# A Step towards Upper-bound of Conflict of Belief Functions based on Non-conflicting Parts

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**Abstract**

This study compares the size of conflict based on non-conflicting parts of belief functions *Conf* with the sum of all multiples of bbms of disjoint focal elements of belief functions in question. In general, we make an effort to reach a simple upper bound function for *Conf*. (Nevertheless, the maximal value of conflict is, of course, equal to 1 for fully conflicting belief functions). We apply both theoretical research using the recent results on belief functions and also experimental computational approach here.

**Keywords:** Belief functions, Dempster-Shafer theory, Uncertainty, Conflicting belief masses, Conflict between belief functions, Hidden conflict.

## 1 Introduction

Belief functions representing an uncertain and/or incomplete, imperfect information about the object of interest may be, of course, in mutual conflict. The classic definition of conflict between belief functions is equivalent to the sum of all multiples of conflicting belief masses of individual belief functions [17]; i.e. the belief mass assigned to the empty set when non-normalized conjunction combination rule is considered (frequently denoted by $m_{\textcircled{\cap}}(\emptyset)$). After this measure was observed to be inadequate for a correct representation of conflict between belief functions [1, 14], several different measures were introduced in last dozen years, e.g. [6, 7, 12, 13, 14, 15, 16, 19]. Conflict between belief functions is usually assumed to be less or equal to the belief mass appearing on the empty set $m_{\odot}(\emptyset)$.

One of the progressive current alternative conflict measures of the conflict between belief functions is based on their non-conflicting parts [7]. Despite the original assumption, positive conflict was observed there even in situations when the previously mentioned conflict measures were zero and belief functions in question were considered to be non-conflicting. These so-called hidden conflicts were analyzed and presented in [8, 11]. In this paper we try to give a simple upper-bound function of conflict based on non-conflicting parts and also of previous measures of conflict, to obtain an improved general assumption for conflict measures.

We apply here both theoretical approach using our recent results on degrees of hidden conflicts [11] and of degrees of non-conflictness [10] and also experimental computational approach continuing our computations from [8, 9].

# 2 Preliminaries

We assume classic definitions and basic notion from the theory of *belief functions* [17] on finite exhaustive frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is a *power-set* of $\Omega$.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A\subseteq\Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed.

A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, such that $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. A *plausibility function* $Pl : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Pl(A) = \sum_{\emptyset\neq A\cap X} m(X)$. Because there is a unique correspondence among $m$ and corresponding $Bel$ and $Pl$, we often speak about $m$ as of a belief function.

A *focal element* is a subset of the frame of discernment $X \subseteq \Omega$, such that $m(X) > 0$; if $X \subsetneq \Omega$ then it is a *proper focal element*. If all focal elements are *singletons* (i.e. one-element subsets of $\Omega$), then we speak about a *Bayesian belief function (BBF)*; in fact, it is a probability distribution on $\Omega$. If there are only focal elements such that $|X| = 1$ or $|X| = n$ we speak about *quasi-Bayesian BF (qBBF)*. In the case of $m(\Omega) = 1$ we speak about *vacuous BF*. In the case of $m(X) = 1$ for $X \subset \Omega$ we speak about *categorical BF*. If $m(X) > 0$ for $X \subset \Omega$ and $m(\Omega) = 1 - m(X)$ we speak about *simple support BF*. If all focal elements have a non-empty intersection, we speak about *a consistent BF*; and if all of them are nested, about *a consonant BF*.

*Dempster's (normalized conjunctive) rule of combination* $\oplus$: $(m_1 \oplus m_2)(A) = \sum_{X\cap Y=A} K m_1(X)\, m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X\cap Y=\emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [17]. Putting $K = 1$ and $(m_1 \odot m_2)(\emptyset) = \kappa = m_{\odot}(\emptyset)$ we obtain the *non-normalized conjunctive rule of combination* $\odot$ , see e. g. [18].

Smets' *pignistic probability* is given by $BetP(\omega_i) = \sum_{\omega_i\in X\subseteq\Omega} \frac{1}{|X|} \frac{m(X)}{1-m(\emptyset)}$, see e.g. [18]. *Normalized plausibility of singletons*[1] of $Bel$ is a probability distribution

[1]Plausibility of singletons is called *contour function* by Shafer in [17], thus $Pl\_P(Bel)$ is a

$Pl\_P$ such that $Pl\_P(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega\in\Omega} Pl(\{\omega\})}$ [3, 4].

# 3 Conflicts of Belief Functions

Original Shafer's definition of the conflict measure between two belief functions [17] is the following: $\kappa = \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y) = (m' \odot m'')(\emptyset) = m_{\odot}(\emptyset)$, more precisely its transformation $log(1/(1-\kappa))$.

After appearing that $m_{\odot}(\emptyset)$ does not correctly represent conflict between BFs [1, 14] a series of alternative approaches and measures of conflicts have appeared in last dozen years, e. g. [2, 6, 7, 12, 14, 15]. Alternative approaches are often somehow related to $m_{\odot}(\emptyset)$ or use it as one of its components [14].

In 2010, Daniel distinguished internal conflict inside an individual BF from the conflict between them [5] and pointed out that $m_{\odot}(\emptyset)$ contains both individual internal conflicts of BFs and conflict between them. Thus the usual assumption or property of measures of conflict to be less or equal to $m_{\odot}(\emptyset)$ seemed to be natural.

Finally, Daniel's *conflict based on non-conflicting parts of BFs* was introduced in [7]. This last-mentioned measure motivated our research of hidden conflict [9], hidden auto-conflict [8] and also current research of degrees of non-conflictness [10].

*A conflict of BFs* $Bel', Bel''$ *based on their non-conflicting parts* $Bel'_0, Bel''_0$ is defined by the expression $Conf(Bel', Bel'') = (m'_0 \odot m''_0)(\emptyset)$, where non-conflicting part $Bel_0$ (of a BF $Bel$) is unique consonant BF such that $Pl\_P_0 = Pl\_P$ (normalized plausibility of singletons corresponding to $Bel_0$ is the same as that corresponding to $Bel$); $m_0$ is a bba related to $Bel_0$. For an algorithm to compute $Bel_0$ see [7].

This measure of conflict between BFs in correspondence to Daniel's approach from [5] does not include internal conflict of individual BFs. And Theorem 4 from [7] claims that

$$Conf(Bel^i, Bel^{ii}) \leq (m^i \odot m^{ii})(\emptyset) \qquad (*)$$

holds true for arbitrary BFs $Bel^i, Bel^{ii}$ given by bbas $m^i, m^{ii}$ on any finite frame of discernment $\Omega_n$. Nevertheless, during later analysis of $Conf$ properties counter-examples against general validity of $(*)$ have appeared, for some of them see the next Section,

Similarly to plausibility conflict, measure $Conf$ respects plausibilities equivalent to the BFs; and it better generalises the original idea to general frame $\Omega_n$.

# 4 Counter-Examples against General Validity of Inequality $Conf \leq m_{\odot}(\emptyset)$

There are plenty of counter-examples against general validity of inequality $(*)$, thus against $Conf \leq m_{\odot}(\emptyset)$. Counter-examples have started to appear when the

normalization of contour function in fact.

first hidden conflicts had been observed. Any hidden conflict is a counterexample against it. We can start with the first and simple Introductory Example from [8, 9] on $\Omega_3$ and Little Angel example from [9], for both the examples see also [11]).

*Example 1.* **Introductory example.** Let us assume two simple consistent belief functions $Bel'$ and $Bel''$ on $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ given by the bbas $m'(\{\omega_1, \omega_2\}) = 0.6$, $m'(\{\omega_1, \omega_3\}) = 0.4$, and $m''(\{\omega_2, \omega_3\}) = 1.0$.

We can display focal elements of BFs $Bel'$ and $Bel''$ on Figure 1.

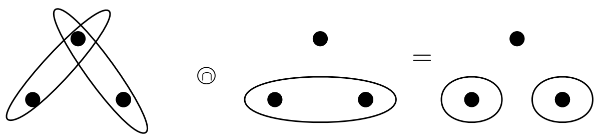

Figure 1: Introductory Example: focal elements of $m', m''$, and of $m' \circledcap m''$.

$\omega_1$ is in both the focal elements of $Bel'$, thus $Pl'(\{\omega_1\}) = 0.6+0.4 = 1$, and the other two singletons each in the only focal element, thus simply $Pl'(\{\omega_2\}) = 0.6$, $Pl'(\{\omega_3\}) = 0.4$ and after the normalization $Pl\_P' = (0.5, 0.3, 0.2)$. For $Bel''$ analogously $Pl''(\{\omega_2\}) = Pl''(\{\omega_3\}) = 1.0$ and $Pl\_P'' = (0.0, 0.5, 0.5)$. Thus non-conflicting parts of the BFs are given by the following bbms: $m'_0(\{\omega_1\}) = \frac{0.5-0.3}{0.5} = \frac{2}{5} = 0.4$, $m'_0(\{\omega_1, \omega_2\}) = \frac{0.3-0.2}{0.5} = \frac{1}{5} = 0.2$, and $m'_0(\{\omega_1, \omega_2, \omega_3\}) = \frac{0.2}{0.5} = 0.4$, and $m''_0(\{\omega_2, \omega_3\}) = m''(\{\omega_2, \omega_3\}) = 1.0$.
Hence we obtain $Conf(Bel', Bel'') = m'_0(\{\omega_1\}) m''_0(\{\omega_2, \omega_3\}) = 0.4 \cdot 1.0 = 0.4 > 0 = (m' \circledcap m'')(\emptyset)$.

*Example 2.* **Little Angel example** Let assume belief functions $Bel^i$ and $Bel^{ii}$ on $\Omega_5 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ given by the bbas $m^i(\{\omega_1, \omega_2, \omega_3\}) = 0.1$, $m^i(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 0.3$, $m^i(\{\omega_1, \omega_3, \omega_4, \omega_5\}) = 0.6$, and $m^{ii}(\{\omega_2, \omega_3 . \omega_4, \omega_5\}) = 1.0$.

Analogously to the previous case, we can display focal elements of BFs $Bel^i$ and $Bel^{ii}$ on Figure 2.

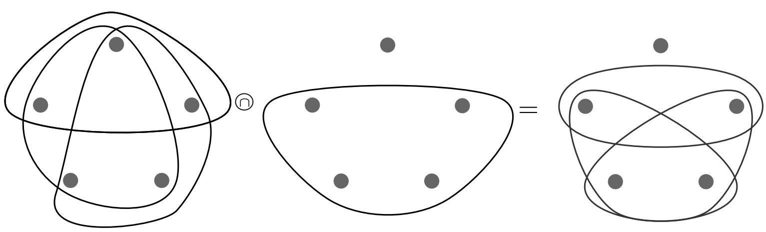

Figure 2: Litte Angel Example: focal elements of $m^i, m^{ii}$, and of $m^i \circledcap m^{ii}$.

Analogously to the previous case we obtain the following plausibility of singletons: $Pl^i(1.0, 0.4, 0.9, 0.9, 0.7)$, $Pl\_P^i(\frac{10}{39}, \frac{4}{39}, \frac{9}{39}, \frac{9}{39}, \frac{7}{39})$ and $m^i_0(\{\omega_1\}) = \frac{10-9}{10} = 0.1$, $m^i_0(\{\omega_1, \omega_3, \omega_4\}) = \frac{9-7}{10} = 0.2$, $m^i_0(\{\omega_1, \omega_3, \omega_4, \omega_5\}) = \frac{7-4}{10} = 0.3$, $m^i_0(\{\omega_1, \omega_2,$

$\omega_3, \omega_4, \omega_5\}) = \frac{4}{10} = 0.4$. For $Bel^{ii}$ there is $m_0^{ii}(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = m^{ii}(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = 1.0$. Hence we obtain $Conf(Bel^i, Bel^{ii}) = m_0^i(\{\omega_1\})m_0^{ii}(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = 0.1 \cdot 1 = 0.1$.

Both these examples are simple with a few focal elements only. Nevertheless, we can find plenty of the other examples moving small belief masses from the original focal elements to the other subset of the frame and create new ones:

*Example 3.* **Modified Introductory example.** Let us suppose belied functions $Bel'$, $Bel''$ given by the modified bbas $m'$ and $m''$, moving parts of the original bbms to singletons and to entire $\Omega_3$ as it follows:

| | | | |
|---|---|---|---|
| $m'(\{\omega_1\})$ | $= 0.1$ | $m''(\{\omega_1\})$ | $= 0.1$ |
| $m'(\{\omega_2\})$ | $= 0.1$ | $m''(\{\omega_2\})$ | $= 0.1$ |
| $m'(\{\omega_3\})$ | $= 0.1$ | $m''(\{\omega_3\})$ | $= 0.1$ |
| $m'(\{\omega_1, \omega_2\})$ | $= 0.4$ | - | |
| $m'(\{\omega_1, \omega_3\})$ | $= 0.2$ | - | |
| - | | $m''(\{\omega_2, \omega_3\})$ | $= 0.6$ |
| $m'(\{\omega_1, \omega_2, \omega_3\})$ | $= 0.1$ | $m''(\{\omega_1, \omega_2, \omega_3\})$ | $= 0.1$ |

After this modification we obtain $Pl\_P' = (\frac{8}{18}, \frac{6}{18}, \frac{4}{18})$, $Pl\_P'' = (\frac{2}{18}, \frac{8}{18}, \frac{8}{18})$, and further $m_0'(\{\omega_1\}) = \frac{8-6}{8} = \frac{2}{8} = 0.25$, $m_0'(\{\omega_1, \omega_2\}) = \frac{6-4}{8} = \frac{2}{8} = 0.25$, and $m_0'(\{\omega_1, \omega_2, \omega_3\}) = \frac{4}{8} = 0.5$, and $m_0''(\{\omega_2, \omega_3\}) = \frac{8-2}{8} = 0.75$, $m_0''(\{\omega_1, \omega_2, \omega_3\}) = \frac{2}{8} = 0.25$.
Hence we obtain $Conf(Bel', Bel'') = m_0'(\{\omega_1\})m_0''(\{\omega_2, \omega_3\}) = 0.25 \cdot 0.75 = 0.1875$.
$(m' \circledcirc m'')(\emptyset) = 6 \times 0.1 \cdot 0.1 + 0.1 \cdot 0.6 + 0.4 \cdot 0.1 + 0.2 \cdot 0.1 = 0.06 + 0.06 + 0.04 + 0.02 = 0.18$.
Hence $Conf(Bel', Bel'') = 0.1875 > 0.1800 = (m' \circledcirc m'')(\emptyset)$.

*Example 4.* **Modified Little Angel example** For following modification of Little Angel BFs we obtain counter-example again:

| | | | |
|---|---|---|---|
| $m^i(\{\omega_1\})$ | $= 0.05$ | - | |
| $m^i(\{\omega_2\})$ | $= 0.05$ | $m^{ii}(\{\omega_2\})$ | $= 0.05$ |
| $m^i(\{\omega_1, \omega_2\})$ | $= 0.05$ | - | |
| $m^i(\{\omega_2, \omega_4\})$ | $= 0.05$ | - | |
| - | | $m^{ii}(\{\omega_3, \omega_4\})$ | $= 0.10$ |
| $m^i(\{\omega_1, \omega_2, \omega_5\})$ | $= 0.10$ | - | |
| $m^i(\{\omega_2, \omega_3, \omega_4\})$ | $= 0.05$ | - | |
| $m^i(\{\omega_1, \omega_2, \omega_3, \omega_4\})$ | $= 0.20$ | - | |
| $m^i(\{\omega_1, \omega_3, \omega_4, \omega_5\})$ | $= 0.40$ | - | |
| - | | $m^{ii}(\{\omega_2, \omega_3, \omega_4, \omega_5\})$ | $= 0.80$ |
| $m^i(\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\})$ | $= 0.05$ | $m^{ii}(\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\})$ | $= 0.05$ |

where $Conf(Bel', Bel'') = 0.1114551$ while $(m' \circledcirc m'')(\emptyset) = 0.0875$.

After observation of the original examples, we had a working hypothesis of validity of equality $(*)$ for all quasi Bayesian BFs on a general finite frame of discernment, unfortunately, instead of proving the hypothesis we have found several counterexamples for both qBBF and even for Bayesian BFs already on four-element frame of discernment $\Omega_4$. We have used a method described in Section 8. The first counterexample found on $\Omega_4$ is shown in the next more general Example with $\varepsilon = 0$ in which case both BFs are Bayesian and $Conf(Bel^i, Bel^{ii}) = 0.984375$ while $(m^i \circledcirc m^{ii})(\emptyset) = 0.98$.

*Example 5.* **8-1-1-0** Assume the following class of BFs on $\Omega_4$:

$$
\begin{array}{rlrl}
m^i(\{\omega_1\}) & = 0.1 & m^{ii}(\{\omega_1\}) & = 0.1 \\
 & & m^{ii}(\{\omega_2\}) & = 0.8 - \varepsilon \\
m^i(\{\omega_3\}) & = 0.8 - \varepsilon & & \\
m^i(\{\omega_4\}) & = 0.1 & m^{ii}(\{\omega_4\}) & = 0.1 \\
m^i(\{\omega_1, \omega_2, \omega_3, \omega_4\}) & = \varepsilon & m^{ii}(\{\omega_1, \omega_2, \omega_3, \omega_4\}) & = \varepsilon
\end{array}
$$

Because the inequality $(*)$ holds for the majority of qBBFs on $\Omega_4$, i.e. for $n = 4$, size of belief mass moved to the entire frame must be rather small. Indeed, note that in case of $\varepsilon < 0.008$ the inequality $(*)$ does not hold (the exact bound is slightly higher). For $\varepsilon = 0.008$ we obtain $Conf(Bel^i, Bel^{ii}) = 0.964475 > 0.964064 = (m^i \circledcirc m^{ii})(\emptyset)$.

Any of the above-presented examples can be easily extended for a greater frame of discernment. For an extension of Example 8-1-1-0 to 10-element frame of discernment $\Omega_{10}$ see $m^{i-0}, m^{ii-0}$ in Example 8-small-small: Example 6, Section 6.

# 5 Validity of $Conf \leq m_{\circledcirc}(\emptyset)$

We can start from the simplest case of 2-element frame of discernment, which had motivated too strong version of the statement about $(*)$ in Belief'14 [7]:

**Lemma 1** *Inequality $Conf \leq m_{\circledcirc}(\emptyset)$ holds true for arbitrary BFs on a 2-element frame of discernment $\Omega_2$.*

*Proof.* Let us denote $(a_i, b_i) = (m_i(\{\omega_1\}), m_i(\{\omega_2\})$, thus $m_i(\{\omega_1, \omega_2\}) = 1 - a_i - b_i$. $m(\emptyset) = (m_1 \circledcirc m_2)(\emptyset) = a_1 b_2 + a_2 b_1$. Let us suppose $a_1 \geq b_1$. If also $a_2 \geq b_2$ then both maximal plausibilities are higher for $\omega_1$, thus $Conf(m_1, m_2) = 0 \leq m(\emptyset)$. Hence there still remain to prove the case $a_2 \leq b_2$. There it holds: $Pl_i = (1 - b_i, 1 - a_i)$ and $Pl\_P_i = (\frac{1-b_i}{2-a_i-b_i}, \frac{1-a_i}{2-a_i-b_i})$. $m_{01}(\{\omega_1\}) = a_{01} = \frac{a_1-b_1}{1-b_1}$, $m_{01} = (a_{01}, b_{01}) = (\frac{a_1-b_1}{1-b_1}, 0)$, and analogously $m_{02} = (a_{02}, b_{02}) = (0, \frac{b_2-a_2}{1-a_2}.)$. Thus $Conf(m_1, m_2) = (\frac{a_1-b_1}{1-b_1})(\frac{b_2-a_2}{1-a_2})$. Hence it remains to verify $(\frac{a_1-b_1}{1-b_1})(\frac{b_2-a_2}{1-a_2}) \leq a_1 b_2 + a_2 b_1$. We can show that $(\frac{a_1-b_1}{1-b_1}) \leq a_1$: $(a_1 - b_1) \geq a_1 1 - b_1) = a_1 - a_1 b_1$, $-b_1 \leq -a_1 b_1$, $0 \leq b_1(1 - a_1)$, what follows $0 \leq a, b \leq 1$, analogously we can show that $(\frac{b_2-a_2}{1-a_2}) \leq b_2$. Hence $(\frac{a_1-b_1}{1-b_1})(\frac{b_2-a_2}{1-a_2}) \leq a_1 b_2 \leq a_1 b_2 + a_2 b_1$.

**Lemma 2** *Inequality $Conf \leq m_{\circledcirc}(\emptyset)$ holds for any pair of consonant BFs on any finite $\Omega_n$.*

*Proof.* The statement follows the fact that there is the unique consonant non-conflicting part of a BF. Consonant BFs are because of this uniqueness equal to their non-conflicting parts. Hence $Conf(Bel^i, Bel^{ii}) = (m_0^i \textcircled{\cap} m_0^{ii})(\emptyset) = (m^i \textcircled{\cap} m^{ii})(\emptyset)$, thus $Conf = m_{\textcircled{\cap}}(\emptyset)$, hence inequality holds true.

**Corollary 3** *(i) Inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds for any pair of categorical BFs on any finite frame of discernment $\Omega_n$.*

*(ii) Inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds for any pair of simple support BFs on any finite frame of discernment $\Omega_n$.*

*Proof.* Both categorical and simple support belief functions are consonant, thus both (i) and (ii) are special cases of Lemma 2.

Unfortunately, as we have seen in Example 8-1-1-0, inequality $(*)$ does not holds either for two arbitrary quasi Bayesian BFs on $\Omega_4$. Nevertheless, We have no counter-example against validity of $(*)$ for qBBFs on $\Omega_3$ but we also do not have a complete proof of its validity yet. That is why we moved the issue of qBBFs on $\Omega_3$ to the next section about hypotheses. To complete this Section, we have to mention the following trivial observation:

**Observation 4** *If one of the belief functions in question is vacuous, inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ always holds. ($Conf = 0 = m_{\textcircled{\cap}}(\emptyset)$ in that cases.)*

# 6 Hypotheses

**Hypothesis 5** *Inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds true for any couple of quasi Bayesian belief functions on any 3-element frame $\Omega_3$.*

*Partial proof.* If we want to find a proof analogous to that for $\Omega_2$, there is no problem with $(m^i \textcircled{\cap} m^{ii})(\emptyset)$: formula for its computation from input bbms is always the same for given cardinality $n$ of $\Omega_n$. This is different for computation of value $Conf(Bel^i, Bel^{ii})$, where different focal elements appear in corresponding $Bel_0^i, Bel_0^{ii}$, also focal element of cardinality 2, thus there is not only higher complexity of formula for higher $n$, but the number of different formulas for different orders of values of plausibility of singletons (for qBBFs equal to order of input bbms). Moreover, neither analogy of proof on $\Omega_2$ has not been found for any of the cases of formulas on $\Omega_3$. Nevertheless, we have some kind of proof for some special cases of input qBBFs.

Two simplified cases have been already proven with the usage of *WolframAplha tool: https://www.wolframalpha.com* just by checking if the formula corresponding to the inequality $(*)$ has a solution in the $[0, 1]$ interval of respective variables. Nevertheless, a complete list of formulae for the general case has not been formulated yet, thus either proved.

Let us start from the case of Bayesian BFs with only two positive values: $(a, 1-a, 0|0)$, $(0, 1-a, a|0)$ where the notation corresponds to $(m(\omega_1), m(\omega_2), m(\omega_3)|m(\Omega_3))$, with maximal values assigned to different singletons (otherwise $Conf = 0$). If $a \geq 1-a$, i.e. $a \geq 1/2$, then we obtain: $Conf : \left(\frac{2a-1}{a}\right)^2 + 2\left(\frac{2a-1}{a}\right)\left(\frac{1-a}{a}\right)$ and $m(\emptyset) = a^2 + 2a(1-a) = 2a - a^2$.

Nevertheless, the inequality corresponding to the opposite of $(*)$: $\left(\frac{2a-1}{a}\right)^2 + 2\left(\frac{2a-1}{a}\right)\left(\frac{1-a}{a}\right) - 2a + a^2 > 0$ has a solution only for $a \notin [-1, 1]$ and therefore no counter-example can exists in this case.

Analogously we can consider BBFs one with three different values and the other with opposite order of the same bbms, thus $(a, b, 1-a-b|0)$, $(1-a-b, b, a|0)$. In case of $a > b > 1-a-b$ we obtain: $\left(\frac{a-b}{a}\right)^2 + 2\left(\frac{a+2b-1}{a}\right)\left(\frac{a-b}{a}\right) - 2a^2 - 2ab + b^2 + 2a - 1 > 0$ that is necessary to hold for a counter-example, but again, it has not solution for $0 <= b <= a <= 1-b$. Therefore NO counter-example can exist here.

In the completely general case for qBBFs on $\Omega_3$ it is necessary to verify several inequalities with 6 variables. As an example we can present the inequality $(\frac{a-b}{1-b-c})(\frac{f-e}{1-d-e}) + (\frac{a-b}{1-b-c})(\frac{e-d}{1-d-e}) + (\frac{b-c}{1-b-c})(\frac{1-d-2e}{1-d-e}) - (ae+af+bd+bf+cd+ce) \leq 0$, which should be verified for the case $(a, b, c|1-a-b-c)$ and $(c, d, e|1-c-d-e)$ where $a > b > c$ and $c < d < e$.

In the future, we would like to analogously check all the possible $\Omega_3$ cases.

**Hypothesis 6** *Inequality $Conf \leq m_{\bigcirc\!\!\!\cap}(\emptyset)$ holds true for any couple of quasi Bayesian belief functions, having all singletons ($m(\{\omega_i\}) > 0$ for any $\omega_i \in \Omega_n$) on any finite frame of discernment $\Omega_n$.*

Arguments for this hypothesis are as follows:

(i) We have not found any counter-example on $\Omega_n$ for $n <= 5$ yet.

(ii) When moving some positive mass to any singletons in Example 8-small-small, $Conf$ decreased bellow $m(\emptyset)$, thus counter-example disappears. This was checked both for BBF and qBBF counter-examples with BFs without some singletons on $\Omega_{10}$.

(iii) It is a sort of generalization of the previous hypothesis.

*Example 6.* **Example 8-small-small** Let suppose BBFs $m^{i-0}$ and $m^{ii-0}$ which are extensions of BFs from Example 8-1-1-0 to $\Omega_{10}$. And their further extension to qBBFs (thus qBBFs 'with 0', i.e. without some singleton focal elements, $m^{i-2}$ and $m^{ii-2}$; (zeros are not typed to be more visible). We can verify, that these are really counter-examples against general validity of $Conf \leq m_{\bigcirc\!\!\!\cap}(\emptyset)$:
$Conf(m^{i-0}, m^{ii-0}) = 0.9982937 > 0.995608 = (m^{i-0} \bigcirc\!\!\!\cap\, m^{ii-0})(\emptyset)$ and also
$Conf(m^{i-2}, m^{ii-2}) = 0.9884347 > 0.98788 = (m^{i-2} \bigcirc\!\!\!\cap\, m^{ii-2})(\emptyset)$.
When we remove some belief masses to missing singletons we obtain BBFs 'without zero' $m^{i-1}$, $m^{ii-1}$ and qBBFs 'without zero' $m^{i-3}$, $m^{ii-3}$ in both these cases counter-examples against $Conf \leq m_{\bigcirc\!\!\!\cap}(\emptyset)$ disappear:

$Conf(m^{i-1}, m^{ii-1}) = 0.9841313 < 0.986424 = (m^{i-1} \textcircled{\cap} m^{ii-1})(\emptyset)$and also
$Conf(m^{i-3}, m^{ii-3}) = 0.9744128 > 0.978536 = (m^{i-3} \textcircled{\cap} m^{ii-3})(\emptyset)$.

| $X \subseteq \Omega_{10}$ | BBF with 0 | | BBF without 0 | | qBBF with 0 | | qBBF without 0 | |
|---|---|---|---|---|---|---|---|---|
| $X \subseteq \Omega_{10}$ | $m^{i-0}$ | $m^{ii-0}$ | $m^{i-1}$ | $m^{ii-1}$ | $m^{i-2}$ | $m^{ii-2}$ | $m^{i-3}$ | $m^{ii-3}$ |
| $\{\omega_1\}$ | 0.800 | | 0.800 | 0.006 | 0.800 | | 0.800 | 0.006 |
| $\{\omega_2\}$ | 0.040 | 0.012 | 0.040 | 0.010 | 0.040 | 0.012 | 0.038 | 0.010 |
| $\{\omega_3\}$ | 0.034 | 0.016 | 0.034 | 0.014 | 0.034 | 0.014 | 0.032 | 0.014 |
| $\{\omega_4\}$ | 0.030 | 0.020 | 0.030 | 0.018 | 0.030 | 0.018 | 0.030 | 0.018 |
| $\{\omega_5\}$ | 0.026 | 0.022 | 0.026 | 0.022 | 0.026 | 0.022 | 0.026 | 0.022 |
| $\{\omega_6\}$ | 0.022 | 0.026 | 0.022 | 0.026 | 0.022 | 0.026 | 0.022 | 0.026 |
| $\{\omega_7\}$ | 0.020 | 0.030 | 0.018 | 0.030 | 0.018 | 0.030 | 0.018 | 0.030 |
| $\{\omega_8\}$ | 0.016 | 0.034 | 0.014 | 0.034 | 0.014 | 0.034 | 0.014 | 0.032 |
| $\{\omega_9\}$ | 0.012 | 0.040 | 0.010 | 0.040 | 0.012 | 0.040 | 0.010 | 0.028 |
| $\{\omega_{10}\}$ | | 0.800 | 0.006 | 0.800 | | 0.800 | 0.006 | 0.800 |
| $\Omega_{10}$ | | | | | 0.004 | 0.004 | 0.004 | 0.004 |

Thus we have a couple of similar BFs, one without some singleton, which is counter-example and the other with all singletons (positive bbms for all elements of the frame of discernment), which is not a counterexample. The same we have both for general qBBFs and BBFs. Note that there are also extensions of $m^{i-0}, m^{ii-0}$ to qBBFs which are not counter-examples either 'with 0' or 'without 0', nevertheless these are not interesting for us w.r.t. to Hypothesis 6.

# 7 Open Problems

There are plenty of open problems related to this topic, especially (i) to decide whether the hypotheses from the previous section hold true or not, and (ii) for which sets of belief functions inequality $(*)$ holds true and for which does not.

In the context of Lemma 2 it seems to be interesting to decide, whether the inequality holds also for any couple of consonant BFs. Nevertheless, this is not the case, as in both Examples 1 and 2 both BFs are consonant and therefore we already have counter-examples on $\Omega_3$ and $\Omega_5$.

A completely different question we did not study so far is whether inequality $(*)$ holds for separable support belief functions (i.e. Dempster's $\oplus$-sum of simple support functions). Note that it has a relation to statement (ii) in Corollary 3.

Because the inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds only for some types of BFs, we have to weaken the inequality for more general validity. Perspective/prospective seems to be question of validity $Conf \leq (\textcircled{\cap}_1^k (m_1 \textcircled{\cap} m_2))(\emptyset)$ for a convenient $k$. $(\textcircled{\cap}_1^k (m_1 \textcircled{\cap} m_2))(\emptyset)$ is related to the hidden conflicts and looking for full non-conflictness [8, 9]. We can see that $(*)$ is this inequality for $k = 1$. Due to a new parameter $k$ in the inequality this is rather complex challenging topic for our future research.

# 8 Computation

We have performed many computational experiments to find a counterexample to $(*)$ for different classes of BFs on different $\Omega_n$. To do so, we have used R-Project, a free tool for statistical computing and we have implemented all procedures needed to calculate various conflicts of BFs. BFs are represented using an object based on several database tables and we have employed the effective implementation of the join operation for relational databases.

The plan was to systematically search the whole space of BFs of a certain class. In our search for counterexamples we took BFs being on a grid defined using a fixed step for bbm values taken in consideration (e.g. 0.1). Then we generated all BFs with bbms which are multiples of the step. As an example of such BFs you can take $m'$ and $m''$ from Example 1.

The idea of the grid is quite simple. In the case of $n = 3$ and general BFs you have 8 possible focal elements, i.e. up to 8 bbms. Assume step 0.1. Then we can divide the total belief mass of 1 into 10 pieces. Find all sets of integers that sum to 10 and limit your search to sets of cardinality 8 and less. Then, to find all BFs with bbms divisible by the step, you just take all permutations of respective sets of integers multiplied by the step and use them as bbms.

Because of the exponential increase in the number of focal elements and the number of permutations, the amount of different BFs is overwhelming even for small $n$ regardless of the step. We were only able to go through a few classes. In the case of the other classes, we will have to run through the grid at random. Another method, which we successfully applied in Examples 5 and 6, is to identify an area with a chance of success and pass it with another step.

To illustrate the calculations, let us provide several numbers. Note that in case of $n = 3$ and step 0.1, there are 8.046 general BFs and 552 qBBFs. By decreasing the step to 0.01 we have $10^{11}$ BFs and 8.037 qBBFs. Note that out of these $64 \cdot 10^6$ combinations of qBBFs ($n = 3$, step: 0.01) 706.751 represent counterexamples.

In case of $n = 4$, there are $5 \cdot 10^{13}$ general BFs with step 0.1 and 3.600 qBBFs. In case of qBBFs we can decrease the step to 0.025 (136.824 qBBFs), 0.02 (318.269 qBBFs), or 0.01 (4.598.126 qBBFs).

In case of $n = 5$ and step 0.1 there are $10^{37}$ general BFs and 4.200 qBBFs.

# 9 Conclusion

Motivated by appearing of counter-examples against general validity of inequality $Conf \leq m_{\bigcirc\!\!\!\!\cap}(\emptyset)$, relations of value of conflict between belief functions based on non-conflicting parts $Conf$ and of sum of all multiples of disjoint focal elements of belief functions in question $m_{\bigcirc\!\!\!\!\cap}(\emptyset)$ have been analysed.

It has been proven that inequality $Conf \leq m_{\bigcirc\!\!\!\!\cap}(\emptyset)$ holds for any couple of BFs on $\Omega_2$, it was partially proved that it holds for any quasi Bayesian BFs on $\Omega_3$ and hypothesis that the inequality holds for any couple of quasi Bayesian BFs with positive values for all singletons on any finite frame $\Omega_n$ was formulated. Further, it was proven that it holds for any couple of consonant BFs on any finite frame $\Omega_n$.

Besides it was shown, where the inequality does not holds: e.g. general BFs, on $\Omega_3$, quasi Bayesian BFs without some singleton on $\Omega_4$, etc. Several still open issues were formulated.

This study enables a better understanding of the measure of conflict $Conf$ and to understanding of conflicts between belief function in general.

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