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# Model Predictive Control with Reference Anisotropic Control for Robot Motion with Disturbed Measured Outputs

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***Abstract*— This paper introduces specific extension of model predictive control (MPC) with reference anisotropic control ($\mathcal{H}_a$ control) for a robot motion with disturbed measured outputs. The purpose is to exploit advance of flexible multi-step MPC and stabilizing (robust) properties of $\mathcal{H}_a$ control serving for disturbance attenuation of disturbed measured outputs. The linking of MPC and $\mathcal{H}_a$ control is derived as both a simple improvement of MPC by $\mathcal{H}_a$ control providing disturbance attenuation only and a modification of a cost function in MPC design by an additional tunable term weighting the proximity of MPC design to $\mathcal{H}_a$ control. Considered novel $\mathcal{H}_a$ control represents adjustable transition between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control i.e. between excited (trusting) and too conservative design. The proposed control design is based on state-space formulation that run in output feedback configuration complemented by state estimation based on anisotropic theory again. The theoretical achievements are demonstrated by simulations using a state-space model describing dynamics of one specific over-actuated planar parallel kinematic machine, robot-manipulator.**

## I. INTRODUCTION

Efficient, accurate and safe motion control in various applications is the present issue for increasing robotisation in industrial enterprises. Exploitation of industrial robots influences all branches of production from automotive to every day life products. The robotic appliances are still predominantly controlled by various PID controllers and their modifications, usually implemented as a set of local controllers of drives. This concept does not consider dynamic linkages through the robot construction caused by inertia and gravity effects or friction. This drawback is overcome by surplus of input energy.

Advanced model based control strategies, such as model predictive control (MPC) [1], [2], [3], [4], $\mathcal{H}_2$ or $\mathcal{H}_\infty$ [5], [6], offers attractive properties. However, the strategies are not considerably robust against surrounding disturbances or they are able to reduce only energy consumption without increasing the production quality, specifically increasing of the motion accuracy.

The robustness property of MPC respecting imprecisely known model parameters and stochastic disturbances is not fully solved for real-time control of industrial machines [7].

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Problems of an insufficient disturbance attenuation of $\mathcal{H}_2$ or excessive conservatism of $\mathcal{H}_\infty$ can be solved by a specific control design based on anisotropic control theory which measures statistic disturbance uncertainty in terms of a relative entropy rate. The disturbance attenuation capabilities of the feedback control for specific controlled system are quantified by a specific anisotropic norm [8], firstly studied by I. G. Vladimirov [9], which represents a stochastic counterpart of the $\mathcal{H}_\infty$ norm. Minimisation of such norm leads to the control, which is less conservative and efficient in disturbance attenuation than $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms respectively.

The paper aims at compact introduction of specific interconnection ways of MPC and anisotropic control ($\mathcal{H}_a$ control). The reason for exploration of such interconnection is to improve robustness of MPC but without increase of computational complexity and to avoid inadequate behaviour $\mathcal{H}_a$ control in an abrupt transitions of reference signals, which is caused by one-ahead character of $\mathcal{H}_a$ control. Due to the flexibility of MPC design, the $\mathcal{H}_a$ control may be incorporated as a robust reference control or as a specific disturbance attenuation component in the MPC design. The basic properties of MPC and $\mathcal{H}_a$ control in motion control were described in [10]. The control proposed here will be demonstrated with a dynamic model of the parallel kinematic machine, robot-manipulator, depicted in Fig. 1.

The paper is organized as follows. In section II, there is a formulation of dual goal problem that consists in the effort to meet reference signals simultaneously with the solution of the disturbance attenuation. Sections III, IV and V introduce anisotropic control and anisotropic estimation. Section VI deals with MPC design concept and its modifications using reference anisotropic control. Section VII demonstrates proposed control ways by simulation examples. Finally, section VIII summarizes the proposed ways and their practical use in motion control.

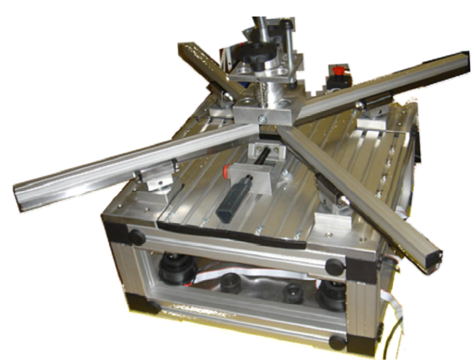

Fig. 1. Over-actuated planar parallel kinematic machine [11].

## II. DUAL GOAL PROBLEM

Let us consider the control task formulated as a dual goal problem. Specifically, the meeting desired reference trajectories and the disturbance attenuation in the state, used in control design, should be ensured simultaneously. This problem can be solved starting with the tracking control along a given reference trajectory, and then using the disturbance attenuation via a specific state estimation.

In this paper, the model is considered as general discrete linear-like state-space model (simplifying notation: $A(x_k^*) \to A_k$, $B(x_k^*) \to B_k$, considering ideal deterministic state $x_k^*$)

$$\begin{cases} x_{k+1}^* = A_k\, x_k^* + B_k\, u_{r,\,k} \\ z_k^* = C_z\, x_k^* + D_{zr}\; r_k + D_{zu}\, u_{r,\,k} \\ y_k^* = C\, x_k^* \end{cases} \tag{1}$$

where $A_k$, $B_k$, $C$, $y_k^*$, $r_k$ and $u_{r,\,k}$ are state, input and output matrices, and vectors of system outputs, reference inputs and control actions (system control inputs) respectively. Note that the matrices are obtained, in case of robotic systems, e.g. by specific decompositions of nonlinear dynamic models based on mathematical-physical analysis [12], [13]. Finally, $C_z$, $D_{zr}$ and $D_{zu}$ and $z_k^*$ are matrices of controlled output, reference and input and vector of controlled outputs respectively. The vector of control actions $u_{r,\,k}$ corresponds to the required vector $z_k^*$.

In the real environment, a corrective control $u_{a,\,k}$ should be considered to minimise the effects of the disturbances to the output deviation. Then, the system with a real stochastic state $x_k$ can be generally defined as follows

$$\begin{cases} x_{k+1} = A_k\; x_k + B_w\;\; w_k + B_k\;\; u_k \\ z_k \;= C_z\;\; x_k + D_{zw}\, w_k + D_{zu} u_k \\ \qquad + D_{zr}\; r_k \\ y_k \;= \; C\;\; x_k + D_{yw} w_k,\;\; u_k = f(u_{r,\,k},\, u_{a,\,k}) \end{cases} \tag{2}$$

## III. BASIC ANISOTROPIC CONTROL LAW

A control task of the robot motion can be specified such that a given robot should perform the desired user motion trajectories represented by reference signals. The tracking of the reference signals is provided by a controller that takes into account feedback from the system, reference signals and the available mathematical model in the real (stochastic) environment. A suitable controller set can generally be expressed as follows

$$u_{\mathcal{H}_a,\,k} = f(u_{r,\,k},\, u_{a,\,k}) = K_{x\mathcal{H}_a} x_k + K_{r\mathcal{H}_a} r_k \tag{3}$$

To describe the whole closed-loop system, let us consider the following matrix transfer function

$$T_{zw}(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} \tag{4}$$

that will be used in further explanation. It represents the closed-loop system from the external disturbance input $W$ to the controlled output $Z$. Involved matrices in (4) are defined by the following way

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{c|c} \widetilde{A} + \widetilde{B}K & \widetilde{B}_x \\ \hline \widetilde{C}_z + D_{zu}K & 0 \end{array}\right] \tag{5}$$

Individual submatrices arise from the generalized state-space model description (2) with parameters in the context of the motion control. They are defined as follows

$$\widetilde{A} := \begin{bmatrix} A & 0 \\ 0 & I_{m_r} \end{bmatrix}, \quad \widetilde{B}_w := \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \quad \widetilde{B} := \begin{bmatrix} B \\ 0 \end{bmatrix},$$
$$\widetilde{C}_z := \begin{bmatrix} C_z & D_{zr} \end{bmatrix}, \quad \widetilde{C} := \begin{bmatrix} C & 0 \end{bmatrix} \tag{6}$$

where means $A_k \to A$ and $B_k \to B$ and $K := [\,K_x\; K_r\,]$ represents the gain with respect to the control law (3) [10].

## IV. ANISOTROPIC CONTROL (DUAL PROBLEM)

At first, let us define deviation system:

$$\begin{cases} x_{k+1} - x_{k+1}^* = \delta x_{k+1} = A\,\delta x_k + B_w\;\; w_k + B u_{a,k} \\ \quad z_k - z_k^* \;\; = \;\; \delta z_k \;\; = C_z \delta x_k + D_{zw} w_k + D_{zu} u_{a,k} \\ \quad y_k - y_k^* \;\; = \;\; \delta y_k \;\; = C\,\delta x_k + D_{yw} w_k \\ \qquad\qquad\qquad\qquad e_k \;\; = C_e \delta x_k + D_{ew} w_k + D_{eu} u_{a,k} \end{cases} \tag{7}$$

Thus, a linear discrete time invariant system (LDTI), valid for specific time interval, is considered: $A(x_k) \simeq A(x_k^*) \to A_k \to A$ and $B(x_k) \simeq B(x_k^*) \to B_k \to B$. Here, $k$ denotes the discrete time instances, $k = 0, 1, \ldots, N$; $x_k$ denotes the $\mathbb{R}^{n_x}$-valued state; $w_k$ denotes the $\mathbb{R}^{m_w}$-valued input disturbances (more on this later); $u_{a,k}$ denotes the $\mathbb{R}^{m_u}$-valued control input; $z_k$ denotes the $\mathbb{R}^{p_z}$-valued controlled output; $e_k$ denotes the $\mathbb{R}^{p_e}$-valued output to be estimated; $y_k$ denotes the $\mathbb{R}^{p_y}$-valued measurements. All the matrices are known and have appropriate dimensions. In addition, we suppose that the input disturbances $w_k$ are random directionally generic vectors, i.e. $\mathbf{P}(w_k = 0) = 0\,\forall\, k$; moreover, the finite fragment $w_0, w_1, \ldots, w_N$ of the sequence $\{w_k\}_{k>0}$ forms the extended vector $W_{0:N} = (w_0^{\mathrm{T}}, \ldots, w_N^{\mathrm{T}})^{\mathrm{T}}$ which is supposed to be square integrable vector distributed absolutely continuously with respect to $l$-dimensional Lebesgue measure $\mathrm{mes}_l$ where $l = m_w(N+1)$, i.e. $W_{0:N} \in \mathbb{L}_2^l$. The simple, yet pretty fair way one can treat the vector $W_{0:N}$ is to assume that it is a Gaussian distributed random vector with zero mean and nonsingular covariance matrix. Finally, we assume for this vector that the inequality $\mathbf{A}(W_{0:N}) \le a$ is fulfilled with some number $a > 0$.

The first (control) problem is to find the unknown matrices $A_c$, $B_c$, $C_c$, $D_c$ of the control law

$$\begin{cases} \xi_{k+1} = A_c\, \xi_k\, + B_c\, \delta y_k, \quad \xi_0 = 0 \\ u_{a,\,k} = C_c\, \xi_k\, + D_c\, \delta y_k \end{cases} \tag{8}$$

in order to minimise the upper bound $\gamma_c$ of anisotropic norm of the input-to-output matrix of the corresponding closed loop system

$$\begin{cases} \delta x_{k+1} = A_{xx} \delta x_k\, + A_{x\xi} \xi_k\, + B_x w_k, \quad \delta x_0 = 0 \\ \xi_{k+1} \;= A_{\xi x} \delta x_k\, + A_{\xi\xi} \xi_k\, + B_\xi w_k, \qquad \xi_0 = 0 \\ \delta z_k \;\;= C_{zx} \delta x_k\, + C_{z\xi} \xi_k\, + D_z w_k \end{cases} \tag{9}$$

i.e. to ensure $\|F_{cl}^c\|_a \le \gamma_c \to \min$. Here, $A_{xx} = A + BD_cC$, $A_{x\xi} = BC_c$, $A_{\xi x} = B_cC$, $A_{\xi\xi} = A_c$, $B_x = B_w + BD_cD_{yw}$, $B_\xi = B_cD_{yw}$, $C_{zx} = C_z + D_{zu}D_cC$, $C_{z\xi} = D_{zu}C_c$, $D_z = D_{zw} + D_{zu}D_cD_{yw}$.

The details of the dynamical output feedback controller design can be found in [14, Theorem 2], the similar one for static output controller is described in [15]. Although the design has been done for time invariant systems on infinite time horizon, the same can be obtained for the case under the study. It leads to the following convex optimisation problem:

$$\gamma_c^2 \to \min_{\eta,\gamma_c^2,\Phi,X,Y,\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}} \quad \text{subject to} \tag{10}$$

$$X \succ 0, \quad Y \succ 0, \quad \eta > \gamma_c^2, \qquad \begin{bmatrix} X & * \\ I_{n_x} & Y \end{bmatrix} \succ 0,$$
$$(\det \Phi)^{1/m_w} > e^{2a/l}(\eta - \gamma_c^2),$$

$$\begin{bmatrix} \eta I_{m_w} - \Phi & * & * & * \\ B_w + B\mathbf{D}D_{yw} & Y & * & * \\ XB_1 + \mathbf{B}D_{yw} & I_{n_x} & X & * \\ D_1 + D_{zu}\mathbf{D}D_{yw} & 0 & 0 & I_{p_z} \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} Y & * & * & * & * & * \\ I_{n_x} & X & * & * & * & * \\ 0 & 0 & \eta I_{m_w} & * & * & * \\ AY+B\mathbf{C} & A+B\mathbf{D}C & B_w+B\mathbf{D}D_{yw} & Y & * & * \\ \mathbf{A} & XA+\mathbf{B}C & XB_1+\mathbf{B}D_{yw} & I_{n_x} & X & * \\ C_zY+D_{zu}\mathbf{C} & C_z+D_{zu}\mathbf{D}D_{yw} & D_{zw}+D_{zu}\mathbf{D}D_{yw} & 0 & 0 & I_{p_z} \end{bmatrix} \succ 0$$

where unknown matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ contain the controller matrices.

If the solution of the problem (10) is feasible, then the controller is to be set according to the expression

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} U^{-1} & 0 \\ 0 & I_{p_y} \end{bmatrix} \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} V^{-\mathrm{T}} & 0 \\ 0 & I_{p_y} \end{bmatrix} \tag{11}$$

where $\Pi_4 = \mathbf{D}$, $\Pi_3 = \mathbf{C} - D_cC_3Y$, $\Pi_2 = \mathbf{B} - XBD_c$, $\Pi_1 = \mathbf{A} - UB_cCY - XBC_cV^{\mathrm{T}} - X(A + BD_cC)Y$. Nonsingular $\mathbb{R}^{n_x \times n_x}$-valued matrices $U$ and $V$ can be chosen arbitrarily but the equality $VU^{\mathrm{T}} = I_{n_x} - YX$ has to be fulfilled.

Once the controller has been found, the estimation problem can be stated as follows.

## V. ANISOTROPIC ESTIMATION (DUAL PROBLEM)

Given the dynamic part of the system (9) and additional output $e_k$ from the system (7)

$$\begin{cases} \delta x_{k+1} = A_{xx}\delta x_k + A_{x\xi}\xi_k + B_x w_k, & \delta x_k = 0 \\ \xi_{k+1} = A_{\xi x}\delta x_k + A_{\xi\xi}\xi_k + B_\xi w_k, & \xi_k = 0 \\ e_k = C_{ex}\delta x_k + C_{e\xi}\xi_k + D_e w_k \end{cases} \tag{12}$$

where $C_{ex} = C_e + D_{eu}D_cC$, $C_{e\xi} = D_{eu}C_c$, $D_e = D_{ew} + D_{eu}D_cD_{yw}$, the estimation problem is to find the matrices $H$, $G$ of the estimator

$$\begin{cases} \widehat{\delta x}_{k+1} = A_{xx}\widehat{\delta x}_k + A_{x\xi}\xi_k + H(y_k - C\widehat{\delta x}_k), \; \widehat{\delta x}_0 = 0 \\ \widehat{e}_k = C_{ex}\widehat{\delta x}_k + C_{e\xi}\xi_k + G(y_k - C\widehat{\delta x}_k) \end{cases} \tag{13}$$

in order to minimise the upper bound $\gamma_e$ of anisotropic norm of the input-to-output matrix of the corresponding error system

$$\begin{cases} \widetilde{\delta x}_{k+1} = \mathcal{A}\widetilde{\delta x}_k + \mathcal{B}w_k, & \widetilde{\delta x}_0 = 0 \\ \widetilde{e}_k = \mathcal{C}\widetilde{\delta x}_k + \mathcal{D}w_k \end{cases} \tag{14}$$

i.e. to ensure $|||F_{cl}^e|||_a \le \gamma_e \to \min$. Here, $\widetilde{\delta x}_k = \delta x_k - \widehat{\delta x}_k$ is the state estimation error, $\widetilde{e}_k = e_k - \widehat{e}_k$ is output estimation error, and matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are defined as follows: $\mathcal{A} = A_{xx} - HC_3$, $\mathcal{B} = B_x - HD_{yw}$, $\mathcal{C} = C_{ex} - GC$ and $\mathcal{D} = D_e - GD_{yw}$.

A convex optimization approach was considered to solve the filtration problem, e.g. in [16], [17]. The anisotropy based bounded real lemma guaranties that if exist positive defined matrices $R$, $S$ and $L$ and scalar parameter $q \in \left[0, \|F_{cl}^e\|_\infty^{-2}\right)$ satisfying

$$R \succ \mathcal{A}^{\mathrm{T}}R\mathcal{A} + q\mathcal{C}^{\mathrm{T}}\mathcal{C} + L^{\mathrm{T}}S^{-1}L \tag{15}$$

$$S = \left(I_{m_w} - q\mathcal{D}^{\mathrm{T}}\mathcal{D} - q\mathcal{B}^{\mathrm{T}}R\mathcal{B}\right)^{-1} \tag{16}$$

$$L = S\left(\mathcal{B}^{\mathrm{T}}R\mathcal{A} + q\mathcal{D}^{\mathrm{T}}\mathcal{C}\right) \tag{17}$$

$$\ln\det S^{-1} \ge 2a + l\ln(1 - q\gamma_e^2) \tag{18}$$

then anisotropic norm is bounded by $\gamma_e$. To reduce the filtration problem to convex optimisation problem, let us define new matrix variable $\Psi \succ 0$ and change variables as follows: $\zeta = q^{-1} \succ 0$, $\mathcal{R} = \zeta R$. It allows to avoid nonlinearity in (15) - (18) after applying Schur complement formula. The final inequalities receive the following form:

$$\begin{bmatrix} \mathcal{R} & * & * & * \\ 0 & \zeta I_{m_w} & * & * \\ \mathcal{R}A_{xx} - \mathcal{X}C & -\mathcal{R}B_x + \mathcal{X}D_1 & \mathcal{R} & * \\ \mathcal{C} & -\mathcal{D} & 0 & I_{p_e} \end{bmatrix} \succ 0, \tag{19}$$

$$\begin{bmatrix} \zeta I_{m_w} - \Psi & * & * \\ \mathcal{R}B_x - \mathcal{X}D_{yw} & \mathcal{R} & * \\ \mathcal{D} & 0 & I_{p_e} \end{bmatrix} \succ 0, \tag{20}$$

$$(\det\Psi)^{1/m_w} > e^{2a/l}(\zeta - \gamma_e^2), \tag{21}$$

where $\mathcal{X} = \mathcal{R}H$, matrices $\mathcal{R}$, $\Psi$ are positive definite and inequality $\zeta - \gamma_e^2 > 0$ holds true. The convex optimisation filtering problem takes the following form:

$$\gamma_e^2 \to \min_{\zeta,\gamma_e^2,\mathcal{R},\Psi,\mathcal{X},G,H} \tag{22}$$

under conditions (19) - (21). After solving the system of convex inequalities, the matrix $H$ in (13) is defined accordingly the inverse change of variable $H = \mathcal{R}^{-1}\mathcal{X}$.

## VI. MODIFICATION OF MPC DESIGN BY $\mathcal{H}_a$ CONTROL

This section introduces two ways how to take advantages of multi-step property of online MPC and robustness property of one-off $\mathcal{H}_a$ control designs. The first one considers usual MPC design with disturbance attenuation complement by $\mathcal{H}_a$ control and the second way takes $\mathcal{H}_a$ control law including reference directly into multi-step MPC design.

### A. MPC with Anisotropic Disturbance Attenuation

MPC design [1] represents multi-step control concept that can be performed online. Optimisation of control actions takes into account future reference signals and varying state-space matrices that can respect e.g. nonlinear dynamics of industrial robots [18]. The optimisation is performed within a finite time horizon ahead. In each discrete time instant, a specific quadratic function is minimised considering regularly updated equations of predictions. These equations express future outputs in relation to searched control actions.

Let us start with brief overview of MPC design suitable for its interconnection with $\mathcal{H}_a$ control that serves for disturbance attenuation only. MPC design shapes the control actions with respect to reference signals, i.e. it represents powerful feed-forward and complementary feed-back part. Specifically, the quadratic cost function is defined as follows

$$\begin{aligned} J &= \sum_{j=1}^{N} \left\{ ||Q_{yr}(\hat{y}_{k+j} - r_{k+j})||_2^2 + ||Q_u u_{r,\,k+j-1}||_2^2 \right\} \\ &= ||Q_{YR}(\hat{Y}_{k+1} - R_{k+1})||_2^2 + ||Q_U U_{r,\,k}||_2^2 \end{aligned} \tag{23}$$

where predictions $\hat{Y}_{k+1}$ with respect to unknown control actions $U_{r,\,k}$ are the following

$$\hat{Y}_{k+1} = [\,\hat{y}_{k+1}^T, \cdots, \hat{y}_{k+N}^T]^T = F\hat{x}_k + GU_{r,\,k} \tag{24}$$

$$F = \begin{bmatrix} CA \\ \vdots \\ CA^{N-1} \\ CA^N \end{bmatrix}, G = \begin{bmatrix} CB & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ CA^{N-2}B & \cdots & CB & 0 \\ CA^{N-1}B & \cdots & CAB & CB \end{bmatrix} \tag{25}$$

$$R_{k+1} = [\,r_{k+1}^T, \cdots, r_{k+N}^T]^T \tag{26}$$

$$U_{r,\,k} = [\,u_{r,\,k}^T, \cdots, u_{r,\,k+N-1}^T]^T \tag{27}$$

and $Q_{YR}$ and $Q_U$ represent square-roots of weights – control parameters, selected e.g. as diagonal matrices with identical scalar parameters relating to $Y$, $R$ and $U$ in the cost function (23), respectively.

Then, according to [10], a deterministic part of control actions for prediction horizon $N$ can usually be given as follows

$$\begin{aligned} u_{r,\,k} = M\,U_{r,\,k} = M\,\left(G^T Q_{YR}^T Q_{YR} G + Q_U^T Q_U\right)^{-1} \\ \times G^T Q_{YR}^T Q_{YR}\,(R_{k+1} - F\hat{x}_k) \end{aligned} \tag{28}$$

where matrix M is defined as follows

$$M = [I_{mu}, 0_{mu}, \cdots, 0_{mu}] \tag{29}$$

and it serves for selection of appropriate control actions in relation to the corresponding discrete time instant $k$.

Sequentially, the resultant control law is given as follows

$$u_k = u_{r,\,k} + u_{a,\,k} \tag{30}$$

where $u_{r,\,k}$ is given by (28) and $u_{a,\,k}$ by (8). This modification represents fixed interconnection of MPC and $\mathcal{H}_a$ control for reference tracking and attenuation problem, respectively.

### B. Modified MPC Design with Reference $\mathcal{H}_a$ Control

This subsection introduces a specific modification of MPC design as an extension of the cost function by one additional term including weighted difference between computed MPC control actions and robust reference actions from off-line $\mathcal{H}_a$ control design. Such a modification enables user tuning proximity of MPC actions towards $\mathcal{H}_a$ control actions following from the control law (3). This ideas is propagated through the cost function or equations of predictions respectively. Hence, the quadratic cost function is expressed in compliance with this tuning as follows

$$\begin{aligned} J = \sum_{j=1}^{N} &\left\{ ||Q_{yr}(\hat{y}_{k+j} - r_{k+j})||_2^2 \right. \\ &+ ||Q_{\Delta u}(u_{k+j-1} - u_{\mathcal{H}_a,\,k+j-1})||_2^2 \\ &\left. + ||Q_u u_{k+j-1}||_2^2 \right\} \\ = \;\; & ||Q_{YR}(\hat{Y}_{k+1} - R_{k+1})||_2^2 \\ &+ ||Q_{\Delta U}(U_k - K_{X\mathcal{H}_a}\hat{X}_k - K_{R\mathcal{H}_a} R_k)||_2^2 \\ &+ ||Q_U U_k||_2^2 \end{aligned} \tag{31}$$

where

$$\hat{X}_k = [\hat{x}_k^T, \cdots, \hat{x}_{k+N-1}^T]^T \tag{32}$$

$$R_k = [r_k^T, \cdots, r_{k+N-1}^T]^T \tag{33}$$

$$K_{X\mathcal{H}_a}\hat{X}_k = L\,\hat{x}_k + MU_k \tag{34}$$

$$L = \begin{bmatrix} K_{x\mathcal{H}_a} I \\ \vdots \\ K_{x\mathcal{H}_a} A^{N-2} \\ K_{x\mathcal{H}_a} A^{N-1} \end{bmatrix}, M = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ CA^{N-2}B & \cdots & CB & 0 \\ CA^{N-1}B & \cdots & CAB & CB \end{bmatrix} \tag{35}$$

This quadratic cost function can be written as a product of its square-roots:

$$J = \mathbb{J}^T \mathbb{J} \tag{36}$$

where square-root $\mathbb{J}$ of the cost function $J$ is as follows

$$\begin{aligned} \mathbb{J} &= \begin{bmatrix} Q_{YR} & 0 & 0 \\ 0 & Q_{\Delta U} & 0 \\ 0 & 0 & Q_U \end{bmatrix} \begin{bmatrix} \hat{Y}_{k+1} - R_{k+1} \\ U_k - K_{X\mathcal{H}_a}\hat{X}_k - K_{R\mathcal{H}_a} R_k \\ U_k \end{bmatrix} \\ &= \begin{bmatrix} Q_{YR} F\hat{x}_k + Q_{YR} G U_k - Q_{YR} R_{k+1} \\ Q_{\Delta U}(I - M)U_k - Q_{\Delta U}\,L\,\hat{x}_k - Q_{\Delta U}\,K_{R\mathcal{H}_a} R_k \\ Q_U U_k \end{bmatrix} \end{aligned} \tag{37}$$

Considering minimisation of the square-root $\mathbb{J}$ as a specific solution of least-squares problem [19] then, let us take into account the following algebraic equation:

$$\begin{bmatrix} Q_{YR}\,G \\ Q_{\Delta U}(I - M) \\ Q_U \end{bmatrix} U_k = \begin{bmatrix} Q_{YR}\,(R_{k+1} - F\hat{x}_k) \\ Q_{\Delta U}\,(K_{R\mathcal{H}_a} R_k + L\,\hat{x}_k) \\ 0 \end{bmatrix} \tag{38}$$

or

$$\begin{bmatrix} Q_{YR}\,G & Q_{YR}\,(R_{k+1} - F\hat{x}_k) \\ Q_{\Delta U}(I - M) & Q_{\Delta U}\,(K_{R\mathcal{H}_a} R_k + L\,\hat{x}_k) \\ Q_U & 0 \end{bmatrix} \begin{bmatrix} U_k \\ -I \end{bmatrix} = 0 \tag{39}$$

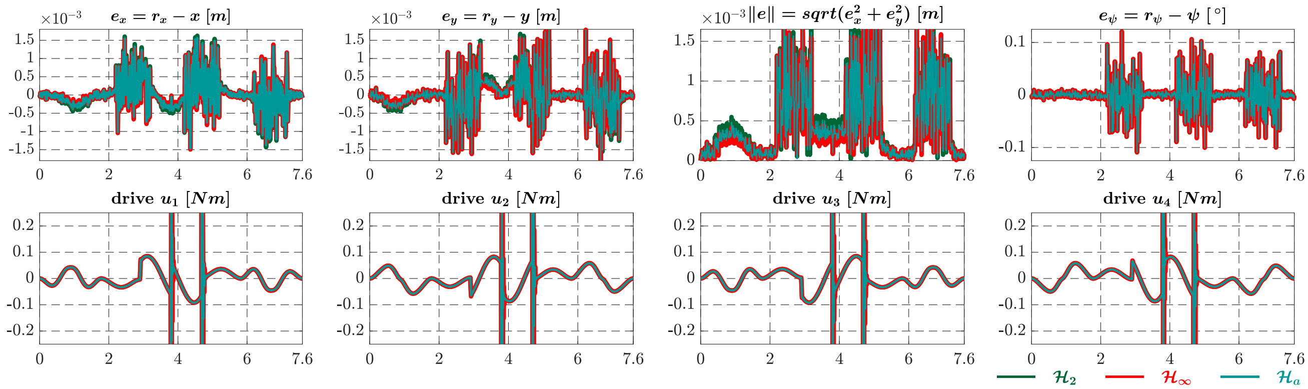


Fig. 2. Time histories [s] of errors of system outputs and realized control actions for $\mathcal{H}_2$, $\mathcal{H}_\infty$ and $\mathcal{H}_a$ controls.

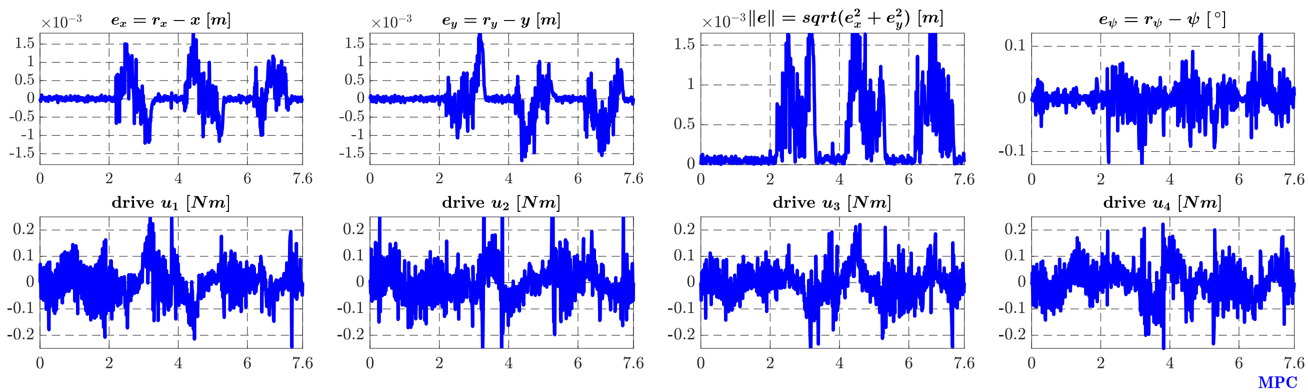


Fig. 3. Time histories [s] of errors of system outputs and control actions for conventional MPC.

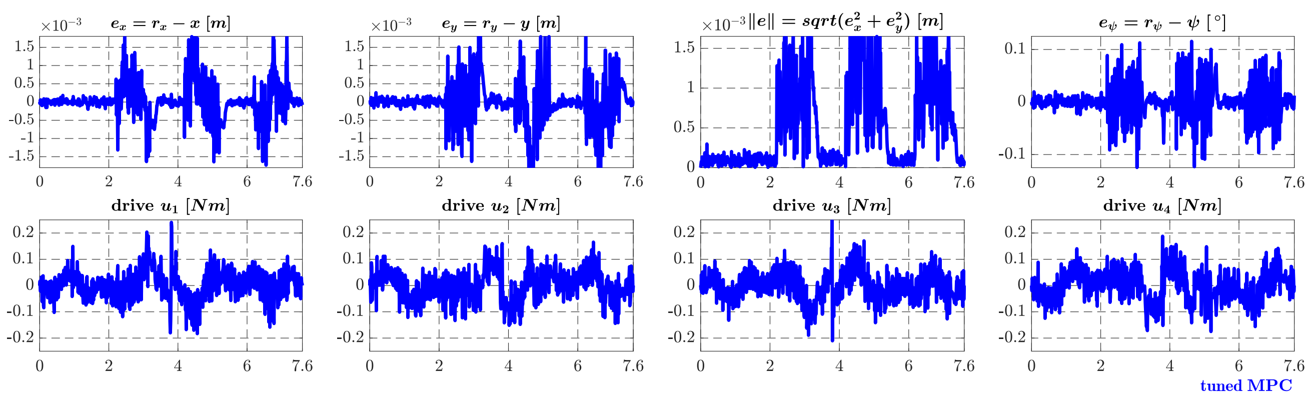


Fig. 4. Time histories [s] of errors of system outputs and control actions for tuned MPC [10].

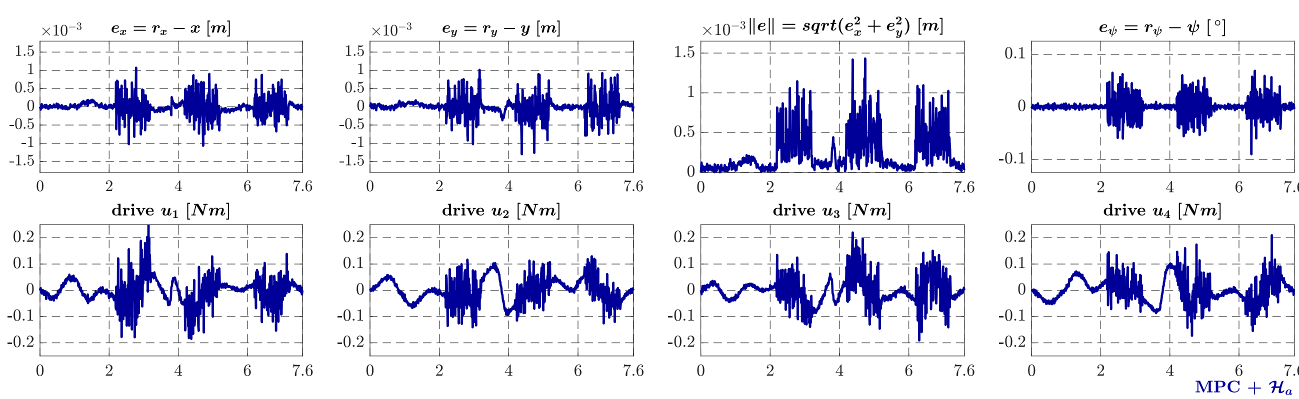


Fig. 5. Time histories [s] of errors of system outputs and control actions for proposed interconnection of MPC and $\mathcal{H}_a$ control.

The over-determined system (38) or (39) respectively can be written in condensed general form (40). It can be transformed to another form (41) by orthogonal-triangular decomposition [20] and solved for unknown $U_k$

$$\mathcal{A}U_k = b \tag{40}$$

$$Q^T\mathcal{A}U_k = Q^T b \;\text{ assuming that }\; \mathcal{A} = QR$$

$$R_1 U_k = c_1 \tag{41}$$

where $Q^T$ is an orthogonal matrix that transforms matrix $\mathcal{A}$ into upper triangle $R_1$ as it is indicated by the following equation diagram

$$\begin{bmatrix} \mathcal{A} \end{bmatrix} \begin{bmatrix} U_k \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \Rightarrow \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \begin{bmatrix} U_k \end{bmatrix} = \begin{bmatrix} c_1 \\ c_z \end{bmatrix} \tag{42}$$

Vector $c_z$ represents a loss vector, Euclidean norm $||c_z||$ of which equals to the square-root of the optimal cost function minimum, i.e. scalar value $\sqrt{J}$, where $J = c_z^T c_z$. Only the first elements corresponding to $u_k$ are used from computed vector $U_k$, i.e. $u_k = MU_k = M \times f(U_{r,k}, U_{\mathcal{H}_a,k})$, where matrix $M$ is defined as in previous subsection by (29).

## VII. SIMULATIONS

Simulations in Fig. 2 – Fig. 5 demonstrate behavior of individual control approaches $\mathcal{H}_2$, $\mathcal{H}_\infty$, $\mathcal{H}_a$, conventional MPC, tuned MPC and proposed interconnection: modified MPC with reference $\mathcal{H}_a$ control. All these control approaches were applied to the model of the robot shown in Fig. 1, considering reference motion trajectory depicted in Fig. 6.

The differences both in control errors and in control actions are noticeable for all shown control approaches. The proposed interconnection depicted in Fig. 5 reaches the smallest control actions at the smoothest trends of control actions. The abrupt increase in errors and in control action chattering is caused by artificial increase in disturbance signal, see Fig. 7.

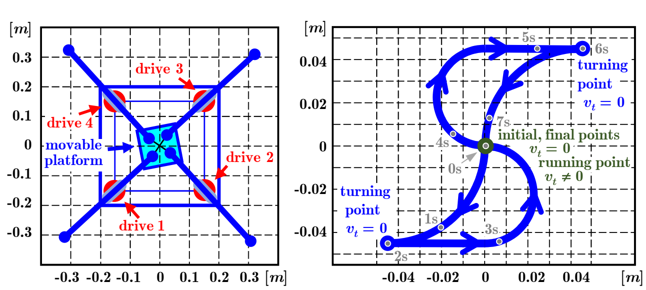


Fig. 6. Wireframe robot model, testing 'S'-shape trajectory.

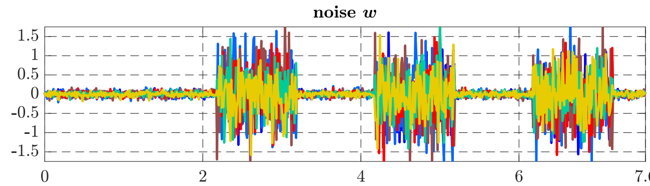


Fig. 7. Time histories [s] of the disturbance signals $w_k$.

## VIII. CONCLUSION

The paper introduces the novel promising interconnection of online flexible multi-step MPC design for reference signal tracking and off-line robust $\mathcal{H}_a$ control for disturbance attenuation. State-space estimation based on anisotropic theory is explained as well. The complexity of the design or computation is reasonable for real-time use and does not increase compared to the individual design approaches used.

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