



Uniqueness of the nonlinear Schrödinger equation driven by jump processes

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Abstract. In a recent paper by the first two authors, existence of martingale solutions to a stochastic nonlinear Schrödinger equation driven by a Lévy noise was proved. In this paper, we prove pathwise uniqueness, uniqueness in law and existence of strong solutions to this problem using an abstract uniqueness result of Kurtz.

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1. Introduction

This paper is a natural continuation of [11], where the first two authors proved the existence of global solutions to a stochastic non-linear Schrödinger equation driven by a time homogeneous Poisson random measure. The existence of solutions was proved in the weak probabilistic sense, i.e., on an undetermined stochastic basis. The present paper aims to give sufficient conditions under which these solutions are also strong and unique. We proceed first by proving pathwise uniqueness of the solutions, and then we apply a result of Yamada–Watanabe–Kurtz to show that these solutions are strong and unique in law.

The Yamada–Watanabe theory has been well developed for stochastic equations driven by Wiener processes, see e.g. [8, 12, 14, 21, 23, 25, 30] or [1] for forward-backward stochastic differential equations. There are also analogous results for equations driven by Poisson random measures. For instance, in [2], the authors develop the Yamada–Watanabe theory for stochastic differential equations driven by both a Wiener process and a Poisson random

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measure (where the latter is defined on a locally compact space) via the original method of Yamada and Watanabe [30]. In [31], the Yamada–Watanabe theory is presented for variational solutions of partial differential equations driven by a Poisson random measure on a locally compact space also by the method of Yamada and Watanabe [30]. Unfortunately, none of these results applies to our problem, mainly because the noise does not live in a locally compact space.

Let us remind the reader that the original proof of Yamada and Watanabe is based on an application of a theorem on the existence of a regular version of a conditional probability. Their idea has proved to be so strong and robust to be applicable not only to stochastic differential equations but also to stochastic partial differential equations driven by various noises. In 2007, Kurtz [17] presented an abstract Yamada–Watanabe theory aiming not only at stochastic equations but also at many other stochastic problems of different nature. In that paper, Kurtz was the first one to abandon the original idea of the proof of Yamada and Watanabe (regular version of a conditional probability) and used the universal Skorokhod representation theorem. This approach made it possible to raise the Yamada–Watanabe theory to an abstract level (see also [18]) where details of particular problems, to which it is applicable, play no role. On the other hand, this abstract approach has one disadvantage for applications that everyone who wishes to apply the result must translate his particular problem to the language of [17] which itself is not straightforward.

In this paper, we recourse to [17] due to its generality. We consider mild solutions to stochastic partial differential equations in Banach spaces driven by time homogeneous Poisson random measures on a Polish space (which is not in general locally compact, so we cannot apply the results [2,31]), we translate this problem to the language of [17] and prove the standard existence of unique strong solutions. This result is then applied to the stochastic nonlinear Schrödinger equation driven by a time homogeneous Poisson random measure.

To be more precise, let $A = \Delta$ be the Laplace operator with $D(A) = \{u \in L^2(\mathbb{R}^d) : \Delta u \in L^2(\mathbb{R}^d)\}$. We are interested in the solution of the following equation

$$\begin{cases} i \, du(t, x) - \Delta u(t, x) \, dt + \lambda |u(t, x)|^{\alpha-1} u(t, x) \, dt \\ \quad = \int_S u(t, x) g(z(x)) \tilde{\eta}(dz, dt) + \int_S u(t, x) h(z(x)) \nu(dz) \, dt, \quad t \in [0, T], \\ u(0) = u_0, \end{cases} \quad (1.1)$$

where η denotes the Poisson random measure corresponding to L and $\tilde{\eta}$ the compensated Poisson random measure, ν the intensity of the Poisson random measure. This equation can be rewritten in terms of a Lévy process having characteristic measure ν . For more details on the connection of Lévy processes and Poisson random measure we refer to section 2.3 in [7] and the references therein. We would like to remark that Poisson random measures are more general than Lévy processes.

If the stochastic perturbation is a Wiener process, the equation is well treated, and the existence and uniqueness of the solution are known. For more

information see [9, 10]. In case the stochastic perturbation is replaced by a Lévy process with infinite activity, de Bouard and Hausenblas could only show in [11] the existence of a solution, without uniqueness. Here in this work, we are interested in conditions under which a unique solution exists.

Since we will use it later on, we will introduce some notations.

Notation 1.1. \mathbb{R} denotes the real numbers, $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. By \mathbb{N} we denote the set of natural numbers (including 0) and by $\bar{\mathbb{N}}$ we denote the set $\mathbb{N} \cup \{\infty\}$.

Notation 1.2. If $(\mathcal{F}_t)_{t \in [0, T]}$ is a filtration and θ a measure then we denote by \mathcal{F}_t^θ the augmentation of \mathcal{F}_t by the θ -null sets in \mathcal{F}_∞ .

Notation 1.3. The set of all finite non-negative measures on a Polish space (S, \mathcal{S}) will be denoted by $M_+(S)$ and $\mathcal{P}_1(S)$ will stand for probability measures on S . If a family of sets $\{S_n \in \mathcal{S} : n \in \mathbb{N}\}$ satisfy $S_n \uparrow S$ then $M_{\bar{\mathbb{N}}}(\{S_n\})$ denotes the family of all $\bar{\mathbb{N}}$ -valued measures θ on S such that $\theta(S_n) < \infty$ for every $n \in \mathbb{N}$. By $\mathcal{M}_{\bar{\mathbb{N}}}(\{S_n\})$ we denote the σ -field on $M_{\bar{\mathbb{N}}}(\{S_n\})$ generated by the functions $i_B : M_{\bar{\mathbb{N}}}(\{S_n\}) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}, B \in \mathcal{S}$.

2. Time homogeneous Poisson random measures

Since the definition of time homogeneous Poisson random measure is introduced in many, not always equivalent ways, we give here our definition.

Definition 2.1. (See [13], Def. I.8.1) Let (S, \mathcal{S}) be a Polish space, ν a σ -finite measure on (S, \mathcal{S}) , $\{S_n \in \mathcal{S}\}$ such that $S_n \uparrow S$ and $\nu(S_n) < \infty$ for every $n \in \mathbb{N}$. A time homogenous Poisson random measure η over a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, is a measurable function

$$\eta : (\Omega, \mathcal{F}) \rightarrow (M_{\bar{\mathbb{N}}}(\{S_n \times (0, T]\}), \mathcal{M}_{\bar{\mathbb{N}}}(\{S_n \times (0, T]\})),$$

such that

- (i) for each $B \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R}^+)$ with $\mathbb{E}\eta(B) < \infty$ $\eta(B) := i_B \circ \eta : \Omega \rightarrow \bar{\mathbb{N}}$ is a Poisson random variable with parameter $\mathbb{E}\eta(B)$, otherwise $\eta(B) = \infty$ a.s.
- (ii) η is independently scattered, i.e. if the sets $B_j \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R}^+), j = 1, \dots, n$, are disjoint, then the random variables $\eta(B_j), j = 1, \dots, n$, are mutually independent;
- (iii) for each $U \in \mathcal{S}$, the $\bar{\mathbb{N}}$ -valued process $(N(t, U))_{t \in [0, T]}$ defined by

$$N(t, U) := \eta(U \times (0, t]), \quad t \in [0, T]$$

is \mathbb{F} -adapted and its increments are stationary and independent of the past, i.e. if $t > s \geq 0$, then $N(t, U) - N(s, U) = \eta(U \times (s, t])$ is independent of \mathcal{F}_s .

Remark 2.2. In the framework of Definition 2.1 the assignment

$$\nu : \mathcal{S} \ni A \mapsto \mathbb{E}[\eta(A \times (0, 1))]$$

defines a uniquely determined measure, called in the following *intensity measure*.

Besides, the term Poisson random measure is sometimes defined in another way, i.e., by starting with the intensity measure and defining the Poisson random measure with a given intensity measure. However, we prove in the next lemma the equivalence between both definitions.

Lemma 2.3. *A measurable mapping $\eta : \Omega \rightarrow M_{\mathbb{N}}(\{S_n \times (0, T]\})$ is a time homogeneous Poisson random measure with intensity ν iff*

- (a) *for any $U \in \mathcal{S}$ with $\nu(U) < \infty$, the random variable $N(t, U)$ is Poisson distributed with parameter $t\nu(U)$, otherwise $\mathbb{P}(N(t, U) = \infty) = 1$;*
- (b) *for any n and disjoint sets $U_1, U_2, \dots, U_n \in \mathcal{S}$, and any $t \in [0, T]$, the random variables $N(t, U_1), N(t, U_2), \dots, N(t, U_n)$ are mutually independent;*
- (c) *the $M_{\mathbb{N}}(\{S_n\})$ -valued process $(N(t, \cdot))_{t \in (0, T]}$ is adapted to \mathbb{F} ;*
- (d) *for any $t \in [0, T]$, $U \in \mathcal{S}$, $\nu(U) < \infty$, and any $r, s \geq t$, the random variables $N(r, U) - N(s, U)$ are independent of \mathcal{F}_t .*

Proof. To see the equivalence of (i) and (a), first observe, that if S is a separable metric space, the Borel space $\mathcal{B}(S \times (0, T])$ of the cartesian product $S \times \mathbb{R}^+$ is the product of the Borel spaces $\mathcal{B}(S)$ and $\mathcal{B}(\mathbb{R}^+)$, see [22, p. 6, Theorem 1.10]. This implies the equivalence from (i) to (a). To show the equivalence of (ii) and (b), one has, in addition, to take into account that the Borel σ -algebra can be generated by intervals of the form $\{(0, t] : t \in (0, T]\}$. The equivalence of (iii), and (c) and (d) follows by the definition of $N(t, U)$. □

Usually, one starts with specifying the measurable space (S, \mathcal{S}) and the intensity measure ν on (S, \mathcal{S}) . Given this, then there exists a Poisson random measure on (S, \mathcal{S}) having the intensity measure ν .

In order to define a stochastic integral with respect to the Poisson random measure, S has to be related to a topological vector space, and the measure ν has either to be finite or has to be a Lévy measure. The definition of Lévy measures in Banach spaces can be found in [20, Chapter 5.4]. Here, we give a characterization of Lévy measures, which is sufficient for our purpose.

Remark 2.4. Let S be a separable Banach space, and \mathcal{S} its Borel σ -algebra. If the intensity measure $\nu : \mathcal{S} \rightarrow \mathbb{R}$ satisfies the integrability condition

$$\sup_{\substack{a \in S^* \\ |a| \leq 1}} \int_S 1 \wedge |\langle z, a \rangle|^2 \nu(dz) < \infty.$$

then ν is a Lévy measure (see [20, Proposition 5.4.1, p. 70]).

For some Banach spaces, one can characterize the Lévy measures in a more precise way. Therefore, let us introduce the following definition. Let $\{\varepsilon_k : k \in \mathbb{N}\}$ be a sequence of independent, identically distributed random variables with $\mathbb{P}(\varepsilon_1 = 1) = \mathbb{P}(\varepsilon_1 = -1) = \frac{1}{2}$. Then a Banach space with norm $|\cdot|$ is of R -type p , (Rademacher type p), where $1 \leq p \leq 2$, if for any sequence $\{x_j : j \in \mathbb{N}\}$ belonging to $l_p(E)$, we have (compare [20, p. 40])

$$\mathbb{P} \left(\left| \sum_{j=1}^{\infty} \varepsilon_j x_j \right| < \infty \right) = 1.$$

The Minkowski inequality implies that each Banach space is of R -type 1.

Remark 2.5. Let S be a Polish space, \mathcal{S} the Borel σ -algebra over S (in the sequel we call (S, \mathcal{S}) just a Polish space), let $\{S_n \in \mathcal{S}\}$ satisfy $S_n \uparrow S$, let ν be a σ -finite measure with $\nu(S_n) < \infty$ for every $n \in \mathbb{N}$ and fix $p \in [1, 2]$. We assume that E is a separable Banach space of R -type p and that $\xi : (S, \mathcal{S}) \rightarrow (E, \mathcal{B}(E))$ is a measurable mapping. In addition, we assume that the intensity measure $\nu : \mathcal{S} \rightarrow \mathbb{R}_0^+$ satisfies the integrability condition

$$\int_S 1 \wedge |\xi(z)|_E^p \nu(dz) < \infty, \quad \text{and} \quad \nu(\{0\}) = 0. \quad (2.1)$$

Then, the measure ν_E induced by ξ on E is a Lévy measure (and $\nu_E(\{0\}) := 0$) (compare [20, p. 75]). In addition, if η is a Poisson random measure with intensity ν over a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, the process

$$L : [0, T] \ni t \mapsto \int_0^t \int_S \xi(z) (\eta - \nu \times \lambda)(dz, ds)$$

is a Lévy process over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Here λ denotes the Lebesgue measure on \mathbb{R} .

Hence, from now on we will assume during the whole paper that the following convention is valid.

Convention 2.1. We convene that (S, \mathcal{S}) is a Polish space, ν a σ -finite measure on (S, \mathcal{S}) and $S_n \in \mathcal{S}$ such that $S_n \uparrow S$ and $\nu(S_n) < \infty$ for every $n \in \mathbb{N}$.

Let us consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ denotes a filtration. A process $\xi : [0, T] \times \Omega \rightarrow X$ is progressively measurable, or simply, progressive, if its restriction to $\Omega \times [0, t]$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$ -measurable for any $t \geq 0$. The predictable random field \mathcal{P} on $\Omega \times \mathbb{R}_+$ is the σ -field generated by all continuous \mathbb{F} -adapted processes (see e.g., Kallenberg [15, Chapter 25, p. 491]).

A real valued stochastic process $\{x(t) : t \in [0, T]\}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called predictable, if the mapping $x : \Omega \times (0, T] \rightarrow \mathbb{R}$ is $\mathcal{P}/\mathcal{B}(\mathbb{R})$ -measurable. A random measure γ on $\mathcal{S} \otimes \mathcal{B}((0, T])$ over $(\Omega; \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called predictable, iff for each $U \in \mathcal{S}$, the \mathbb{R} -valued process $(0, T] \ni t \mapsto \gamma(U \times (0, t])$ is predictable.

Definition 2.6. Assume that (S, \mathcal{S}) is a measurable space and ν is a non-negative σ -finite measure on (S, \mathcal{S}) . Assume that η is a time homogeneous Poisson random measure with intensity measure ν on (S, \mathcal{S}) over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The compensator of η is the unique predictable random measure, denoted by γ , on $\mathcal{S} \otimes \mathcal{B}((0, T])$ over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, such that for each $T < \infty$ and $A \in \mathcal{S}$ with $\mathbb{E}\eta(A \times (0, T]) < \infty$, the \mathbb{R} -valued processes $\{\tilde{N}(t, A)\}_{t \in (0, T]}$ defined by

$$\tilde{N}(t, A) := \eta(A \times (0, t]) - \gamma(A \times (0, t]), \quad 0 < t \leq T,$$

is a martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Remark 2.7. Assume that η is a time homogeneous Poisson random measure with intensity ν on (S, \mathcal{S}) over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. It turns out that the compensator γ of η is uniquely determined and moreover

$$\gamma : \mathcal{S} \times \mathcal{B}(\mathbb{R}^+) \ni (A, I) \mapsto \nu(A) \times \lambda(I).$$

The difference between a time homogeneous Poisson random measure η and its compensator γ , i.e. $\tilde{\eta} = \eta - \gamma$, is called a *compensated Poisson random measure*.

Let (S, \mathcal{S}) be a measurable space and let η be a time homogenous Poisson random measure on S with intensity measure ν being a positive σ -finite measure over \mathfrak{A} satisfying Convention 2.1. We will denote by $\tilde{\eta}$ the *compensated Poisson random measure* defined by $\tilde{\eta} := \eta - \gamma$.

Lemma 2.8. *Let ν be a non-negative σ -finite measure on S satisfying Convention 2.1. Then the following holds*

- (i) *there exists a probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{P})$ and a time homogenous Poisson random measure $\eta : \Omega \rightarrow M_{\mathbb{N}}(\{S_n \times (0, T]\})$ with the intensity measure ν ;*
- (ii) *Denote by Θ_ν the law of η on $\mathcal{M}_{\mathbb{N}}(\{S_n \times (0, T]\})$. If η^\sharp is a time homogenous Poisson random measure defined possibly on different stochastic base $\mathfrak{A}^\sharp = (\Omega^\sharp, \mathcal{F}^\sharp, \mathbb{P}^\sharp)$ and ν is the intensity measure for η^\sharp then Θ_ν is the law of η^\sharp on $\mathcal{M}_{\mathbb{N}}(\{S_n \times (0, T]\})$.*

Proof. Part (i) is given by Theorem 8.1 [13, p. 42]. It remains to show (ii). Since ν is σ -finite, there exists an increasing family $\{S_n : n \in \mathbb{N}\}$ with $S_{n+1} \supseteq S_n$, $S_n \uparrow S$, and $\nu(S_n) < \infty$. To show that η and η^\sharp have the same law on $\mathcal{M}_{\mathbb{N}}(S \times (0, T])$, we have to show that for all $f : S \times (0, T] \rightarrow \mathbb{R}$, bounded and continuous, the random variable $\eta(f) := \int_{S_n} \int_0^T f(s, t) \eta(ds, dt)$ and $\eta^\sharp(f) := \int_{S_n} \int_0^T f(s, t) \eta^\sharp(ds, dz)$ have the same law, see [22, Theorem 5.8, p. 38]. Since $S \times \mathbb{R}^+$ is a Polish space, the σ algebra generated by the family of bounded continuous functions coincides with the Borel- σ -algebra, see [26, Proposition 1.4, p.5]. Therefore, it is sufficient to show for all $n \in \mathbb{N}$, $U \in \mathcal{B}(S_n)$ and $I \in \mathcal{B}((0, T])$, that the random variables $\eta(U \times I)$ and $\eta^\sharp(U \times I)$ have the same law. Let Θ_ν^\sharp be the law of η^\sharp and let us assume $\nu(U), \lambda(I) < \infty$. Let $k \in \mathbb{N}_0$. Then, by the definition of the Poisson random measure and its intensity measure ν we know that

$$\begin{aligned} \Theta_\nu(\eta(U \times I) = k) &= e^{-\lambda(I)\nu(U)} \frac{(\nu(U) \lambda(I))^k}{k!} \\ &= \Theta_\nu^\sharp(\eta^\sharp(U \times I) = k). \end{aligned}$$

If $\nu(U) = \infty$ or $\lambda(I) = \infty$, then $\Theta_\nu(\eta(U \times I) = \infty) = 1 = \Theta_\nu^\sharp(\eta(U \times I) = \infty)$. □

Now, one can define the stochastic integral with respect to the Poisson random measure for progressively measurable integrands, introduced, e.g., in [5] in M -type p Banach spaces.

Definition 2.9. Let $0 < p \leq 2$. A Banach space E is of martingale type p iff there exists a constant $C > 0$ such that for all E -valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$\sup_{0 \leq n \leq N} \mathbb{E}|M_n|_E^p \leq C \mathbb{E} \sum_{n=0}^N |M_n - M_{n-1}|_E^p, \tag{2.2}$$

where as usually, we put $M_{-1} = 0$.

Examples of M -type p Banach spaces are, e.g., $L^q(\mathcal{O})$ spaces, where \mathcal{O} is a bounded domain. $L^q(\mathcal{O})$ is of M -type p for any $p \leq q$ (see e.g. [29, Chapter 2, Example 2.2]). If a Banach space E is of M -type p and A is the generator of an analytic semigroup on E , then the complex interpolation spaces between $D(A)$ and E are of M -type p . Similar facts also hold for real interpolation spaces, but not in this generality, for more details we refer to ‘‘Appendix A’’ of [4]. In particular, in [5] it is proven that for any Banach space E of M -type p there exists a unique continuous linear operator I which associates to each progressively measurable process $\xi : \mathbb{R}_+ \times \Omega \rightarrow L^p(S, \nu; E)$ with \mathbb{P} -a.s.

$$\int_0^T \int_S |\xi(r, x)|_E^p \nu(dx) dr < \infty \tag{2.3}$$

for every $T > 0$, an adapted E -valued càdlàg process

$$I_{\xi, \tilde{\eta}}(t) := \int_0^t \int_S \xi(r, x) \tilde{\eta}(dr, dx), \quad t \geq 0$$

such that if a process ξ satisfying the above condition (2.3) is a random step process with representation

$$\xi(r, x) = \sum_{j=1}^n 1_{(t_{j-1}, t_j]}(r) \xi_j(x), \quad x \in S, \quad r \in [0, T], \tag{2.4}$$

where $\{t_0 = 0 < t_1 < \dots < t_n < \infty\}$ is a finite partition of $[0, \infty)$ and for all $j \in \{1, \dots, n\}$, ξ_j is an E -valued $\mathcal{F}_{t_{j-1}}$ -measurable p -summable simple random variable, then

$$I_{\xi, \tilde{\eta}}(t) = \sum_{j=1}^n \int_S \xi_j(x) \tilde{\eta}((t_{j-1} \wedge t, t_j \wedge t], dx), \quad t \in [0, T]. \tag{2.5}$$

This definition can be extended to all progressively measurable mappings $\xi : \Omega \times [0, T] \times S \rightarrow E$ with \mathbb{P} -a.s.

$$\int_0^T \int_S \min(1, |\xi(r, z)|_E^p) \nu(dz) dr < \infty.$$

Some information on the different settings is given in [24]. Here, in this work we will only use $p = 2$, however, for the sake of completeness, we assume $p \in [1, 2]$.

Besides, we would like to point out in the following Proposition, that we do not need to suppose that the filtration of the given probability space is right continuous. In particular, given a Poisson random measure η over a

filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F})$, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, with an arbitrary filtration, a progressively measurable $L^2(S, \nu)$ -valued process ξ , one can pass to the right continuous augmentation of the filtration without losing the necessary properties. The following remark can be easily shown.

Remark 2.10. Let us assume that η is also a Poisson random measure for the augmented right continuous filtration $\bar{\mathcal{F}}_t := \wedge_{h>0} \mathcal{F}_{t+h}^{\mathbb{P}}$. Then, we can construct the stochastic integral on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and the stochastic integral on $(\Omega, \mathcal{F}, (\bar{\mathcal{F}}_t), \mathbb{P})$. In particular, let

$$I_1 = (\mathcal{F}_t) - \int_0^t \int_S \xi(r, x) \tilde{\eta}(dr, dx), \quad \text{resp.} \quad I_2 = (\bar{\mathcal{F}}_t) - \int_0^t \int_S \xi(r, x) \tilde{\eta}(dr, dx),$$

where in the integral on the left hand, we took for the predictable sequence of simple functions (ξ_n) converging to ξ the underlying filtration $(\mathcal{F}_t)_{t \geq 0}$ and in the integral on the right hand we took for predictable sequence of simple functions (ξ_n) converging to ξ the underlying filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$. Then, $I_1 = I_2$.

3. Pathwise uniqueness of the stochastic Schrödinger equation

We are interested in the uniqueness of the stochastic Schrödinger equation driven by a Lévy noise. The nonlinear Schrödinger equation is an example of a universal nonlinear model that describes many physical nonlinear systems. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena. The Schrödinger equation also arises in the context of water waves. In 1968 V.E. Zakharov derived the Nonlinear Schrödinger equation for the two-dimensional water wave problem in the absence of surface tension, that is, for the evolution of gravity-driven surface water waves. More recently, Villarroel, et all [27, 28] considered the nonlinear Schrödinger equation with randomly distributed, but isolated jumps. Dealing with jumps one may model sudden changes in the field that occur randomly.

In order to model abrupt changes in the medium, as it can, e.g., be the case for the propagation of light in optical fibers, or in other media, one can use Lévy noise. In [11] the first and second authors investigated the existence of a solution if the Lévy measure has infinite activity. However, no uniqueness was proven.

From now on, S will be a Borel subset of a separable Banach function space continuously embedded in the Sobolev space $W_{\infty}^1(\mathbb{R}^d)$. As mentioned in the introduction, the equation we are considering is given by

$$\begin{cases} i du(t, x) - \Delta u(t, x) dt + \lambda |u(t, x)|^{\alpha-1} u(t, x) dt \\ = \int_S u(t, x) g(z(x)) \tilde{\eta}(dz, dt) + \int_S u(t, x) h(z(x)) \gamma(dz, dt), \quad t \in [0, T], \\ u(0) = u_0. \end{cases} \quad (3.1)$$

with $\lambda \geq 0$. Here, $g : \mathbb{R} \rightarrow \mathbb{C}$ and $h : \mathbb{R} \rightarrow \mathbb{C}$ are two functions satisfying :

- (i) There exist some constants C_g and C_h such that

$$|g(\xi)| \leq C_g |\xi|, \quad |g'(\xi)| \leq C_g |\xi| \quad \text{and} \quad |h(\xi)| \leq C_h |\xi|, \quad |h'(\xi)| \leq C_h |\xi|.$$

Note that this implies :

- (ii) $g(0) = 0$ and $h(0) = 0$,
- (iii) $|\Im(h(\xi))| \lesssim |\xi|^2$ and $|\Im(g(\xi))| \lesssim |\xi|^2$.

Here, \Im denotes the imaginary part of a number. The integrability condition (2.1) for the intensity measure ν and condition (i) gives the necessary conditions for the corresponding Nemytskii operator induced by g .

Let us denote by $(\mathcal{T}(t))_{t \geq 0}$ the group of isometries generated by the operator $-iA$. In particular, for any $t \in \mathbb{R}$ and $u_0 \in L^2(\mathbb{R}^d)$ let us denote the solution of the following Cauchy problem

$$\begin{cases} i \dot{u}(t) = Au(t), \\ u(0) = u_0, \end{cases}$$

by $\mathcal{T}(t)u_0$. Observe, $(\mathcal{T}(t))_{t \geq 0}$ forms a unitary group on $L^2(\mathbb{R}^d)$ and $H_2^\gamma(\mathbb{R}^d)$ for any $\gamma \in \mathbb{R}$. In the framework of evolution equations, one considers the mild solution of Eq. (3.1), which is given by the following integral equation for $t \in [0, T]$

$$\begin{aligned} u(t) &= \mathcal{T}(t)u_0 + i\lambda \int_0^t \mathcal{T}(t-s)(|u(s)|^{\alpha-1}u(s)) ds \\ &\quad - i \int_0^t \int_S \mathcal{T}(t-s) [u(s)G(z)] \tilde{\eta}(dz, ds) \\ &\quad - i \int_0^t \int_S \mathcal{T}(t-s) [u(s)H(z)] \nu(dz) ds. \end{aligned}$$

Here, we used the Nemytskii operators corresponding to g and h . In particular, the mappings $G : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ and $H : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ denote the Nemytskii operators associated to the functions g and h defined by

$$(G(y))(x) := g(y(x)), \quad \text{and} \quad (H(y))(x) := h(y(x)), \quad y \in S, x \in \mathbb{R}^d.$$

Definition 3.1. Let $T > 0$. We call u an $L^2(\mathbb{R}^d)$ -valued mild solution to Eq. (3.1) on the time interval $[0, T]$, iff u is an adapted càdlàg process in $L^2(\mathbb{R}^d)$, the terms

$$\begin{aligned} &\int_0^t |\mathcal{T}(t-s)[|u(s)|^{\alpha-1}u(s)]|_{L^2} ds + \int_0^t \int_S |\mathcal{T}(t-s)[u(s)H(y)]|_{L^2} \nu(dy) ds \\ &\quad + \int_0^t \int_{\{y \in S: |\mathcal{T}(t-s)u(s)G(y)|_{L^2} \geq 1\}} |\mathcal{T}(t-s)[u(s)G(y)]|_{L^2} \nu(dy) ds, \\ &\quad + \int_0^t \int_{\{y \in S: |\mathcal{T}(t-s)u(s)G(y)|_{L^2} < 1\}} |\mathcal{T}(t-s)[u(s)G(y)]|_{L^2}^2 \nu(dy) ds \\ &< \infty \quad \mathbb{P}\text{-a.s for every } t \in [0, T] \end{aligned}$$

and for any $t \in [0, T]$, the process u solves \mathbb{P} -a.s. the integral equation

$$\begin{aligned}
 u(t) &= \mathcal{T}(t)u_0 + i\lambda \int_0^t \mathcal{T}(t-s)[|u(s)|^{\alpha-1}u(s)] ds \\
 &\quad - i \int_0^t \int_S \mathcal{T}(t-s) [u(s)G(y)] \tilde{\eta}(dy, ds) \\
 &\quad - i \int_0^t \int_S \mathcal{T}(t-s) [u(s)H(y)] \nu(dy) ds.
 \end{aligned} \tag{3.2}$$

Since in the proof of the existence of solution compactness arguments are used, the underlying probability space gets lost. Hence, a concept of probabilistic weak solutions has to be introduced, which is done in the following definition.

Definition 3.2. Let $u_0 \in L^2(\mathbb{R}^d)$. A *martingale solution* on $L^2(\mathbb{R}^d)$ to the Problem (3.1) is a system

$$(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, \eta, u) \tag{3.3}$$

such that

- (i) $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$,
- (ii) η is a time homogeneous Poisson random measure on (S, \mathcal{S}) over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with intensity measure ν satisfying Convention 2.1,
- (iii) u is an $L^2(\mathbb{R}^d)$ -valued mild solution to Eq. (3.1).

Let us remind that S is a function space continuously embedded in the Sobolev space $W^1_\infty(\mathbb{R}^d)$. Besides, we assume that the intensity measure satisfies the following integrability conditions:

- (i) $C_0(\nu) := \int_S |z|^2_{L^\infty} \nu(dz) < \infty$;
- (ii) $C_1(\nu) := \int_S |z|^2_{W^1_\infty} \nu(dz) < \infty$;
- (iii) $C_2(\nu) := \int_S \int_{\mathbb{R}^d} |x|^2 |z(x)|^2 dx \nu(dz) < \infty$;
- (iv) $C_3(\nu) := \int_S |z|^4_{L^\infty} \nu(dz) < \infty$.

Remark 3.3. One can see from the proof of Theorem 2.7 in [11] that the large jumps do not affect the uniqueness result and one can easily generalize our Theorem 3.4 to the case where one has large jumps without any bounded moments.

Let

$$1 \leq \alpha < \begin{cases} 1 + 4/(d-2) & \text{for } d > 2, \\ \infty & \text{for } d = 1 \text{ or } 2, \end{cases}$$

In [11] we have shown that under the conditions stated above, there exists a martingale solution. For the sake of completeness, we state here the main result of the article. Before, since we will need it later on, let us introduce the mass defined by

$$\mathcal{E}(u) := \int_{\mathbb{R}^d} |u(x)|^2 dx, \tag{3.4}$$

and the energy by

$$\mathcal{H}(u) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx + \frac{\lambda}{\alpha + 1} \int_{\mathbb{R}^d} |u(x)|^{\alpha-1} u(x) \overline{u(x)} dx. \quad (3.5)$$

Theorem 3.4. *Let η be a time homogenous Poisson random measure on S with Lévy measure ν satisfying the integrability conditions (i), (ii), (iii), and (iv) given above. If $u_0 \in H^1_2(\mathbb{R}^d)$, $\lambda \geq 0$, and*

$$\int_{\mathbb{R}^d} |x|^2 |u_0(x)|^2 dx < \infty,$$

then there exists a $H^1_2(\mathbb{R}^d)$ -valued martingale solution to (3.1), which is càdlàg in $H^\gamma_2(\mathbb{R}^d)$ for any $\gamma < 1$.

In addition, there exists a constant $C = C(T, C_0(\nu), C_3(\nu), C_g, C_h)$ such that

$$\mathbb{E} \sup_{0 \leq t \leq T} |u(t)|^2_{L^2} \leq C (1 + \mathbb{E}|u_0|^2_{L^2})$$

and for any $T > 0$ there exists a constant $C = C(T, C_0(\nu), C_1(\nu), C_g, C_h) > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} \mathcal{H}(u(t)) \leq C (1 + \mathbb{E}\mathcal{H}(u_0)).$$

The proof of Theorem 3.4 uses compactness arguments, hence, as mentioned before, the existence of a solution is shown, but no uniqueness of the solution. Here, we are interested in the uniqueness of the solution to Eq. (3.1). However, similar to the concept of solutions, there exist several concepts of uniqueness.

Definition 3.5. The Eq. (3.1) is *pathwise unique* if, whenever $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}, \eta, u_i)$, $i = 1, 2$ are solutions to (3.2) such that $\mathbb{P}(u_1(0) = u_2(0)) = 1$, then $\mathbb{P}(u_1(t) = u_2(t)) = 1$ for every $0 \leq t \leq T$.

Under certain conditions pathwise uniqueness of the stochastic Schrödinger equation driven by Lévy noise can be shown.

Theorem 3.6. *Let us assume that $g, h : \mathbb{R} \rightarrow \mathbb{C}$ are Lipschitz continuous. Let us assume that*

$$\begin{cases} 1 \leq \alpha < \infty & \text{if } d = 1, \text{ or } 2, \\ 1 \leq \alpha < \frac{d}{d-2} & \text{if } d > 2. \end{cases} \quad (3.6)$$

Let be given a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, a Poisson random measure defined on \mathfrak{A} adapted to the filtration \mathbb{F} , and two mild solutions u_1 and u_2 to Eq. (3.1) over \mathfrak{A} , on $[0, T]$ such that u_1 and u_2 are càdlàg in $L^2(\mathbb{R}^d)$. If there exists some $\delta \in \mathbb{R}$ with

$$\delta > \begin{cases} \frac{d}{2} - \frac{d}{2(\alpha-1)}, & \text{if } d = 1, 2, \\ \frac{d}{2} - \frac{1}{\alpha-1}, & \text{if } d > 2, \end{cases} \quad (3.7)$$

and the solutions u_1 and u_2 are belonging \mathbb{P} -a.s. to $\mathbb{D}([0, T]; H^\delta_2(\mathbb{R}^d))$, then u_1 and u_2 are indistinguishable in $L^2(\mathbb{R}^d)$.

Remark 3.7. Condition (3.6) is needed to apply the Strichartz estimate and Sobolev embedding theorems, in order to handle the nonlinearity. The restriction compared to usual conditions is due to the fact that we have to estimate the difference of solutions in $L^2(\mathbb{R}^d)$ to tackle the stochastic terms.

Proof of Theorem 3.6. Note that the solutions given by Theorem 1.2 of [11] satisfy the assumption of Theorem 3.6. Moreover, if u_1 and u_2 are as above, then they belong a.s. to $L^\infty([0, T]; H_2^\delta(\mathbb{R}^d))$.

In the first step we will introduce a family of stopping times $\{\tau_m : m \in \mathbb{N}\}$ and show that on the time interval $[0, \tau_m]$ the solutions u_1 and u_2 are indistinguishable. In the second step, we will show that $\mathbb{P}(\tau_m < T) \rightarrow 0$ for $m \rightarrow \infty$. From this follows that u_1 and u_2 are indistinguishable on the time interval $[0, T]$. \square

Step I Let us introduce the stopping times $\{\tau_m^1 : m \in \mathbb{N}\}$ and $\{\tau_m^2 : m \in \mathbb{N}\}$ given by

$$\tau_m^i := \sup\{s > 0 : |u_i(s)|_{H_2^\delta} < m\} \wedge T, \quad i = 1, 2.$$

The aim is to show that u_1 and u_2 are indistinguishable on the time interval $[0, \tau_m]$, with $\tau_m = \inf(\tau_m^1, \tau_m^2)$.

Fix $m \in \mathbb{N}$. To get uniqueness on $[0, \tau_m]$ we first stop the original solution processes at time τ_m and extend the processes u_1 and u_2 by other processes to the whole interval $[0, T]$. For this propose, let y_1 be a solution to

$$\begin{aligned} y_1(t) &= \mathcal{T}(t - \tau_m)u_1(\tau_m) - i \int_{\tau_m}^t \int_S \mathcal{T}(t - s)y_1(s) G(z) \tilde{\eta}(dz, ds) \\ &\quad - i \int_{\tau_m}^t \int_S \mathcal{T}(t - s)y_1(s)H(z)\gamma(dz, ds), \quad t \geq \tau_m, \end{aligned} \tag{3.8}$$

and let y_2 be a solution to

$$\begin{aligned} y_2(t) &= \mathcal{T}(t - \tau_m)u_2(\tau_m) - i \int_{\tau_m}^t \int_S \mathcal{T}(t - s)y_2(s) G(z) \tilde{\eta}(dz, ds) \\ &\quad - i \int_{\tau_m}^t \int_S \mathcal{T}(t - s)y_2(s)H(z)\gamma(dz, ds), \quad t \geq \tau_m. \end{aligned} \tag{3.9}$$

Since u_1 and u_2 are càdlàg in $L^2(\mathbb{R}^d)$, $u_1(\tau_m)$ and $u_2(\tau_m)$ are well defined and belong \mathbb{P} -a.s. to $L^2(\mathbb{R}^d)$. Since, in addition, $(\mathcal{T}(t))_{t \in \mathbb{R}}$ is a strongly continuous group on $L^2(\mathbb{R}^d)$, the existence of unique solutions y_1 and y_2 to (3.8) and (3.9) in $L^2(\mathbb{R}^d)$ can be shown by standard methods. Now, let us define two processes \bar{u}_1 and \bar{u}_2 which are equal to u_1 and u_2 on the time interval $[0, \tau_m]$ and follow the linear Schrödinger equation y_1 and y_2 afterwards. In particular, let

$$\bar{u}_1(t) = \begin{cases} u_1(t) & \text{for } 0 \leq t < \tau_m, \\ y_1(t) & \text{for } \tau_m \leq t \leq T, \end{cases}$$

and

$$\bar{u}_2(t) = \begin{cases} u_2(t) & \text{for } 0 \leq t < \tau_m, \\ y_2(t) & \text{for } \tau_m \leq t \leq T. \end{cases}$$

Note, that \bar{u}_1 and \bar{u}_2 solve the truncated equation corresponding to (3.2), that is

$$\begin{aligned} u(t) &= \mathcal{T}(t)u_0 + i\lambda \int_0^t 1_{[0,\tau_m)}(s) \mathcal{T}(t-s) \left(|u(s)|^{\alpha-1} u(s) \right) ds \\ &\quad - i \int_0^t \int_S \mathcal{T}(t-s) u(s) G(z) \tilde{\eta}(dz, ds) \\ &\quad - i \int_0^t \int_S \mathcal{T}(t-s) u(s) H(z) \gamma(dz, ds). \end{aligned} \tag{3.10}$$

For $u_0 \in L^2(\mathbb{R}^d)$ and $\xi \in \mathcal{N}^2(\Omega; L^2([0, T]; L^2(\mathbb{R}^d)))$ progressively measurable with respect to the filtration \mathbb{F} , let us define the integral operator

$$(\mathfrak{J}\xi)(t) := \mathcal{T}(t)u_0 + (\mathfrak{F}_{\tau_m}\xi)(t) + (\mathfrak{G}\xi)(t) + (\mathfrak{H}\xi)(t), \quad t \in [0, T], \tag{3.11}$$

where $u_0 \in L^2(\mathbb{R}^d)$, the integral operator \mathfrak{F}_{τ_m} with respect to the nonlinear term is defined by

$$(\mathfrak{F}_{\tau_m}\xi)(t) = i\lambda \int_0^t \mathcal{T}(t-s) (|\xi(s)|^{\alpha-1} \xi(s)) 1_{[0,\tau_m)}(s) ds, \quad t \in [0, T],$$

\mathfrak{G} is defined by

$$(\mathfrak{G}\xi)(t) = -i \int_0^t \int_S \mathcal{T}(t-s) \xi(s) G(z) \tilde{\eta}(dz, ds), \quad t \in [0, T],$$

and \mathfrak{H} is defined by

$$(\mathfrak{H}\xi)(t) := -i \int_0^t \int_S \mathcal{T}(t-s) \xi(s) H(z) \gamma(dz, ds), \quad t \in [0, T].$$

In the next step we will calculate the difference $\bar{u}_1 - \bar{u}_2$. Fix $0 \leq t \leq T$. Similarly as before we have

$$\begin{aligned} \bar{u}_1(t) - \bar{u}_2(t) &= (\mathfrak{G}\bar{u}_1)(t) - (\mathfrak{G}\bar{u}_2)(t) \\ &\quad + (\mathfrak{H}\bar{u}_1)(t) - (\mathfrak{H}\bar{u}_2)(t) + (\mathfrak{F}_{\tau_m}\bar{u}_1)(t) - (\mathfrak{F}_{\tau_m}\bar{u}_2)(t). \end{aligned}$$

Note, that \mathfrak{G} and \mathfrak{H} are linear. In addition, $(\mathcal{T}(t))_{t \in \mathbb{R}}$ is a unitary group on $L^2(\mathbb{R}^d)$. Therefore,

$$\mathbb{E} |\mathfrak{G}u_1(t) - \mathfrak{G}u_2(t)|_{L^2}^2 \leq \int_0^t \int_S \mathbb{E} |(\bar{u}_1(s) - \bar{u}_2(s))G(z)|_{L^2}^2 \nu(dz) ds.$$

Due to the integrability conditions on page 3, i.e. the integrability condition (i), and the fact that g is Lipschitz continuous, we know that for any $v \in L^2(\mathbb{R}^d)$ we have $\int_S |v G(z)|_{L^2} \nu(dz) \leq C_g C_0(\nu) |v|_{L^2}$ and we can proceed

$$\mathbb{E} |\mathfrak{G}\bar{u}_1(t) - \mathfrak{G}\bar{u}_2(t)|_{L^2}^2 \leq C_g C_0(\nu) \int_0^t \mathbb{E} |\bar{u}_1(s) - \bar{u}_2(s)|_{L^2}^2 ds.$$

Similarly, we get for \mathfrak{H} by the Minkowski inequality and the Lipschitz continuity of h ,

$$\begin{aligned} \mathbb{E} |\mathfrak{H}\bar{u}_1(t) - \mathfrak{H}\bar{u}_2(t)|_{L^2}^2 &\leq C_h t \int_0^t \int_Z \mathbb{E} |(\bar{u}_1(s) - \bar{u}_2(s))H(z)|_{L^2}^2 \nu(dz) ds \\ &\leq C C_h C_0(\nu) T \int_0^t \mathbb{E} |\bar{u}_1(s) - \bar{u}_2(s)|_{L^2}^2 ds. \end{aligned}$$

The only term, which has to be carefully analysed is the nonlinear term given by

$$\begin{aligned} & (\mathfrak{F}_{\tau_m} \bar{u}_1)(t) - (\mathfrak{F}_{\tau_m} \bar{u}_2)(t) \\ &= i\lambda \int_0^t \mathcal{T}(t-s) (|\bar{u}_1(s)|^{\alpha-1} \bar{u}_1(s) - |\bar{u}_2(s)|^{\alpha-1} \bar{u}_2(s)) \mathbf{1}_{[0, \tau_m)}(s) ds. \end{aligned}$$

Let γ' and σ be given such that $\frac{1}{\sigma} + \frac{1}{2} = \frac{1}{\gamma'}$,

$$-\delta + \frac{d}{2} < \frac{d}{\sigma(\alpha-1)}, \tag{3.12}$$

and

$$2 \geq \gamma' \begin{cases} \geq 1 & \text{if } d = 1, \\ > 1 & \text{if } d = 2, \\ \geq \frac{2d}{d+2} & \text{if } d > 2. \end{cases} \tag{3.13}$$

Due to our assumption on α and δ , such a couple (γ', σ) exists. Let (γ, ρ) be an admissible pair, where γ is the conjugate exponent to γ' , i.e. $2/\rho = d(1/2 - 1/\gamma)$. Let ρ' be the conjugate exponent to ρ . Then, we have

$$2 \leq \gamma \begin{cases} \leq \infty & \text{if } d = 1, \\ < \infty & \text{if } d = 2, \\ \leq \frac{2d}{d-2} & \text{if } d > 2, \end{cases} \text{ and } 1 \leq \rho' \begin{cases} \leq \frac{4}{3} & \text{if } d = 1, \\ < 2 & \text{if } d = 2, \\ \leq 2 & \text{if } d > 2. \end{cases} \tag{3.14}$$

Before continuing, let us shortly introduce the Strichartz estimate. Let us define the convolution operator

$$\mathfrak{I}u(t) := \int_0^t \mathcal{T}(t-r)u(r) dr, \quad t \geq 0. \tag{3.15}$$

Let $(p_0, q_0), (p_1, q_1) \in [2, \infty) \times [1, \infty)$ be two admissible pairs. Then for all $T > 0$ we have by the Strichartz estimate (see [19, p. 64])

$$|\mathfrak{I}u|_{L^{q_0}(0, T; L^{p_0})} \leq C |u|_{L^{q'_1}(0, T; L^{p'_1})}.$$

Applying the Strichartz estimate we have, for any $r \geq 1$,

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left| \int_0^t \mathcal{T}(t-s) (|\bar{u}_2(s)|^{\alpha-1} \bar{u}_2(s) - |\bar{u}_1(s)|^{\alpha-1} \bar{u}_1(s)) \mathbf{1}_{[0, \tau_m)}(s) ds \right|_{L^2}^{\rho' r} \\ & \leq \left| (|\bar{u}_2(s)|^{\alpha-1} \bar{u}_2(s) - |\bar{u}_1(s)|^{\alpha-1} \bar{u}_1(s)) \mathbf{1}_{[0, \tau_m)}(s) \right|_{L^{\rho'}([0, T]; L^{\gamma'})}^{\rho' r}. \end{aligned}$$

By Hölder's inequality, recalling that σ is such that $\frac{1}{\sigma} + \frac{1}{2} = \frac{1}{\gamma'}$, so that

$$\sigma \begin{cases} \geq 2 & \text{if } d = 1, \\ > 2 & \text{if } d = 2, \\ \geq d & \text{if } d > 2, \end{cases} \tag{3.16}$$

the right hand side of the above expression may be bounded by

$$\begin{aligned} & \left(\int_0^T \left(|\bar{u}_2(s)|^{\alpha-1} + |\bar{u}_1(s)|^{\alpha-1} \right) 1_{[0, \tau_m]} \Big|_{L^\sigma}^{\rho'} \left| (\bar{u}_1(s) - \bar{u}_2(s)) 1_{[0, \tau_m]}(s) \Big|_{L^2}^{\rho'} ds \right)^r \\ & \leq \left(\int_0^T \left(|\bar{u}_2(s)| + |\bar{u}_1(s)| \right) 1_{[0, \tau_m]} \Big|_{L^{(\alpha-1)\sigma}}^{(\alpha-1)\rho'} \left| (\bar{u}_1(s) - \bar{u}_2(s)) 1_{[0, \tau_m]}(s) \Big|_{L^2}^{\rho'} ds \right)^r. \end{aligned}$$

By inequality (3.12) and Sobolev embeddings Theorems we know that $H_2^\delta(\mathbb{R}^d) \hookrightarrow L^{\sigma(\alpha-1)}(\mathbb{R}^d)$ continuously. This implies that the right hand side above is in turn bounded by

$$\left(\int_0^T \left(|\bar{u}_2(s)| + |\bar{u}_1(s)| \right) 1_{[0, \tau_m]} \Big|_{H_2^\delta}^{(\alpha-1)\rho'} \left| (\bar{u}_1(s) - \bar{u}_2(s)) 1_{[0, \tau_m]}(s) \Big|_{L^2}^{\rho'} ds \right)^r,$$

and, using the Hölder inequality, by

$$\begin{aligned} & C(T) \sup_{s \in [0, T]} \left| \left(|\bar{u}_2(s)| + |\bar{u}_1(s)| \right) 1_{[0, \tau_m]}(s) \Big|_{H_2^\delta}^{(\alpha-1)\rho'} \right. \\ & \quad \times \left. \int_0^T \left| (\bar{u}_1(s) - \bar{u}_2(s)) 1_{[0, \tau_m]}(s) \Big|_{L^2}^{r\rho'} ds \right. \end{aligned}$$

The definition of the stopping time allows to bound the above term by

$$2C(T)m^{(\alpha-1)\rho' r} \int_0^T \left| (\bar{u}_1(s) - \bar{u}_2(s)) 1_{[0, \tau_m]}(s) \Big|_{L^2}^{r\rho'} ds.$$

Now, taking $r = 2/\rho'$, an application of the Grownwall Lemma gives $\mathbb{E}|\bar{u}_1(t) - \bar{u}_2(t)|_{L^2}^2 = 0$. Since \bar{u}_1 and \bar{u}_2 are càdlàg on $L^2(\mathbb{R}^d)$, both processes \bar{u}_1 and \bar{u}_2 are indistinguishable in $L^2(\mathbb{R}^d)$ on the time interval $[0, \tau_m]$.

Step II: We show that $\mathbb{P}(\tau_m < T) \rightarrow 0$ as $m \rightarrow \infty$. Observe that, thanks to the definition of τ_m ,

$$\{\tau_m < T\} \subset \{|u_1|_{L^\infty([0, T]; H_2^\delta)} \geq m \text{ or } |u_2|_{L^\infty([0, T]; H_2^\delta)} \geq m\}.$$

Therefore,

$$\mathbb{P}(\tau_m < T) \leq \mathbb{P}\left(|u_1|_{L^\infty([0, T]; H_2^\delta)} \geq m\right) + \mathbb{P}\left(|u_2|_{L^\infty([0, T]; H_2^\delta)} \geq m\right).$$

Since u_1 and u_2 are càdlàg in $H_2^\delta(\mathbb{R}^d)$, and \mathbb{P} -a.s. $\sup_{0 \leq s \leq T} |u_1(s)|_{H_2^\delta} < \infty$ and $\sup_{0 \leq s \leq T} |u_2(s)|_{H_2^\delta} < \infty$, it follows

$$\mathbb{P}\left(|u_i|_{L^\infty([0, T]; H_2^\delta)} \geq m\right) \longrightarrow 0,$$

as $m \rightarrow \infty$, for $i = 1, 2$. This implies $\mathbb{P}(\tau_m < T) \rightarrow 0$ as $m \rightarrow \infty$. Hence, both processes u_1 and u_2 are undistinguishable on $[0, T]$. \square

Due to pathwise uniqueness, one can show that a unique strong solution exists.

Theorem 3.8. *If the conditions of Theorems 3.4 and 3.6 are satisfied, then there exists a unique strong solution to Eq. (3.2) in $\mathbb{D}([0, T]; L^2(\mathbb{R}^d))$.*

Proof. Let $\delta < 1$ be the constant given in Theorem 3.6. In order to apply Theorem 5.14 below, we put $Y = H_2^{-30d}(\mathbb{R}^d)$, $Y_0 = \mathcal{D}(\mathbb{R}^d)$, $X = H_2^1(\mathbb{R}^d)$, and

$$\mathcal{X} := \mathbb{D}([0, T]; L^2(\mathbb{R}^d)).$$

We have chosen $Y = H_2^{-30d}(\mathbb{R}^d)$, since Y can be arbitrarily large, we only have to be sure that the following points are satisfied and these points are satisfied by this choice. Let us define the functions

- $a : [0, T] \times X \rightarrow Y$
 $(t, x) \mapsto \mathcal{T}(t)x;$
- $b : [0, T] \times [0, T] \times X \rightarrow Y$
 $(t, s, x) \mapsto \mathcal{T}(t-s) 1_{[0,t)}(s) F_\alpha(x) + \int_S \mathcal{T}(t-s) 1_{[0,t)}(s) [xH(z)]\nu(dz),$

in where $F_\alpha(x) = |x|^{\alpha-1}x$, for $x \in X \subset L^{\alpha+1}(\mathbb{R}^d)$, so that $F_\alpha(x) \in L^{(\alpha+1)/\alpha}(\mathbb{R}^d)$;

- $c : [0, T] \times [0, T] \times S \times X \rightarrow Y$
 $(t, s, z, x) \mapsto \mathcal{T}(t-s) 1_{[0,t)}(s) (xG(z));$
- $\theta_i^{\alpha_i} : \mathbb{D}([0, T], Y) \rightarrow [0, \infty]$, $\alpha_i \in A_i$, $i \in \{0, 1\}$, with $A_0 = [0, T]$, $A_1 = \emptyset$,

$$\theta_0^0(u) = \begin{cases} \infty & \text{if } u \notin \mathbb{D}(0, T; H_2^\delta(\mathbb{R}^d)) \\ 0 & \text{if } u \in \mathbb{D}(0, T; H_2^\delta(\mathbb{R}^d)); \end{cases}$$

if $t \in (0, T]$ then

$$\theta_0^t(u) = \infty \text{ if } u \notin \mathbb{D}([0, T]; L^2(\mathbb{R}^d))$$

and, for $u \in D([0, T]; L^2(\mathbb{R}^d))$,

$$\begin{aligned} \theta_0^t(u) &= \int_0^t \{|\mathcal{T}(t-s)F_\alpha(u(s))|_{L^2} \\ &+ \int_{\{y \in S: |\mathcal{T}(t-s)[u(s)G(y)]|_{L^2} < 1\}} |\mathcal{T}(t-s)[u(s)G(y)]|_{L^2}^2 \\ &+ \int_{\{y \in S: |\mathcal{T}(t-s)[u(s)G(y)]|_{L^2} \geq 1\}} |\mathcal{T}(t-s)[u(s)G(y)]|_{L^2} \\ &+ |\mathcal{T}(t-s)[u(s)H(y)]|_{L^2} \nu(dy)\} ds \end{aligned}$$

where $F_\alpha(x)$ is as above. □

Remark 3.9. Roughly speaking, the setting in Theorem 3.8 has to fit the setting in Sect. 4 and has to be chosen as follows. The space X has to be a space such that for any $t \geq 0$, the random variable $u(t)$ is X valued. Note, that this space need not coincide with the space where the process is cádlág. The space Y has to be chosen such that the mappings a , b and c are well defined, $\mathbb{D}([0, T]; Y)$ corresponds to the Kurtz space (denoted in his paper by Z_1), moreover, \mathcal{X} is the path space. Additional regularity properties of the solution can be incorporated in the family of functions $\{\theta_0^{\alpha_0} : \alpha_0 \in A_0\}$ and $\{\theta_1^{\alpha_1} : \alpha_1 \in A_1\}$, where A_0 and A_1 are index sets. In our case, we choose $A_0 = [0, T]$ and $A_1 = \emptyset$.

Remark 3.10. The setting need not to be unique. Thus, instead of the setting above, we could choose $X = H_2^1(\mathbb{R}^d)$, $\mathcal{X} = \mathcal{X}_0 = D([0, T]; H_2^\delta(\mathbb{R}^d))$, $A_0 = \emptyset$, and $A_1 = \emptyset$. Indeed, due to the fact that given pathwise uniqueness in $L^2(\mathbb{R}^d)$ with the condition $\mathbb{P}(u \in \mathbb{D}([0, T]; H_2^\delta(\mathbb{R}^d))) = 1$ one has pathwise uniqueness in $H_2^\delta(\mathbb{R}^d)$ and we could take directly $\mathcal{X} = D([0, T]; H_2^\delta(\mathbb{R}^d))$.

In the next step, we have to verify that the mappings a, b and c are measurable. In fact, first note that \mathcal{T} is a strongly continuous unitary group on $H_2^{-30d}(\mathbb{R}^d)$, therefore, a is measurable. To show that b is measurable, we first investigate the measurability of $\mathcal{T}(t-s) 1_{[0,t)}(s) F_\alpha(x)$. Here, we show that $F_\alpha : H_2^1(\mathbb{R}^d) \rightarrow Y$ is continuous, from which follows that $F_\alpha : H_2^1(\mathbb{R}^d) \rightarrow Y$ is measurable. Indeed, using Hölder inequality, it is easily seen that for any $x, y \in L^{\alpha+1}(\mathbb{R}^d)$,

$$|F_\alpha(x) - F_\alpha(y)|_{L^{(\alpha+1)/\alpha}(\mathbb{R}^d)} \leq C \left(|x|_{L^{\alpha+1}(\mathbb{R}^d)}^{\alpha-1} + |y|_{L^{\alpha+1}(\mathbb{R}^d)}^{\alpha-1} \right) |x - y|_{L^{\alpha+1}(\mathbb{R}^d)},$$

and the continuity result follows from the embeddings $H_2^1(\mathbb{R}^d) \hookrightarrow L^{\alpha+1}(\mathbb{R}^d)$ and $L^{(\alpha+1)/\alpha}(\mathbb{R}^d) \hookrightarrow Y$. Since $1_{[0,t)}(s)$ is measurable, and $(\mathcal{T}(t))_{t \in \mathbb{R}}$ is pointwise strongly continuous, we are done. It remains to show that the second term of b is measurable, but this follows since, by the Lipschitz continuity of h , and the fact that $h(0) = 0$, the Nemytsky operator H is continuous from $L^2(\mathbb{R}^d)$ to Y . Similarly the last mapping c can be handled. It remains to verify that θ_0^0 is a measurable mapping. But this is given, since $\mathbb{D}([0, T]; H_2^\delta(\mathbb{R}^d))$ is a Borel subset of $\mathbb{D}([0, T]; Y)$. Now, the existence of the strong solutions follows by an application of Theorem 5.14. □

4. The abstract uniqueness result

Let X and Y be separable Fréchet spaces, $Y_0 \subseteq Y^*$ separates points in Y , \mathcal{A} a Borel subset in $\mathbb{D}([0, T], Y)$, A_1 and A_2 be two index sets and

- $a : [0, T] \times X \rightarrow Y$, $b : [0, T] \times [0, T] \times X \rightarrow Y$,
- $c : [0, T] \times [0, T] \times S \times X \rightarrow Y$,
- $\theta_i^{\alpha_i} : \mathbb{D}([0, T], Y) \rightarrow [0, \infty]$, $\alpha_i \in A_i$, $i \in \{0, 1\}$

be measurable mappings, ν a σ -finite measure on (S, \mathcal{S}) , and, finally, $S_n \in \mathcal{S}$ such that $S_n \uparrow S$ and $\nu(S_n) < \infty$.

Let η be a time homogeneous Poisson random measure with intensity measure ν on the space (S, \mathcal{S}) defined over a probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where \mathbb{F} denotes a filtration $(\mathcal{F}_t)_{t \in [0, T]}$. Given is an abstract evolution equation of the following form:

$$\begin{aligned} \langle u(t), \varphi \rangle &= \langle a(t, u(0)), \varphi \rangle + \int_0^t \langle b(t, s, u(s)), \varphi \rangle ds \\ &\quad + \int_0^t \int_S \langle c(t, s, x, u(s)), \varphi \rangle \tilde{\eta}(dx, ds), \end{aligned} \tag{4.1}$$

for every $t \in [0, T]$ and $\varphi \in Y_0$. We define next the concept of solution in the way we will use it in the following pages of the article.

Definition 4.1. We say that a 6-tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, u, \eta)$, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, consisting of a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a time homogeneous Poisson random measure η on (S, \mathcal{S}) over \mathfrak{A} with intensity measure ν and a process u on $[0, T]$, being \mathbb{F} -adapted and càdlàg in Y , is a solution of (4.1) provided that

$$\mathbb{P}(u(t) \in X) = 1, \quad \forall t \in [0, T], \quad \mathbb{P}(u \in \mathcal{X}) = 1,$$

there exists $p \in [1, 2]$ such that, for all $t \in [0, T]$,

$$\begin{aligned} & \int_0^t |\langle b(t, s, u(s)), \varphi \rangle| ds \\ & + \int_0^t \int_{\{x \in S: |\langle c(t, s, x, u(s)), \varphi \rangle| < 1\}} |\langle c(t, s, x, u(s)), \varphi \rangle|^p \nu(dx) ds \\ & + \int_0^t \int_{\{x \in S: |\langle c(t, s, x, u(s)), \varphi \rangle| \geq 1\}} |\langle c(t, s, x, u(s)), \varphi \rangle| \nu(dx) ds \\ & < \infty \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{4.2}$$

hold for every $t \in [0, T]$, $\varphi \in Y_0$, and u solves Eq. (4.1).

Additional regularity properties, which are not part of the definition of the solution, but which are essential for the pathwise uniqueness, are incorporated by the additional mappings $\theta_0^{\alpha_0}$ and $\theta_1^{\alpha_1}$, $\alpha_0 \in A_0$, $\alpha_1 \in A_1$.

Hypothesis 4.1. *The solution u satisfies additionally*

$$\mathbb{P}(\theta_0^{\alpha_0}(u) < \infty) = 1 \quad \text{and} \quad \mathbb{E} \theta_1^{\alpha_1}(u) < \infty \quad \text{for all} \quad \alpha_0 \in A_0, \quad \alpha_1 \in A_1. \tag{4.3}$$

Note that in the particular case of the Nonlinear Schrödinger equation of Sect. 3, $\{\theta_0^{\alpha_0}\}$ were defined in the proof of Theorem 3.8 and represent sufficient conditions for existence and uniqueness of the solution u .

Definition 4.2. If $\eta \in M_{\mathbb{N}}(\{S_n \times \mathbb{R}_+\})$ then we define

$$\eta_t(V) = \eta(V \cap (S \times (0, t))), \quad \eta^t(V) = \eta(V \cap (S \times (t, T])), \quad V \in \mathcal{S} \otimes \mathcal{B}((0, T]). \tag{4.4}$$

Lemma 4.3. *If η is a time homogeneous Poisson random measure over a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, then, for every $t \in [0, T]$, η_t is an \mathcal{F}_t -measurable $M_{\mathbb{N}}(\{S_n \times \mathbb{R}_+\})$ -valued random variable and η^t is independent of \mathcal{F}_t .*

Since the proof is obvious, we omit it.

Lemma 4.4. *Let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, u a $\mathbb{D}([0, T]; Y)$ -valued random variable over \mathfrak{A} , and η an $M_{\mathbb{N}}(\{S_n \times (0, T]\})$ -valued random variable over \mathfrak{A} . In addition, we assume that for any $t \geq 0$, $u(t)$ and η_t are \mathcal{F}_t -measurable and η^t is independent of \mathcal{F}_t . If there exists a solution $\bar{\mathfrak{A}} = (\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t \geq 0}, \bar{\mathbb{P}}, \bar{u}, \bar{\eta})$ of Eq. (4.1) satisfying Eq. (4.1) and the assumption of Definition 4.1, such that the law of (u, η) coincides with the law of $(\bar{u}, \bar{\eta})$ on $\mathbb{D}([0, T]; Y) \times M_{\mathbb{N}}(\{S_n \times [0, T]\})$, then $(\Omega, \mathcal{F}, (\mathcal{F}_t^{\mathbb{P}})_{t \geq 0}, \mathbb{P}, u, \eta)$ is a solution to Eq. (4.1) satisfying Hypothesis 4.1.*

Before giving the proof of Lemma 4.4, let us introduce some notations. For jointly Borel measurable mappings $\check{a} : \mathbb{R}_0^+ \times X \rightarrow \mathbb{R}$, $\check{b} : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times X \rightarrow \mathbb{R}$ and $\check{c} : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times Z \times X \rightarrow \mathbb{R}$, a time homogeneous Poisson random measure η with intensity measure ν and an adapted càdlàg process v in Y with $v(t) \in X$ a.s. for every $t \in \mathbb{R}_0^+$, both defined on a probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, satisfying

$$\begin{aligned} & \int_0^\infty |\check{b}(t, s, v(s))|_Y ds \\ & + \int_0^\infty \int_{\{x \in S : |\check{c}(t, s, x, v(s))|_Y < 1\}} |\check{c}(t, s, x, v(s))|_Y^p \nu(dx) ds \\ & + \int_0^\infty \int_{\{x \in S : |\check{c}(t, s, x, v(s))|_Y \geq 1\}} |\check{c}(t, s, x, v(s))|_Y \nu(dx) ds < \infty \quad \text{a.s.} \end{aligned}$$

for some $p \in [1, 2]$ and every $t \in (0, T]$, define a nonlinear map $\mathcal{K}_{\mathfrak{A}}$

$$\begin{aligned} \mathcal{K}_{\mathfrak{A}}(v, \eta)(t) &= \check{a}(t, v(0)) + \int_0^\infty \check{b}(t, s, v(s)) ds \\ &+ \int_0^\infty \int_S \check{c}(t, s, x, v(s)) \tilde{\eta}(dx, ds). \end{aligned} \tag{4.5}$$

Observe that $\mathcal{K}_{\mathfrak{A}}(v, \eta)$ actually depends via the compensator of $\tilde{\eta}$, also on the probability measure \mathbb{P} .

Proof of Lemma 4.4. It is rather standard to prove that η is a time homogeneous Poisson random measure with intensity ν for the augmented filtration $(\mathcal{F}_t^{\mathbb{P}})_{t \geq 0}$, cf. Lemma 4.3, and that all the measure and integrability assumptions in Definition 4.1 are satisfied for \mathbb{P} , u and η . So it just remains to prove that the actual equation (4.1) holds, i.e., that for any $t \in \mathbb{R}_0^+$ and $\varphi \in Y^*$ we have

$$\mathbb{P}(\mathcal{K}_{\mathfrak{A}}(u, \eta)(t) - \langle u(t), \varphi \rangle = 0) = 1,$$

where $\mathcal{K}_{\mathfrak{A}}$ is defined with $\check{a} = \langle a, \varphi \rangle$, $\check{b} = \langle b, \varphi \rangle$ and $\check{c} = \langle c, \varphi \rangle$;

Fix $t \in \mathbb{R}_0^+$ and $\varphi \in Y^*$. Let us remind, that the mappings

- $a : [0, T] \times X \rightarrow Y$, $b : [0, T] \times [0, T] \times X \rightarrow Y$,
- $c : [0, T] \times [0, T] \times S \times X \rightarrow Y$,
- $\theta_i^{\alpha_i} : \mathbb{D}([0, T], Y) \rightarrow [0, \infty]$, $\alpha_i \in A_i$, $i \in \{0, 1\}$

are measurable. Hence, the mapping

$$X \ni v_0 \mapsto \mathbf{a}_t := \langle a(t, v_0), \varphi \rangle \in \mathbb{R}$$

is Borel measurable. Since u and \bar{u} belong a.s. to $\mathbb{D}([0, T]; Y)$ and $u(0)$, $\bar{u}(0)$ belong a.s. to X , $u(0)$ and $\bar{u}(0)$ have the same law on X , it follows by Lemma 1.22 [15] that the triplets $(\mathbf{a}_t(u(0)), u, \eta)$ and $(\mathbf{a}_t(\bar{u}(0)), \bar{u}, \bar{\eta})$ have the same law on $X \times \mathbb{D}([0, T]; Y) \times M_{\mathbb{N}}(\{S_n \times (0, T]\})$.

Since $b : [0, T] \times [0, T] \times X \rightarrow Y$, is measurable, for any $s \in [0, T]$ $u(s)$ and $\bar{u}(s)$ are X -valued random variables, $\mathcal{L}aw(\bar{u}(s)) = \mathcal{L}aw(u(s))$, it follows

that for any $s, t \in [0, T]$, the processes $\{\bar{\mathbf{b}}_t(s) : s \geq 0\}$ and $\{\mathbf{b}_t(s) : s \geq 0\}$, defined by

$$\bar{\mathbf{b}}_t(s, \bar{\omega}) := \langle b(t, s, \bar{u}(s, \bar{\omega}), \varphi), \quad s \in [0, T], \quad \bar{\omega} \in \bar{\Omega},$$

and

$$\mathbf{b}_t(s, \omega) := \langle b(t, s, u(s, \omega), \varphi), \quad s \in [0, T], \quad \omega \in \Omega,$$

have the same law for each $t \in [0, T]$. In addition, we know by the definition of the solution that

$$\bar{\mathbb{P}} \left(\int_0^t |\bar{\mathbf{b}}_t(s)| ds < \infty \right) = 1 \quad \text{and} \quad \mathbb{P} \left(\int_0^t |\mathbf{b}_t(s)| ds < \infty \right) = 1.$$

By Theorem 8.3 of [21], it follows that for any $t \in [0, T]$

$$\mathcal{L}aw \left(\bar{u}, \int_0^t \bar{\mathbf{b}}_t(s) ds \right) \quad \text{and} \quad \mathcal{L}aw \left(u, \int_0^t \mathbf{b}_t(s) ds \right)$$

are equal on $\mathbb{D}([0, T]; L^2(\mathbb{R}^d)) \times Y$. Finally, since $c : [0, T] \times [0, T] \times S \times X \rightarrow Y$ is measurable, for any $s \in [0, T]$, the random variable $\bar{u}(s)$ is $\bar{\mathcal{F}}_s$ -measurable and the random variable $u(s)$ is \mathcal{F}_s -measurable. It follows that the processes $\{\bar{\mathbf{c}}_t(s) : s \geq 0\}$ and $\{\mathbf{c}_t(s) : s \geq 0\}$, defined by

$$\bar{\mathbf{c}}_t(s, \bar{\omega}) := \langle c(t, s, \bar{u}(s, \bar{\omega}), \varphi), \quad s \in [0, T],$$

and

$$\mathbf{c}_t(s, \omega) := \langle c(t, s, u(s, \omega), \varphi), \quad s \in [0, T],$$

are adapted to the filtrations $(\bar{\mathcal{F}}_s)_{s \in [0, T]}$ and $(\mathcal{F}_s)_{s \in [0, T]}$, respectively. Moreover

$$\bar{\mathbb{P}} \left(\int_0^t \int_{\{x \in S : |\langle c(t, s, x, \bar{u}(s)), \varphi \rangle| < 1\}} |\bar{\mathbf{c}}_t(x, s)|^p \nu(dx) ds < \infty \right) = 1$$

holds because \bar{u} is a solution, and since the law of $(\bar{u}, \bar{\eta})$ coincides with the law of (u, η) , we get that

$$\mathbb{P} \left(\int_0^t \int_{\{x \in S : |\langle c(t, s, x, u(s)), \varphi \rangle| < 1\}} |\mathbf{c}_t(x, s)|^p \nu(dx) ds < \infty \right) = 1$$

holds. Hence, $\bar{\mathbb{P}}$ -a.s. the process $[0, T] \ni s \mapsto \bar{\mathbf{c}}_t(s) \in \mathbb{R}$ and \mathbb{P} -a.s. the process $[0, T] \ni s \mapsto \mathbf{c}_t(s) \in \mathbb{R}$ are belonging (but of the large jumps) to $L^p([0, T]; \mathbb{R})$. It follows by Proposition B.1(ii) that they are progressively measurable. Hence, Theorem A.1 is applicable and we know that $(\bar{\mathcal{I}}(t), \bar{u}, \bar{\eta})$ and $(\mathcal{I}(t), u, \eta)$ have the same law on $\mathbb{R} \times \mathbb{D}([0, T]; Y) \times M_{\bar{\mathbb{N}}}(\{S_n \times (0, T]\})$, where

$$\bar{\mathcal{I}}(t) := \int_0^T \int_S \check{c}(t, s, \bar{u}(s), z) \tilde{\eta}(dz, ds), \quad t \in [0, T],$$

and

$$\mathcal{I}(t) := \int_0^T \int_S \check{c}(t, s, u(s), z) \tilde{\eta}(dz, ds), \quad t \in [0, T].$$

To deal with the large jumps, we use the fact that

$$\bar{\mathbb{P}} \left(\int_0^t \int_{\{x \in S: |\bar{c}_t(x,s)| \geq 1\}} |\bar{c}_t(x,s)| \nu(dx) ds < \infty \right) = 1,$$

and

$$\mathbb{P} \left(\int_0^t \int_{\{x \in S: |c_t(x,s)| \geq 1\}} |c_t(x,s)| \nu(dx) ds < \infty \right) = 1.$$

and proceed as above with $p = 1$.

Summing up, it follows that, if (u, η) and $(\bar{u}, \bar{\eta})$ have the same law on $\mathbb{D}([0, T]; Y) \times M_{\bar{\mathbb{N}}}(\{S_n \times (0, T]\})$, then $\mathcal{K}_{\bar{\mathfrak{A}}}(\bar{u}, \bar{\eta})(t), \bar{u}, \bar{\eta}$ and $(\mathcal{K}_{\mathfrak{A}}(u, \eta)(t), u, \eta)$ have the same law on $Y \times \mathbb{D}([0, T]; Y) \times M_{\bar{\mathbb{N}}}(\{S_n \times (0, T]\})$. Since for all $t \in [0, T]$

$$\bar{\mathbb{P}}(\mathcal{K}_{\bar{\mathfrak{A}}}(\bar{u}, \bar{\eta})(t) - \bar{u}(t) = 0) = 1,$$

it follows that

$$\mathbb{P}(\mathcal{K}_{\mathfrak{A}}(u, \eta)(t) - u(t) = 0) = 1.$$

In particular, the six tuple $(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, u, \eta)$ is a solution to (4.1). □

5. Uniqueness

Throughout this section, the notation of Sect. 4 will be kept. We are going to prove here that the abstract result of Kurtz [17] can be applied to the problem (4.1). Or, in other words, that pathwise uniqueness for the Eq. (4.1) implies joint uniqueness in law and strong existence for the Eq. (4.1).

In order to show this, let us define the Kurtz's *compatibility structure* and *C-compatibility* according to [17, Definition 3.3].

Throughout this section we fix an intensity measure ν and sets $S_n \in \mathcal{S}$ such that $S_n \uparrow S$ and $\nu(S_n) < \infty$.

Definition 5.1. Let us denote

$$Z_1 = \mathbb{D}([0, T], Y), \quad Z_2 = M_{\bar{\mathbb{N}}}(\{S_n \times (0, T]\}) \times X, \\ \mathcal{B}_t^{Z_1} = \sigma(\pi_s : s \leq t), \quad \mathcal{B}_t^{Z_2} = \sigma(R_t) \otimes \mathcal{B}(X),$$

where π_t and R_t are the canonical mappings, $\pi_t : \mathbb{D}([0, T], Y) \ni \pi \mapsto \pi(t) \in Y$,

$$R_t : M_{\bar{\mathbb{N}}}(\{S_n \times (0, T]\}) \rightarrow M_{\bar{\mathbb{N}}}(\{S_n \times (0, T]\}) : \mu \mapsto \mu(\cdot \cap (S \times (0, t])),$$

and denote by \mathcal{C} the Kurtz compatibility structure $\{(\mathcal{B}_t^{Z_1}, \mathcal{B}_t^{Z_2}) : t \in [0, T]\}$.

Definition 5.2. If A is an Z_1 -valued random variable over some probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{P})$, B a Z_2 -valued random variable over \mathfrak{A} and $t \in [0, T]$, then \mathcal{F}_t^A and \mathcal{F}_t^B are the coarsest σ -algebras such that the mappings

$$A : (\Omega, \mathcal{F}_t^A) \rightarrow (Z_1, \mathcal{B}_t^{Z_1}) \quad \text{and} \quad B : (\Omega, \mathcal{F}_t^B) \rightarrow (Z_2, \mathcal{B}_t^{Z_2})$$

are measurable.

Remark 5.3. If A is a Z_1 -valued random variable over \mathfrak{A} and $B = (\eta, \xi)$ a Z_2 -valued random variable over \mathfrak{A} , it is rather standard to see that

$$\mathcal{F}_t^A = \sigma(A_s : s \leq t) \quad \text{and} \quad \mathcal{F}_t^B = \sigma(\eta_t) \vee \sigma(\xi)$$

hold for every $t \in [0, T]$, as defined in (4.4).

Definition 5.4. We say that Z_1 -valued random variables A_1, \dots, A_n are \mathcal{C} -compatible with a Z_2 -valued random variable B provided that

$$\mathbb{E}[h(B)|\mathcal{F}_t^{A_1} \vee \dots \vee \mathcal{F}_t^{A_n} \vee \mathcal{F}_t^B] = \mathbb{E}[h(B)|\mathcal{F}_t^B] \quad \text{a.s.} \quad (5.1)$$

holds for every $t \in [0, T]$ and every real bounded Borel measurable function h on Z_2 .

Remark 5.5. \mathcal{C} -compatibility of random variables A_1, \dots, A_n with a random variable B is actually a property of the joint law of (A_1, \dots, A_n, B) , as follows from [17, Remark 3.5]. Hence we can introduce the notion of \mathcal{C} -compatibility for Borel probability measures on $Z_1^n \times Z_2$, see [17, Definition 3.6] and Definition 5.6 below.

Definition 5.6. A probability measure μ on $Z_1^n \times Z_2$ is called \mathcal{C} -compatible provided that, for any (A_1, \dots, A_n, B) distributed according to μ , then A_1, \dots, A_n are \mathcal{C} -compatible with B in the sense of Definition 5.4.

Lemma 5.7. *Let η be a time homogeneous Poisson random measure with intensity measure ν on S for some filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and let u_0 be an \mathcal{F}_0 -measurable X -valued random variable. Then Z_1 -valued random variables A_1, \dots, A_n are \mathcal{C} -compatible with $B = (\eta, u_0)$ if and only if $\mathcal{F}_t^{A_1} \vee \dots \vee \mathcal{F}_t^{A_n} \vee \mathcal{F}_t^B$ is \mathbb{P} -independent of $\sigma(\eta^t)$.*

Proof. Since $\eta = \eta_t + \eta^t$, (5.1) holds if and only if

$$\mathbb{E}[h(\eta^t)|\mathcal{F}_t^{A_1} \vee \dots \vee \mathcal{F}_t^{A_n} \vee \mathcal{F}_t^B] = \mathbb{E}[h(\eta^t)|\mathcal{F}_t^B].$$

But $\sigma(\eta^t)$ and $\mathcal{F}_t^B = \sigma(\eta_t) \vee \sigma(u_0)$ are \mathbb{P} -independent by Lemma 4.3, hence (5.1) holds if and only if $\mathcal{F}_t^{A_1} \vee \dots \vee \mathcal{F}_t^{A_n} \vee \mathcal{F}_t^B$ is \mathbb{P} -independent of $\sigma(\eta^t)$. \square

Now we are ready to define a Kurtz convexity constraint Γ_ν , see [17, p. 958].

Definition 5.8. If μ is a Borel probability measure on $Z_1 \times Z_2$ and a random vector (u, η, u_0) over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has the distribution μ , we say that μ satisfies a convexity constraint Γ_ν provided that

- (a) $u(0) = u_0$ almost surely,
- (b) there exists a solution $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\bar{\mathcal{F}}_t)_{t \in [0, T]}, \bar{\mathbb{P}}, \bar{u}, \bar{\eta})$ to Eq. (4.1) satisfying Hypothesis 4.1 such that ν is the intensity measure of $\bar{\eta}$ and the law of $(\bar{u}, \bar{\eta})$ coincides with the law of (u, η) .

Remark 5.9. Paradoxically, despite of the notion, we need not prove here that the Kurtz’s convexity constraint Γ_ν really defines a convex set of probability measures on $Z_1 \times Z_2$. For us, Γ_ν is viewed as a mere constraint with no convexity properties. We will explain more in the proof of Theorem 5.14.

Finally, we define the Kurtz set $\mathcal{S}_{\Gamma_\nu, \mathcal{C}, \Theta_\nu \otimes \beta}$, see [17, p. 958].

Definition 5.10. Let β be a Borel probability measure on X . We denote by $\mathcal{S}_{\Gamma_\nu, \mathcal{C}, \Theta_\nu \otimes \beta}$ the set of probability measures μ on $Z_1 \times Z_2$ such that

- (a) μ satisfies the convexity constraint Γ_ν ,
- (b) μ is \mathcal{C} -compatible,
- (c) $\mu(Z_1 \times \cdot) = \Theta_\nu \otimes \beta$ on $\mathcal{B}(Z_2)$,

where Θ_ν is the law of the time homogeneous Poisson random measure with intensity measure ν , cf. Lemma 2.8.

Remark 5.11. Observe that the condition (b) in Definition 5.10 is redundant as it follows from (a) due to Lemma 5.7. We however present Definition 5.10 as it is, to be conformal with Kurtz’s notation in [17].

Remark 5.12. The Kurtz set $\mathcal{S}_{\Gamma_\nu, \mathcal{C}, \Theta_\nu \otimes \beta}$ is in fact convex. But since we do not need the convexity in this paper, we do not prove it.

We can now give a full description of the set $\mathcal{S}_{\Gamma_\nu, \mathcal{C}, \Theta_\nu \otimes \beta}$.

Remark 5.13. Let β be a Borel probability measure on X . Then $\mu \in \mathcal{S}_{\Gamma_\nu, \mathcal{C}, \Theta_\nu \otimes \beta}$ if and only if

- there exists a solution $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, (\bar{\mathcal{F}}_t)_{t \in [0, T]}, \bar{\mathbb{P}}, \bar{u}, \bar{\eta})$ to the Eq. (4.1) satisfying Hypothesis 4.1,
- ν is the intensity measure of $\bar{\eta}$,
- μ is the law of $(\bar{u}, \bar{\eta}, \bar{u}(0))$,
- β is the law of $\bar{u}(0)$.

Proof. Comparing Definition 5.8 with Definition 5.10, taking into account Remark 5.11 and the fact that $\bar{\eta}$ is independent from $\bar{u}(0)$ by Lemma 4.3, we get the equivalence immediately. \square

Theorem 5.14. Let β be a Borel probability measure on X and assume that

- there exists a solution $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, (\bar{\mathcal{F}}_t)_{t \in [0, T]}, \bar{\mathbb{P}}, \bar{u}, \bar{\eta})$ to Eq. (4.1) satisfying Hypothesis 4.1 such that ν is the intensity measure of $\bar{\eta}$ and β is the law of $\bar{u}(0)$,
- whenever $(\Omega, \mathcal{F}, \mathbb{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}, u^1, \eta)$ and $(\Omega, \mathcal{F}, \mathbb{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}, u^2, \eta)$ are solutions to Eq. (4.1) satisfying Hypothesis 4.1 such that ν is the intensity measure of η , β is the law of $u^1(0)$ and $u^1(0) = u^2(0)$ a.s. then $u^1 = u^2$ a.s.

Then there exists a Borel measurable mapping

$$F : M_{\mathbb{N}}(\{S_n \times (0, T]\}) \times X \rightarrow \mathbb{D}([0, T]; Y)$$

depending on ν and β such that

- (1) if $(\Omega, \mathcal{F}, \mathbb{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P}, u, \eta)$ is a solution to the Eq. (4.1) satisfying Hypothesis 4.1 such that ν is the intensity measure of η and β is the law of $u(0)$ then $u = F(\eta, u(0))$ a.s. and u is adapted to the \mathbb{P} -augmentation of the filtration $(\sigma(\eta_t), \sigma(u(0)))_{t \in [0, T]}$,

- (2) if $(\Omega, \mathcal{F}, \mathbb{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ is a stochastic basis, ξ^* is an X -valued \mathcal{F}_0 -measurable random variable with law β and η is a time homogeneous $(\mathcal{F}_t)_{t \in [0, T]}$ -Poisson random measure with intensity ν then $u = F(\eta, \xi)$ is $(\mathcal{F}_t^{\mathbb{P}})_{t \in [0, T]}$ -adapted, $u(0) = \xi$ a.s. and $(\Omega, \mathcal{F}, \mathbb{F}, (\mathcal{F}_t^{\mathbb{P}})_{t \in [0, T]}, \mathbb{P}, u, \eta)$ is a solution to (4.1) satisfying Hypothesis 4.1.

Consequently, if $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, (\mathcal{F}_t^i)_{t \in [0, T]}, \mathbb{P}^i, u^i, \eta^i)$, $i = 1, 2$ are solutions to Eq. (4.1) satisfying Hypothesis 4.1 such that ν is the intensity measure of η^1 and η^2 , β is the law of $u^1(0)$ and $u^2(0)$ then the law of (u^1, η^1) coincides with the law of (u^2, η^2) .

Proof. By the assumptions in Theorem 5.14, the Kurtz set $\mathcal{S}_{\Gamma_\nu, \mathcal{C}, \Theta_\nu \otimes \beta}$ is non-empty and pointwise uniqueness holds for \mathcal{C} -compatible solutions of $(\Gamma_\nu, \Theta_\nu \times \beta)$ in the sense of [17, p. 959]. Hence by the implication (a) \Rightarrow (b) in [17, Theorem 3.14], joint uniqueness in law holds for compatible solutions, i.e. $\mathcal{S}_{\Gamma_\nu, \mathcal{C}, \Theta_\nu \otimes \beta}$ contains exactly one measure, and there exists a strong compatible solution in the sense of [17, p. 959] and [17, Lemma 3.11], i.e. (1) holds.

To prove (2), once $u = F(\eta, \xi)$, we have that the law of (u, η, ξ) coincides with the law of $(\bar{u}, \bar{\eta}, \bar{u}(0))$. Hence $u(0) = \xi$ a.s. and u is compatible with $(\eta, u(0))$ by [17, Remark 3.5]. Thus u is adapted to the \mathbb{P} -augmentation of the filtration $(\sigma(\eta_t) \vee \sigma(u(0)))_{t \in [0, T]}$ by [17, Lemma 3.11]. The rest then follows from Lemma 4.4.

We must point out here that the constraint Γ_ν need not define a convex subset of Borel probability measures on $Z_1 \times Z_2$ if we apply just the implication (a) \Rightarrow (b) in [17, Theorem 3.14]. The convexity of the Kurtz set $\mathcal{S}_{\Gamma_\nu, \mathcal{C}, \Theta_\nu \otimes \beta}$ is needed just for the implication (a) \Leftarrow (b) in [17, Theorem 3.14] which we do not apply in our case. □

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Appendix A: Uniqueness of the stochastic integral

Let X and E be two separable Banach spaces. Later on we will take X to be one of the spaces E or $L^p(S, \nu, E)$. Let $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be an arbitrary filtered probability space and η be a Poisson random measure defined over \mathfrak{A} . Let $\mathcal{N}(\Omega \times [0, T]; X)$ be the space of (equivalence classes of) progressively measurable functions $\xi : \Omega \times [0, T] \rightarrow X$.

For $q \in (1, \infty)$ we set

$$\mathcal{N}^q(\Omega \times [0, T], \mathcal{F}; X) = \left\{ \xi \in \mathcal{N}(\Omega \times [0, T], \mathcal{F}; X) : \int_0^\infty |\xi(t)|^q dt < \infty \text{ a.s.} \right\}, \tag{A.1}$$

$$\mathcal{M}^q(\Omega \times [0, T], \mathcal{F}; X) = \left\{ \xi \in \mathcal{N}(\Omega \times [0, T], \mathcal{F}; X) : \mathbb{E} \int_0^\infty |\xi(t)|^q dt < \infty \right\}. \tag{A.2}$$

Let E be a space of martingale type p , put $X = L^p(S, \nu; E)$, and let $\xi \in \mathcal{N}^p(\Omega \times [0, T], \mathcal{F}; X)$. In particular, $\xi : \Omega \times [0, T] \times S \rightarrow E$ is a progressively measurable process such that \mathbb{P} -a.s.

$$\int_0^T \int_S |\xi(r, z)|_E^p \nu(dz) dr < \infty. \tag{A.3}$$

Let us consider the law of the triplet (η, ξ, I) , where I is the Itô integral of ξ with respect to η as defined on page 5. It was proved in Theorem 2.4 of [6] that the law is unique in case

$$\mathbb{E} \int_0^T \int_S |\xi(r, z)|_E^p \nu(dz) dr < \infty.$$

In this ‘‘Appendix’’ we want to extend this result to all progressively processes ξ satisfying only (A.3). However, before stating the Theorem we want to define uniqueness in law. Here, it is important that evaluating the integral, the underlying probability space can be changed with having any consequence for the laws.

Theorem A.1. *Let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, $i = 1, 2$, be two probability spaces and $(\mathcal{F}_t^i)_{t \in [0, T]}$ a filtration of $(\Omega_i, \mathcal{F}_i)$. Assume that $\{(\xi_i, \eta_i), n \in \mathbb{N}\}$, $i = 1, 2$, are two $L^p([0, T]; L^p(S, \nu, E)) \times M_{\mathbb{N}}(\{S_n \times (0, T]\})$ valued random variables defined on $(\Omega_i, \mathcal{F}_i, \mathcal{F}_t^i, \mathbb{P}_i)$, $i = 1, 2$, respectively. Assume that η_1 is a time homogeneous Poisson random measure over $(\Omega_1, \mathcal{F}_1, (\mathcal{F}_t^1)_{t \in [0, T]}, \mathbb{P}_1)$ with intensity ν . Furthermore, assume that $\xi_1 \in \mathcal{N}(\Omega_1 \times [0, T]; L^p(S, \nu, E))$ with respect to $(\mathcal{F}_t^1)_{t \in [0, T]}$.*

Let

$$I_i(t) := I(\xi_i, \eta_i)(t) = \int_0^t \int_S \xi_i(s, z) \tilde{\eta}_i(dz, ds).$$

If $\mathcal{L}aw((\xi_1, \eta_1)) = \mathcal{L}aw((\xi_2, \eta_2))$ on

$$L^p([0, T]; L^p(S, \nu; E)) \times M_{\mathbb{N}}(\{S_n \times (0, T]\}),$$

then $\mathcal{L}aw((I_1, \xi_1, \eta_1)) = \mathcal{L}aw((I_2, \xi_2, \eta_2))$ on

$$L^p([0, T]; E) \cap \mathbb{D}([0, T]; E) \times L^p(\mathbb{R}_+; L^p(S, \nu; E)) \times M_{\mathbb{N}}(\{S_n \times (0, T]\}).$$

Proof. In fact, Theorem A.1 follows from Theorem 2.4 in [6] by localization. First, for a fixed $R > 0$ let us introduce the stopping times

$$\tau_i^R = \inf_{t > 0} \left\{ \int_0^t |\xi(s)|_X^p ds \geq R \right\}.$$

Put $\xi_i^R := 1_{[0, \tau_i^R]} \xi$. Observe, using the shifted Haar projection defined in (B.2) one obtains a sequence of simple functions $\{\mathfrak{h}_n^s \xi_i^R : n \in \mathbb{N}\}$ such that \mathbb{P}^i -a.s. $\mathfrak{h}_n^s \xi_i^R \rightarrow \xi_i^R$ in $L^p([0, T]; E)$. In addition

$$\mathbb{E} \int_0^T \int_S |\xi_i^R(r, z)|_E^p \nu(dz) dr \leq R.$$

Thus, Theorem 2.4 in [6] is applicable and we have $\mathcal{L}aw((I_1^R, \xi_1^R, \eta_1)) = \mathcal{L}aw((I_2^R, \xi_2^R, \eta_2))$ on

$$(\mathbb{D}([0, T]; X) \cap L^p([0, T]; X)) \times L^p([0, T]; L^p(Z, \nu; E)) \times M_{\mathbb{N}}(\{S_n \times (0, T]\}),$$

where

$$I_i^R(t) = \int_0^t \int_S \xi_i^R(s, z) \tilde{\eta}_i(ds, dz), \quad t \in [0, T]. \tag{A.4}$$

Let

$$A_i^R := \left\{ \omega \in \Omega_i : \int_0^T |\xi_i(s)|_{L^p(Z, \nu; E)}^p ds \leq R \right\}.$$

Then, first, on A_i^R , $\xi_i^R = \xi_i$, secondly, $A^{R_1} \supset A^{R_2}$ for $R_1 > R_2$, and, thirdly, by Lemma 1.14 [15], $\lim_{R \rightarrow \infty} \mathbb{P}(A_i^R) = \mathbb{P}(\Omega) = 1$. Take a set

$$B_1 \times B_2 \times B_3 \in \mathcal{B}(\mathbb{D}([0, T]; X) \cap L^p([0, T]; B)) \times \mathcal{B}(L^p([0, T]; L^p(Z, \nu; E))) \times \mathcal{B}(M_{\mathbb{N}}(\{S_n \times (0, T]\})).$$

Since $\xi_i^R \leq \xi_i$, we have by the dominated convergence Theorem

$$\begin{aligned} & \mathbb{P}_1((u_1, \xi_1, \eta) \in B_1 \times B_2 \times B_3) \\ &= \lim_{R \rightarrow \infty} \mathbb{P}_1((u_1, \xi_1, \eta) \in B_1 \times B_2 \times B_3, \tau_1^R > T) \\ &= \lim_{R \rightarrow \infty} \mathbb{P}_1((u_1^R, \xi_1^R, \eta) \in B_1 \times B_2 \times B_3) \\ &= \lim_{R \rightarrow \infty} \mathbb{P}_2((u_2^R, \xi_2^R, \eta) \in B_1 \times B_2 \times B_3) \\ &= \lim_{R \rightarrow \infty} \mathbb{P}_2((u_2^R, \xi_2^R, \eta) \in B_1 \times B_2 \times B_3, \tau_1^R > T) \\ &= \mathbb{P}_2((u_2, \xi_2, \eta) \in B_1 \times B_2 \times B_3). \end{aligned}$$

Now, the assertion follows. □

Appendix B: the Haar projection

B.1. The Haar projection onto L^q -spaces

For $n \in \mathbb{N}$, let $\Pi^n = \{s_0^n = 0 < s_1^n < \dots < s_{2^n}^n\}$ be a partition of the interval $[0, T]$ defined by $s_j^n = j 2^{-n} T$, $j = 1, \dots, 2^n$. Each interval of the form $(s_{j-1}^n, s_j^n]$, where $n \in \mathbb{N}$ and $j = 1, \dots, 2^n$ is called a *dyadic interval*. For $n \in \mathbb{N}$, the j^{th} element, for $j = 1, \dots, 2^n$, of the Haar system of order n is the indicator function of the interval $(s_{j-1}^n, s_j^n]$, i.e. $1_{(s_{j-1}^n, s_j^n]}$. First, given a function $x : [0, T] \rightarrow Y$, Y a Banach space, let us define the averaging operator $\iota_{j,n} : L^p(0, T, Y) \rightarrow Y$ over the interval $(s_{j-1}^n, s_j^n]$ by

$$\iota_{j,n}(x) := \frac{1}{s_j^n - s_{j-1}^n} \int_{s_{j-1}^n}^{s_j^n} x(s) ds, \quad x \in L^p([0, T], Y). \tag{B.1}$$

For $n \in \mathbb{N}$, let $\mathfrak{h}_n^s : L^p([0, T], Y) \rightarrow L^p([0, T], Y)$ be the *shifted Haar projection* of order n , i.e.

$$\mathfrak{h}_n^s x = \sum_{j=1}^{2^n-1} 1_{(s_j^n, s_{j+1}^n]} \otimes \iota_{j,n}(x), \quad x \in L^p([0, T]; Y), \tag{B.2}$$

where we put $\iota_{0,n} = 0$ for every $n \in \mathbb{N}$. In the above, for $f \in L^p([0, T], \mathbb{R})$ and $y \in Y$, by $f \otimes y$ we mean an element of $L^p([0, T], Y)$ defined by $[0, T] \ni t \mapsto f(t)y \in Y$. For completeness, let us cite the following results taken from [6, Appendix B, Theorem B.2].

Proposition B.1. *The following holds:*

- (i) For any $n \in \mathbb{N}$, the shifted Haar projection $\mathfrak{h}_n^s : L^p([0, T]; Y) \rightarrow L^p([0, T]; Y)$ is a continuous operator.
- (ii) For all $x \in L^p([0, T]; Y)$, $\mathfrak{h}_n^s x \rightarrow x$ in $L^p([0, T]; Y)$.

Remark B.1. Observe, for any $\xi \in \mathcal{N}_p([0, T]; X)$, the process $[0, T] \ni t \mapsto \mathfrak{h}_n^s x(t)$ is simple, left continuous and predictable and the sequence $\{\mathfrak{h}_n^s \xi : n \in \mathbb{N}\}$ converges to ξ in $L^p([0, T]; Y)$.

Appendix C: Polish measure spaces

Lemma C.1. *Let (S, \mathcal{S}) be a Polish space and the family $\{S_n \in \mathcal{S}\}$ satisfy $S_n \uparrow S$. Then $(M_{\mathbb{N}}(\{S_n\}), \mathcal{M}_{\mathbb{N}}(\{S_n\}))$ is a Polish space.*

Let (S, \mathcal{S}) be a Polish space and let $S_n \in \mathcal{S}$ satisfy $S_n \uparrow S$. Then there exists a metric ϱ on S such that (S, ϱ) is a complete separable metric space, $\mathcal{B}(S, \varrho) = \mathcal{S}$ and S_n is closed for every $n \in \mathbb{N}$, see, e.g. [16, (13.5) p. 83]. Consider the Lévy-Prokhorov metric on $M_+(S, \varrho)$

$$\pi(\mu, \nu) = \inf \{ \varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon, \nu(A) \leq \mu(A^\varepsilon) + \varepsilon, \forall A \in \mathcal{S} \}$$

where $A^\varepsilon = \{x \in S : \exists a \in A, \varrho(x, a) < \varepsilon\}$. Then $(M_+(S, \varrho), \pi)$ is a complete separable metric space and $\pi(\mu_n, \mu) \rightarrow 0$ iff

$$\int_S f d\mu_n \rightarrow \int_S f d\mu, \quad \forall f \in C_b(S, \varrho),$$

see, e.g., [3, pp. 72–73], where the proof for probability measures can be quite easily adapted to finite non-negative measures.

Lemma C.2. *The σ -algebra $\mathcal{M}_+(S)$ on $M_+(S)$ generated by the mappings $\mu \mapsto \mu(A)$, $A \in \mathcal{S}$ coincides with the Borel σ -algebra $\mathcal{B}(M_+(S), \pi)$.*

Proof. The mapping $\mu \mapsto \mu(A)$ is upper semicontinuous on $(M_+(S), \pi)$ for every $A \subseteq S$ closed and lower semicontinuous for every $A \subseteq S$ open, hence Borel measurable for every $A \in \mathcal{S}$. In particular, $\mathcal{M}_+(S) \subseteq \mathcal{B}(M_+(S), \pi)$. On the other hand, let \mathcal{G} be a countable basis of open sets in (S, ϱ) closed under finite unions. Then

$$\begin{aligned} \{ \mu : \pi(\mu, \theta) < r \} &= \bigcup_{\varepsilon \in \mathbb{Q} \cap (0, r)} \bigcap_{A \in \mathcal{G}} \{ \mu : \mu(A) \leq \theta(A^\varepsilon) \\ &\quad + \varepsilon, \theta(A) \leq \mu(A^\varepsilon) + \varepsilon \} \in \mathcal{M}_+(S). \end{aligned}$$

Hence open balls in $(M_+(S), \pi)$ belong to $\mathcal{M}_+(S)$ and since $(M_+(S), \pi)$ is separable, every open set is a countable union of open balls. Consequently, every open set in $(M_+(S), \pi)$ belong to $\mathcal{M}_+(S)$, hence $\mathcal{B}(M_+(S), \pi) \subseteq \mathcal{M}_+(S)$. \square

Lemma C.3. *The set of integer-valued measures $M_{\mathbb{N}}(S)$ is closed in $(M_+(S, \rho), \pi)$.*

Proof. Let $\pi(\mu_n, \mu) \rightarrow 0$, μ_n be integer-valued and $k < \mu(A) < k + 1$ for some integer k and some $A \in \mathcal{S}$. By regularity, we can find a compact $C \subseteq A$ such that $k < \mu(C) \leq \mu(A) < k + 1$ and $\delta > 0$ such that $k < \mu(C) - \delta$ and $\mu(C^{2\delta}) + \delta < k + 1$. If $\pi(\mu_n, \mu) < \delta$ then

$$k < \mu(C) - \delta \leq \mu_n(C^\delta) \leq \mu(C^{2\delta}) + \delta < k + 1,$$

which cannot happen as $\mu_n(C^\delta)$ is an integer. \square

Lemma C.4. *$(M_{\mathbb{N}}(\{S_n\}), \mathcal{M}_{\mathbb{N}}(\{S_n\}))$ is a Polish space.*

Proof. Consider the metric

$$\rho(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} \min \{1, \pi(\mu(\cdot \cap S_n), \nu(\cdot \cap S_n))\}, \quad \mu, \nu \in M_{\mathbb{N}}(\{S_n\}).$$

Then $(M_{\mathbb{N}}(\{S_n\}), \rho)$ is a metric space and the mapping

$$I : (M_{\mathbb{N}}(\{S_n\}), \rho) \rightarrow (M_{\mathbb{N}}(S), \pi)^{\mathbb{N}} : \mu \mapsto (\mu(\cdot \cap S_n))_{n \in \mathbb{N}}$$

is a homeomorphism onto a closed set in $(M_{\mathbb{N}}(S), \pi)^{\mathbb{N}}$, hence $(M_{\mathbb{N}}(\{S_n\}), \rho)$ is a complete separable metric space by Lemma C.3. To show closedness, let $\lim_{j \rightarrow \infty} \pi(\mu_j(\cdot \cap S_n), \theta_n) = 0$ for every $n \in \mathbb{N}$, and let $m < k$. If $f \in C_b(S_m)$ and we extend f by zero on $S \setminus S_m$ then $f \in C_b(S)$ since S_m is clopen. So $\partial S_n = \emptyset$,

$$\theta_n(S \setminus S_n) = \lim_{j \rightarrow \infty} \mu_j((S \setminus S_n) \cap S_n) = 0, \quad \forall n \in \mathbb{N},$$

and

$$\int_S f d\theta_k = \lim_{j \rightarrow \infty} \int_S f d\mu_j(\cdot \cap S_k) = \lim_{j \rightarrow \infty} \int_S f d\mu_j(\cdot \cap S_m) = \int_S f d\theta_m$$

so $\theta_m(\cdot) = \theta_k(\cdot \cap S_m)$. In particular, $\theta_n(A) \uparrow$ for every $A \in \mathcal{S}$ and $\theta(A) = \lim_n \theta_n(A)$ is a σ -additive, \mathbb{N} -valued measure on \mathcal{S} and $\theta_m(\cdot) = \theta(\cdot \cap S_m)$.

Now, by Lemma C.2, the mapping $\mu \mapsto \mu(A \cap S_k)$ is $\mathcal{B}(M_{\mathbb{N}}(\{S_n\}), \rho)$ measurable for every $k \in \mathbb{N}$ and every $A \in \mathcal{S}$ since I is Borel measurable. Hence $\mathcal{M}_{\mathbb{N}}(\{S_n\}) \subseteq \mathcal{B}(M_{\mathbb{N}}(\{S_n\}), \rho)$. On the other hand, the mapping

$$(M_{\mathbb{N}}(\{S_n\}), \mathcal{M}_{\mathbb{N}}(\{S_n\})) \rightarrow (M_{\mathbb{N}}(S), \mathcal{B}(M_{\mathbb{N}}(S))) : \mu \mapsto \mu(\cdot \cap S_k)$$

is measurable for every $k \in \mathbb{N}$ by Lemma C.2. So, if $\theta \in M_{\mathbb{N}}(\{S_n\})$ is fixed, the mapping

$$(M_{\mathbb{N}}(\{S_n\}), \mathcal{M}_{\mathbb{N}}(\{S_n\})) \rightarrow \mathbb{R} : \mu \mapsto \pi(\mu(\cdot \cap S_k), \theta(\cdot \cap S_k))$$

is measurable for every $k \in \mathbb{N}$. Consequently, the mapping

$$(M_{\mathbb{N}}(\{S_n\}), \mathcal{M}_{\mathbb{N}}(\{S_n\})) \rightarrow \mathbb{R} : \mu \mapsto \rho(\mu, \theta)$$

is measurable. Since $(M_{\mathbb{N}}(\{S_n\}), \rho)$ is a separable metric space and every open set is a countable union of open balls, we conclude that $\mathcal{B}(M_{\mathbb{N}}(\{S_n\}), \rho) \subseteq \mathcal{M}_{\mathbb{N}}(\{S_n\})$. \square

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