1/s} + (\mathbb{E}^{g(\mu)}_{C}[(g \circ Y)^{s}])^{1/s}\big)\\ &\quad= g^{-1}\big(g(g^{-1}((\mathbb{E}^{g(\mu)}_{C}[(g \circ X)^{s}])^{1/s})) + g(g^{-1}((\mathbb{E}^{g(\mu)}_{C}[(g \circ Y)^{s}])^{1/s}))\big)\\ &\quad= g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[(g \circ X)^{s}])^{1/s}\big) \oplus g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[(g \circ Y)^{s}])^{1/s}\big)\\ &\quad= \big[g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g(g^{-1}((g \circ X)^{s}))])^{1/s}\big)\\ &\qquad\oplus g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g(g^{-1}((g \circ Y)^{s}))])^{1/s}\big)\big]\\ &\quad= g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g(X^{(s)}_{\otimes})])^{1/s}\big) \oplus g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[g(Y^{(s)}_{\otimes})])^{1/s}\big)\\ &\quad= \big[g^{-1}\big((g(g^{-1}(\mathbb{E}^{g(\mu)}_{C}[g(X^{(s)}_{\otimes})])))^{1/s}\big) \oplus g^{-1}\big((g(g^{-1}(\mathbb{E}^{g(\mu)}_{C}[g(Y^{(s)}_{\otimes})])))^{1/s}\big)\big]\\ &\quad= g^{-1}\big((g(\mathbb{E}^{\mu}_{Cl,g}[X^{(s)}_{\otimes}]))^{1/s}\big) \oplus g^{-1}\big((g(\mathbb{E}^{\mu}_{Cl,g}[Y^{(s)}_{\otimes}]))^{1/s}\big)\\ &\quad= (\mathbb{E}^{\mu}_{Cl,g}[X^{(s)}_{\otimes}])^{(1/s)}_{\otimes} \oplus (\mathbb{E}^{\mu}_{Cl,g}[Y^{(s)}_{\otimes}])^{(1/s)}_{\otimes}.\end{aligned}$$

Hence, (2.5) is valid. This completes the proof. □

Theorem 2.6. *Let X, Y be two comonotonic nonnegative random variables defined on a real monotone measure space $(\Omega, \mathcal{F}, \mu)$. For $s \ge 1$ and the Choquet-like expectation (1.2), the Minkowski inequality*

$$(\mathbb{E}^{\mu}_{Cl,g}[(X \oplus Y)^{(s)}_{\otimes}])^{(1/s)}_{\otimes} \le (\mathbb{E}^{\mu}_{Cl,g}[X^{(s)}_{\otimes}])^{(1/s)}_{\otimes} \oplus (\mathbb{E}^{\mu}_{Cl,g}[Y^{(s)}_{\otimes}])^{(1/s)}_{\otimes}$$

holds.

Proof. Using the Minkowski inequality for Choquet expectation (Zhu and Ouyang, 2011), we complete the proof via a similar argument as in the proof of Theorem 2.5. □

2.1.3 *Concentration inequalities.*

Theorem 2.7 (Markov's inequality). *Let X be a nonnegative random variable defined on a real monotone measure space $(\Omega, \mathcal{F}, \mu)$. For two arbitrary positive* *constants p, k and Choquet-like expectation (1.2), the Markov inequality*

$$k_{\otimes}^{(p)} \otimes \mu(\{\omega \in A : X(\omega) \geq k\}) \leq \mathbb{E}_{Cl,g}^{\mu}[X_{\otimes}^{(p)} \mathbb{I}_A] \tag{2.6}$$

holds for all $A \in \mathcal{F}$.

Proof. It suffices to define $A^* = \{\omega \in A : X(\omega) \geq k\}$. Therefore,

$$\begin{aligned}\mathbb{E}_{Cl,g}^{\mu}[X_{\otimes}^{(p)} \mathbb{I}_A] &\geq g^{-1}(\mathbb{E}_{C}^{g(\mu)}[(g(X))^p \mathbb{I}_{A^*}]) \geq g^{-1}((g(k))^p \mathbb{E}_{C}^{g(\mu)}[\mathbb{I}_{A^*}]) \\ &= g^{-1}((g(k))^p g(\mu(A^*))) = k_{\otimes}^{(p)} \otimes \mu(A^*).\end{aligned}$$

If Φ is a strictly monotonically increasing nonnegative-valued function and t is a real number, then by using the Markov type inequality (2.6), we have

$$\Phi(t) \otimes \mu(X \geq t) \leq \mathbb{E}_{Cl,g}^{\mu}[\Phi(X)].$$

Also, if X is an arbitrary random variable and $t > 0$, then

$$t_{\otimes}^{(2)} \otimes \mu(|X - \mathbb{E}_{Cl,g}^{\mu}[X]| \geq t) \leq \mathbb{E}_{Cl,g}^{\mu}[|X - \mathbb{E}_{Cl,g}^{\mu}[X]|_{\otimes}^{(2)}],$$

which is a generalization of *Chebyshev's inequality*. More generally, for any $p > 0$ we have

$$t_{\otimes}^{(p)} \otimes \mu(|X - \mathbb{E}_{Cl,g}^{\mu}[X]| \geq t) \leq \mathbb{E}_{Cl,g}^{\mu}[|X - \mathbb{E}_{Cl,g}^{\mu}[X]|_{\otimes}^{(p)}].$$

A related idea is at the basis of Chernof's bounding method. For an arbitrary positive number s and any $t > 0$,

$$e^{st} \otimes \mu(X \geq t) \leq \mathbb{E}_{Cl,g}^{\mu}[e^{sX}]. \qquad \square$$

Theorem 2.8. *Let Y be a random variable with $\mathbb{E}_C^{\mu}[Y] = 0$ and $a \leq Y \leq b$ defined on a real monotone measure space $(\Omega, \mathcal{F}, \mu)$. Then for $s > 0$*

$$\mathbb{E}_C^{\mu}[e^{sY}] \leq \|\mu\| \exp\left(\frac{s^2(b-a)^2}{8}\right). \tag{2.7}$$

In particular, if μ is a probability measure, we get the classical Hoeffding type inequality (Hoeffding, 1963).

Proof. Since e^{sy} is a convex function, then we have

$$e^{sy} \leq \frac{y-a}{b-a} e^{sb} + \frac{b-y}{b-a} e^{sa}, \qquad \text{for } a \leq y \leq b.$$

Then

$$\frac{1}{\|\mu\|} \mathbb{E}_C^{\mu}[e^{sX}] \leq \frac{1}{\|\mu\|} \mathbb{E}_C^{\mu}\left[\frac{e^{sb} - e^{sa}}{b-a} Y + \frac{be^{sa} - ae^{sb}}{b-a}\right] = \frac{be^{sa} - ae^{sb}}{b-a} = e^{\eta(u)},$$

where $\eta(u) = -\theta u + \log(1 - \theta + \theta e^u)$, $\theta = \frac{-a}{b-a}$, $u = s(b-a)$. So,

$$\log\left(\frac{1}{\|\mu\|}\mathbb{E}_C^{\mu}[e^{sY}]\right) \le \eta(u) \le \frac{u^2}{8}$$

by taking the Taylor series expansion of $\eta(u)$ about 0. And the proof is completed. □

Theorem 2.9 (Hoeffding's inequality). *Let X be a random variable with $\mathbb{E}_C^{g(\mu)}[\frac{1}{s}\ln(g(e^{sX}))] = 0$ and $a \le \frac{1}{s}\ln g(e^{sX}) \le b$ defined on a real monotone measure space $(\Omega, \mathcal{F}, \mu)$. Then for $s > 0$*

$$\mathbb{E}_{Cl,g}^{\mu}[e^{sX}] \le \|\mu\| \otimes g^{-1}\left(\exp\left(\frac{s^2(b-a)^2}{8}\right)\right).$$

Proof. Let $Y = \frac{1}{s}\ln(g(e^{sX}))$. We apply Hoeffding's inequality for Choquet expectation (2.7) and then we obtain

$$\mathbb{E}_C^{g(\mu)}[e^{sY}] \le g(\|\mu\|)\exp\left(\frac{s^2(b-a)^2}{8}\right).$$

Hence,

$$\begin{aligned}
&\mathbb{E}_{Cl,g}^{\mu}[e^{sX}] \\
&\quad = g^{-1}(\mathbb{E}_C^{g(\mu)}[g(e^{sX})]) = g^{-1}(\mathbb{E}_C^{g(\mu)}[e^{\ln(g(e^{sX}))}]) = g^{-1}(\mathbb{E}_C^{g(\mu)}[e^{sY}]) \\
&\quad \le g^{-1}\left(g(\|\mu\|)\exp\left(\frac{s^2(b-a)^2}{8}\right)\right) = \|\mu\| \otimes g^{-1}\left(\exp\left(\frac{s^2(b-a)^2}{8}\right)\right).
\end{aligned}$$ □

2.1.4 *Fatou's lemma.*

Theorem 2.10. *Let $\{X_n\}$ be a sequence of nonnegative random variables. For Choquet-like expectation (1.2), the Fatou lemma*

$$\mathbb{E}_{Cl,g}^{\mu}\left[\liminf_{n\to\infty} X_n\right] \le \liminf_{n\to\infty} \mathbb{E}_{Cl,g}^{\mu}[X_n]$$

holds if $g(\mu)$ is continuous from below and $\mathbb{E}_{Cl,g}^{\mu}[\lim_{n\to\infty}\inf X_n] < \infty$.

Proof. Recall the Monotone convergence theorem for Choquet expectation (Denneberg, 1994) which asserts that for an increasing sequence of nonnegative random variables $\{X_n\}$,

$$\lim_{n\to\infty} \mathbb{E}_C^{g(\mu)}[X_n] = \mathbb{E}_C^{g(\mu)}\left[\lim_{n\to\infty} X_n\right], \tag{2.8}$$

where $g(\mu)$ is continuous from below. Let X denote the limit inferior of the X_n and $Y_k = \inf_{n\ge k} X_n$. Then, the sequence $Y_1, Y_2, \ldots$ is monotonically increasing and converges pointwise to X. Now from the monotonicity of Choquet-like expectation, we have

$$\mathbb{E}^{\mu}_{Cl,g}[Y_k] \le \inf_{n\ge k} \mathbb{E}^{\mu}_{Cl,g}[X_n]. \tag{2.9}$$

So, (2.8) and (2.9) imply that

$$\begin{aligned}
\mathbb{E}^{\mu}_{Cl,g}\Big[\liminf_{n\to\infty} X_n\Big] &= \mathbb{E}^{\mu}_{Cl,g}[X] = g^{-1}\Big(\mathbb{E}^{g(\mu)}_{C}\Big[g\Big(\lim_{k\to\infty} Y_k\Big)\Big]\Big) \\
&= g^{-1}\Big(\mathbb{E}^{g(\mu)}_{C}\Big[\lim_{k\to\infty} g(Y_k)\Big]\Big) = g^{-1}\Big(\lim_{k\to\infty} \mathbb{E}^{g(\mu)}_{C}[g(Y_k)]\Big) \\
&= \lim_{k\to\infty} g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}[g(Y_k)]\big) = \lim_{k\to\infty} \mathbb{E}^{\mu}_{Cl,g}[Y_k] \\
&\le \lim_{k\to\infty} \inf_{n\ge k} \mathbb{E}^{\mu}_{Cl,g}[X_n] = \liminf_{n\to\infty} \mathbb{E}^{\mu}_{Cl,g}[X_n]. \qquad \square
\end{aligned}$$

3 Convergence of sequences of random variables

Below we consider a sequence of random variables $\{X_n : n \in \mathbb{N}\}$ be a sequence of random variables defined on a real monotone measure space $(\Omega, \mathcal{F}, \mu)$.

3.1 Modes of convergence

Definition 3.1. (I) Let $\otimes$ be generated by a generator g. We say that $\{X_n\}$ converges to X in $L^p_{Cl,g}$ and write $X_n \xrightarrow{L^p_{Cl,g}} X$ if

$$\mathbb{E}^{\mu}_{Cl,g}\big[(|X_n - X|)^{(p)}_{\otimes}\big] \to 0 \qquad \text{as } n \to \infty.$$

In particular, if $g = i$ (the identity mapping), then we say that $\{X_n\}$ converges in L^p_C to X and write $X_n \xrightarrow{L^p_C} X$.

(II) We say that $\{X_n\}$ converges to X in μ and write $X_n \xrightarrow{\mu} X$ if for every $\varepsilon > 0$,

$$\mu\big[|X_n - X| > \varepsilon\big] \to 0 \qquad \text{as } n \to \infty.$$

(III) We say that $\{X_n\}$ is μ-almost convergent to X and write $X_n \xrightarrow{\mu\text{-a.e.}} X$ if there exists a subset $N \subset \Omega$ such that $\mu(N) = 0$ and

$$X_n(\omega) \to X(\omega) \qquad \text{as } n \to \infty$$

for all $\omega \in \Omega \setminus N$.

3.2 Relationships among the modes

Theorem 3.2. *Suppose that $X, X_1, X_2, \ldots$ belong to $\zeta^p_{Cl,g}$. Then*

(a) *If $1 \le q \le p$ and $X_n \xrightarrow{L^p_{Cl,g}} X$, then $X_n \xrightarrow{L^q_{Cl,g}} X$.*

(b) *If $X_n \xrightarrow{L^p_{Cl,g}} X$, then $X_n \xrightarrow{\mu} X$.*

Proof. (a) Using Lyapunov's inequality (2.4), we have:

$$\begin{aligned}&(\mathbb{E}^{\mu}_{Cl,g}[(|X_n - X|)^{(q)}_{\otimes}])^{(1/q)}_{\otimes}\\ &\quad \le (\|\mu\|)^{((p-q)/qp)}_{\otimes} \otimes (\mathbb{E}^{\mu}_{Cl,g}[(|X_n - X|)^{(p)}_{\otimes}])^{(1/p)}_{\otimes} \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

(b) Using Markov's inequality (2.6), then for each $\varepsilon > 0$:

$$\varepsilon^{(p)}_{\otimes} \otimes \mu\big(|X_n - X| > \varepsilon\big) \le \mathbb{E}^{\mu}_{Cl,g}[|X_n - X|^{(p)}_{\otimes}] \to 0 \qquad \text{as } n \to \infty.$$

Therefore, Definition 1.3(M3) implies that $\mu(|X_n - X| > \varepsilon) \to 0$. This completes the proof. □

Theorem 3.3. *Let $\oplus, \otimes$ be generated by a subadditive generator g. Suppose that $g(\mu)$ is submodular and $X_n, Y_n, X, Y \in \zeta^p_{Cl,g}$ for each n. If $X_n \xrightarrow{L^p_{Cl,g}} X$ and $Y_n \xrightarrow{L^p_{Cl,g}} Y$, then $X_n + Y_n \xrightarrow{L^p_{Cl,g}} X + Y$.*

Proof. Using Minkowski's inequality, we have:

$$\begin{aligned}&(\mathbb{E}^{\mu}_{Cl,g}[(|X_n + Y_n - (X+Y)|)^{(p)}_{\otimes}])^{(1/p)}_{\otimes}\\ &\quad \le g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[(g(|X_n - X|) + g(|Y_n - Y|))^{p}])^{1/p}\big)\\ &\quad \le g^{-1}\big((\mathbb{E}^{g(\mu)}_{C}[(g(|X_n - X|))^{p}])^{1/p} + (\mathbb{E}^{g(\mu)}_{C}[(g(|Y_n - Y|))^{p}])^{1/p}\big)\\ &\quad = (\mathbb{E}^{\upsilon}_{Cl,g}[(|X_n - X|)^{(p)}_{\otimes}])^{(1/p)}_{\otimes} \oplus (\mathbb{E}^{\upsilon}_{Cl,g}[(|Y_n - Y|)^{(p)}_{\otimes}])^{(1/p)}_{\otimes}\\ &\quad \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $(\oplus, \otimes)$ be a magnifying pair of pseudo-arithmetic operations generated by a generator g. Suppose that $g(\mu)$ is submodular and $X_n, Y_n, X, Y \in \zeta^p_{Cl,g}$ for each n.*

(a) *If $X_n \xrightarrow{L^p_{Cl,g}} X$ and $Y_n \xrightarrow{L^p_{Cl,g}} Y$, then $X_nY_n \xrightarrow{L^1_{Cl,g}} XY$.*

(b) *If $X_n \xrightarrow{\mu} X$ and g is uniformly bounded, then $X_n \xrightarrow{L^p_{Cl,g}} X$.*

Proof. (a) Using Minkowski's inequality (2.5) and Hölder's inequality (2.1), we have:

$$\begin{aligned}
&\mathbb{E}^{\upsilon}_{Cl,g}[|X_nY_n - XY|] \\
&\quad = \mathbb{E}^{\upsilon}_{Cl,g}[|(X_nY_n - X_nY) + (X_nY - XY)|] \\
&\quad \leq (\mathbb{E}^{\upsilon}_{Cl,g}[|X_n| \otimes |Y_n - Y|]) \oplus (\mathbb{E}^{\upsilon}_{Cl,g}[(|Y| \otimes |X_n - X|)]) \\
&\quad \leq [((\mathbb{E}^{\upsilon}_{Cl,g}[(|X_n|)^{(q)}_{\otimes}])^{(1/q)}_{\otimes} \otimes (\mathbb{E}^{\upsilon}_{Cl,g}[(|Y_n - Y|)^{(p)}_{\otimes}])^{(1/p)}_{\otimes}) \\
&\qquad \oplus ((\mathbb{E}^{\upsilon}_{Cl,g}[(|Y|)^{(q)}_{\otimes}])^{(1/q)}_{\otimes} \otimes (\mathbb{E}^{\upsilon}_{Cl,g}[(|X_n - X|)^{(p)}_{\otimes}])^{(1/p)}_{\otimes})] \\
&\quad \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

(b) For each $\varepsilon > 0$:

$$\begin{aligned}
&\mathbb{E}^{\mu}_{Cl,g}[(|X_n - X|)^{(p)}_{\otimes}] \\
&\quad = g^{-1}(\mathbb{E}^{g(\mu)}_{C}[(g(|X_n - X|))^p(\mathbb{I}_{[|X_n - X| \leq \varepsilon]} + \mathbb{I}_{[|X_n - X| > \varepsilon]})]) \\
&\quad \leq g^{-1}(\mathbb{E}^{g(\mu)}_{C}[(g(|X_n - X|))^p \mathbb{I}_{[|X_n - X| \leq \varepsilon]}] \\
&\qquad\qquad + \mathbb{E}^{g(\mu)}_{C}[(g(|X_n - X|))^p \mathbb{I}_{[|X_n - X| > \varepsilon]}]) \\
&\quad \leq g^{-1}((g(\varepsilon))^p g(\|\mu\|) + M^p g(\mu(|X_n - X| > \varepsilon))) \\
&\quad \leq (\varepsilon^{(p)}_{\otimes} \otimes \|\mu\|) \oplus (g^{-1}(M^p) \otimes \mu(|X_n - X| > \varepsilon)) \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

This completes the proof. □

Note 3.5. Notice that if both $\otimes$ and $\oplus$ are generated by a generator g, the semiring $([0,\infty], \oplus, \otimes)$ is isomorphic with the classical semiring $([0,\infty], +, \cdot)$, and thus some properties valid for standard $+$ and $\cdot$ based integrals such as the Choquet expectation can be deduced directly (for example, Hölder inequality in Theorem 2.1). However, this is not the case of all results (see, Theorem 3.4 where we need the pair $(\oplus, \otimes)$ to be magnifying). So, in several cases we need some additional constraints not automatically present by a general semiring.

3.3 Uniform integrability

Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of random variables defined on a real monotone measure space $(\Omega, \mathcal{F}, \mu)$.

Definition 3.6. The sequence $\{X_n\}$ is g-uniformly integrable if $X_n \in \zeta^1_{Cl,g}$ for each n and if

$$\lim_{b\to\infty} \sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n| \mathbb{I}_{[|X_n| > b]}] = 0. \tag{3.1}$$

In particular, if $g = i$ (the identity mapping), then we say that $\{X_n\}$ is uniformly integrable.

Proposition 3.7. *If $g(\mu)$ is submodular, then the sequence $\{X_n\}$ is g-uniformly integrable if and only if*

(I) $\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n|] < \infty$,
(II) $\forall \varepsilon > 0, \exists \delta > 0$ *s.t.* $\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n|\mathbb{I}_{[A]}] < \varepsilon$ *if* $A \in \mathcal{F}$, $\mu(A) < \delta$.

Proof. Let the sequence $\{X_n\}$ be g-uniformly integrable. Then, for an $\varepsilon > 0$, there is a finite $b > 0$ such that $\mathbb{E}^{\mu}_{Cl,g}(|X_n|\mathbb{I}_{[|X_n|>b]}) < \varepsilon$. Hence,

$$\begin{aligned}
\mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\big] &= g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\big]\big) = g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)(\mathbb{I}_{[|X_n|>b]} + \mathbb{I}_{[|X_n|\le b]})\big]\big)\\
&= g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[|X_n|>b]} + g\big(|X_n|\big)\mathbb{I}_{[|X_n|\le b]}\big]\big)\\
&\le g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[|X_n|>b]}\big] + \mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[|X_n|\le b]}\big]\big)\\
&\le g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[|X_n|>b]}\big] + g(b)\mathbb{E}^{g(\mu)}_C[\mathbb{I}_{[|X_n|\le b]}]\big)\\
&\le g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[|X_n|>b]}\big] + g(b)g\big(\|\mu\|\big)\big)\\
&\le g^{-1}\big(g\big(\mathbb{E}^{\mu}_{Cl,g}\big(|X_n|\mathbb{I}_{[|X_n|>b]}\big)\big) + g(b)g\big(\|\mu\|\big)\big)\\
&\le g^{-1}\big(g(\varepsilon) + g(b)g(1)g\big(\|\mu\|\big)\big) < \infty.
\end{aligned}$$

Therefore, $\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n|] < \infty$.

(II) Let $\varepsilon > 0$. Take $b_\varepsilon \in (0, \infty)$ such that $\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n|\mathbb{I}_{[|X_n|>b]}] < g^{-1}(\frac{g(\varepsilon)}{2})$, $\forall b > b_\varepsilon$. Let $\delta_\varepsilon = g^{-1}(\frac{g(\varepsilon)}{2g(b_\varepsilon)})$. If $\mu(A) < \delta_\varepsilon$ for any $A \in \mathcal{F}$, then we have

$$\begin{aligned}
&\sup_n \mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[A]}\big]\\
&\quad= \sup_n g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[A]}\big]\big)\\
&\quad\le \sup_n g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[A\cap\{|X_n|>b_\varepsilon\}]} + g\big(|X_n|\big)\mathbb{I}_{[A\cap\{|X_n|\le b_\varepsilon\}]}\big]\big)\\
&\quad\le \sup_n g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[A\cap\{|X_n|>b_\varepsilon\}]}\big] + \mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[A\cap\{|X_n|\le b_\varepsilon\}]}\big]\big)\\
&\quad\le \sup_n g^{-1}\big(\mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[|X_n|>b_\varepsilon]}\big] + \mathbb{E}^{g(\mu)}_C\big[g\big(|X_n|\big)\mathbb{I}_{[A\cap\{|X_n|\le b_\varepsilon\}]}\big]\big)\\
&\quad= \sup_n g^{-1}\big(g\big(\mathbb{E}^{\mu}_{Cl,g}\big[\big(|X_n|\mathbb{I}_{[|X_n|>b]}\big)\big]\big) + g(b_\varepsilon)\mathbb{E}^{g(\mu)}_C[\mathbb{I}_{[A]}]\big)\\
&\quad\le g^{-1}\bigg(\frac{g(\varepsilon)}{2} + g(b_\varepsilon)g\big(\mu(A)\big)\bigg) \le g^{-1}\bigg(\frac{g(\varepsilon)}{2} + g(b_\varepsilon)g(\delta_\varepsilon)\bigg)\\
&\quad\le g^{-1}\bigg(\frac{g(\varepsilon)}{2} + \frac{g(\varepsilon)}{2}\bigg) = \varepsilon.
\end{aligned}$$

Conversely, if $\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n|] \le M < \infty$, then by using Markov's inequality (2.6) and part (I), we have for all $b > 0$,

$$\mu(A_n) = \mu(\{|X_n| > b\}) \le g^{-1}\left(\frac{g(\mathbb{E}^{\mu}_{Cl,g}[|X_n|])}{g(b)}\right) \le g^{-1}\left(\frac{g(M)}{g(b)}\right).$$

Clearly, $\lim_{b\to\infty} \mu(\{|X_n| > b\}) = 0$. Consider a $\delta \ge g^{-1}(\frac{g(M)}{g(b)})$. Then by the fact of $\mu(A_n) < \delta$ and part (II), we have $\sup_n \mathbb{E}^{\mu}_{Cl,g}(|X_n|\mathbb{I}_{[A_n]}) < \varepsilon$ and the proof is completed. □

Corollary 3.8. *If μ is submodular, then the sequence $\{X_n\}$ is uniformly integrable if and only if*

(I) $\sup_n \mathbb{E}^{\mu}_{C}[|X_n|] < \infty$,
(II) $\forall \varepsilon > 0, \exists \delta > 0$ *s.t.* $\sup_n \mathbb{E}^{\mu}_{C}[|X_n| \cdot \mathbb{I}_{[A]}] < \varepsilon$ *if* $A \in \mathcal{F}$, $\mu(A) < \delta$.

Theorem 3.9. *Let $\oplus, \otimes$ be generated by a submultiplicative generator g. If $g(\mu)$ is submodular and $\{X_n\}$ and $\{Y_n\}$ are g-uniformly integrable, then $\{X_n + Y_n\}$ is also g-uniformly integrable.*

Proof. By using the Minkowski inequality (2.5), we have

$$\begin{aligned}
&\mathbb{E}^{\mu}_{Cl,g}\big[|X_n + Y_n|\mathbb{I}_{[|X_n+Y_n|>2b]}\big]\\
&\quad= g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}\big[g\big(|X_n + Y_n|\big)\mathbb{I}_{[|X_n+Y_n|>2b]}\big]\big)\\
&\quad\le g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}\big[g\big(2\max\{|X_n|, |Y_n|\}\big)\mathbb{I}_{[2\max\{|X_n|,|Y_n|\}>2b]}\big]\big)\\
&\quad= g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}\big[g\big(2\max\{|X_n|, |Y_n|\}\big)\big(\mathbb{I}_{[\{\max\{|X_n|,|Y_n|\}>b\}\cap\{|X_n|>|Y_n|\}]}\\
&\qquad\qquad + \mathbb{I}_{[\{\max\{|X_n|,|Y_n|\}>b\}\cap\{|X_n|\le|Y_n|\}]}\big)\big]\big)\\
&\quad= g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}\big[g\big(2\max\{|X_n|, |Y_n|\}\big)\mathbb{I}_{[\{\max\{|X_n|,|Y_n|\}>b\}\cap\{|X_n|>|Y_n|\}]}\\
&\qquad\qquad + g\big(2\max\{|X_n|, |Y_n|\}\big)\mathbb{I}_{[\{\max\{|X_n|,|Y_n|\}>b\}\cap\{|X_n|\le|Y_n|\}]}\big]\big)\\
&\quad\le g^{-1}\big(g(2)\mathbb{E}^{g(\mu)}_{C}\big[g\big(|X_n|\big)\mathbb{I}_{[\{|X_n|>b\}]}\big] + g(2)\mathbb{E}^{g(\mu)}_{C}\big[g\big(|Y_n|\big)\mathbb{I}_{[\{|Y_n|>b\}]}\big]\big)\\
&\quad= g^{-1}\big(g(2)g\big(\mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[|X_n|>b]}\big]\big) + g(2)g\big(\mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[|X_n|>b]}\big]\big)\big)\\
&\quad= g^{-1}\big(g(2)g\big(\mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[|X_n|>b]}\big]\big) + g(2)g\big(\mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[|X_n|>b]}\big]\big)\big)\\
&\quad= g^{-1}\big(g\big(2 \otimes \mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[|X_n|>b]}\big]\big) + g\big(2 \otimes \mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[|X_n|>b]}\big]\big)\big)\\
&\quad= \big(2 \otimes \mathbb{E}^{\mu}_{Cl,g}\big[|Y_n|\mathbb{I}_{[|Y_n|>b]}\big]\big) \oplus \big(2 \otimes \mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[|X_n|>b]}\big]\big).
\end{aligned}$$

Then $\lim_{b\to\infty} \sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n + Y_n|\mathbb{I}_{[|X_n+Y_n|>2b]}] = 0$ follows from the g-uniform integrability of $\{X_n\}$ and $\{Y_n\}$ and the continuity of $\otimes, \oplus$. Therefore, $\{X_n + Y_n\}_{n=1}^{\infty}$ is also g-uniformly integrable and the proof is completed. □

Theorem 3.10. *Let* $\oplus, \otimes$ *be generated by a submultiplicative generator* g. *If* $g(\mu)$ *is submodular and* $\{|X_n|_\otimes^{(p)}\}$ *and* $\{|Y_n|_\otimes^{(q)}\}$ $(\frac{1}{p}+\frac{1}{q}=1,\ p\geq 1)$ *are* g*-uniformly integrable, then* $\{X_nY_n : n\in\mathbb{N}\}$ *is also* g*-uniformly integrable.*

Proof. By using the Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}^{\mu}_{Cl,g}\big[|X_nY_n|\mathbb{I}_{[|X_nY_n|>b]}\big]\\
&\quad\leq g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_nY_n|\big)(\mathbb{I}_{[|Y_n|>\sqrt{b}]}+\mathbb{I}_{[|X_n|>\sqrt{b}]})\big]\big)\\
&\quad= g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_nY_n|\big)\mathbb{I}_{[|Y_n|>\sqrt{b}]}+g\big(|X_nY_n|\big)\mathbb{I}_{[|X_n|>\sqrt{b}]}\big]\big)\\
&\quad\leq g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n|\big)g\big(|Y_n|\big)\mathbb{I}_{[|Y_n|>\sqrt{b}]}\big]\\
&\qquad\qquad+\mathbb{E}_C^{g(\mu)}\big[g\big(|Y_n|\big)g\big(|X_n|\big)\mathbb{I}_{[|X_n|>\sqrt{b}]}\big]\big)\\
&\quad\leq g^{-1}\big(\big(\mathbb{E}_C^{g(\mu)}\big[\big(g\big(|X_n|\big)\big)^p\big]\big)^{1/p}\big(\mathbb{E}_C^{g(\mu)}\big[\big(g\big(|Y_n|\big)\mathbb{I}_{[|Y_n|>\sqrt{b}]}\big)^q\big]\big)^{1/q}\\
&\qquad\qquad+\big(\mathbb{E}_C^{g(\mu)}\big[\big(g\big(|Y_n|\big)\big)^q\big]\big)^{1/q}\big(\mathbb{E}_C^{g(\mu)}\big[\big(g\big(|X_n|\big)\mathbb{I}_{[|X_n|>\sqrt{b}]}\big)^p\big]\big)^{1/p}\big)\\
&\quad= g^{-1}\big(\big(\mathbb{E}_C^{g(\mu)}\big[\big(g\big(|X_n|\big)\big)^p\big]\big)^{1/p}\big(\mathbb{E}_C^{g(\mu)}\big[\big(g\big(|Y_n|\big)\big)^q\mathbb{I}_{[|Y_n|>\sqrt{b}]}\big]\big)^{1/q}\\
&\qquad\qquad+\big(\mathbb{E}_C^{g(\mu)}\big[\big(g\big(|Y_n|\big)\big)^q\big]\big)^{1/q}\big(\mathbb{E}_C^{g(\mu)}\big[\big(g\big(|X_n|\big)\big)^p\mathbb{I}_{[|X_n|>\sqrt{b}]}\big]\big)^{1/p}\big)\\
&\quad=\big[\big(\big(\mathbb{E}^{\mu}_{Cl,g}\big[|X_n|_\otimes^{(p)}\big]\big)_\otimes^{(1/p)}\otimes\big(\mathbb{E}^{\mu}_{Cl,g}\big[\big(|Y_n|\big)_\otimes^{(q)}\mathbb{I}_{[|Y_n|>\sqrt{b}]}\big]\big)_\otimes^{(1/q)}\big)\\
&\qquad\oplus\big(\big(\mathbb{E}^{\mu}_{Cl,g}\big[|Y_n|_\otimes^{(q)}\big]\big)_\otimes^{(1/q)}\otimes\big(\mathbb{E}^{\mu}_{Cl,g}\big[\big(|X_n|\big)_\otimes^{(p)}\mathbb{I}_{[|X_n|>\sqrt{b}]}\big]\big)_\otimes^{(1/p)}\big)\big].
\end{aligned}$$

Then $\lim_{b\to\infty}\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_nY_n|\mathbb{I}_{[|X_nY_n|>b]}]=0$ follows from the uniform integrability of $\{|X_n|_\otimes^{(p)}\}$ and $\{|Y_n|_\otimes^{(q)}\}$ and the continuity of $\otimes$, $\oplus$. And the proof is completed. □

The following theorem is a sufficient condition for uniform integrability.

Theorem 3.11. *If* $\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n|_\otimes^{(1+\delta)}]<\infty$ *for some* $\delta>0$, *then* $\{X_n\}$ *is* g*-uniformly integrable.*

Proof. If $\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n|_\otimes^{(1+\delta)}]\leq M<\infty$ for some $\delta>0$, then we have,

$$\begin{aligned}
&\mathbb{E}^{\mu}_{Cl,g}\big[|X_n|\mathbb{I}_{[|X_n|>b]}\big]\\
&\quad= g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n|\big)\mathbb{I}_{[|X_n|>b]}\big]\big)\\
&\quad\leq \mathbb{E}_C^{g(\mu)}\bigg[g\big(|X_n|\big)\mathbb{I}_{[|X_n|>b]}\frac{g^{\delta}(|X_n|)}{g^{\delta}(b)}\bigg]
\end{aligned}$$

$$\leq \frac{1}{g^{\delta}(b)}\mathbb{E}_C^{g(\mu)}\big[g^{\delta+1}\big(|X_n|\big)\big] = \frac{1}{g^{\delta}(b)}\mathbb{E}_C^{g(\mu)}\big[g\big(g^{-1}\big(g^{\delta+1}\big(|X_n|\big)\big)\big)\big]$$
$$= \frac{1}{g^{\delta}(b)}g\big(g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n|_{\otimes}^{(1+\delta)}\big)\big]\big)\big) \leq \frac{g(1)}{g^{\delta}(b)}g(M).$$

So, $\lim_{b\to\infty}\sup_n \mathbb{E}_{Cl,g}^{\mu}[|X_n| \otimes \mathbb{I}_{[|X_n|>b]}] = \lim_{b\to\infty} \frac{g(1)}{g^{\delta}(b)}g(M) = 0.$ □

Theorem 3.12. *Let $\oplus, \otimes$ be generated by a subadditive generator g. Suppose that $g(\mu)$ is submodular and $X, X_n \in \zeta_{Cl,g}^r$ for each n. Then the following are equivalent*:

(a) $X_n \xrightarrow{\mu} X$ *and $\{X_n\}$ is g-uniformly integrable.*

(b) $X_n \xrightarrow{L_{Cl,g}^1} X$.

Proof. (a) ⇒ (b) For any $\varepsilon > 0$:

$$\begin{aligned}
&\mathbb{E}_{Cl,g}^{\mu}\big[|X_n - X|\big]\\
&\quad= g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n - X|\big)\big]\big)\\
&\quad= g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n - X|\big)(\mathbb{I}_{[|X_n-X|>\varepsilon]} + \mathbb{I}_{[|X_n-X|\leq\varepsilon]})\big]\big)\\
&\quad= g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n - X|\big)\mathbb{I}_{[|X_n-X|>\varepsilon]} + g\big(|X_n - X|\big)\mathbb{I}_{[|X_n-X|\leq\varepsilon]}\big]\big)\\
&\quad\leq g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n - X|\big)\mathbb{I}_{[|X_n-X|>\varepsilon]}\big] + \mathbb{E}_C^{g(\mu)}\big[g\big(|X_n - X|\big)\mathbb{I}_{[|X_n-X|\leq\varepsilon]}\big]\big)\\
&\quad\leq g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n - X|\big)\mathbb{I}_{[|X_n-X|>\varepsilon]}\big] + g(\varepsilon)\mathbb{E}_C^{g(\mu)}[\mathbb{I}_{[|X_n-X|\leq\varepsilon]}]\big)\\
&\quad\leq g^{-1}\big(\mathbb{E}_C^{g(\mu)}\big[g\big(|X_n|\big)\mathbb{I}_{[|X_n-X|>\varepsilon]}\big]\\
&\qquad\qquad + \mathbb{E}_C^{g(\mu)}\big[g\big(|X|\big)\mathbb{I}_{[|X_n-X|>\varepsilon]}\big] + g(\varepsilon)g\big(\|\mu\|\big)\big)\\
&\quad= g^{-1}\big(g\big(\mathbb{E}_{Cl,g}^{\mu}\big(|X_n|\mathbb{I}_{[|X_n-X|>\varepsilon]}\big)\big)\\
&\qquad\qquad + g\big(\mathbb{E}_{Cl,g}^{\mu}\big(|X|\mathbb{I}_{[|X_n-X|>\varepsilon]}\big)\big) + g\big(\varepsilon \otimes \|\mu\|\big)\big)\\
&\quad= \big[\mathbb{E}_{Cl,g}^{\mu}\big(|X_n|\mathbb{I}_{[|X_n-X|>\varepsilon]}\big)\big] \oplus \big[\mathbb{E}_{Cl,g}^{\mu}\big(|X|\mathbb{I}_{[|X_n-X|>\varepsilon]}\big)\big] \oplus \big(\varepsilon \otimes \|\mu\|\big).
\end{aligned}$$

By using Proposition 3.7, the first term converges to zero because $\mu(|X_n - X| > \varepsilon) \to 0$. The second term converges to zero because $X \in L_{Cl,g}^1$, since this means $\{X\}$ is g-uniformly integrable.

(b) ⇒ (a): $X_n \xrightarrow{\mu} X$ immediately holds from the Theorem 3.2 (part (b)). Given $\varepsilon > 0$ choose N such that $\mathbb{E}_{Cl,g}^{\mu}[|X_n - X|] < g^{-1}(\frac{g(\varepsilon)}{2})$ for $n \geq N$. For a fixed N choose $\delta > 0$ such that $\mu(A) < \delta$, $\sup_n \mathbb{E}_{Cl,g}^{\mu}[|X_n|\mathbb{I}_{[A]}] < \delta$ and $\mathbb{E}_{Cl,g}^{\mu}[|X|\mathbb{I}_{[A]}] < \delta$. Then if $\mu(A) < \delta$ and n is large enough

$$\sup_n \mathbb{E}_{Cl,g}^{\mu}\big[|X_n|\mathbb{I}_{[A]}\big] \leq \sup_n\big(\big[\mathbb{E}_{Cl,g}^{\mu}\big[|X_n - X|\mathbb{I}_{[A]}\big] \oplus \mathbb{E}_{Cl,g}^{\mu}\big[|X|\mathbb{I}_{[A]}\big]\big]\big) < \varepsilon.$$

Furthermore, $\sup_n \mathbb{E}^{\mu}_{Cl,g}[|X_n|] < \infty$. Therefore, the proof is completed by Proposition 3.7. $\square$

Acknowledgments

The authors are very grateful to the anonymous reviewers for their insightful and constructive comments and suggestions that have led to an improved version of this paper. The third author was supported by Grant VEGA 1/0171/12.

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